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Closed embeddings of $\mathbb{C}^*$ in $\mathbb{C}^2$, part I

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Abstract

We consider closed curves $C \simeq \mathbb{C}^*$ in the affine plane $S \simeq \mathbb{C}^2$ that admit a good or very good asymptote. By this we mean a curve $L \simeq \mathbb{C}$ in S that in suitable coordinates for S is linear and tangent to C at infinity, and meets C once or not at all at finite distance. We classify these curves up to automorphism of S. Relying on the theory of open algebraic surfaces we first determine the possibilities for the singularities of C at infinity and then proceed to give explicit equations.

Keywords: Closed plane embeddings of once punctured lines; Affine plane; Rational curves; Open algebraic surfaces; Affine algebraic geometry

This paper is part of a project to describe all closed embeddings of $U \simeq \mathbb{C}^*$ into $S \simeq \mathbb{C}^2$. Using techniques from the theory of open algebraic surfaces developed in [KR2] we classify, giving equations, the embeddings that admit what we call a *good* or *very good asymptote*, that is a line tangent to U at infinity and meeting U once or not at all at finite distance. In a subsequent paper [KR3] it will be shown that in a sense most U have this property and the few exceptions will be listed.

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The U with a very good asymptote were classified already in [Kal]. Most of those with a good asymptote were found experimentally by the first author [C-N]. Our final complete list agrees with that in [BZ], where possible parametrizations for U are found by different methods. Their list (it covers all curves homeomorphic to $\mathbb{C}^*$) depends on some quite plausible, but as yet unproved, local estimates. We ask for the readers indulgence if we have to pull quite a few stops of open surface theory to arrive at an unconditional classification.

In Section 1 we use certain quite natural birational transformations to transform U into what we call a *special one place curve* V in $X \simeq \mathbb{C}^2$. See 1.6 for the definition.

Sections 2 to 5 are devoted to the classification of these curves. We do this by listing all possibilities for the characteristic pairs of their singularities. This classification task is simpler than the original one since a parameter in the description of U is eliminated that tends to make U very "negative" in the sense of making the square of a minimal resolution of the closure $\bar{U}$ very negative.

In Section 6 we construct equations for V from the characteristic pairs and then those for U by reversing the above birational transformation. In Section 7 we describe the equivalence classes up to automorphism of $\mathbb{C}^2$ of the embeddings defined by our equations. We summarize our results in Section 8, listing the equations for *special one place curves* in Theorem 8.1 and the equations for $\mathbb{C}^*$*-embeddings with a good or very good asymptote* in Theorem 8.2.

1. Asymptotes and special one place curves

1.1. Definition. Let U be a closed algebraic curve in $S = \mathbb{C}^2$ isomorphic to $\mathbb{C}^*$ and $\bar{U}$ its closure in $\mathbb{P}^2$. We put $L_\infty = \mathbb{P}^2 \setminus \mathbb{C}^2$. We denote by λ and $\tilde{\lambda}$ the branches at infinity of $\bar{U}$. An *asymptote* of U is a line L in $\mathbb{P}^2$, $L \neq L_\infty$, that is tangent to a branch at infinity of $\bar{U}$, say to λ. Let ξ denote the total intersection of L and U in $\mathbb{C}^2$. We say that an asymptote L is *good* (resp. *very good)* if $\xi = 1$ (resp. 0), i.e. if L meets U *precisely once normally* (resp. *not at all)* at finite distance.

1.2. We note that if L is an asymptote and if the branch $\tilde{\lambda}$ is also located at $L \cap L_\infty$, then it is tangent to L_∞ since otherwise $\bar{U}$ has a point with multiplicity equal to $\deg(\bar{U})$.

1.3. We find it convenient in studying $U \simeq \mathbb{C}^*$ with an asymptote L by first subjecting U to a birational transformation ψ which we now describe.

Stage 1. We blow up $\mathbb{P}^2$ at $L \cap L_\infty$. Let H be the exceptional curve. λ has separated from L_∞ and will be k-times tangent to H, $k \geqslant 1$.

Stage 2. We next blow up k-times along λ (until it separates from H), creating exceptional curves $E_1, \dots, E_k$.

$$L - - E_1 - - E_2 - - \cdots - - E_{k-1} - - E_k - - H - - L_\infty$$

is a chain with

$$L_\infty^2 = 0, \quad H^2 = -1 - k, \quad L^2 = E_k^2 = -1 \quad \text{and} \quad E_j^2 = -2 \quad \text{for } j \neq k.$$

Here we distinguish two possibilities regarding the center of λ on E_k.

1.3.1. λ meets E_{k-1}, or L if $k = 1$. We call this a *bad* case.

1.3.2. λ does not meets E_{k-1}. We call this a *good* case.

Stage 3. We successively contract $L, E_1, \dots, E_{k-1}$ to a point $q_0 \in E_k$. Note that intersection points of U with L will produce branches centered at q_0 for the transform of $\bar{U}$. $\mathbb{P}^2$ has been transformed into a Hirzebruch surface $\overline{F} \longrightarrow \mathbb{P}^1$ with negative section H and $\tilde{f} = L_\infty$, $f = E_k$ as fibers. Let $\bar{V}$ be the transform of $\bar{U}$ on $\overline{F}$. We put

$$X = \overline{F} \setminus H \cup \tilde{f}.$$

Then $X \simeq \mathbb{C}^2$, $f \cap X \simeq \mathbb{C}$ and $V = \bar{V} \cap X$ is a curve in X with one place at infinity. We denote by

$$\psi : S \longrightarrow X$$

the birational map induced by the above procedure. Our strategy is to describe all possibilities for V, giving explicit equations in suitable coordinates, and to reconstitute all possible $U \simeq \mathbb{C}^*$ by reversing ψ.

We note that $\bar{V} \cdot E_k = \xi + \lambda \cdot E_k$.

Stage 4. We have two possibilities.

1.3.3. Assume that $\bar{V} \cdot \tilde{f} = \tilde{\lambda} \cdot \tilde{f} = 1$. Then $\bar{V}$ is a 1-section in $\overline{F}$. So $\bar{V} \simeq \mathbb{P}^1$ and $V \simeq \mathbb{C}$. Moreover $\xi = 0$, so L is a very good asymptote, and $V \cdot f = 1$. We make elementary transformations in the fiber $\tilde{f}$ of $\overline{F}$ (with centers on $H \cdot \tilde{f}$ or $\bar{V} \cdot \tilde{f}$ as needed) until

(i) $\tilde{\lambda}$ separates from H and
(ii) $H^2 = -1$.

1.3.4. Assume that $\bar{V} \cdot \tilde{f} = \tilde{\lambda} \cdot \tilde{f} > 1$. We make elementary transformations in the fiber $\tilde{f}$ of $\overline{F}$ until

(i) $\tilde{\lambda}$ separates from H and
(ii) $\tilde{\lambda}$ is tangent to (the new fiber) $\tilde{f}$.

1.3.5. In either case we arrive at another Hirzebruch surface which we denote $\overline{M}'$. H is a section and $f, \tilde{f}$ are fibers of $\overline{M}'$. We put

$$H^2 = -n.$$

Note that n may differ from $k+1$ due to the elementary transformations performed. Also, f has a distinguished point $q_0 \notin H$, the image of L.

We denote by E_0' the transform of $\bar{U}$ in $\overline{M}'$. By construction, E_0' is disjoint from H.

$$X = \overline{F} \setminus H \cup \tilde{f} = \overline{M}' \setminus H \cup \tilde{f}$$

has not been touched in this process. We put $\tilde{q}_0 = E_0' \cap \tilde{f}$.

1.4. We assume now that U has a good or very good asymptote L and that we are in the situation of 1.3.5. Then E_0' is a rational curve in $\overline{M}'$ with the following properties:

(i) E'_0 is disjoint from H.

(ii) E'_0 meets $\tilde{f}$ in one point $\tilde{q}_0$ with one irreducible branch $\tilde{\lambda}$. Either $\tilde{\lambda} \cdot \tilde{f} = 1$ and $H^2 = -1$ or $\tilde{\lambda}$ is tangent to $\tilde{f}$.

(iii)

(iii vga) (*very good asymptote case*) E'_0 meets f in one irreducible branch λ, either at a point $q \neq q_0$ (good case), or at q_0 with λ tangent to f (bad case)

or

(iii ga) (*good asymptote case*) E'_0 has two irreducible branches, λ_0 and λ, on f, with λ_0 simple and transversal to f at q_0. The center q of λ is different from q_0 (good case) or it is q_0 and λ is tangent to F (bad case).

(iv) $E'_0 \setminus f \cup \tilde{f}$ is smooth.

1.5. Lemma. *We have $n \geqslant 1$.*

Proof. If $\tilde{\lambda} \cdot \tilde{f} = 1$ this is 1.4(ii). So suppose that $\tilde{\lambda} \cdot \tilde{f} \geqslant 2$ and $H^2 \geqslant 0$. Blow up over $f \cap H$ until $H'^2 = 0$ where H' is the proper transform of H. H' induces a $\mathbb{P}^1$-ruling with E'_0 in a fiber and the proper transform of $\tilde{f}$ is a 1-section of the ruling. Hence $\tilde{\lambda}$ is transversal to $\tilde{f}$; contradiction. □

1.6. Suppose conversely that a curve satisfying the conditions of 1.4 is given. We call such a curve a *special one place curve*. We choose an integer $k \geqslant 1$. In a good asymptote case we blow up k-times following the branch λ_0. In a very good asymptote case we instead blow up along a virtual simple branch normal to f at q_0. (This will involve a choice of parameters.) If from the blown up surface we delete f, H, $\tilde{f}$ and all exceptional curves except the last one, L say, we obtain a surface $S \simeq \mathbb{C}^2$ and the intersection of the transform of E'_0 with S is a curve $U \simeq \mathbb{C}^*$ with L as good or very good asymptote. For explicit equations see Section 6 below.

Before proceeding further we establish some notation and recall a number of results needed later.

1.7. Let $\overline{M}$ be a complete, non-singular surface. Since this is the only case of interest in this paper, we assume for simplicity that $\overline{M}$ is rational.

(i) We write $K_{\overline{M}}$ for a canonical divisor on $\overline{M}$. We write "$\sim$" for linear equivalence of integral divisors on $\overline{M}$ and "$\equiv$" for numerical equivalence of divisors, both over $\mathbb{Z}$ and over $\mathbb{Q}$.

(ii) A (b)-curve on $\overline{M}$ is a curve $L \simeq \mathbb{P}^1$ with $L^2 = b$.

(iii) Let

$$T = \sum_{i=1}^{n} m_i T_i$$

be a divisor on $\overline{M}$ with $T_1, \ldots, T_n$ distinct, irreducible curves. (Usually $m_i \in \mathbb{Z}$ and some times $m_i \in \mathbb{Q}$, this will be clear from the context.)

(iii.1) We call the T_j with $m_j \neq 0$ the *components* of T and denote by $\operatorname{Supp} T$ their union and by $\#T$ their number. We say T is *reduced* if $m_i = 1$ for each component T_i of T. If T is reduced we call $Q(T) = (T_i \cdot T_j)_{1 \leqslant i,j \leqslant n}$ the *intersection matrix* of T and put $d(T) = \det(-Q(T))$.

(iii.2) We call T a *normal crossing divisor* (*NC-divisor*) if it is reduced, each component of T is non-singular and at most two components meet at any point, and if so, transversally. The *dual graph of* T then is the graph with the T_i for vertices and T_i, T_j joined by an edge if and only if $T_i \cdot T_j \neq 0$. A component of T is called *branching in* T (*a tip of* T) if it meets at least three (at most one) other component of T. We say T is *NC-minimal* if each (-1)-component of T is branching. A *maximal twig* of T is a *maximal chain* (*connected divisor without branching component*) in T that is either a connected component of T or has one end that is a tip of T, with the other attached to a unique component of T that is branching.

(iii.3) We say T is a *tree of rational curves,* or more briefly a *rational tree,* if it is an *NC*-divisor with connected and simply connected support, i.e. if each component is isomorphic to $\mathbb{P}^1$ and the dual graph is a tree. A *rational chain* is a rational tree without branching component.

1.8. (i) Let T be a disjoint union of rational trees. We say that T is *contractible* if the intersection matrix $Q(T)$ is negative definite.

(ii) A rational chain

$$T_1 - - \cdots - - T_n$$

is called *admissible* if $T_i^2 \leqslant -2$, $i = 1, \dots, n$. An NC-minimal rational chain is contractible if and only if it is admissible. A non-contractible rational chain T can be modified (by a succession of blow-ups over T and contractions of (-1)-components in T) so that T_1, say, is a (0)-curve. T_1 then induces a $\mathbb{P}^1$-ruling of $\overline{M}$, i.e. is a fiber in a morphism $\overline{M} \longrightarrow \mathbb{P}^1$ with general fiber $\mathbb{P}^1$.

(iii) We recall that the support of any fiber g in a $\mathbb{P}^1$-ruling of $\overline{M}$ is a tree of rational curves that can be contracted to a (0)-curve. We denote by $mult_g(C)$ the multiplicity of a component C in the (scheme-theoretic) fiber. We note

(1) If C is a (-1)-component of g, then C meets one other component of g if $mult_g(C) = 1$ and if C meets two other components then $mult_g(C) \geqslant 2$.
(2) If C is the only (-1)-component of g, then $mult_g(C) \geqslant 2$.
(3) If $mult_g(C) = 1$, then there is a contraction process that makes C into a (0)-curve.

1.8.1. Lemma. *Let D be a divisor and T a disjoint union of rational trees such that $T \cap \operatorname{Supp} D = \emptyset$.*

(a) *If $K_{\overline{M}} + D + T$ is effective, then $K_{\overline{M}} + D$ is effective.*
(b) *If all components of T are (-2)-curves and $m(K_{\overline{M}} + D + T)$ is effective, $m \geqslant 1$, then $m(K_{\overline{M}} + D)$ is effective.*

Proof. (a) If C is a tip of T, then $(K_{\overline{M}} + D + T) \cdot C \leqslant -1$ and C is a fixed component of $K_{\overline{M}} + D + T$. We delete it and proceed by induction on $\#T$.

(b) We argue that T is in the fixed part of $m(K_{\overline{M}} + D) + (m - i)T$, $i = 0, \dots, m - 1$. □

1.9. We refer to [I] for the notion of the *Kodaira dimension* $\kappa(D)$ of a divisor D on $\overline{M}$. If T is an NC-divisor on $\overline{M}$, the *logarithmic Kodaira dimension* of $M = \overline{M} \setminus T$ is

$$\bar{\kappa}(M) = \kappa(K_{\overline{M}} + T).$$

It depends on M only. We recall the following:

(i) If $\bar{\kappa}(M) = -\infty$, then T is a disjoint union of rational trees [Ru2, 2.1, 2.2].
(ii) If T has a non-contractible maximal twig consisting of rational curves, then $\bar{\kappa}(M) = -\infty$.

(This follows from 1.8(ii).)

1.10. We refer to [Fu1, §6] for the notion of the *Zariski decomposition*

$$D = D^+ + D^-$$

of an effective divisor D. We recall that $\operatorname{Supp} D^-$ is contractible. If $D = K_{\overline{M}} + T$, where T is a union of rational trees, then any maximal twig of T is contained in $\operatorname{Supp} D^-$.

The following is a key numerical result for our investigation.

1.11. Lemma. *Let T be an NC-divisor on $\overline{M}$ with all components rational. Put $M = \overline{M} \setminus T$. Suppose that $\bar{\kappa}(M) \geqslant 0$ and that T has at least one maximal twig. Then*

$$(K_{\overline{M}} + T)^2 < 3\chi(M),$$

where "χ" denotes topological Euler characteristic.

Proof. By the Miyaoka inequality [Mi, 1.1] $(K_{\overline{M}} + T)^2 \leqslant 3\chi(M) + \frac{1}{4}N^2$ where $N = (K_{\overline{M}} + T)^-$ is the negative part in the Zariski decomposition of the divisor $K_{\overline{M}} + T$. Every maximal twig of T is contained in $\operatorname{Supp} N$, hence $N \neq 0$ and $N^2 < 0$. □

We return now to the situation of 1.6. The resolution of E_0', more precisely the NC-resolution of $E_0' \cup f \cup \tilde{f}$, determines sequences of Hamburger–Noether pairs. For convenience of a reader we briefly recall the notion. For more details we refer to [Ru1] and [KR1, Appendix].

1.12. Let M be a non-singular algebraic surface and let λ_1 be an analytically irreducible branch of a curve at $q_1 \in M$. Let a local coordinate x_1 at q_1 be given and let L_{x_1} be the branch at q_1 of the curve defined by it. We put

$$c_1 = (\lambda_1 \cdot L_{x_1})_{q_1}.$$

Here $(T \cdot Z)_p$ denotes the local intersection index of two curves T, Z at a point p. We then pick y_1 so that $\{x_1, y_1\}$ is a system of parameters at q_1 and

$$p_1 = (\lambda_1 \cdot L_{y_1})_{q_1}$$

is the multiplicity of λ_1 at q_1. This forces $c_1 \geqslant p_1$. If $c_1 = 1$, we do nothing. Otherwise we blow up successively over q_1 until the proper transform λ_2 of λ_1 meets the inverse image of the divisor $L_{x_1} + L_{y_1}$ not in a node. The exceptional curves form a chain called *the chain produced by the pair* $\binom{c_1}{p_1}$. Let C_1 be the last exceptional curve. We then say that C_1 is the *exceptional curve produced by the pair* $\binom{c_1}{p_1}$. Now λ_2 meets C_1 in a point q_2 and does not meet any other component of the inverse image of $L_{x_1} + L_{y_1}$. We choose for x_2 a local coordinate for C_1 at

q_2 and continue the process, noting that $c_2 = \lambda_2 \cdot C_1 = GCD(c_1, p_1)$. We continue this process until the proper transform of λ_1 meets the last exceptional curve transversally. This is the *NC-resolution of* $\lambda_1 \cup L_{x_1}$ and

$$\binom{c_1}{p_1}, \binom{c_2}{p_2}, \dots, \binom{c_h}{p_h},$$

are called the *HN-pairs of* $\lambda_1 \cup L_{x_1}$. We note that $c_{i+1} = GCD(c_i, p_i)$, $i = 1, \dots, h-1$, and $GCD(c_h, p_h) = 1$.

If $p_1 \nmid c_1$, the HN-pairs depend on λ_1 only. Otherwise they depend on x_1 as well, even if λ_1 is singular. For instance, the sequences

$$\left\{\binom{4}{2}, \binom{2}{2}, \binom{2}{1}\right\}, \quad \left\{\binom{6}{2}, \binom{2}{1}\right\}, \quad \left\{\binom{7}{2}\right\}$$

all describe the same singularity.

1.12.1. (i) In a blow-up process as above, a blow-up is called *subdivisional* w.r.t. to an NC-divisor T if its center is at the intersection of two components of T, and otherwise *sprouting*. In the reverse direction, the contraction of a non-branching (-1)-component of T is called *sprouting* if it is a tip of T and *subdivisional* otherwise.

(ii) We say a curve J is *touched by a blow-up* (*blow-down*) *process* if a point on J is blown up (a curve is contracted to a point in J).

(iii) If φ is the induced morphism and D is a reduced effective divisor in the target surface, we denote by $\varphi^{-1}(D)$ the reduced effective divisor supported by the set-theoretic inverse image of Supp D.

(iv) A blow-up process $\alpha = \alpha_k \circ \cdots \circ \alpha_1$ is called a *connected sequence of blow-ups* if, as above in the resolution of an irreducible branch λ, the center of α_i is on the exceptional curve produced by α_{i-1}, $i = 2, \dots, k$. The reverse process is called a *connected sequence of blow-downs*.

1.13. Let $\Pi : \overline{M} \to \overline{M}'$ be the resolution of $E_0' \cup f \cup \tilde{f}$. Let $T = \Pi^{-1}(f + H + \tilde{f})$. Let E_0 be the proper transform of E_0' in $\overline{M}$. Otherwise, if there is no danger of confusion, we will use the same notation for a curve and its proper transforms at various stages of the blow-up process. The $\mathbb{P}^1$-ruling of $\overline{M}'$ induces a ruling $\overline{M} \to \mathbb{P}^1$. Let $F = \Pi^{-1}(f)$ and $\tilde{F} = \Pi^{-1}(\tilde{f})$. Hence $T = F + H + \tilde{F}$.

Let $\binom{c_1}{p_1}, \dots, \binom{c_h}{p_h}$ (resp. $\binom{\tilde{c}_1}{\tilde{p}_1}, \dots, \binom{\tilde{c}_{\tilde{h}}}{\tilde{p}_{\tilde{h}}}$) be the sequence of HN-pairs of $\lambda \cup f$ (resp. $\tilde{\lambda} \cup \tilde{f}$). The last curve produced in the blow up process determined by the sequence $\binom{c_1}{p_1}, \dots, \binom{c_i}{p_i}$ (resp. $\binom{\tilde{c}_1}{\tilde{p}_1}, \dots, \binom{\tilde{c}_i}{\tilde{p}_i}$) will be called *the exceptional curve produced by the pair* $\binom{c_i}{p_i}$ (resp. $\binom{\tilde{c}_i}{\tilde{p}_i}$) and denoted by T_i (resp. $\tilde{T}_i$). We put $T_0 = f$, $\tilde{T}_0 = \tilde{f}$.

Let R_i be the chain produced by the pair $\binom{c_i}{p_i}$. It is attached to T_{i-1} and we write it as a chain

$$G_i' - - T_i - - G_i,$$

with G_i' meeting T_{i-1}. We note that $G_i' = \emptyset$ if and only if $p_i | c_i$ and that $G_i = \emptyset$ if and only if $p_i = c_i$. We use analogous notation on the tilde-side.

1.13.1. T is a tree of rational curves, E_0 is not a component of T and meets T normally in $2+\xi$ points. Hence E_0+T is an NC-divisor with rational components that is not a tree. We calculate

$$\chi(\overline{M} \setminus (T \cup E_0) = \xi.$$

Suppose that $c_1 > 1$ or $\xi = 1$. Then E_0+T has a maximal twig. By 1.9(i) and 1.11 we therefore have

$$(K_{\overline{M}} + T + E_0)^2 < 3\xi.$$

We define ε_0 by

$$(K_{\overline{M}} + T + E_0)^2 = 3\xi - 1 - \varepsilon_0.$$

Then $\varepsilon_0 \geqslant 0$. $(K_{\overline{M}} + T + E_0)^2$ is not changed by subdivisional blow-ups or contractions. From this it is easy to deduce that ε_0 is an invariant of $\overline{M} \setminus (T \cup E_0)$, but we will not use this.

We have $(K_{\overline{M}}+T+E_0)^2 = K_{\overline{M}} \cdot (K_{\overline{M}}+T+E_0) - 4 + 2E_0 \cdot T = K_{\overline{M}} \cdot (K_{\overline{M}}+E_0+T) + 2\xi$. Thus

$$K_{\overline{M}} \cdot (K_{\overline{M}} + E_0 + T) = \xi - 1 - \varepsilon_0.$$

1.13.2. We put $\gamma = -E_0^2$.

1.13.3. Lemma. *If $c_1 > 1$, then $h + \tilde{h} = n + \varepsilon_0 + \gamma - \xi + 1$. If $c_1 = 1$ (resp. $\tilde{c}_1 = 1$) the same formula holds if we put $h = 0$ (resp. $\tilde{h} = 0$).*

Proof. We have $K_{\overline{M'}} \cdot (K_{\overline{M'}} + f + H + \tilde{f}) = 2 + n$ and $K_{\overline{M}} \cdot (K_{\overline{M}} + T) = \xi - \varepsilon_0 - \gamma + 1$ by 1.13.1. The number of sprouting blow ups in Π (see 1.13) equals $h + \tilde{h}$. Hence

$$h + \tilde{h} = K_{\overline{M'}} \cdot (K_{\overline{M'}} + f + H + \tilde{f}) - K_{\overline{M}} \cdot (K_{\overline{M}} + T) = n + \varepsilon_0 + \gamma - \xi + 1. \qquad \square$$

1.13.4. We define $l \geqslant 0$ to be the smallest integer so that $p_{l+1} < c_{l+1}$.

1.14. Lemma. *We keep the notation of* 1.13. *We then have the following*:

- (i) $\tilde{c}_1 > \tilde{p}_1$ *unless* $c_1 = 1$, $\xi = 0$, $\tilde{c}_1 = 1$; $c_1 > p_1$ *in a bad case.*
- (ii) $c_{i+1} = GCD(c_i, p_i)$, $i = 1, 2, \ldots, h-1$, *and* $\tilde{c}_{i+1} = GCD(\tilde{c}_i, \tilde{p}_i)$, $i = 1, 2, \ldots, \tilde{h} - 1$.
- (iii) $GCD(c_h, p_h) = GCD(\tilde{c}_{\tilde{h}}, \tilde{p}_{\tilde{h}}) = 1$.
- (iv) *Let $\mu_1, \mu_2, \ldots$ (resp. $\tilde{\mu}_1, \tilde{\mu}_2, \ldots$) be the sequence of multiplicities of all singular points of E_0' infinitely near $q = \lambda \cap f$ (resp. $\tilde{q} = \tilde{\lambda} \cap \tilde{f}$). (We always count a point as infinitely near to itself.) Then*

$$\sum_{i \geqslant 1} \tilde{\mu}_i = \tilde{c}_1 + \tilde{p}_1 + \tilde{p}_2 + \cdots + \tilde{p}_{\tilde{h}} - 1,$$

$$\sum_{i \geqslant 1} \tilde{\mu}_i^2 = \tilde{c}_1 \tilde{p}_1 + \tilde{c}_2 \tilde{p}_2 + \cdots + \tilde{c}_{\tilde{h}} \tilde{p}_{\tilde{h}}.$$

In a bad case with $\xi = 1$

$$\sum_{i \geqslant 1} \mu_i = c_1 + p_1 + p_2 + \cdots + p_h \quad \textit{and}$$

$$\sum_{i \geqslant 1} \mu_i^2 = c_1 p_1 + c_2 p_2 + \cdots + c_h p_h + 2p_1 + 1.$$

In all other cases

$$\sum_{i \geqslant 1} \mu_i = c_1 + p_1 + p_2 + \cdots + p_h - 1 \quad \textit{and}$$

$$\sum_{i \geqslant 1} \mu_i^2 = c_1 p_1 + c_2 p_2 + \cdots + c_h p_h.$$

Proof. (i) follows from 1.4(iii). (ii)–(iv) are well-known properties of HN-pairs, see [Ru1], [KR1, Appendix]. Notice that $\mu_1 = p_1 + 1$ in a bad case with $\xi = 1$. $\square$

1.15. Lemma. *Let* $d = E_0 \cdot f_t$ *where* f_t *is the general fiber of the ruling* $f : \overline{M} \to \mathbb{P}^1$. *Recall that* $n = -H^2$. *Suppose that* $\xi = 1$ *and* $c_1 > 1$. *Then*:

(a) $d = c_1 + 1 = \tilde{c}_1$.

(b1) *In a bad case*

$$\gamma + nd = \sum_{i=1}^{h} p_i + \sum_{i=1}^{\tilde{h}} \tilde{p}_i.$$

(b2) *In a good case*

$$\gamma + 1 + nd = \sum_{i=1}^{h} p_i + \sum_{i=1}^{\tilde{h}} \tilde{p}_i.$$

(c1) *In a bad case*

$$\gamma + nd^2 = \sum_{i=1}^{h} p_i c_i + \sum_{i=1}^{\tilde{h}} \tilde{p}_i \tilde{c}_i + 2p_1 + 1.$$

(c2) *In a good case*

$$\gamma + nd^2 = \sum_{i=1}^{h} p_i c_i + \sum_{i=1}^{\tilde{h}} \tilde{p}_i \tilde{c}_i.$$

(d1) *In a bad case*

$$\gamma(d-1) = -p_1 - 1 + \sum_{i=2}^{h} p_i(d-c_i) + \sum_{i=2}^{\tilde{h}} \tilde{p}_i(d-\tilde{c}_i).$$

(d2) *In a good case*

$$\gamma(d-1) + d = p_1 + \sum_{i=2}^{h} p_i(d-c_i) + \sum_{i=2}^{\tilde{h}} \tilde{p}_i(d-\tilde{c}_i).$$

Proof. (a) is obvious. We have $E_0' = dH + ndf$ in $Pic\,\overline{M}'$. Hence $K_{\overline{M}'} \cdot E_0' = -2d - nd$ and $E_0'^2 = nd^2$. Also $K_{\overline{M}} \cdot E_0 = -2 + \gamma$ and $E_0^2 = -\gamma$. Since $K_{\overline{M}} \cdot E_0 - K_{\overline{M}'} \cdot E_0' = \sum_{i\geqslant 1} \mu_i + \sum_{i\geqslant 1} \tilde{\mu}_i$ and $E_0'^2 - E_0^2 = \sum_{i\geqslant 1} \mu_i^2 + \sum_{i\geqslant 1} \tilde{\mu}_i^2$, we obtain (b) and (c) from 1.14(iv) and 1.15(a). In order to get (d) multiply (b) by d and subtract (c). □

1.16. Lemma. *Suppose we have a bad case and $\xi = 1$.*

(i) *If $h \geqslant 2$ or $\tilde{h} \geqslant 2$, then*

$$\sum_{i=2}^{h} p_i + \sum_{i=2}^{\tilde{h}} \tilde{p}_i \leqslant 2\gamma + 1.$$

(ii) *If $h \geqslant 2$ or $\tilde{h} \geqslant 2$, then $\gamma \geqslant 0$.*

(iii) *If $h \geqslant 2$ and*

$$\sum_{i=2}^{h} p_i + \frac{1}{2}\sum_{i=2}^{\tilde{h}} \tilde{p}_i > 1 \quad \text{or} \quad c_1 - p_1 > c_2, \quad \text{then} \sum_{i=2}^{h} p_i + \sum_{i=2}^{\tilde{h}} \tilde{p}_i \leqslant 2\gamma.$$

The conclusion holds in particular if $h \geqslant 2$ and $\tilde{h} \geqslant 2$.

Proof. From 1.15(d1), since $p_1 + 1 \leqslant c_1 = d - 1$, we get

$$(\gamma+1)(d-1) \geqslant \sum_{i\geqslant 2} (c_1 - c_i)p_i + \sum_{i\geqslant 2} (\tilde{c}_1 - \tilde{c}_i)\tilde{p}_i + \sum_{i\geqslant 2} p_i.$$

It follows that $\gamma \geqslant -1$. Suppose that $\gamma = -1$. Then $h = 1 = \tilde{h}$ since $c_1 > c_i$ and $\tilde{c}_1 > \tilde{c}_i$ for $i > 1$ by 1.14(i), (ii). So we have (ii). Since $c_1 - c_i \geqslant \frac{c_1}{2}$ and $\tilde{c}_1 - \tilde{c}_i \geqslant \frac{\tilde{c}_1}{2}$ for $i \geqslant 2$, we get

$$\begin{aligned}(\gamma+1)c_1 &\geqslant \frac{c_1}{2}\sum_{i\geqslant 2} p_i + \frac{\tilde{c}_1}{2}\sum_{i\geqslant 2} \tilde{p}_i + \sum_{i\geqslant 2} p_i \\ &= \frac{c_1}{2}\left(\sum_{i\geqslant 2} p_i + \sum_{i\geqslant 2} \tilde{p}_i\right) + \sum_{i\geqslant 2} p_i + \frac{1}{2}\sum_{i\geqslant 2} \tilde{p}_i.\end{aligned}$$

When $h > 1$ or $\tilde{h} > 1$ we get

$$2\gamma + 2 > \sum_{i \geqslant 2} p_i + \sum_{i \geqslant 2} \tilde{p}_i,$$

that is (i). Suppose that $h \geqslant 2$. By 1.14(ii), $c_1 - p_1 - c_2 \geqslant 0$. Put $p'_2 = p_2 - 1$ and $p'_i = p_i$ for $i > 2$. Now 1.15(d1) gives

$$\begin{aligned} \gamma c_1 &= \sum_{i \geqslant 2} (c_1 - c_i) p'_i + \sum_{i \geqslant 2} (\tilde{c}_1 - \tilde{c}_i) \tilde{p}_i + \sum_{i \geqslant 2} p_i + c_1 - p_1 - c_2 - 1 \\ &\geqslant \frac{c_1}{2} \left(\sum_{i \geqslant 2} p'_i + \sum_{i \geqslant 2} \tilde{p}_i \right) + \frac{1}{2} \sum_{i \geqslant 2} \tilde{p}_i + \sum_{i \geqslant 2} p_i - 1. \end{aligned}$$

Under the assumptions of (iii)

$$2\gamma > \sum_{i \geqslant 2} p'_i + \sum_{i \geqslant 2} \tilde{p}_i,$$

which gives the result. $\square$

In a good case it may happen that $c_1 = p_1$.

1.17. Lemma. *Suppose that $\xi = 1$ and $c_1 > 1$ in a good case.*

(a) *Suppose that $c_1 = p_1$. Let l be as in* 1.13.4. *Then*

$$\sum_{i \geqslant l+2} p_i + \sum_{i \geqslant 2} \tilde{p}_i \leqslant 2\gamma \quad \textit{and} \quad \gamma \geqslant 0.$$

(b) *Suppose that $c_1 > p_1$ and that $h + \tilde{h} \geqslant 3$. Then*

$$\sum_{i \geqslant 2} p_i + \sum_{i \geqslant 2} \tilde{p}_i \leqslant 2\gamma + 1 \quad \textit{and} \quad \gamma \geqslant 0.$$

Proof. (a) From 1.15(d2) we obtain, since $c_{l+1} = c_1$,

$$\begin{aligned} \gamma(d-1) + d &= l c_1 (d - c_1) + p_1 + \sum_{i \geqslant l+1} p_i (d - c_i) + \sum_{i \geqslant 2} \tilde{p}_i (d - \tilde{c}_i) \\ &\geqslant l(d-1) + \sum_{i \geqslant l+1} p_i + \frac{c_1}{2} \sum_{i \geqslant l+2} p_i + \frac{d}{2} \sum_{i \geqslant 2} \tilde{p}_i, \\ &= l(d-1) + p_{l+1} + \frac{1}{2} \sum_{i \geqslant l+2} p_i + \frac{d}{2} \left(\sum_{i \geqslant l+2} p_i + \sum_{i \geqslant 2} \tilde{p}_i \right). \end{aligned}$$

Since $p_{l+1} \geqslant 1$ and $l \geqslant 1$ we get

$$\gamma(d-1) \geqslant \frac{d}{2}\left(\sum_{i \geqslant l+2} p_i + \sum_{i \geqslant 2} \tilde{p}_i\right).$$

This gives $\gamma \geqslant 0$ and

$$\sum_{i \geqslant l+2} p_i + \sum_{i \geqslant 2} \tilde{p}_i \leqslant 2\gamma.$$

(b) 1.15(d2) gives

$$(\gamma+1)(d-1)+1 \geqslant p_1 + \frac{d}{2}\left(\sum_{i \geqslant 2} p_i + \sum_{i \geqslant 2} \tilde{p}_i\right) + \frac{1}{2}\sum_{i \geqslant 2} p_i.$$

Since

$$\sum_{i \geqslant 2} p_i + \sum_{i \geqslant 2} \tilde{p}_i \geqslant 1$$

we get $\gamma \geqslant 0$ and

$$(\gamma+1)d > \frac{d}{2}\left(\sum_{i \geqslant 2} p_i + \sum_{i \geqslant 2} \tilde{p}_i\right),$$

which gives the statement. □

1.18. Lemma. *Assume that* $\xi = 0$ *and* $c_1 > 1$. *Then*

(a1) $$\gamma + nd = \sum_{i \geqslant 1} p_i + \sum_{i \geqslant 1} \tilde{p}_i,$$

(a2) $$\gamma + nd^2 = \sum_{i \geqslant 1} p_i c_i + \sum_{i \geqslant 1} \tilde{p}_i \tilde{c}_i,$$

(a3) $$\gamma(d-1) = \sum_{i \geqslant 2} p_i(d-c_i) + \sum_{i \geqslant 2} \tilde{p}_i(d-\tilde{c}_i).$$

(b) *Suppose that* $c_1 = p_1$. *Let* l *be as in* 1.13.4. *Then*

$$\sum_{i \geqslant l+2} p_i + \sum_{i \geqslant 2} \tilde{p}_i \leqslant 2\gamma \quad \textit{and} \quad \gamma \geqslant 0.$$

(c) *Suppose that* $c_1 > p_1$. *Then*

$$\sum_{i \geqslant 2} p_i + \sum_{i \geqslant 2} \tilde{p}_i \leqslant 2\gamma \quad \textit{and} \quad \gamma \geqslant 0.$$

Proof. (a1) and (a2) we prove in the same way as 1.15(b), (c). (a3) we get subtracting (a2) from (a1) multiplied by d. (b) and (c) we prove as 1.17. □

1.18.1. Lemma. *Suppose that $c_1 > 1$. Then $\gamma \geqslant 0$ unless we have a bad case where $h = \tilde{h} = 1$, $p_1 = c_1 - 1$, $\gamma = -1$, $\tilde{p}_1 = 1$, $n = 1$, $\varepsilon_0 = 2$ or a good case where $\xi = 1$, $h = \tilde{h} = 1$, $p_1 = 1$, $\gamma = -1$, $n = 1$, $d = \tilde{p}_1 + 1$, $\varepsilon_0 = 2$.*

Proof. By 1.17, 1.18, 1.16(ii), $\gamma \geqslant 0$ if $h + \tilde{h} \geqslant 3$. Suppose that $\gamma < 0$ in a bad case. Then $h = \tilde{h} = 1$ and 1.15(d1) gives $\gamma c_1 = -p_1 - 1$. Thus $\gamma = -1$ and $p_1 = c_1 - 1$. From 1.15(b1) $-1 + nd = p_1 + \tilde{p}_1 = d - 2 + \tilde{p}_1$. So $d(n-1) + 1 = \tilde{p}_1$ which implies $n = 1$ and $\tilde{p}_1 = 1$. In the same way we get the second case. □

2. The basic inequality

In this section we prove the inequality 2.7 which provides bounds for γ and ε_0 (defined in 1.13.1 and 1.13.2).

2.1. Throughout Section 2 we assume that

(i) $\varepsilon_0 \leqslant 1$ and $c_1 > 1$,
(ii) a bad case with $\xi = 1$,

or

(iii) a good case with $\xi = 1$, $p_1 = 1$, $\tilde{h} \geqslant 3$.

2.2. We use the notation of 1.13. Let $\pi : \overline{M} \to \overline{P}$ be the NC-minimalization of the divisor T with respect to E_0. By this we mean that we successively contract curves in T so that $Z + E_0$ is a *NC*-divisor, where

$$Z = \pi(T),$$

and any (-1)-curve in Z is branching in $T + E_0$. Let us note for later use that this definition remains meaningful if we replace E_0 by a collection of disjoint smooth branches of curves.

Let $T_1' = T_1$ in a bad case and let T_1' be the component of T which meets E_0 and H in the case 2.1(iii). Since $p_1 < c_1$ and $\tilde{p}_1 < \tilde{c}_1$, T_1' and $\tilde{T}_1$ are the first components (seen from H) in F and $\tilde{F}$ that are branching in $T + E_0$. The part of T between T_1' and $\tilde{T}_1$ is a chain which we call the *H-chain of T*. The contractions of π take place inside this chain. The process is not necessarily unique, but becomes unique if we agree that in case of a choice (the image of) H is to be contracted first, and then a component of (the image of) F before a component of (the image of) $\tilde{F}$.

2.2.1. By 1.13.1, since π does not involve sprouting contractions with respect to $T + E_0$,

$$K_{\bar{P}} \cdot (K_{\bar{P}} + E_0 + Z) = \xi - 1 - \varepsilon_0 = -\varepsilon_0.$$

2.3. Let $\pi' : \overline{P} \to \overline{P}'$ be a 2-reduction of Z with respect to E_0. Let $Z' = \pi'(Z)$, $\overline{E}_0 = \pi'(E_0)$. By definition π' is a minimal sequence of successive contractions in Z such that Z' has the following properties:

(a) Z' is an NC-divisor.
(b) Every (-1)-component Z'_i of Z' is a branching component of Z' or $Z'_i \cdot \overline{E}_0 \geqslant 2$.

Minimal here means that a curve J with $J \cdot E_0 \geqslant 2$ is not contracted. It follows that $\overline{E}_0$ is smooth. To obtain uniqueness, we agree to make necessary contractions in $\pi(R_h)$ and $\pi(\tilde{R}_{\tilde{h}})$ first and then adopt the same convention as in 2.2. The contractions in $\pi(\tilde{F})$ then all take place in $\pi(\tilde{R}_{\tilde{h}})$ (see 1.13). A sprouting contraction occurs precisely when $\tilde{p}_{\tilde{h}} = 1$. The situation is similar for $\pi(F)$ except that in a bad case with $h = 1$ no sprouting contraction occurs due to the presence of the branch λ_0.

The next is obvious.

2.3.1. Lemma. $K_{\bar{P}'} \cdot (K_{\bar{P}'} + \overline{E}_0) = K_{\bar{P}} \cdot (K_{\bar{P}} + E_0)$.

2.4. Proposition. *Suppose that $(\overline{E}_0 + 2K_{\bar{P}'}) \cdot (K_{\bar{P}'} + Z') \leqslant 0$ and that $\overline{P}'$ is not isomorphic to a Hirzebruch surface or $\mathbb{P}^2$. Then there exists a smooth rational curve A in $\overline{P}'$ such that $A^2 = -1$ and $A \cdot \overline{E}_0 \leqslant 1$.*

Proof. Suppose that such a curve does not exists. This will be in force until 2.5.

2.4.1. Lemma. *There is no curve B in $\overline{P}'$ such that $(\overline{E}_0 + 2K_{\bar{P}'}) \cdot B < 0$.*

Proof. Suppose B exists. Suppose first that $|B + K_{\bar{P}'} + Z'| \neq \emptyset$. Let m be the greatest integer such that $|B + m(K_{\bar{P}'} + Z')| \neq \emptyset$ (see [Fu2] for the existence of m). Write $B + m(K_{\bar{P}'} + Z') = \sum B_i$. Then $|B_i + K_{\bar{P}'} + Z'| = \emptyset$ for every i. Since $(\overline{E}_0 + 2K_{\bar{P}'}) \cdot (K_{\bar{P}'} + Z') \leqslant 0$, $\sum B_i \cdot (\overline{E}_0 + 2K_{\bar{P}'}) \leqslant B \cdot (\overline{E}_0 + 2K_{\bar{P}'}) < 0$. Hence there exists B_i such that $B_i \cdot (\overline{E}_0 + 2K_{\bar{P}'}) < 0$. Thus we may assume that $|B + K_{\bar{P}'} + Z'| = \emptyset$. It follows from the Riemann–Roch theorem that B is a smooth rational curve and that $B \cdot Z' \leqslant 1$, see [Ru2, 2.1, 2.2] for example. In particular $B \neq \overline{E}_0$. Hence $K_{\bar{P}'} \cdot B < 0$, i.e. $B^2 \geqslant -1$. Suppose that $B^2 \geqslant 0$. Let $\sum B_j$ be a singular member of $|B|$ such that $B_j^2 < 0$ for every j (this exists since $\overline{P}'$ is not a relatively minimal surface, see [KR1, 4.1]). There exists B_j such that $B_j \cdot (\overline{E}_0 + 2K_{\bar{P}'}) < 0$. It follows that $B_j \cdot K_{\bar{P}'} < 0$ and hence $B_j^2 = -1$. We may assume therefore that $B^2 = -1$. But then $B \cdot \overline{E}_0 < 2$ and we have a contradiction. □

We will use the following easily verifiable fact:

2.4.2. Let C be a (-1)-curve on a smooth complete surface $\overline{M}$. Let A, E be reduced curves on $\overline{M}$ such that C is not a component of $A + E$. Let A', E' be their images under the contraction $\overline{M} \to \overline{M}'$ of C and let r be a non-negative integer. Then

$$A' \cdot (E' + rK_{\overline{M}'}) = A \cdot (E + rK_{\overline{M}}) + A \cdot C(E \cdot C - r).$$

2.4.3. Lemma. *Let* $C = \pi(T_h)$, $\tilde{C} = \pi(\tilde{T}_{\tilde{h}})$.

(a) $K_{\bar{P}} \cdot Z_i \geqslant 0$ *for every* $Z_i \subset Z$, $Z_i \neq C, \tilde{C}$.
(b) $C^2 = \tilde{C}^2 = -1$. *Both* $C, \tilde{C}$ *are contracted by* π'.

Proof. (a) Suppose that $K_{\bar{P}} \cdot Z_i < 0$. Then $Z_i^2 \geqslant -1$. Clearly Z_i is not contracted by π' (if $Z_i^2 = -1$, then it is a branching component of Z and at the moment it is touched during the contraction process π' it becomes a 0-curve). Let $Z_i' = \pi'(Z_i)$. Then, by 2.4.2, $Z_i' \cdot (\overline{E}_0 + 2K_{\bar{P}'}) \leqslant Z_i \cdot (E_0 + 2K_{\bar{P}}) < 0$ and we reach a contradiction with 2.4.1.

(b) We have $C^2 \geqslant -1$. If $C^2 = -1$, then C is contracted by π' since $C \cdot E_0 = 1$. Otherwise $C^2 \geqslant 0$, $C \cdot (E_0 + 2K_{\bar{P}'}) < 0$ and we reach a contradiction as in (a). The argument for $\tilde{C}$ is the same. □

The next result follows from 2.4.2 (for $r = 1$).

2.4.4. Remark. For every $Z_i \subset Z$, $Z_i \cdot (E_0 + K_{\bar{P}}) = Z_i' \cdot (\overline{E}_0 + K_{\bar{P}'})$, where $Z_i' = \pi'(Z_i)$.

We need the following result on the Kodaira dimension on the complement of a single smooth rational curve from [KM].

2.4.5. Proposition. *Let* E *be a smooth rational curve on a smooth complete rational surface* $\overline{M}$. *Suppose that each* (-1)*-curve on* $\overline{M}$ *meets* E *at least twice. Then*

(a) *if* $\kappa(K_{\overline{M}} + E) \geqslant 0$, *then* $E^2 \leqslant -4$ (*Corollary* 2.5 *in* [KM]),
(b) $\kappa(K_{\overline{M}} + E) = 0$ *or* 1 *if and only if* $h^0(2K_{\overline{M}} + E) = 1$ (*Corollary* 3.2 *in* [KM]).

2.4.6. Assume that $\kappa(K_{\bar{P}'} + \overline{E}_0) = -\infty$.

Suppose that $K_{\bar{P}'} \cdot (K_{\bar{P}'} + \overline{E}_0) \leqslant 0$. Let L be a (-1)-curve in $\overline{P}'$. Since $L \cdot \overline{E}_0 \geqslant 2$, $|L + K_{\bar{P}'} + \overline{E}_0| \neq \emptyset$ [Ru2]. We argue as in the proof of 2.4.1. There exists $m \geqslant 1$ such that $|L + m(K_{\bar{P}'} + \overline{E}_0)| \neq \emptyset$ and $|L + (m+1)(K_{\bar{P}'} + \overline{E}_0)| = \emptyset$. If $R = L + m(K_{\bar{P}'} + \overline{E}_0) = \sum A_i$, A_i irreducible, then $|A_i + K_{\bar{P}'} + \overline{E}_0| = \emptyset$ and hence $A_i \simeq \mathbb{P}^1$ and $A_i \cdot \overline{E}_0 \leqslant 1$. We may assume that $A_i^2 < 0$ for every i. Since $R \cdot K_{\bar{P}'} < 0$, there exists A_j such that $A_j \cdot K_{\bar{P}'} < 0$. Hence $A_j^2 = -1$ and we get a contradiction.

Suppose that $K_{\bar{P}'} \cdot (K_{\bar{P}'} + \overline{E}_0) \geqslant 1$. Then $-K_{\bar{P}'} - \overline{E}_0 \geqslant 0$ by the Riemann–Roch theorem and, in fact, $-K_{\bar{P}'} - \overline{E}_0 > 0$ since $\overline{E}_0 \cdot (-K_{\bar{P}'} - \overline{E}_0) = 2$. Let again L be a (-1)-curve in $\overline{P}'$. Write $L = L + K_{\bar{P}'} + \overline{E}_0 + (-K_{\bar{P}'} - \overline{E}_0)$. Since $h^0(L) = 1$ and again $L + K_{\bar{P}'} + \overline{E}_0 \geqslant 0$, $L + K_{\bar{P}'} + \overline{E}_0 = 0$. Since $\overline{P}' \setminus Z'$ is affine, there exists a component $Z_i' = \pi'(Z_i)$ of Z' such that $Z_i' \neq L$ and $Z_i' \cdot L > 0$. Then $Z_i' \cdot (K_{\bar{P}'} + \overline{E}_0) = Z_i' \cdot (-L) < 0$. It follows that $Z_i' \cdot K_{\bar{P}'} < 0$. We obtain $Z_i' \cdot (\overline{E}_0 + 2K_{\bar{P}'}) = Z_i' \cdot (\overline{E}_0 + K_{\bar{P}'}) + Z_i' \cdot K_{\bar{P}'} < 0$. This contradicts Lemma 2.4.1.

2.4.7. Assume that $\kappa(K_{\bar{P}'} + \overline{E}_0) \geqslant 0$. Since every exceptional curve meets $\overline{E}_0$ at least twice, the pair $(\overline{P}', \overline{E}_0)$ is almost minimal, [M, 3.11]. Hence

$$K_{\bar{P}'} + \overline{E}_0 = (K_{\bar{P}'} + \overline{E}_0)^+ + \frac{2}{\gamma'}\overline{E}_0$$

is the Zariski decomposition of the divisor $K_{\bar{P}'} + \overline{E}_0$, where $\gamma' = -(\overline{E}_0)^2$ [Fu1, 6.20]. By 2.4.5, $\gamma' \geqslant 4$. Since $((K_{\bar{P}'} + \overline{E}_0)^+)^2 \geqslant 0$,

$$-2 + K_{\bar{P}'} \cdot (K_{\bar{P}'} + \overline{E}_0) = (K_{\bar{P}'} + \overline{E}_0)^2 = \big((K_{\bar{P}'} + \overline{E}_0)^+\big)^2 - \frac{4}{\gamma'} \geqslant -1.$$

2.4.7.1. Assume that $((K_{\bar{P}'} + \overline{E}_0)^+)^2 > 0$. Then $K_{\bar{P}'} \cdot (K_{\bar{P}'} + \overline{E}_0) \geqslant 2$ by 2.4.7. From 2.3.1 and 2.2.1 we get $K_{\bar{P}} \cdot Z \leqslant -2 - \varepsilon_0$. Since $K_{\bar{P}} \cdot Z = -2 + K_{\bar{P}} \cdot (Z - C - \tilde{C})$ by 2.4.3(b) we obtain

$$K_{\bar{P}} \cdot (Z - C - \tilde{C}) \leqslant -\varepsilon_0.$$

In view of 2.4.3(a) this implies that $\varepsilon_0 = 0$ and $K_{\bar{P}} \cdot (Z - C - \tilde{C}) = 0$, i.e. every component $Z_i \neq C, \tilde{C}$ of Z is a (-2)-curve. Since $\xi = 1$, $E_0 \cdot Z = 3$. Let Z_0 be the component of Z such that $Z_0 \cdot E_0 = 1$ and $Z_0 \neq C, \tilde{C}$. So $(K_{\bar{P}} + E_0) \cdot Z_i = 0$ for every $Z_i \subset Z$, $Z_i \neq C, \tilde{C}, Z_0$. The components of Z generate $Pic\,\overline{P}$ and we may therefore write

$$E_0 \sim x_0 Z_0 + \sum x_i Z_i + xC + \tilde{x}\tilde{C}.$$

Suppose that $x_0 = 0$. Then $-2 = E_0 \cdot (K_{\bar{P}} + E_0) = (K_{\bar{P}} + E_0) \cdot \sum x_i Z_i = 0$; contradiction. It follows that E_0 and the components Z_i', $i \neq 0$, generate $Pic\,\overline{P}' \otimes \mathbb{Q}$. By 2.4.4,

$$Z_i' \cdot (K_{\bar{P}'} + \overline{E}_0) = Z_i \cdot (K_{\bar{P}} + E_0) = 0 = Z_i' \cdot (K_{\bar{P}'} + \overline{E}_0)^+ + Z_i' \cdot \frac{4}{\gamma'} E_0'.$$

Thus $Z_i' \cdot (K_{\bar{P}'} + \overline{E}_0)^+ = 0$. Also $\overline{E}_0 \cdot (K_{\bar{P}'} + \overline{E}_0)^+ = 0$ by the properties of the Zariski decomposition. It follows that $(K_{\bar{P}'} + \overline{E}_0)^+ \equiv 0$; contradiction.

2.4.7.2. We may assume therefore that $((K_{\bar{P}'} + \overline{E}_0)^+)^2 = 0$. Then $\kappa(K_{\bar{P}'} + \overline{E}_0) < 2$ and $h^0(2K_{\bar{P}'} + \overline{E}_0') = 1$ by 2.4.5. This implies $K_{\bar{P}'} \cdot (K_{\bar{P}'} + E_0') < 2$ (by the Riemann–Roch theorem). Now 2.4.7 gives $K_{\bar{P}'} \cdot (K_{\bar{P}'} + E_0') = 1$ and $\gamma' = 4$, $K'^2_{\bar{P}'} = -1$. Thus $(\overline{E}_0 + 2K_{\bar{P}'}) \cdot K_{\bar{P}'} = 0$ and $(\overline{E}_0 + 2K_{\bar{P}'}) \cdot Z' \leqslant 0$ by the assumption in 2.4. It follows that $(\overline{E}_0 + 2K_{\bar{P}'}) \cdot Z_i' = 0$ for every $Z_i' \subset Z'$ since otherwise we have a contradiction with 2.4.1. We have $K_{\bar{P}'} \cdot (K_{\bar{P}'} + \overline{E}_0) = 1$. By 2.3.1 and 2.2.1, $K_{\bar{P}} \cdot Z + \varepsilon_0 = -1$. Hence $K_{\bar{P}} \cdot Z \leqslant -1$.

Suppose Z_0 is a $(\geqslant -1)$-component of Z other than $C, \tilde{C}$. Then $(E_0 + 2K_{\bar{P}}) \cdot Z_0 < 0$. Z_0 is not contracted by π' since it becomes a $(\geqslant 0)$-curve as soon as it is touched in the contraction process. Hence $(\overline{E}_0 + 2K_{\bar{P}'}) \cdot Z_0' < 0$ by 2.4.2 and we have a contradiction to 2.4.1. It now follows from $K_{\bar{P}} \cdot Z \leqslant -1$ that there exists at most one (< -2)-component of Z, Z_0 say, and that $Z_i^2 = -2$ for every component $Z_i \neq Z_0, C, \tilde{C}$. (Actually $Z_0^2 = -3$ if Z_0 exists.)

Let $Z_{1,1}$, $Z_{1,2}$ be the components of Z adjacent to C. If $Z_{1,1}^2 = -2 = Z_{1,2}$, put $J = Z_{1,1} + 2C + Z_{1,2}$. Then $J \cdot E_0 \leqslant 3$ and J induces a $\mathbb{P}^1$-ruling of $\overline{P}$. A general fiber J_0 of the ruling satisfies $J_0 \cdot E_0 \leqslant 3$ and hence $(E_0 + 2K_{\bar{P}}) \cdot J_0 \leqslant -1$. J_0 is not contracted by $\pi' : \overline{P} \to \overline{P}'$ so, in view of 2.4.2, we reach a contradiction with 2.4.1. We may assume therefore that, say, $Z_{1,1} = Z_0$, i.e. Z_0 exists and is adjacent to C. By symmetry, it is also adjacent to $\tilde{C}$, i.e. C and $\tilde{C}$ are connected by Z_0 in Z. Since $C^2 = \tilde{C}^2 = -1$ by 2.4.3, the map $\pi : \overline{M} \to \overline{P}$ does not contract a component of T adjacent to T_h or $\tilde{T}_{\tilde{h}}$. Therefore the shortest chain connecting C and $\tilde{C}$ in Z has at least two components and we have reached a contradiction. □

2.5. Lemma. *If a curve J on $\overline{M}$ is not a component of T and satisfies $J \cdot T = 1$, then J is a fiber of the $\mathbb{P}^1$-ruling of $\overline{M}$ given in* 1.13. *In particular $J \simeq \mathbb{P}^1$ and $J^2 = 0$.*

Proof. If J meets one of F, $\tilde{F}$, then it meets both and $J \cdot T \geqslant 2$. Hence J is a component of a fiber. Since F, $\tilde{F}$ are the only reducible fibres, J is a fiber. $\square$

2.5.1. Corollary. *There is no curve J' in $\overline{P}'$ such that J' is not a component of Z', $J' \simeq \mathbb{P}^1$, $J' \cdot Z' = 1$ and $J'^2 < 0$.*

Proof. Consider J, the proper transform of J' in $\overline{M}$ and apply 2.5. $\square$

2.5.2. Let t denote the number of sprouting contractions in π'. Then $t \leqslant 2$ (see 2.3, 1.12.1). We compute

$$\begin{aligned}
(\overline{E}_0 + 2K_{\bar{P}'}) \cdot (K_{\bar{P}'} + Z') &= (E_0 + 2K_{\bar{P}}) \cdot (K_{\bar{P}} + Z) + t \\
&= E_0 \cdot K_{\bar{P}} + E_0 \cdot Z + 2K_{\bar{P}} \cdot (K_{\bar{P}} + Z) + t \\
&= E_0 \cdot K_{\bar{P}} + 3 + 2(-\varepsilon_0 - E_0 \cdot K_{\bar{P}}) + t \quad \text{(see 2.2.1)} \\
&= 3 - 2\varepsilon_0 - E_0 \cdot K_{\bar{P}} + t \\
&= 5 - 2\varepsilon_0 - \gamma + t.
\end{aligned}$$

The following is well known.

2.5.3. Lemma. *Let P be a Hirzebruch surface with fiber Φ and non-positive section Δ, $-\Delta^2 = \nu \geqslant 0$. Suppose $J \in |a\Delta + b\Phi|$, $a \geqslant 0$, is irreducible and has arithmetic genus* 0. *If $\nu = 0$, then $a = 1$ or $b = 1$. If $\nu \geqslant 1$, then $a = 1$ and $b = 0$ or $b \geqslant \nu$, or $\nu = 1$ and $a = b = 2$.*

2.5.3.1. Lemma. *Let $H_1 - - H_2 - - H_3$ be a rational chain in P with $H_1^2 < 0$. Then $\nu > 0$, $H_1 = \Delta$, $H_2 \in |\Phi|$ and $H_3 \in |\Delta + \nu\Phi|$.*

2.5.4. Lemma. *Suppose that $(\overline{E}_0 + 2K_{\bar{P}'}) \cdot (K_{\bar{P}'} + Z') \leqslant 0$. Then $\overline{P}'$ is not isomorphic to a Hirzebruch surface or $\mathbb{P}^2$.*

Proof. This is clear in the case 2.1(iii) since then $\overline{P}'$ contains at least two negative curves, namely the tip of the chain $\tilde{R}_1$ and $\tilde{T}_{\tilde{h}-1}$. Assume that we have the case 2.1(ii). Suppose $\overline{P}'$ is isomorphic to a Hirzebruch surface or $\mathbb{P}^2$. Since $\overline{P}' \setminus Z' \simeq \mathbb{C}^* \times \mathbb{C}^1$, Z' has three components, at most one negative, if $\overline{P}'$ is a Hirzebruch surface, and has two components, none of them negative, if $\overline{P}'$ is $\mathbb{P}^2$. Put $\pi'' = \pi' \circ \pi$. We make the following observations.

(a) F and $\tilde{F}$ are not completely contracted by π''.

In fact, T_h is not contracted by π. If it is touched by π, $C^2 \geqslant 0$ and C is not contracted by π'. Otherwise there are two components of $\pi(F)$ adjacent to C and C is contracted. If one of these components is further contracted, the other acquires intersection at least 2 with E_0 and is not contracted. The argument for $\tilde{F}$ is the same.

(b) Suppose $\tilde{h} > 1$. Then $\tilde{T}_1$ is branching in $\tilde{F}$. In view of (a), it remains branching in Z and is not contracted by π'. Hence $\tilde{G}_1$ (see 1.13) is not touched by π'' and has one component only which becomes a tip of Z' with $\tilde{G}_1^2 \leqslant -2$. In view of (a), $\tilde{R}_2 + \cdots + \tilde{R}_{\tilde{h}}$ are contracted by π' and we have $E_0 \cdot \tilde{T}_1 = c_2 \geqslant 2$. Since $E_0 \cdot \tilde{G}_1 = 0$, 2.5.3 and 2.5.3.1 imply that $\tilde{G}_1^2 = -1$.

Hence $\tilde{h} = 1$. We argue in a similar way that $h = 1$.

Combining 1.13.3, 2.5.2 and our assumption we find

$$2n + \gamma + t + 5 \leqslant 2(h + \tilde{h}) = 4.$$

This is not possible since $t \geqslant 0$ and, by 1.5 and 1.18.1, $n > 0$ and $\gamma \geqslant -1$. □

The following now is our basic numerical result for embeddings of $\mathbb{C}^*$ with a good or very good asymptote.

2.6. Proposition. *We have* $(\overline{E}_0 + 2K_{\bar{P}'}) \cdot (K_{\bar{P}'} + Z') > 0$.

Proof. Suppose the opposite. By 2.5.4, $\overline{P}'$ is not isomorphic to a Hirzebruch surface or $\mathbb{P}^2$. Let $A \subset \overline{P}'$ be as in 2.4. If A is not contained in Z', then $A \cdot Z' \geqslant 2$ by 2.5.1. If $A \subset Z'$, then A is a branching component since Z' is 2-reduced w.r.t. $\overline{E}_0$. Again $A \cdot Z' \geqslant 2$. It follows that $|A + K_{\bar{P}'} + Z'| \neq \emptyset$. We argue as in the proof of 2.4.1.

Let m be the greatest integer such that $|A + m(K_{\bar{P}'} + Z')| \neq \emptyset$. Write

$$A + m(K_{\bar{P}'} + Z') \sim \sum A_i.$$

For any i, since $|A_i + K_{\bar{P}'} + Z'| = \emptyset$ by the choice of m, A_i is a smooth rational curve and $A_i \cdot Z' \leqslant 1$. We may assume that $A_i^2 < 0$. Since $0 > (\overline{E}_0 + 2K_{\bar{P}'}) \cdot (A + m(K_{\bar{P}'} + Z'))$ there exists A_{i_0} such that $A_{i_0} \cdot (\overline{E}_0 + 2K_{\bar{P}'}) < 0$. So $A_{i_0} \cdot \overline{E}_0 \leqslant 1$. Clearly $A_{i_0} \neq \overline{E}_0$ since $\overline{E}_0 \cdot Z' \geqslant 2$. Hence $A_{i_0} \cdot K_{\bar{P}'} < 0$. It follows that $A_{i_0}^2 = -1$. Therefore A_{i_0} is not a component of Z' by the definition of 2-reduction. We now have a contradiction with 2.5.1. □

2.7. Corollary. *Under the assumptions of* 2.1 *we have* $5 + t > 2\varepsilon_0 + \gamma$.

Proof. This follows from 2.6 and 2.5.2. □

3. The case $\epsilon_0 \leqslant 1$

Throughout this section we assume $\xi = 1$. We prove that $\varepsilon_0 \geqslant 2$ under the assumptions specified in 3.2. We return to the notation of 1.13.

3.1. Lemma. *Assume that we have a bad case and* $h + \tilde{h} \geqslant 3$. *Then* $\tilde{h} = 1$ *implies* $p_h = 1$ *and* $h = 1$ *implies* $\tilde{p}_{\tilde{h}} = 1$.

Proof. Suppose $\tilde{h} = 1$. From 1.15(d1) we obtain

$$\gamma c_1 = -p_1 - 1 + \sum_{i=2}^{h} (d - c_i) p_i = -p_1 - 1 + \sum_{i=2}^{h} (c_1 - c_i) p_i + \sum_{i=2}^{h-1} p_i + p_h.$$

Since c_h divides c_i for every i and divides p_i for $i = 1, \dots, h-1$, we obtain $c_h | p_h - 1$. Since $p_h < c_h$, $p_h = 1$.

Suppose $h = 1$. From 1.15(d1) we get $\gamma d - \gamma = -p_1 - 1 + \sum_{i \geqslant 2} \tilde{p}_i (d - \tilde{c}_i)$. From 1.15(b1) we get $\gamma + nd = p_1 + \sum_{i \geqslant 1} \tilde{p}_i$.

Adding these two equalities we obtain

$$\gamma d + nd = -1 + \sum_{i \geqslant 2} \tilde{p}_i (d - \tilde{c}_i) + \sum_{i=1}^{\tilde{h}-1} \tilde{p}_i + \tilde{p}_{\tilde{h}}.$$

Thus $\tilde{c}_{\tilde{h}} | \tilde{p}_{\tilde{h}} - 1$ and therefore $\tilde{p}_{\tilde{h}} = 1$. □

3.2. We assume

(i) $\varepsilon_0 \leqslant 1$,
(ii) $h + \tilde{h} \geqslant 3$,
(iii) a bad case with $\xi = 1$

or

(iv) a good case with $\xi = 1$, $n = 1$, $h = 1$, $p_1 = 1$, $\tilde{h} \geqslant 3$, $\tilde{c}_1 = \tilde{p}_1 + \tilde{c}_2$.

3.2.1. We consider $\overline{M}$ and the divisor T, see 1.13. The divisor $W = E_0 + T - T_h - \tilde{T}_h$ has three connected components W_0, W_1, W_2 which we label so that

$$W_0 \supset H, \quad W_1 \supset G_h, \quad W_2 \supset \tilde{G}_{\tilde{h}}.$$

Then $W_2 = \tilde{G}_{\tilde{h}}$. If $h = 1$ in (iii) then $W_1 = G_1 + E_0$. Otherwise $E_0 \subset W_0$. Let

$$\nu : \overline{M} \to \overline{X}$$

be the NC-minimalization of $W = W_0 + W_1 + W_2$. Let

$$V = \nu(W), \qquad V_i = \nu(W_i), \quad i = 1, 2, 3.$$

We have $K_{\overline{M}} \cdot (K_{\overline{M}} + W) = -\varepsilon_0 + 2$ by 1.13.1.

These assumptions and notations will be in force until 3.10.

3.2.2. Lemma. *Assume that $n = 1$. Then T_1 or $\tilde{T}_1$ is a branching component in W_0.*

Proof. This is clear if we have 3.2(iv). So assume 3.2(iii) and assume the contrary. Then we have one of the following:

(i) $h = 1$, $\tilde{h} = 2$ and $\tilde{p}_2 = 1$,
(ii) $h = 2$, $p_2 = 1$, $\tilde{h} = 1$,
(iii) $h = 2 = \tilde{h}$, $p_2 = \tilde{p}_2 = 1$.

Assume (i). From 1.15(d1) we obtain $(\gamma-1)(d-1)=-p_1-\tilde{c}_2$. By 1.16(ii) $\gamma \geqslant 0$. It follows that $\gamma=0$. By 1.13.3, $3=h+\tilde{h}=1+\varepsilon_0 \leqslant 2$; contradiction.

Assume (ii). From 1.15(d1) we obtain $(\gamma-1)(d-1)=-p_1-c_2$. Again $\gamma=0$ and we reach contradiction in the same way.

Assume (iii). From 1.15(d1) we obtain $(\gamma-2)(d-1)=-p_1-c_2-\tilde{c}_2+1$. This implies $\gamma \leqslant 1$. From 1.13.3, $\varepsilon_0+\gamma=3$. This gives $\varepsilon_0 \geqslant 2$; contradiction. □

3.2.3. Lemma. *Suppose that* $\gamma \leqslant 2$. *Then*:

(i) *If* $\varepsilon_0=0$ *or if* $\gamma \leqslant 1$, *then* $p_h>1$ *and* $\tilde{p}_{\tilde{h}}>1$.
(ii) *If* $\varepsilon_0=1$ *and* $p_h=1$, $\tilde{p}_{\tilde{h}}=1$, *then* $\gamma=2$ *and* $h=1$.

Proof. Let $T_i^\flat=W_i \cap T$.

(i) Suppose that $p_h=1$ or $\tilde{p}_{\tilde{h}}=1$. Then $T_1^\flat$ or $T_2^\flat$ consists of (-2)-curves, i.e. we have $\{1,2\}=\{i_0,i_1\}$ such that $K_{\overline{M}} \cdot T_{i_0}^\flat=0$. We have

$$\begin{aligned} K_{\overline{M}} \cdot \left(K_{\overline{M}}+T_0^\flat+T_1^\flat+T_2^\flat\right) &= K_{\overline{M}} \cdot (K_{\overline{M}}+W_0+W_1+W_2-E_0) \\ &= -\varepsilon_0+2-K_{\overline{M}} \cdot E_0 \geqslant 2 \end{aligned}$$

by 1.13.1 and the assumptions of (i). Hence

$$K_{\overline{M}} \cdot \left(K_{\overline{M}}+T_0^\flat+T_{i_1}^\flat\right) \geqslant 2.$$

From the Riemann–Roch theorem we get

$$h^0\left(-K_{\overline{M}}-T_0^\flat-T_{i_1}^\flat\right)+h^0\left(2K_{\overline{M}}+T_0^\flat+T_{i_1}^\flat\right)>0.$$

Since $\kappa(K_{\overline{M}}+T_0^\flat+T_{i_1}^\flat) \leqslant \kappa(K_{\overline{M}}+T)=-\infty$, $h^0(2K_{\overline{M}}+T_0^\flat+T_{i_1}^\flat)=0$ and hence

$$-K_{\overline{M}}-T_0^\flat-T_{i_1}^\flat \geqslant 0.$$

The Riemann–Roch theorem also gives

$$h^0(-K_{\overline{M}}-T-E_0)+h^0(2K_{\overline{M}}+T+E_0)>0.$$

If $-K_{\overline{M}}-T-E_0 \geqslant 0$, then $-K_{\overline{M}}-T-E_0=0$ since $\kappa(K_{\overline{M}}+T+E_0) \geqslant 0$. But $T_h \cdot (K_{\overline{M}}+T+E_0)=1$, for instance. So

$$2K_{\overline{M}}+T+E_0 \geqslant 0.$$

We have

$$\begin{aligned} 2K_{\overline{M}}+T+E_0 &= 2K_{\overline{M}}+W_0+W_1+W_2+T_h+\tilde{T}_{\tilde{h}} \\ &= 2K_{\overline{M}}+T_0^\flat+T_1^\flat+T_2^\flat+E_0+T_h+\tilde{T}_{\tilde{h}}. \end{aligned}$$

Thus

$$K_{\overline{M}} + T_{i_0}^\flat + T_h + \tilde{T}_{\tilde{h}} + E_0 = -K_{\overline{M}} - T_0^\flat - T_{i_1}^\flat + 2K_{\overline{M}} + T_0^\flat + T_1^\flat + T_2^\flat + E_0 + T_h + \tilde{T}_{\tilde{h}} \geqslant 0.$$

If $h \geqslant 2$, or if $W_{i_0} = W_2$, or if we have the case 3.2(iv), then $T_{i_0}^\flat + T_h + \tilde{T}_{\tilde{h}} + E_0$ is a chain and we obtain $K_{\overline{M}} \geqslant 0$, see 1.8.1(a); contradiction. If $h = 1$ in the case 3.2(iii), then $\tilde{p}_{\tilde{h}} = 1$ by 3.1, so we may assume $W_{i_0} = W_2$.

(ii) Now $K_{\overline{M}} \cdot T_1^\flat = K_{\overline{M}} \cdot T_2^\flat = 0$ and, in view of (i), $\gamma = 2$. We have

$$K_{\overline{M}} \cdot \left(K_{\overline{M}} + T_0^\flat\right) = K_{\overline{M}} \cdot \left(K_{\overline{M}} + T_0^\flat + T_1^\flat + T_2^\flat\right) = -\varepsilon_0 + 2 - K_{\overline{M}} \cdot E_0 = 1.$$

Again using the Riemann–Roch theorem we argue as in (i) first that $K_{\overline{M}} + T_0^\flat \geqslant 0$, and then that $2K_{\overline{M}} + T + E_0 \geqslant 0$. Hence

$$K_{\overline{M}} + T_1^\flat + T_2^\flat + T_h + \tilde{T}_{\tilde{h}} + E_0 = 2K_{\overline{M}} + T + E_0 + \left(-K_{\overline{M}} - T_0^\flat\right) \geqslant 0.$$

If $h \geqslant 2$, then $T_1^\flat + T_2^\flat + T_h + \tilde{T}_{\tilde{h}} + E_0$ is a chain and we get $K_{\overline{M}} \geqslant 0$; contradiction. □

3.2.4. Corollary. *We have* $\gamma \geqslant 2$.

Proof. Suppose that $\gamma \leqslant 1$. Suppose we have a bad case. If $h \geqslant 2$ and $\tilde{h} \geqslant 2$ then $p_h = 1$ or $\tilde{p}_{\tilde{h}} = 1$ by 1.16(iii). We obtain the same by 3.1 if $h = 1$ or $\tilde{h} = 1$. In the case 3.2(iv), $p_1 = 1$. In all cases we reach contradiction with 3.2.3(i). □

3.2.5. Let $\alpha : \overline{M}_0 \to \overline{M}$ be an arbitrary sequence of blow-ups over W_0. The proper transform in $\overline{M}_0$ of a curve L will again be denoted by L. Let $Z = \alpha^{-1}(W_0)$ and let $Z_1, \dots, Z_s$ be the (-1)-curves produced by α. We consider the following parts of α:

α_1, the blow-ups over $E_0 \cup G_1$,
α_2, the blow-ups over $W_0 \cap F \setminus E_0 \cup G_1$ and
$\tilde{\alpha}$, the blow-ups over $W_0 \cap \tilde{F}$.

Let $\beta : \overline{M}_0 \to \overline{M}_1$ be a connected sequence of contractions (see 1.12.1(iv)) inside $Z - (Z_1 + \cdots + Z_s)$, starting with the contraction of (the proper transform of) H and such that $\beta(Z)$ is an NC-divisor.

Let A (resp. $\tilde{A}$) be the component of Z which meets T_h (resp. $\tilde{T}_{\tilde{h}}$).

3.2.6. Lemma. *Let things be as in* 3.2.5.

(i) *Suppose we have* 3.2(iii).
  (i.1) *If* $\alpha_2 = \mathrm{id}$*, then* $\beta(A)$ *is not a* (-1)*-curve.*
  (i.2) *If* $\tilde{\alpha} = \mathrm{id}$*, then* $\beta(\tilde{A})$ *is not a* (-1)*-curve.*
(ii) *Suppose we have* 3.2(iv). *Then* $\beta(T_0)$ *is not a* (-1)*-curve.*

Proof. Assume (i.1) and that $\beta(A)$ is a (-1)-curve. Then A is the component of W_0 that meets T_h and $A^2 \leqslant -2$ in W_0. It is, by assumption, not contracted by β. Hence $\beta \neq \mathrm{id}$, $n = 1$ and H is not touched by α. If both T_1 and $\tilde{T}_1$ are branching components in W_0 then they remain branching in Z and all contractions in β take place inside the inverse image of the chain in T connecting $\tilde{T}_1$ and T_1. Since $h \geqslant 2$, A is not touched by β and $\beta(A)$ is a ($\leqslant -2$)-curve.

Suppose that T_1 is not branching in W_0, so $h = 1$ or h=2 and $p_2 = 1$. In view of 3.2.2, $\tilde{T}_1$ is branching in W_0, and hence in Z. Since A is not contracted by β, the contractions of β take place inside the inverse image of the chain connecting $\tilde{T}_1$ and A in W_0. So β is no affected if we undo the blow-ups of α_1 and those of $\tilde{\alpha}$ not over $\tilde{G}'_1 \cup \tilde{T}_1$. Let $V = \beta(A) + T_h$. Then $\beta(A)^2 = T_h^2 = -1$ and the divisor V defines a $\mathbb{P}^1$-ruling of $\overline{M}_1$. Let r be the contraction of the chains $\tilde{R}_{\tilde{h}}, \dots \tilde{R}_2$. They take place inside a fiber of the ruling, and $r(E_0)$ becomes a 1-section of the ruling. In particular $r(E_0)$ is smooth. It follows that $\tilde{h} = 2$ and $\tilde{p}_2 = 1$. But then $\tilde{T}_1$ is not branching in W_0.

So suppose that $\tilde{T}_1$ is not branching in W_0. Then T_1 is branching. Since A is touched by β, $\alpha^{-1}(\tilde{F} \cap W_0)$ must be contracted. Therefore $\alpha^{-1}(\tilde{F} \cap W_0)$ does not contain any Z_i, $i = 1, \dots, s$. It follows that $\tilde{\alpha} = \mathrm{id}$. $\tilde{A}$ is eventually contracted by β, and this before $\tilde{T}_{\tilde{h}}$ is touched or T_1 is contracted. So we have $\beta = \beta'' \circ \beta'$ with $\beta'(\tilde{A})^2 = \tilde{T}_{\tilde{h}}^2 = -1$. Consider $\tilde{V} = \beta'(\tilde{A}) + \beta'(\tilde{T}_{\tilde{h}})$. We may again undo α_1. E_0 is then a 1-section of the ruling induced by $\tilde{V}$. We contract the chains $R_h, \dots, R_2$ and come to contradiction as above.

The argument in case (i.2) is similar.

Now suppose that we have 3.4(iv) and $\beta(T_0)^2 = -1$. Then $\tilde{T}_1$ is branching in W_0. T_0 and $\tilde{T}_1$ are connected by the chain

$$H^\flat\colon \quad H - - \tilde{T}_0 - - \tilde{G}'_1,$$

where $\tilde{G}'_1$ (see 1.13) consists of $\frac{\tilde{c}_1}{\tilde{c}_2} - 2$ (-2)-curves, $\tilde{T}_0^2 = -2$ and $H^2 = -n$. We have $T_0^2 = -c_1$ and E_0 is a tip of W_0 meeting T_0. After the blow-ups of α we have $T_0^2 \leqslant -c_1$ and the number u of contractions in β which touch T_0 is at most $\frac{\tilde{c}_1}{\tilde{c}_2}$, the length of $H^\flat$. Hence $-1 = -c_1 + u \leqslant -c_1 + \frac{\tilde{c}_1}{\tilde{c}_2}$. From this we obtain $\tilde{c}_2(d-2) \leqslant d$, which implies $d \leqslant 4$. So $d = 4$ and $\tilde{c}_2 = 2$. From 1.15(d1) we obtain $3\gamma + 1 = (\tilde{h} - 2)4$. By 1.13.3, $\tilde{h} = \varepsilon_0 + \gamma$. Hence $\gamma = 9$ if $\varepsilon_0 = 0$ or $\gamma = 13$ if $\varepsilon_0 = 1$. In both cases we reach contradiction since $\gamma \leqslant 6$ by 2.7. □

3.2.7. Lemma. *Let the situation be as in* 3.2.5. *Then Z does not contain a component B such that*

(i) $\beta(B)^2 = 0$,
(ii) $\beta(B)$ *meets* $\beta(Z_1), \dots, \beta(Z_s)$.

Proof. Suppose that B exists. Since Z consists of negative curves, $\beta \neq \mathrm{id}$, $n = 1$, H is not touched by α and B is touched by β.

Suppose first that we have 3.2(iii). We claim that $\beta(B)$ and $\beta(E_0)$ do not meet.

Suppose $\tilde{T}_1$ is branching in W_0 and hence in Z. Suppose T_1 is not branching. If $h = 1$, E_0 is not in W_0 and the claim is clear. So suppose $h = 2$, $p_2 = 1$. If $\alpha_2 \neq \mathrm{id}$, let $Z' = T_1 + Z''$, where Z'' is the chain in Z connecting T_1 and H in Z. Then there is a component $Z^\flat$ of Z' that is either branching in Z or is one of the Z_i. $Z^\flat$ is not contracted by β. If $\alpha_2 = \mathrm{id}$ put $Z^\flat = A$. Then $Z^\flat$ is not contracted by β in view of 3.2.6(i.2). If T_1 is branching, put $Z^\flat = T_1$. In all these cases the contractions of β take place in the part $Z^\sharp$ of Z between $\tilde{T}_1$ and $Z^\flat$, B is a component of $Z^\sharp$, and B and E_0 are in different connected components of $Z - Z^\sharp$. This establishes the claim.

It follows from the claim that E_0 is not touched by α since otherwise there is a Z_i with $\beta(Z_i)$ disjoint from $\beta(B)$. We have also that $\tilde{T}_{\tilde{h}} \cdot \beta(B) = 0$ since $\tilde{T}_{\tilde{h}}$ is disjoint from $Z^{\sharp}$. It follows that $E_0 + \tilde{T}_{\tilde{h}} + W_2$ is contained in a fiber of the $\mathbb{P}^1$-ruling of $\overline{M}_1$ induced by $\beta(B)$. Since $\tilde{T}_1$ is branching and not contracted, $\tilde{A} \cdot \beta(B) = 0$ or $\tilde{A} \cdot \beta(B) = 1$ (then $B = \tilde{T}_1$). In the first case $\tilde{A}$ is in the fiber and the (-1)-curve $\tilde{T}_{\tilde{h}}$ meets three other components of the fiber. This is not possible. In the second case the multiplicity of $\tilde{T}_{\tilde{h}}$ in the fiber equals 1 and we have contradiction with 1.8(1).

We apply a similar argument when T_1 is a branching component of W_0.

Suppose we have 3.2(iv). We put $Z^{\flat} = T_0$ and argue as above, using 3.2.6(ii). $\square$

3.3. Let $\overline{Y}$ be a non-singular surface, D an *NC*-divisor on $\overline{Y}$ and $g : \overline{Y} \to \mathbb{P}^1$ a $\mathbb{P}^1$-ruling. Put $Y = \overline{Y} \setminus D$. We recall a few facts about g from [Fu1, §4].

An irreducible component C of a fiber g' of g is called a *D-component* of g' if $C \subset D$, otherwise a *Y-component*. We write the number of Y-components of g' as $\sigma(g')$ and put

$$\Sigma_g = \sum_{\sigma(g')>0} \big(\sigma(g') - 1\big)$$

(of course $\sigma(g') = 1$ for general g').

Let υ_g denote the number of fibers g' entirely contained in D, i.e with $\sigma(g') = 0$. Let h_g denote the number of horizontal components of g, i.e. of components with $g(A) = \mathbb{P}^1$.

3.3.1. Lemma. *(See [Fu1, §4].) The quantity*

$$B(Y) = h_g - \Sigma_g + \upsilon_g - 2$$

can be expressed in terms of the Betti numbers of Y and hence depends on Y only, i.e. it does not depend on the completion $\overline{Y}$ and the ruling g.

3.4. Lemma. *Put*

$$M = \overline{M} \setminus W = \overline{X} \setminus V.$$

We then have the following:

(a) $\chi(M) = -1$ *and* $B(M) = 0$.
(b) W *(and V) is not a contractible divisor.*
(c) *There does not exist a morphism $g : M \to \mathbb{P}^1$ with general fiber an irreducible complete curve.*
(d) $\bar{\kappa}(M) \geqslant 0$.
(e) W_i *(and V_i) is a contractible chain for $i = 1, 2$. W_0 (and V_0) is not contractible.*
(f) W_0 *(and V_0) is not a chain.*

Proof. (a) We use the ruling $f : \overline{M} \to \mathbb{P}^1$ to compute $B(M) = 0$ (there are 2 horizontal components, E_0 and H, $\upsilon = 0$ and $\Sigma = 0$).

(b) Suppose W is contractible, i.e. the intersection matrix of W is negative definite. Then the components of W are independent in $Pic\,\overline{M}$. Since their number equals the rank of $Pic\,\overline{M}$ we reach contradiction with the Hodge Index Theorem.

(c) Suppose g exists. It extends to a morphism $\overline{M} \to \mathbb{P}^1$, which we also denote g, with W_1, W_2, W_3 contained in fibers. Let $\Gamma^{(0)}, \Gamma^{(\infty)}$ be effective divisors without common component and supported on fibers of g. If $\Gamma^{(0)} - \Gamma^{(\infty)}$ is the divisor of a rational function ζ on $\overline{M}$, then ζ has no zero or pole on a general fiber of g, so is constant on fibers and hence the pull-back of a rational function on $\mathbb{P}^1$. Hence both $\operatorname{Supp}\Gamma^{(0)}, \operatorname{Supp}\Gamma^{(\infty)}$ are unions of full fibers. We obtain the following: If $\Gamma_i \subset g_i$, $i = 1 \cdots s$, is a reduced effective divisor, g_i a fiber of g, and if Γ_i is a full fiber for at most one i, then the components of $\bigcup \Gamma_i$ are linearly independent in $Pic\,\overline{M}$. Now W_1, W_2 are contractible and hence not complete fibers. From this and the fact that fibers are connected we conclude that if g_0 is a fiber of g and $g_0 \supset W_1$ or $g_0 \supset W_2$, then $g_0 \cap W \subsetneqq g_0$. By the above argument the components of W are linearly independent in $Pic\,\overline{M}$. Since the components are in fibers, the intersection matrix is negative semi-definite and we reach a contradiction as in (b).

(d) Suppose that $\bar{\kappa}(M) = -\infty$. Since W is not contractible M is either $\mathbb{C}^1$-ruled or it contains an open subset U which has a structure of *Platonic fibration* [MT]. The set $M \setminus U$ is a disjoint union of curves isomorphic to $\mathbb{C}^1$ or to $\mathbb{C}^*$ or to $\mathbb{P}^1$. Thus $\chi(M) \geqslant \chi(U)$. It is well known that $\chi(U) = 0$. Since $\chi(M) < 0$ this case cannot occur. Therefore M is $\mathbb{C}^1$-ruled. This means that there exists a connected sequence of blow-ups (which we assume to be minimal) $\alpha : \overline{M}_0 \to \overline{M}$ over a point in W such that there is a $\mathbb{P}^1$-ruling $g : \overline{M}_0 \to \mathbb{P}^1$ which induces a ruling of M with general fiber $\mathbb{C}^1$ since $\mathbb{P}^1$ is ruled out as a possibility by (c).

Put $Z = \alpha^{-1}(W)$. Then by the above, g has a unique horizontal component $Z_1 \subset Z$ and Z_1 is a section of g.

We have $B(M) = 1 - \Sigma_g + \upsilon_g - 2 = 0$. Hence $\upsilon_g \geqslant 1$ and there exists a fiber $g_0 \subset Z$. We may contract g_0 to a (0)-curve B in such a way that Z_1 is not contracted and meets B transversally. It is clear that $g_0 \subset \alpha^{-1}(W_0)$ and we reach contradiction with 3.2.7.

(e) In view of (d) this follows from 1.9(ii) since W_i, $i = 1, 2$, is a chain, so a maximal twig of W. By (b) W_0 is not contractible and in particular not a chain. So we have (f). □

3.4.1. Lemma. *Suppose that $\gamma \leqslant 2$. If $\varepsilon_0 = 1$, then $p_h > 1$ or $\tilde{p}_{\tilde{h}} > 1$.*

Proof. Suppose not. Then $\gamma = 2$ and $h = 1$ by 3.2.3(ii). It follows that W_1 is a chain of (-2)-curves containing E_0 and we find $\kappa(K_{\overline{M}} + W) = \kappa(K_{\overline{M}} + W_0 + W_2) \leqslant \kappa(K_{\overline{M}} + T) = -\infty$ by 1.8.1(ii), in contradiction to 3.4(d). □

3.4.2. Lemma. *ν does not involve sprouting contractions.*

Proof. Since $\gamma \geqslant 2$ by 3.2.4, ν is a connected sequence of contractions starting with the contraction of H. Now we apply 3.2.6 for $\alpha = \mathrm{id}$. □

3.5. Lemma. *We keep the assumptions of 3.2. Suppose that there exists a (-1)-curve $L \subset \overline{X}$ such that $L \nsubseteq V$, $L \cdot V_i \leqslant 1$, $i = 0, 1, 2$, and L meets at most two connected components of V. Then $\varepsilon_0 = 1$ and $L \cdot V = 2$. Also $(K_{\overline{X}} + V + L)^2 = -4$ and the NC-minimalization of the divisor $V + L$ does not involve a sprouting contraction.*

Proof. In view of 3.4.2 and 1.13.1, $(K_{\overline{X}} + V)^2 = (K_{\overline{M}} + W)^2 = -4 - \varepsilon_0$. We have $(K_{\overline{X}} + V + L)^2 = -7 - \varepsilon_0 + 2L \cdot V$. By 1.11

$$(K_{\overline{X}} + V + L)^2 < 3\chi(M \setminus L) = 3\bigl(-1 - (2 - L \cdot V)\bigr). \tag{$*$}$$

We obtain $2 < \varepsilon_0 + L \cdot V \leqslant \varepsilon_0 + 2$. Hence $\varepsilon_0 = 1$, $L \cdot V = 2$. If the NC-minimalization of $V + L$ involves a sprouting contraction then the left-hand side of $(*)$ increases and the right-hand side stays the same. We find $\varepsilon_0 \geqslant 2$ in contradiction with 3.2(i). □

3.5.1. Remark. The analogue of Lemma 3.5 is true if we replace $\overline{X}$ by $\overline{M}$ and V by W.

3.6. Proposition. *Under the assumptions of* 3.2, $M = \overline{M} \setminus W$ *is not* $\mathbb{C}^*$*-ruled.*

Proof. Suppose the opposite. Then there exists a pencil $\tau^\flat$ of rational curves in $\overline{M}$ with at most two base points such that $\tau' \cap M \simeq \mathbb{C}^*$ for a general member τ' of $\tau^\flat$. Let $\alpha : \overline{M}_0 \to \overline{M}$ be the resolution of the base points. Let $\tau : \overline{M}_0 \to \mathbb{P}^1$ be the induced $\mathbb{P}^1$-ruling. If A is a reduced divisor in $\overline{M}$ we use by A also to denote the strict transform and put $A' = \alpha^{-1}(A)$.

(a) We have $h_\tau = 1$ and τ has one 2-section in W', J say, or $h_\tau = 2$ and τ has two sections in W', J_1 and J_2 say. An M-component C of a fiber τ' meets J_1 or J_2 at most once, and the same is true for J if $mult_{\tau'}(C) > 1$. Any other component of W' is in a fiber of τ, and since τ' is a tree, C meets any connected component of $W' - J$, or $W' - (J_1 + J_2)$, at most once.

(b) If $\upsilon_\tau \geqslant 1$, let τ_0 be a fiber contained in W'. Since all components of W' are negative curves, τ_0 contains a (-1)-curve which must be H in view of 3.2.4. Thus $\upsilon_\tau = 1$ and H is not touched by α. It follows that $\tau_0 \subset W_0'$, that the horizontal components of τ are contained in W_0' and that W_1, W_2 are not touched by α. τ_0 can be contracted to a (0)-curve by a connected sequence of contractions β. If $h_\tau = 2$ we may assume that β satisfies the conditions of 3.2.5. (They could be violated if we contract a (-1)-curve meeting both J_1 and J_2. But in such a case there exists another (-1)-curve in the fiber not meeting J_1 or J_2 that can be contracted first.) By 3.2.7, $h_\tau = 1$ if $\upsilon_\tau \geqslant 1$.

(c) We have $\chi(M_0 \setminus W') = \chi(M) = \sum \chi(\tau' \cap M)$, the sum extended over the singular fibers of τ. Since $\chi(M) = -1$ there exist a fiber τ_1 such that $\chi(\tau_1 \cap M) < 0$. We note that for an M-component C of a fiber, $\chi(C \cap M) = 2 - \mathrm{card}(C \cap W')$.

Suppose that $h_\tau = 1$. We have, by 3.3.1 and 3.4(a), $\Sigma_\tau = \upsilon_\tau - 1 \geqslant 0$. Thus $\upsilon_\tau = 1$ and $\Sigma_\tau = 0$. If τ_1 is as in (c), τ_1 has a unique M-component C_1 which is also the only (-1)-curve in τ_1 by (b). Hence $mult_{\tau_1}(C_1) > 1$ and C_1 is not branching in τ_1. Since $\chi(C_1) \cap M < 0$, C_1 meets at least three components of W'. In view of 1.8(iii.1), it meets J and, by (a), two connected components of $W' - J$ precisely once. None of these is in W_0 since otherwise $\tau_1 \cdot J > 2$. (J meets τ_0, so $J \subset W_0$.) Hence C_1 meets W_1' and W_2'. From this we conclude that W_1', W_2' have one component each and that $\tau_1 = W_1' + 2C_1 + W_2'$ (scheme theoretically) with $W_1'^2 = W_2'^2 = -2$. If $h \geqslant 2$ or if we have 3.2(iv) then, in view of (b), $W_1' = G_h$, $W_2' = \tilde{G}_{\tilde{h}}$ and hence $2 = c_h = \tilde{c}_{\tilde{h}}$. So both $c_1, \tilde{c}_1$ are even in contradiction to 1.15(a). If $h = 1$ in the case 3.2(iii), then W_1' has at least two components, in contradiction to the above.

Suppose that $h_\tau = 2$. Then $\upsilon_\tau = 0$ by (b). Hence $\Sigma_\tau = 0$, and with τ_1 as in (c), C_1 again is the unique M-component of τ_1 and meets at least three components of W'. If $mult_{\tau_1}(C_1) > 1$, in particular if C_1 is the only (-1)-component of τ_1, then C_1 does not meet J_1 or J_2 and C_1 is branching. It follows that $mult_{\tau_1}(C_1) = 1$, $H^2 = -1$ and $H \subset \tau_1$. There is a connected sequence of contractions $\beta : \overline{M}_0 \to \overline{M}_1$, beginning with H and in $\tau_1 \cap W_0'$, such that $\beta(\tau_1) = \beta(C_1)$ is a

(0)-curve. $\beta(J_1)$ and $\beta(J_2)$ meet $\beta(C_1)$ precisely once, and no other component of $\beta(W')$ meets $\beta(C_1)$. It follows that β contracts W_0 to a point (on $\beta(C_1)$). This is not possible, by 3.4(e) for instance. $\square$

3.7. We have two possibilities:

(a) There is no curve L as in 3.5. We then put $\overline{Y} = \overline{X}$, $Q = V$ and $Q^{(0)} = V_0$.

(b) A curve L as in 3.5 exists. Let then $\alpha : \overline{X} \to \overline{Y}$ be the NC-minimalization of the divisor $V + L$ and put $Q = \alpha(V)$. By 3.5, Q has two connected components, $Q^{(0)}$ and $Q^{(1)}$ say, where $Q^{(0)} \supset \alpha(V_0)$.

3.8. Lemma.

(a) *α does not involve a sprouting contraction.*
(b) *In case* 3.7(a), $(K_{\overline{Y}} + Q)^2 = -4 - \varepsilon_0$. *In case* 3.7(b), $(K_{\overline{Y}} + Q)^2 = -4$. *In both cases* $\chi(\overline{Y} \setminus Q) = -1$.
(c) *The pair* $(\overline{Y}, Q)$ *is almost minimal. If* $\bar{\kappa}(\overline{Y} \setminus Q) = 2$, *then the pair* $(\overline{Y}, Q)$ *is strongly minimal, see* [GM], [M, 4.9].
(d) $\bar{\kappa}(\overline{Y} \setminus Q) = 0$.
(e) $Q^{(0)}$ *is not contractible.*

Proof. (a) and (b) follow from 3.5. Note that $\chi(L \setminus V) = 0$ if we have 3.7(b).

(c) Suppose that $(\overline{Y}, Q)$ is not almost minimal or it is not strongly minimal. Then there exists a smooth rational curve $L_1 \nsubseteq Q$ such that

- L_1 meets each connected component of Q at most once.
- L_1 meets at most two connected components of Q.
- $L_1^2 = -1$ or $L_1^2 = -2$.

We have, by (b),

$$\begin{aligned}(K_{\overline{Y}} + Q + L_1)^2 &= (K_{\overline{Y}} + Q)^2 + 2K_{\overline{Y}} \cdot L_1 + 2Q \cdot L_1 + L_1^2 \\ &= (K_{\overline{Y}} + Q)^2 - 2 + K_{\overline{Y}} \cdot L_1 + 2Q \cdot L_1.\end{aligned}$$

By 1.11,

$$(K_{\overline{Y}} + Q + L_1)^2 < 3\bigl(-1 - (2 - L_1 \cdot Q)\bigr).$$

From this

$$(K_{\overline{Y}} + Q)^2 + 7 + K_{\overline{Y}} \cdot L_1 < L_1 \cdot Q.$$

If $\alpha \neq \mathrm{id}$, then $(K_{\overline{Y}} + Q)^2 = -4$ by (b) and we get $2 < L_1 \cdot Q$; contradiction. Hence $\alpha = \mathrm{id}$ and $L_1^2 = -2$ by 3.7. Again we get $L_1 \cdot Q > 2$.

(d) Suppose that $\bar{\kappa}(\overline{Y} \setminus Q) = 2$. Since $(\overline{Y}, Q)$ is strongly minimal and $\chi(\overline{Y} \setminus Q) < 0$, this contradicts the Kobayashi–Sakai inequality [Ko], [M, 6.6.2]. In its simplest form, sufficient here, it asserts that

$$0 < (K_{\overline{Y}} + Q)^{+2} \leqslant 3\chi(\overline{Y} \setminus Q).$$

If $\bar{\kappa}(\overline{Y} \setminus Q) = 1$, then $\overline{Y} \setminus Q$ and therefore $\overline{M} \setminus W$ is $\mathbb{C}^*$-ruled or has a fibration by elliptic curves [Ka]. This is ruled out by 3.6 and 3.4(c).

(e) This follows from 3.4(e). □

3.9. We keep the assumptions of 3.2. Fujita [Fu1, 8.7, 8.8] classifies boundary divisors of minimal surfaces of Kodaira dimension 0. By 3.8(a) and 3.4(f), $Q^{(0)}$ is not a chain. It follows that $Q^{(0)}$ is one of the following:

(1) A rational tree with precisely two branching components B_1, B_2, each meeting two tips of $Q^{(0)}$ which are (-2)-curves.

(2) A rational tree with one branching component B and four tips $S_1, \ldots, S_4$ which are (-2)-curves.

(3) A rational fork, i.e. a rational tree with one branching component B and 3 admissible maximal twigs S_1, S_2, S_3. Moreover, $\sum_{i=1}^{3} \frac{1}{d(S_i)} = 1$ and $(d(S_1), d(S_2), d(S_3))$ is up to permutation one of $(3, 3, 3)$, $(2, 4, 4)$, $(2, 3, 6)$.

3.9.1. Lemma.

(a) *$Q^{(0)}$ is not of the type* 3.9(1) *or* 3.9(2).
(b) *We have case* 3.7(a)*, so $\overline{Y} = \overline{X}$, $Q = V$.*

Proof. (a) Suppose $Q^{(0)}$ is of type 3.9(1). It is readily proved, cf. [K1, 6.2] for instance, that $Q^{(0)}$ is contractible if $B_1^2 \leqslant -2$ and $B_2^2 \leqslant -2$. In view of 3.8(e) we may assume that $B_1^2 \geqslant -1$. Let S_1, S_2 be the (-2)-tips of $Q^{(0)}$ meeting B_1. Blow up along B_1 over the point $B_1 \cap (Q^{(0)} - (B_1 + S_1 + S_2))$ until the proper transform B_1' of B_1 is a (-1)-curve. Then the divisor $S_1 + 2B_1' + S_2$ induces a $\mathbb{C}^*$-ruling of $\overline{Y} - Q$ in contradiction to 3.6.

Suppose that $Q^{(0)}$ is of the type 3.9(2). Since W does not contain a component with branching number 4, α cannot be the identity, i.e. we have case 3.7(b). Notice that then $\#Q = 1 + \mathrm{rank}(Pic\,\overline{Y})$. The intersection matrix of the divisor $Q - B$ is negative definite, in particular the irreducible components of $Q - B$ are independent in $Pic\,\overline{Y}$ and we reach a contradiction with the Hodge Index Theorem.

(b) Since $Q^{(0)}$ is of the type 3.9(3) by (a), the intersection matrix of $Q - B$ is negative definite. If α is not be the identity, we argue as in (a). □

3.10. Proposition. *Assume that we have* 3.2.(ii) *and* 3.2(iii) *or* (iv)*. Then $\varepsilon_0 \geqslant 2$.*

Proof. Suppose that $\varepsilon_0 \leqslant 1$. We keep the notation of 3.2–3.9. By 3.9.1 and 3.8(a), V_0 is a fork. Hence T_1 or $\tilde{T}_1$ is not a branching component of W_0.

3.10.1. Lemma.

(i) *If R is a maximal twig of V_0 and $\# R > 1$, then R consists of (-2)-curves.*

(ii) *If T_1 is not a branching component of W_0, then $\tilde{h} = 2$ and $\tilde{p}_2 > 1$, or $\tilde{h} = 3$ and $\tilde{c}_2 = \tilde{p}_2$ and $\tilde{p}_3 = 1$. Also $h = 1$, or $h = 2$ and $p_2 = 1$.*

(iii) *If $\tilde{T}_1$ is not a branching component of W_0, then $\tilde{h} = 1$, or $\tilde{h} = 2$ and $\tilde{p}_2 = 1$. Also $h = 2$ and $p_2 > 1$, or $h = 3$, $c_2 = p_2$ and $p_3 = 1$.*

Proof. (i) This is clear since $d(R)$ is one of $2, 3, 4, 6$.

(ii) $\tilde{T}_1$ is the unique branching component of W_0. Suppose that $\tilde{h} \geqslant 4$. Then the branch at $\tilde{T}_1$ containing $\tilde{T}_2$ has at least two irreducible components. Since $\tilde{T}_{\tilde{h}-1}$ is a ($\leqslant -3$)-curve we get contradiction with (i). If $\tilde{h} = 3$, then the same argument gives $\tilde{p}_3 = 1$. The rest of the statement is clear.

(iii) We argue as in (ii). $\square$

Suppose that T_1 is not a branching component. We write $V_0 = B + S_1 + S_2 + S_3$ where $B = \nu(\tilde{T}_1)$, S_1 is the twig which contains $\nu(H)$, S_2 is the twig which meets $\tilde{T}_{\tilde{h}}$.

By (ii) above, $h = 2$ and $p_2 = 1$ or $h = 1$.

Suppose $h = 2$. Since T_1 is not contracted by ν by 3.2.6, S_1 has at least 3 irreducible components (one of them is E_0) and (i) above implies that $\gamma = 2$ and also $p_1 | c_1$, so $p_1 = c_2$. Also $d(S_1) \geqslant 4$. From 3.2.3 we obtain that $\varepsilon_0 = 1$, and $\tilde{p}_{\tilde{h}} > 1$. From 3.10.1(ii), $\tilde{h} = 2$. Hence $n = 1$ by 1.13.3. S_1 has at least two components besides G_1, namely T_1 and E_0. Thus $d(S_1) \geqslant \frac{c_1}{c_2} + 2$. Since $d(S_1) = 4$ or 6, we get $\frac{c_1}{c_2} = 2$ or 3 or 4.

Suppose that $c_1 = 2c_2$. From 1.15(d1) we obtain

$$2c_1 = \tilde{p}_2(d - \tilde{c}_2).$$

Since $GCD(c_1, \tilde{c}_1) = 1$ by 1.15(a), $\tilde{c}_2 = 2$. But then $\tilde{p}_2 = 1$; contradiction.

Suppose that $c_1 = 3c_2$. From 1.15(d1) we obtain

$$5c_1 = 3\tilde{p}_2(d - \tilde{c}_2).$$

Thus $\tilde{c}_2 | 5$, so $\tilde{c}_2 = 5$. Hence

$$3\tilde{p}_2(d - 5) = 5d - 5 = 5(d - 5) + 20.$$

It follows that $(d - 5) | 20$. Also $3 | d - 1$. We get $d = 10$, $\tilde{p}_2 = 3$ or $d = 25$, $\tilde{p}_2 = 2$. We now know T explicitly and find that S_1 is not a (-2)-chain.

Suppose that $c_1 = 4c_2$. From 1.15(d1) we obtain

$$7c_1 = \tilde{p}_2(d - \tilde{c}_2).$$

Hence $\tilde{c}_2 = 7$. We find $d = 21$, $c_1 = 20$, $c_2 = 5$ and again S_1 is not a (-2)-chain; contradiction.

Suppose $h = 1$. Then $\tilde{p}_{\tilde{h}} = 1$ by 3.1. By 3.10.1(ii), $\tilde{h} = 3$ and $\tilde{c}_2 = \tilde{p}_2$. Adding up 1.15(b1) and 1.15(d1) we obtain

$$(\gamma + n)d = \tilde{p}_1 + \tilde{c}_2(d - \tilde{c}_2) + d. \qquad (*)$$

By 1.13.3, $4 = n + \varepsilon_0 + \gamma$ and thus $2 \leqslant \gamma \leqslant 3$. If $\gamma = 2$, then $\varepsilon_0 = 1$ by 3.2.3(ii) and therefore $n = 1$. The same holds if $\gamma = 3$. Since $d(S_2) = 1 + \tilde{c}_2$, $\tilde{c}_2$ is one of 2, 3 or 5.

Suppose $\tilde{c}_2 = 5$. From $(*)$ we get $25 = (5-\gamma)d + \tilde{p}_1$. Since $d \geqslant 2\tilde{c}_2 = 10$, $\gamma = 3$ and $d = 10$. So $c_1 = 9$ and 1.15(b1) gives $p_1 = 2$. Now we know T explicitly and it turns out that S_1 does not consist of (-2)-curves.

Suppose that $\tilde{c}_2 = 3$. We obtain $9 = \tilde{p}_1 + (3-\gamma)d$. If $\gamma = 2$, then $d = 6$ and $\tilde{p}_1 = 3$, $c_1 = 5$. From 1.15(b1) we obtain $p_1 = 1$, but this contradicts 3.2.3. If $\gamma = 3$, we get $\tilde{p}_1 = 9$. By 1.15(b1), $d = p_1 + 10$, so $c_1 = p_1 + 9$. Since $d(S_2) = 4$ we have $d(S_3) = \frac{\tilde{c}_1}{\tilde{c}_2} = 2$ or 4. Since $d > 10$ we get $d = 12$, so $c_1 = 11$ and $p_1 = 2$. Now we find $d(S_1) = 3$; this is impossible in view of 3.9(3) (since $d(S_3) = 4$).

Suppose $\tilde{c}_2 = 2$. From $(*)$ we get $4 = \tilde{p}_1 + (2-\gamma)d$. Since $d > \tilde{p}_1$, we have $\gamma = 2$ and $\tilde{p}_1 = 4$. Since $d(S_3) = \frac{\tilde{c}_1}{\tilde{c}_2}$ we have only three possibilities:

(1) $d(S_3) = 2$, $d = 4$, $d(S_1) = 6$, $c_1 = 3$;
(2) $d(S_3) = 3$, $d = 6$, $d(S_1) = 3$, $c_1 = 5$;
(3) $d(S_3) = 6$, $d = 12$, $d(S_1) = 2$, $c_1 = 11$.

From 1.15(b1) we obtain $c_1 = p_1 + 4$. Hence (1) cannot occur. In case (2), $p_1 = 1$, in contradiction with 3.2.3. In case (3), $p_1 = 7$ and we may find T explicitly. It turns out that ν contracts the H-chain to a point, a contradiction in view of 3.2.6.

Now we assume that $\tilde{T}_1$ is not a branching component of W_0. Then T_1 is the unique branching component of W_0. Write $V_0 = B + S_1 + S_2 + S_3$, where $B = \nu(T_1)$, S_3 is the twig which meets (or contains) $\tilde{T}_1$ and S_2 is the twig which meets T_h.

Since $E_0 \subset S_1$, S_1 has at least two components, hence it consists of (-2)-curves by 3.10.1(i). Thus $\gamma = 2$ and $p_1 | c_1$, so $p_1 = c_2$. By 3.10.1(iii), $\tilde{h} = 2$ and $\tilde{p}_2 = 1$ or $\tilde{h} = 1$.

Suppose $\tilde{h} = 1$. By 3.1, $p_h = 1$. Hence $h = 3$ by 3.10.1(iii). Since $p_3 = 1$, $\varepsilon_0 = 1$ by 3.2.3. From 1.13.3, $n = 1$. From 1.15(d1) we obtain $c_1 + c_2 = c_2(c_1 - c_2)$ and from this $c_1 = 6$ and $c_2 = 2$ or $c_2 = 3$. In the first case, $d(S_1) = 4$, $d(S_2) = 3$. In the second case, $d(S_1) = 3$, but $d(S_2) = 4$. We get a contradiction with 3.9(3).

Suppose now that $\tilde{h} = 2$. Since $\tilde{p}_2 = 1$, W_2 consists of (-2)-curves. By 3.2.3 $\varepsilon_0 = 1$ and also $p_h > 1$. Hence $h = 2$ by 3.10.1(iii). From 1.13.3 we get $n = 1$. Since $\tilde{T}_1$ is not contracted by ν, S_2 has at least 2 components and so consists of (-2)-curves. It follows that $\tilde{p}_1 | \tilde{c}_1$, i.e. $\tilde{c}_2 = \tilde{p}_1$. From 1.15(b1) we obtain therefore

$$2 + d = c_2 + p_2 + \tilde{c}_2 + 1 \quad \text{and} \quad \tilde{c}_2 = d - c_2 - p_2 + 1.$$

Plugging this into 1.15(d1) we obtain $2c_1 + 2 = 2p_2 + p_2(c_1 - c_2)$. Hence $c_2 | 2(p_2 - 1)$. We have $d(S_2) = p_2$, hence $p_2 \in \{2, 3, 4, 6\}$. If p_2 is even, then c_2 is odd. Then $c_2 | p_2 - 1$, which implies $p_2 = 1$; contradiction. Therefore $p_2 = 3$, so $c_2 = 4$. We obtain $2c_1 + 2 = 6 + 3(c_1 - 4)$, which gives $c_1 = 8$. From 1.15(b1), $\tilde{p}_1 = 3$. We find S_3 is not a (-2)-chain; contradiction. □

4. The bad good asymptote case

In this section we provide the final classification of E_0' in the bad case with $\xi = 1$. By 1.14(i), $c_1 > 1$. The key to the classification is 3.10.

4.1. Lemma. *If $h + \tilde{h} \geqslant 3$, then $\varepsilon_0 = 2$, $n = 1$ and $0 \leqslant \gamma \leqslant 2$.*

Proof. By 1.16(i) and 1.13.3 we have

$$2\gamma + 1 \geqslant \sum_{i\geqslant 2} p_i + \sum_{i\geqslant 2} \tilde{p}_i \geqslant 2(h-2) + 1 + 2(\tilde{h}-2) + 1 = 2n + 2\gamma + 2\varepsilon_0 - 6.$$

Thus $7 \geqslant 2n + 2\varepsilon_0$, which implies $\varepsilon_0 = 2$ and $n = 1$ in view of 3.10 and 1.13.3. The last statement follows from 1.16(ii) and 2.7. □

In 4.2–4.4 we assume $h + \tilde{h} \geqslant 3$.

4.2. Assume $\gamma = 2$. By 1.13.3, $h + \tilde{h} = 5$. From 2.7 we get $t = 2$. This implies $p_h = \tilde{p}_{\tilde{h}} = 1$ and, since we are in a bad case, $h \geqslant 2$ (if $h = 1$, then $t \leqslant 1$, see 2.3). By 1.16(iii), $\sum_{i\geqslant 2} p_i + \sum_{i\geqslant 2} \tilde{p}_i \leqslant 4$ unless $h = 4$.

4.2.1. Assume $h = 2$. Then $\tilde{p}_2 + \tilde{p}_3 \leqslant 4 - p_2 = 3$. Thus $\tilde{p}_2 = 2 = \tilde{c}_3$. We get $2\tilde{c}_2 + d = p_1 + 1$ from 1.15(d1)and $d = p_1 + \tilde{p}_1 + 2$ from 1.15(b1). It follows that $p_1 + 2\tilde{p}_1 + 3 = c_2 + 2\tilde{c}_2$; contradiction since $p_1 \geqslant c_2$ and $\tilde{p}_1 \geqslant \tilde{c}_2$.

4.2.2. Assume $h = 3$. Then $p_2 \leqslant 2$, so $p_2 = c_3 = 2$. Arguing as above we get $p_1 + 2\tilde{p}_1 + 3 = 2c_2 + \tilde{c}_2$. It follows that $p_1 = c_2$, otherwise we reach contradiction as above. Hence

$$2\tilde{p}_1 + 3 = c_2 + \tilde{c}_2. \tag{$*$}$$

This implies $\tilde{c}_2 | c_2 - 3$. Since c_2 is even (it is divisible by c_3) and $\tilde{c}_2 > 1$, we have $\tilde{c}_2 < c_2$. Now 1.15(b1) gives $c_1 = c_2 + \tilde{p}_1 + 1$. So $\tilde{p}_1 = c_1 - c_2 - 1$. Substituting this into $(*)$ we obtain $\tilde{c}_2 = 2c_1 - 3c_2 + 1$. It follows that $c_2 | \tilde{c}_2$ and therefore $c_2 \leqslant \tilde{c}_2$; contradiction.

4.2.3. Assume that $h = 4$. Then $p_2 + p_3 + 1 \leqslant 5$ by 1.16(i). Thus $p_2 = p_3 = 2 = c_4 = c_3$, $\tilde{p}_1 = p_4 = 1$ since $t = 2$. From 1.15(d1) we get $3c_1 = p_1 + 2c_2 + 2$. Since $p_1 \leqslant c_1 - c_2$ we have $2c_1 \leqslant c_2 + 2$, which gives $3c_2 \leqslant 2$; contradiction.

4.3. Assume $\gamma = 1$. Then $h + \tilde{h} = 4$.

4.3.1. Assume $h = 1$. By 1.16(i), $\tilde{p}_2 + \tilde{p}_3 \leqslant 3$. Thus $\tilde{p}_2 = 2 = \tilde{c}_3$, $\tilde{p}_3 = 1$. 1.15(d1) gives $2 + p_1 = 2d - 2\tilde{c}_2$. Since $p_1 \leqslant c_1 - 1 = d - 2$ and $d \geqslant 2\tilde{c}_2$, this implies $p_1 = c_1 - 1$ and $d = 2\tilde{c}_2$. But now 1.15(b1) gives $d = p_1 + \tilde{p}_1 + 2$ and therefore $2 = \tilde{p}_1 + 2$; contradiction.

4.3.2. Assume $h = 2$. We obtain $p_2 = \tilde{p}_2 = 1$ by 1.15(iii). 1.15(d1) gives $p_1 + c_2 + \tilde{c}_2 = d = c_1 + 1$. This implies $c_2 | \tilde{c}_2 - 1$ and $c_2 < \tilde{c}_2$. From 1.15(b1), $d = p_1 + \tilde{p}_1 + 1$. Hence $p_1 + c_2 + \tilde{c}_2 = p_1 + \tilde{p}_1 + 1$, i.e. $c_2 + \tilde{c}_2 = \tilde{p}_1 + 1$, which implies $\tilde{c}_2 | c_2 - 1$ and so $\tilde{c}_2 < c_2$; contradiction.

4.3.3. Assume $h = 3$. Then $p_2 + p_3 \leqslant 3$ by 1.16(i). Hence $p_2 = 2$, $p_3 = 1$. Also $p_1 = c_1 - c_2$ by 1.16(iii). From 1.15(d1) we obtain $2c_1 = p_1 + 2c_2$ and therefore $c_1 = c_2$; contradiction.

4.4. Assume $\gamma = 0$. Then $h + \tilde{h} = 3$.

4.4.1. Assume $h = 1$. By 1.16(i), $\tilde{p}_2 = 1$. From 1.15(d1) we get $d = p_1 + \tilde{c}_2 + 1$. By 1.15(b1), $d = p_1 + \tilde{p}_1 + 1$. Hence $p_1 + \tilde{c}_2 + 1 = p_1 + \tilde{p}_1 + 1$, which gives $\tilde{c}_2 = \tilde{p}_1$. Put $\tilde{c}_2 = p$. Let $d = (s+1)p$. Then $c_1 = (s+1)p - 1$ and $p_1 = d - \tilde{p}_1 - 1 = sp - 1$. We get the numerical solution

$$\begin{pmatrix} c_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} (s+1)p-1 \\ sp-1 \end{pmatrix}; \quad \begin{pmatrix} \tilde{c}_1 \\ \tilde{p}_1 \end{pmatrix} = \begin{pmatrix} (s+1)p \\ p \end{pmatrix};$$
$$\begin{pmatrix} \tilde{c}_2 \\ \tilde{p}_2 \end{pmatrix} = \begin{pmatrix} p \\ 1 \end{pmatrix}; \quad s, p \geqslant 1,\ sp \geqslant 2,\ n = 1.$$

4.4.2. Assume $h = 2$. We have $p_2 = 1$ by 1.16(i). Also $p_1 = c_1 - c_2$ by 1.16(iii). From 1.15(b1) we get $c_1 = p_1 + \tilde{p}_1$. It follows that $\tilde{p}_1 = c_2$. Put $c_2 = p$ and let $p_1 = sp$. Then $c_1 = sp + p$, $d = sp + p + 1$. We get another numerical solution

$$\begin{pmatrix} c_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} sp+p \\ sp \end{pmatrix}; \quad \begin{pmatrix} c_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} p \\ 1 \end{pmatrix}; \quad \begin{pmatrix} \tilde{c}_1 \\ \tilde{p}_1 \end{pmatrix} = \begin{pmatrix} sp+p+1 \\ p \end{pmatrix}; \quad s, p \geqslant 1,\ n = 1.$$

4.5. Assume that $h = \tilde{h} = 1$. From 1.15(d1) we get $-\gamma(d-1) = p_1 + 1$. This implies $\gamma < 0$. Thus we are in the bad case described in 1.18.1. Put $c_1 = s$. We get the numerical solution

$$\begin{pmatrix} c_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} s \\ s-1 \end{pmatrix}; \quad \begin{pmatrix} \tilde{c}_1 \\ \tilde{p}_1 \end{pmatrix} = \begin{pmatrix} s+1 \\ 1 \end{pmatrix}; \quad s \geqslant 2;\ n = 1,\ \gamma = -1,\ \varepsilon_0 = 2.$$

5. The good case and the bad very good asymptote case

In this section we consider the bad case with $\xi = 0$ and the good case.

Let $X = \overline{M}' \setminus (H \cup \tilde{f})$, see 1.3.5. Then $X \simeq \mathbb{C}^2$ and $E_0' \cap X$ is a contractible curve in X with one place at $H \cup \tilde{f}$.

5.1. Definition. Let E be an irreducible curve in a smooth projective surface $\overline{X}$ and let D be a reduced effective divisor in $\overline{X}$. We say that D is negative minimal with respect to E if the following holds*:*

- D is an NC-divisor and E meets D transversally.
- Every component of D is a $(\leqslant -2)$-curve except possibly for components branching in $D + E$.

5.2. Lemma. *Let E_i be an irreducible curve in a smooth projective irreducible surface $\overline{X}_i$, $i = 1, 2$. Let X_i be non-empty open in $\overline{X}_i$ with $E_i \cap X_1 \neq \emptyset$. Let $f : X_1 \to X_2$ be an isomorphism such that $f(E_1 \cap X_1) = E_2 \cap X_2$. Suppose that $D_1 = \overline{X}_1 \setminus X_1$ is negative minimal w.r.t. E_1. We have the following*:

(a) *If D_2 is an NC-divisor and E_2 meets D_2 transversally then $\#D_1 \leqslant \#D_2$.*
(b) *If $D_2 = \overline{X}_2 \setminus X_2$ is negative minimal w.r.t. E_2 then f extends to an isomorphism $\overline{X}_1 \to \overline{X}_2$.*

Proof. There exists a sequence of blow-ups $\alpha : \overline{Y} \to \overline{X}_1$ over D_1 such that $f \circ \alpha = \beta$ is a morphism. We take α minimal.

(a) Suppose that β contracts a (-1)-curve L. Since α is minimal, L is the proper transform of a component $D_1^{(0)} \subset D_1$. $D_1^{(0)}$ is a branching component of $E_1 + D_1$ since otherwise $L^2 \leqslant -2$. But now $E_2 + D_2 = \beta(E_1 + D_1)$ contains 3 components meeting at one point; contradiction. Hence β does not contract a curve, so β is an isomorphism. In particular $\#D_1 \leqslant \#\alpha^{-1}(D_1) = \#\beta(\alpha^{-1}(D_1)) = \#D_2$.

(b) If α is not the identity, then the divisor $\alpha^{-1}(E_1 + D_1)$ contains a (-1)-component which is not branching and therefore $\alpha^{-1}(D_1)$ is not negative minimal w.r.t. E_1. But then D_2 is not negative minimal w.r.t. E_2 since β maps $\alpha^{-1}(E_1 + D_1)$ isomorphically onto $E_2 + D_2$. □

5.3. Lemma. *Let $\overline{X}$ be a smooth compactification of $X = \mathbb{C}^2$. Let $E \subset \overline{X}$ be an irreducible curve which meets $D = \overline{X} \setminus X$ transversally at one point. Assume that D is NC-minimal w.r.t. δ, where δ is the branch of E at D. If E is singular then D is negative minimal w.r.t. E.*

Proof. Let D_0 be a component of D that is non-branching in $D + \delta$. Then $D_0^2 \neq -1$. Suppose that $D_0^2 \geqslant 0$. After blowing up over a point of $D_0 \setminus E$ we may assume $D_0^2 = 0$. So E is contained in a fiber of the ruling induced by D_0 (if $E \cap D_0 = \emptyset$) or E is a 1-section of the ruling (if $E \cap D_0 = 1$). In both cases E is smooth; contradiction. Thus $D_0^2 \leqslant -2$. □

5.3.1. We recall that a resolution of a singular branch γ is a sequence of blow ups such that the proper transform of γ and all exceptional curves form an NC-divisor.

5.4. Proposition. *Let $\overline{X}$ be a smooth compactification of $X = \mathbb{C}^2$. Let $E \subset \overline{X}$ be an irreducible curve such that $E \cap X$ is a topologically contractible singular curve that does not meet $T = \overline{X} \setminus X$ normally. Let $p : \overline{Y} \to \overline{X}$ be the minimal resolution of $E + T$. Let E_0 be the proper transform of E in $\overline{Y}$. Then $E_0^2 = 0$.*

Proof. Let $\tilde{\lambda}$ denote the branch of E at T. By the Lin–Zaidenberg theorem [ZL], [M, 3.5] there exists an isomorphism $\varphi : X \to \mathbb{C}^2 = X' = Spec\,\mathbb{C}[\zeta, \eta] \subset \overline{X}' = \mathbb{P}^2$ which maps $E \cap X$ onto the curve C: $\zeta^p - \eta^q = 0$, where $1 < p < q$ and $GCD(p, q) = 1$. Let E' be the closure of C in $\overline{X}'$. Then E' has one branch δ' at $T' = \overline{X}' \setminus X'$ and E has one branch δ at T. Let $D = p^{-1}(T)$. Let $p_1 : \overline{Y}_1 \to \overline{X}$ be the minimal NC-resolution of $T + \tilde{\delta}$ and let $D_1 = p_1^{-1}(T)$. Let $E_1, \tilde{\delta}_1$ be the proper transforms of $E, \tilde{\delta}$ in $\overline{Y}_1$. Let $r : \overline{Y}_1 \to \overline{Z}_1$ be an NC-minimalization of D_1 w.r.t. $\tilde{\delta}_1$. We claim that r does not touch E_1. Indeed, let A be the component of D_1 which meets E_1. All blow-ups in the resolution are along $\tilde{\delta}$. Since p_1 is non-trivial and minimal, we find that $A^2 = -1$ and that A is a branching in $D_1 + \tilde{\delta}_1$. In the contraction process A remains branching or becomes a $(\geqslant 0)$-curve and so is not contracted by r. Hence E_1 is not touched by r. We define $r', p_1', \overline{Y}_1', \overline{Z}_1', \delta_1'$ analogously. Again E_1' is not touched by r'. By 5.3 and 5.2, φ extends to an isomorphism $\Phi : \overline{Z}_1 \to \overline{Z}_1'$. Hence $E_1^2 = E_1'^2$. We also have $E_2^2 = E_2'^2$, where E_2, E_2' are obtained by minimally resolving the singularity of E_1, E_1' in X, X'. It is readily calculated that $E_2'^2 = 0$. □

5.5. Lemma. *If the branch λ of E_0' is singular, then $\gamma = 0$.*

Proof. In case λ singular the minimal resolution of $\lambda + f$ coincides with the minimal resolution of λ. Hence the minimal resolution of $E_0' + f + H + \tilde{f}$ coincides with the minimal resolution of $E_0' + H + \tilde{f}$. Now the statement follows from 5.4. □

5.6. Assume that $\gamma = 0$ and $\xi = 1$. So we have a good case by assumption.

5.6.1. Suppose that $c_1 > p_1$. If $h = \tilde{h} = 1$ then 1.15(d2) gives $d = p_1$; contradiction. So $h + \tilde{h} \geqslant 3$ and 1.17(b) gives $\sum_{i\geqslant 2} p_i + \sum_{i\geqslant 2} \tilde{p}_i \leqslant 1$. Hence $h + \tilde{h} = 3$ and $p_2 = 1$ or $\tilde{p}_2 = 1$. By 1.13.3, $3 = n + \varepsilon_0$. Suppose that $n = 2$. Then $\varepsilon_0 = 1$ and $K_{\overline{M}} \cdot (K_{\overline{M}} + T) = 1$ by 1.13.1. From the Riemann–Roch theorem we obtain $-K_{\overline{M}} - T \geqslant 0$. Again by the Riemann–Roch theorem, $2K_{\overline{M}} + T + E_0 \geqslant 0$. It follows that $K_{\overline{M}} + E_0 \geqslant 0$, so $K_{\overline{M}} \geqslant 0$; contradiction. Thus $n = 1$.

5.6.1.1. Suppose that $h = 2$. The formulas 1.15(d2) and 1.15(b2) take the form $p_1 = c_2$ and $d = p_1 + \tilde{p}_1$. Denote $c_2 = p$ and let $c_1 = sp$. Then $\tilde{p}_1 = d - p_1 = sp - p + 1$. We get the numerical solution

$$\binom{c_1}{p_1} = \binom{sp}{p}, \binom{c_2}{p_2} = \binom{p}{1}; \quad \binom{\tilde{c}_1}{\tilde{p}_1} = \binom{sp+1}{sp-p+1}; \quad s, p \geqslant 1;\ n = 1.$$

5.6.1.2. Suppose that $\tilde{h} = 2$. The formulas 1.15(d2) and 1.15(b2) take the form $p_1 = \tilde{c}_2$ and $d = p_1 + \tilde{p}_1$. Let $\tilde{c}_2 = p$, $d = sp$. We get the numerical solution

$$\binom{c_1}{p_1} = \binom{sp-1}{p}; \quad \binom{\tilde{c}_1}{\tilde{p}_1} = \binom{sp}{sp-p}, \binom{\tilde{c}_2}{\tilde{p}_2} = \binom{p}{1}; \quad p \geqslant 1,\ s \geqslant 2;\ n = 1.$$

(We have $s \geqslant 2$ since we assume $c_1 > p_1$).

5.6.2. Suppose that $c_1 = p_1 > 1$. Let l be as in 1.13.4. From 1.15(d2) we get

$$d = lc_1 + p_{l+1} + \sum_{i\geqslant l+2} p_i(d - c_i) + \sum_{i\geqslant 2} \tilde{p}_i(d - \tilde{c}_i).$$

It follows that

$$1 = (l-1)c_1 + p_{l+1} + \sum_{i\geqslant l+2} p_i(d - c_i) + \sum_{i\geqslant 2} \tilde{p}_i(d - \tilde{c}_i).$$

Hence $l = 1$, $p_{l+1} = 1$, $h = 2$, $\tilde{h} = 1$. From 1.15(b2) we get $nd = c_1 + \tilde{p}_1$. It follows that $n = 1$ and $\tilde{p}_1 = 1$. So we have the case 5.6.1.1 with $s = 1$.

5.6.3. Suppose that $c_1 = p_1 = 1$. Then $d = \tilde{c}_1 = 2$. Thus $\tilde{p}_1 = 1$ and $\tilde{h} = 1$. By 1.13.3, $1 = \tilde{h} = n + \varepsilon_0$. Hence $n = 1$ by 1.5. Contraction of H in $\overline{M}'$ leads to $\mathbb{P}^2$ and E_0' becomes a smooth conic. It follows that $E_0^2 = 2$ and $\gamma = -2$. This case is handled in 5.9.2.

5.7. Assume that $\gamma = 0$ and $\xi = 0$.

5.7.1. Suppose that $c_1 > p_1$. Then $h = \tilde{h} = 1$ by 1.18(c). Hence $n = 1$, $\varepsilon_0 = 1$ by 1.13.3. $\lambda, \tilde{\lambda}$ have characteristic pairs $\binom{c_1}{p_1}$, $\binom{\tilde{c}_1}{\tilde{p}_1}$. We have $\tilde{c}_1 = c_1$, and it follows from 1.18(a1) that $\tilde{p}_1 = c_1 - p_1$. So we obtain the numerical solution

$$\binom{c_1}{p_1}; \quad \binom{c_1}{c_1 - p_1}; \quad c_1 > p_1;\ n = 1,$$

with $GCD(c_1, p_1) = 1$.

5.7.2. Suppose that $c_1 = p_1 > 1$. Let l be as in 1.13.4. It follows from 1.18(a3) that $h = l + 1$ and $\tilde{h} = 1$. The formulas 1.18(a1), (a2) give $nd = (h-1)d + p_h + \tilde{p}_1$. From 1.13.3 we get $n = h + 1 - \varepsilon_0$. Thus $d(2 - \varepsilon_0) = p_h + \tilde{p}_1$. Since $p_h < d$ and $\tilde{p}_1 < d$ we obtain $\varepsilon_0 = 1$ and $d = p_h + \tilde{p}_1$. So $n = h$. We get the numerical solution

$$\binom{c_1}{p_1} = \cdots = \binom{c_{h-1}}{p_{h-1}} = \binom{c_1}{c_1}, \quad \binom{c_h}{p_h} = \binom{c_1}{p_h};$$
$$\binom{\tilde{c}_1}{\tilde{p}_1} = \binom{c_1}{c_1 - p_h}; \quad c_1 > p_h; \; n = h,$$

with $GCD(c_1, p_h) = 1$.

5.7.3. Suppose that $c_1 = p_1 = 1$. Then $\tilde{c}_1 = 1$. By 1.4(ii), $n = 1$. Then $H^2 = -1$, see 1.4(ii), and E_0 is a line after the contraction of H and we have $\gamma = -1$. This is the case described in 1.3.3. It will be handled in 5.9.3.

5.8. Assume that $\gamma > 0$. By 5.5 the branch λ is smooth. Recall that $X = \overline{M}' \setminus (H + \tilde{f}) \simeq \mathbb{C}^2$. Then $C = E_0' \cap X$ is an Abhyankar–Moh line (curve isomorphic to $\mathbb{C}$) in X. Let $\pi : \overline{M}_1 \to \overline{M}'$ be the resolution of $\tilde{f} + \overline{C}$. Let $\tilde{C}$ be the proper transform of $\overline{C}$ in $\overline{M}_1$. Suppose that $n \geqslant 2$. Then $H + \tilde{F}$ is negative minimal w.r.t. $\tilde{C}$, where $\tilde{F} = (\pi^{-1}(\tilde{f}))$. We reach a contradiction with 5.2(a) since for the standard linear embedding of C into $\mathbb{C}^2$ the resolution divisor at infinity has one irreducible component. Therefore $n = 1$. By the Abhyankar–Moh theorem, π, regarded as a blow up process, is a sequence $\Delta_1, \dots, \Delta_r, \delta_{r+1}$ where Δ_i comes from an elementary automorphism of X and δ_{r+1} is a blow up process given by a pair $\binom{\tilde{c}_{\tilde{h}}}{\tilde{c}_{\tilde{h}}-1}$. In particular $\tilde{p}_{\tilde{h}} = \tilde{c}_{\tilde{h}} - 1$. Each elementary transformation Δ_i is given by a sequence of HN-pairs of a form

$$\binom{a_i}{a_i - a_{i+1}}, \underbrace{\binom{a_{i+1}}{a_{i+1}}, \dots, \binom{a_{i+1}}{a_{i+1}}}_{\frac{a_i}{a_{i+1}} - 1}$$

where $a_{i+1} | a_i$ and $a_{i+1} < a_i$. We put $r = 0$ if $\tilde{h} = 1$. The quantity $\sum \tilde{p}_i$ for the above sequence equals $2a_i - 2a_{i+1}$ and $\sum \tilde{p}_i \tilde{c}_i = a_i^2 - a_{i+1}^2$. Since $a_{r+1} = \tilde{c}_{\tilde{h}}$, we get $\sum_1^{\tilde{h}} \tilde{p}_i = 2\tilde{c}_1 - \tilde{c}_{\tilde{h}} - 1$ and $\sum_1^{\tilde{h}} \tilde{p}_i \tilde{c}_i = \tilde{c}_1^2 - \tilde{c}_{\tilde{h}}$. Also $\tilde{h} = \frac{a_1}{a_2} + \cdots + \frac{a_r}{a_{r+1}} + 1$. Since λ is smooth, $h = 1$ and $p_1 = 1$.

5.8.1. Assume that $\xi = 0$.

Let $X_1 = \overline{M}' \setminus (H + f)$. Then $X_1 \simeq \mathbb{C}^2$ and $E_0' \cup X_1$ is a contractible curve with one place at $H \cup f$.

Suppose that $c_1 > 1$. Then E_0' does not meet $H \cup f$ normally. Proposition 5.4 implies that $\tilde{\lambda}$ is smooth. Thus $\tilde{h} = 1$ and $\tilde{p}_1 = 1$. By 5.8, $\tilde{c}_1 = \tilde{p}_1 + 1 = 2$. From 1.13.3 we obtain $2 = h + \tilde{h} = 1 + \varepsilon_0 + \gamma$. Hence $\varepsilon_0 = 0$ and $\gamma = 1$. 1.18(a1) gives $d = \tilde{p}_1 = 1$; contradiction.

Suppose that $c_1 = 1$. We then have the situation of 5.7.3.

Suppose that $r = 0$. By 1.13.3, $\varepsilon_0 + \gamma = 1$. Hence $\gamma = 1$. From 1.15(b2) we get $2 + d = 1 + \tilde{p}_1$; contradiction.

Thus $r \geqslant 1$, which implies that $\tilde{h} \geqslant 3$. By 3.10, $\varepsilon \geqslant 2$. (Note that $\tilde{c}_1 = \tilde{p}_1 + \tilde{c}_2$ by the discussion in 5.8, so we have 3.2(iv).)

Suppose that $\gamma \geqslant 2$. From 2.7 we obtain $\gamma = 2$, $\varepsilon_0 = 2$, $\tilde{p}_{\tilde{h}} = 1$. From 1.15(b2) we get $\tilde{c}_{\tilde{h}} = 3$ and $d = 6$. Thus $\tilde{c}_2 = \tilde{c}_{\tilde{h}}$, which implies $r = 1$. So $\tilde{h} = 3$. Now we reach a contradiction with 1.13.3.

So we have $\gamma = 1$. From 1.15(b2) we get $\tilde{c}_{\tilde{h}} = 2$ and $d = 4$. Therefore $r = 1$ and $\tilde{h} = 3$, $\tilde{c}_2 = 2$, $c_1 = 3$. So we have the numerical solution

$$\binom{c_1}{p_1} = \binom{3}{1}; \quad \binom{\tilde{c}_1}{\tilde{p}_1} = \binom{4}{2}, \ \binom{\tilde{c}_2}{\tilde{p}_2} = \binom{2}{2}, \ \binom{\tilde{c}_3}{\tilde{p}_3} = \binom{2}{1}; \quad \gamma = 1, \ n = 1.$$

5.8.2. Assume that $\xi = 1$ and $c_1 = 1$. We have $d = 2$. Hence $\tilde{p}_1 = 1$ and $\tilde{h} = 1$. This is as in 5.7.3.

5.9. Assume that $\gamma < 0$.

5.9.1. Suppose that $c_1 > p_1$. Then we have the good case with $\xi = 1$ described in 1.18.1. Put $c_1 = s$. We obtain the numerical solution

$$\binom{c_1}{p_1} = \binom{s}{1}; \quad \binom{\tilde{c}_1}{\tilde{p}_1} = \binom{s+1}{s}; \quad s > 1, \ n = 1.$$

5.9.2. Suppose that $c_1 = 1$ and $\xi = 1$. Then $d = 2$ and we have the above numerical solution with $s = 1$. This is the situation described in 5.6.3.

5.9.3. Suppose $c_1 = 1$ and $\xi = 0$. This case is described in 5.7.3. We obtain the numerical solution

$$\binom{1}{1}; \quad \binom{1}{1}, \quad n = 1,$$

if we agree to choose local coordinate curves at q and $\tilde{q}$ transversal to f and $\tilde{f}$ that are not tangent to E_0.

6. Equations

We begin by introducing coordinates for $X = \overline{M}' \setminus H \cup \tilde{f} \simeq \mathbb{C}^2$ and then give equations for the curves $V = E_0' \cap X \subset X$ (see 1.3.5), going through the numerical possibilities for the HN-pairs found in the previous sections. We put $f^\bullet = f \cap X$.

6.1. We introduce coordinates $\{u, v\}$ for X so that:

(i) $f^\bullet$ has equation $v = 0$;
(ii) $q = (0, 0)$ and $q_0 = (1, 0)$ in a good case.

v is then determined up to multiplication by a non-zero constant.

6.2. We consider the *very good asymptote (**vga**)* case.

6.2.1. Consider the numerical solution in 5.7.1. We put

$$c_1 = b, \qquad p_1 = a.$$

We have $n = -H^2 = 1$. Upon contracting H, we obtain a surface $\mathbb{P} \simeq \mathbb{P}^2 = X \cup \tilde{f}$. The line $\tilde{H}$ in $\mathbb{P}$ joining q and $\tilde{q}$ meets E_0' in $q, \tilde{q}$ only. We choose for u an equation for $\tilde{H} \cap X$. u is determined up to multiplication by a non-zero constant.

f and $\tilde{H}$ (resp. $\tilde{f}$ and $\tilde{H}$) are suitable local coordinate curves at q (resp. $\tilde{q}$) to define the characteristic pairs of E_0'. Consider the pencil of rational curves given on X by $t_1 v^a - t_2 u^b = 0$, $t_1, t_2 \in \mathbb{C}$ and the $\mathbb{P}^1$-fibration ψ induced by it (on a blow-up of $\mathbb{P}$). The characteristic pairs of a general member at the base points $q, \tilde{q}$ of the pencil are the same as those of E_0', i.e. $\binom{b}{a}$, $\binom{b-a}{a}$. It follows that (the strict transform of) E_0' has zero intersection with a general member of ψ and hence is part of a fiber, clearly with $t_1, t_2 \neq 0$. Free to multiply u, v by non-zero constants, we may assume that the equation for $V = E_0' \cap X$ is

$$\theta(u, v) = v^a - u^b = 0, \quad 1 \leqslant a < b, \qquad GCD(a, b) = 1.$$

6.2.2. Consider the numerical solution in 5.7.2 and 5.9.3. A member $\tilde{H}$ of the linear system $|H + nf|$ on $\overline{M}'$ has $\tilde{H}^2 = n$. So we can find $\tilde{H}$ passing through $\tilde{q}$ and the first n points of E_0' infinitely near to q. Note that then $\tilde{H}$ meets E_0' in these points only. f and $\tilde{H}$ (resp. $\tilde{f}$ and $\tilde{H}$) are suitable local coordinate curves at q (resp. $\tilde{q}$) to define the characteristic pairs of E_0'. We now make elementary transformations in the fiber $\tilde{f}$ until $H^2 = -1$. More precisely, we blow up $(n-1)$-times along $\tilde{H}$ above $\tilde{q}$ and contract $\tilde{f}$ and the first $n-2$ exceptional curves. We again use $\tilde{f}$ to denote the new fiber. We contract H and obtain $\mathbb{P}$ as in 6.2.1. We put

$$b = c_1, c = p_h \quad \text{and} \quad a = (n-1)b + c.$$

In $\mathbb{P}$, the point at infinity of E_0' is at $\tilde{q} = \tilde{f} \cap f$ and $\tilde{H}$ is the line tangent to E_0' at q. We have $(\tilde{H} \cdot E_0')_q = a$, $(f \cdot E_0')_q = b$, $(\tilde{f} \cdot E_0')_{\tilde{q}} = a$, $(f \cdot E_0')_{\tilde{q}} = b - a$. Arguing as in 6.2.1, we find that E_0' is a member of the pencil $v^a - tu^b = 0$, $t \in \mathbb{C}^*$, and that we may assume that the equation for V is

$$\theta(u, v) = v^a - u^b = 0, \quad 1 \leqslant b \leqslant a, \qquad GCD(a, b) = 1.$$

Here $a = b = 1$ covers the case 5.9.3, i.e. 1.3.3.

6.2.3. We can summarize 6.2.1 and 6.2.2 as follows. In the case of a very good asymptote ((**vga**)-case) we find an equation for V in the form

$$\theta(u, v) = v^a - u^b = 0, \qquad GCD(a, b) = 1$$

with $q = (0, 0)$ and, in a good case, $q_0 = (1, 0)$. Moreover, $a < b$ in a bad case.

6.3. We consider numerical solutions in the *good case with a good asymptote* (**gga**-*case*). We note that the problem here is to classify contractible curves V in $X = \mathbb{C}^2$ that meet a line $f^\bullet$ in precisely two points, normally in one of them. The numerical solutions are given in 5.6.1.1, 5.6.1.2, 5.8.2 and 5.9. We have $n = 1$ in these cases and contract H as in 6.2.1. We keep the notation introduced there. u is now uniquely determined since we place q at $u=0$ and q_0 at $u=1$.

6.3.1. Consider 5.6.1.1. Free to replace v by a constant mulptiple, we can assume that the center of λ after blowing up according to $\binom{sp}{p}$ at q is at $u=0$, $v/u^s=-1$ and find that V is a member of the pencil $(v+u^s)^p - tu^{sp+1}=0$. Having q_0 at $u=1$ forces $t=1$. Hence V has equation

$$\theta(u,v)=\left(v+u^s\right)^p - u^{sp+1}=0, \quad s,p\geqslant 1.$$

We call this the *(**gga+**)-case*. With $p=1$ it also covers 5.9. Note that $p=1$ precisely when V is smooth, i.e. an Abhyankar–Moh line. (We had tacitly assumed $p>1$ in 5.6.1.1.)

6.3.2. Consider 5.6.1.2. After blowing up according to $\binom{sp}{(s-1)p}$ at $\tilde{q}$, $1/v$ is a local equation for the last exceptional curve $\tilde{C}_1$ and u^s/v is a parameter along it. So we can assume that the center of $\tilde{\lambda}$ is at $u^s/v=-1$ and find that V is a member of the pencil $t(v+u^s)^p - u^{sp-1}=0$. Hence V has equation

$$\theta(u,v)=\left(v+u^s\right)^p - u^{sp-1}=0, \quad s,p\geqslant 1,\ s\geqslant 2.$$

We call this the *(**gga−**)-case*.

6.3.3. Consider 5.8.2. V is an Abhyankar–Moh line in this case. In contrast to the situation in 6.3.1 with $p=1$ it is not given by one elementary automorphism of X, but is the composite of two elementary automorphisms of degree 2. Let

$$\theta(u,v)=v-16v^2+4uv-8u^2v+u^3-u^4.$$

We have $8\theta(u,v)=u+(8v-u+2u^2)-2(8v-u+2u^2)^2$, so θ is a composite of two elementary automorphisms, and $\theta=0$ meets $v=0$ in $(0,0)$ (3 times) and $(1,0)$ only. As to uniqueness, consider the pencil of rational curves $tv+\theta=0$. E_0' meets a general member 4 times in X and 12 times at infinity (there are 3 common double points, the location of the third one being determined by the choice of v). Hence E_0' is a member of the pencil. On the other hand, $t=0$ gives the only member with one place at infinity. So the equation of V is θ with an appropriate choice of v. We call this the *special quartic (**sq**) case*.

6.4. We consider numerical solutions in the *bad case with a good asymptote (**bga**-case)*. We note that the problem here is to classify rational curves with one place at infinity in X meeting a line $f^\bullet$ in one point only, having two branches there, one tangent to $f^\bullet$ and the other a simple branch meeting $f^\bullet$ normally.

The numerical solutions are given in 4.4.1, 4.4.2 and 4.5. We can consider 4.5 as a subcase of 4.4.1 with $p=1$ and will not consider it separately.

We again contract H and keep the notation of 6.2.1.

6.4.1. Consider 4.4.1. The line $\tilde{H}$ in $\mathbb{P}$ meets E_0' in q and $\tilde{q}$ and is not tangent to λ_0 at q. We can therefore choose u,v so that λ_0 is tangent to $u-v=0$. After we blow up X in q and remove the transform of $\tilde{H}$ we obtain $X'\simeq\mathbb{C}^2$ so that the transform V' of V in X' satisfies the conditions of a **gga**-case w.r.t. the coordinate system $u_1=v/u$, $v_1=u$. The first characteristic pair of V' at

$q' = (0,0)$ is $\binom{p_1}{c_1 - p_1} = \binom{sp-1}{p}$ and so we are in the **(gga−)**-case 6.3.2. Still free to multiply u, v by the same constant, we can arrange for V' to have equation $(v_1 + u_1^s)^p - u_1^{sp-1}$. Let $\theta(u,v) = 0$ be an equation for V. The multiplicity of V at q is sp. So we have

$$\theta(v_1, u_1 v_1) = v_1^{sp}\big(\big(v_1 + u_1^s\big)^p - u_1^{sp-1}\big).$$

Hence the equation of V is

$$\theta(u,v) = \big(v^s + u^{s+1}\big)^p - v^{sp-1}u = 0.$$

We call this the ***(bga−)****-case*.

6.4.2. Consider 4.4.2. Blowing up at q now leads to a **(gga+)**-case. We find that the equation for V is

$$\theta(u,v) = \big(v^s + u^{s+1}\big)^p u - v^{sp+1} = 0.$$

We call this the ***(bga+)****-case*.

6.5. As set out in 1.6, we now choose an integer $k \geqslant 1$ and blow up $\mathbb{P}$ k-times along a simple branch λ^* normal to f at q_0, with $q_0 = q = (0,0)$ in a bad case and $q_0 = (1,0) \neq q = (0,0)$ in a good case. In a **ga**-case, λ^* is the branch λ_0 of E_0' and in a **vga**-case a virtual branch involving a choice of parameters, see 6.6 below. $S \simeq \mathbb{C}^2$ is the complement of T, the union of $\tilde{f}$, f and all exceptional curves except the last one, which we denote L. We denote by U the transform of V in S. It is clear that the branches $\lambda, \tilde{\lambda}$ of E_0' have centers on T and that $L \cdot U = 1$ in a **ga**-case and $L \cdot U = 0$ in a **vga**-case. So L is a good (resp. very good) asymptote.

6.6. Blowing up at $(c_0, 0) \in X = \mathit{Spec}\,\mathbb{C}[u,v]$ and removing the transform of $f^\bullet = \{v = 0\}$ we pass to

$$X' = \mathit{Spec}\,\mathbb{C}[u_1, v_1], \quad u_1 = (u - c_0)/v,\ v_1 = v.$$

The next blow-up (if $k \geqslant 2$) then is at $(c_1, 0) \in X'$, with c_1 being determined by the position of λ^*. Arguing by induction on k we find

6.6.1.

$$S = \mathit{Spec}\,\mathbb{C}[x,y] \quad \text{with } u = xy^k + g(y),\ v = y,$$

where g is a polynomial in y of degree $< k$. Moreover, $g(0) = 1$ in a good case and $g(0) = 0$ in a bad case.

6.6.2. Let μ be the multiplicity of E_0' at q_0. Then the multiplicity of L in $E_0'^*$ (the total transform) is

$$\nu = \mu \quad \text{in a } \textbf{vga}\text{-case}$$

and

$$\nu = \mu + k - 1 \quad \text{in a } \mathbf{ga}\text{-case}.$$

Hence the equation of U in S is

$$G(x, y) = \theta\big(xy^k + g(y), y\big)/y^\nu,$$

where θ is one of the equations for V found above. In a **ga**-case h is uniquely determined by the requirement that G be a polynomial. In a **gvga**-case (resp. **bvga**-case) h can be chosen freely except for the requirement $h(0) = 1$ (resp. $h(0) = 0$).

6.7. We call a branch at infinity of U *asymptotic* (w.r.t. the $\{x, y\}$-coordinate system) if it is not tangent to L_∞, the line at infinity. By construction, one branch, λ, is asymptotic and has $\{y = 0\}$ as asymptote (tangent at infinity). We call λ the *right* branch and the other one, $\tilde{\lambda}$, the *left* branch of U at infinity.

Let $\tilde{\lambda}$ have center at $\{x' = 0\}$. Below, when we give characteristic pairs for $\tilde{\lambda}, \lambda$, the upper entry in the first pair on the left and on the right will be the intersection with L_∞. On the right, the lower entry in the first pair will be intersection with $\{y = 0\}$. On the left, the lower entry will not necessarily be intersection with $\{x' = 0\}$, but rather with another simple branch, usually a branch of maximal contact with $\tilde{\lambda}$.

6.8. We consider the **vga**-case 6.2.3 with

$$\theta(u, v) = v^a - u^b = 0, \qquad GCD(a, b) = 1.$$

6.8.1. Suppose we have a good case. Then

$$G(x, y) = y^a - \big(xy^k + g(y)\big)^b,$$

with g(0)=1 and g otherwise arbitrary of degree at most $k - 1$.

6.8.1.1. Suppose $b = 1$. If

$$a \geqslant k, \quad \text{put } x' = -x + y^{a-k}, \qquad g'(y) = g(y).$$

If

$$a < k, \quad \text{put } x' = -x, \qquad g'(y) = g(y) - y^a.$$

Then in the (x', y)-coordinates the equation of U is

$$G'(x', y) = x'y^k - g'(y),$$

with $g'(0) = 1$ and g' otherwise arbitrary of degree at most $k - 1$.

We have

$$\deg(G') = k + 1.$$

$\tilde{\lambda}$, λ have centers on $\{x'=0\}$, $\{y=0\}$ with characteristic pairs

$$\binom{1}{k+1}; \quad \binom{k}{k+1}.$$

$\tilde{\lambda}$ is asymptotic with $\{x'=0\}$ as asymptote. It is a very good asymptote precisely when $\tilde{g}=1$ and

$$G'(x',y)=x'y^k-1$$

and a good asymptote precisely when $\deg(\tilde{g})=1$ and

$$G'(x',y)=x'y^k-cy-1, \quad c\neq 0.$$

6.8.1.2. Suppose $b(k+1)>a, b>1$. Then

$$\deg(G)=(k+1)b$$

and $\tilde{\lambda}$, λ have centers on $\{x=0\}$, $\{y=0\}$ with characteristic pairs

$$\binom{b}{(k+1)b-a}; \quad \binom{kb}{(k+1)b}, \binom{b}{a}.$$

We consider two possibilities.

(a) We have $b(k+1)-a>b$. Then the multiplicity of $\tilde{\lambda}$ is b. Put $g^{\dagger}(y)=G(0,y)=y^a-g(y)^b$. a is not a multiple of b since we assume $b>1$. Hence $0<\deg(g^{\dagger})\leqslant(k-1)b<\deg(G)-b$. So $\tilde{\lambda}$ is asymptotic with $\{x=0\}$ as asymptote and $\{x=0\}$ cannot be a very good asymptote. It is a good asymptote precisely when $a=1$, $g(y)=1$ and

$$G(x,y)=y-\left(xy^k+1\right)^b.$$

(b) We have $b(k+1)-a=c<b$. Now $\tilde{\lambda}$ is tangent to L_∞ and not asymptotic. We remark that we can make an automorphism of S that makes $\tilde{\lambda}$ asymptotic precisely when $c=1$. λ, however, will then not be asymptotic and the asymptote for $\tilde{\lambda}$ will not be good or very good.

6.8.1.3. Suppose $b(k+1)<a, b>1$. Then

$$\deg(G)=a.$$

Now both $\tilde{\lambda}$ and λ have center on $\{y=0\}$, with $\tilde{\lambda}$ tangent to L_∞. The characteristic pairs are

$$\binom{a-kb}{a-(k+1)b}; \quad \binom{kb}{(k+1)b}, \binom{b}{a}.$$

We remark that the two branches cannot be separated by an automorphism of S since $b>1$.

6.8.2. Suppose we have a bad case. Note that then $b > a$ and $g(0) = 0$. We write $g = yg^\star$.

(a) Suppose $k > 1$. Then

$$G(x, y) = 1 - y^{b-a}\bigl(xy^{k-1} + g^\star(y)\bigr)^b,$$

where $g^\star$ is arbitrary of degree at most $k - 2$.

We have

$$\deg(G) = (k+1)b - a$$

and $\tilde{\lambda}$, λ, with centers on $\{x = 0\}$, $\{y = 0\}$, have characteristic pairs

$$\begin{pmatrix} b \\ (k+1)b - a \end{pmatrix}; \quad \begin{pmatrix} kb - a \\ (k+1)b - a \end{pmatrix}.$$

Again $\tilde{\lambda}$ is asymptotic with $\{x = 0\}$ as asymptote. This is a very good asymptote if $g^\star = 0$, so

$$G(x, y) = 1 - x^b y^{kb-a},$$

and a good asymptote if $g^\star = 1$ and $b - a = 1$, so

$$G(x, y) = 1 - y\bigl(xy^{k-1} + 1\bigr)^b.$$

(b) Suppose $k = 1$. We put

$$x' = x + g^\star(y).$$

Then in the (x', y)-coordinates the equation of U is

$$G'(x', y) = 1 - x'^b y^{b-a}.$$

We note that this case will be covered if we allow $k = 1$ in (a) with $g^\star = 0$.

6.9. We consider the **gga**-cases from 6.3. Here $g(0) = 1$ and g, of degree at most $k - 1$, will be uniquely determined. By construction, λ is asymptotic with $\{y = 0\}$ as a good asymptote.

6.9.1. In the **(gga+)**-case 6.3.1 we have, with $s, p \geqslant 1$,

$$G(x, y) = \frac{(y + (xy^k + g(y))^s)^p - (xy^k + g(y))^{sp+1}}{y^k}$$

and

$$\deg(G) = (k+1)sp + 1.$$

$\tilde{\lambda}$, λ have centers on $\{x = 0\}$, $\{y = 0\}$ and characteristic pairs

$$\begin{pmatrix} sp + 1 \\ (k+1)(sp+1) - p \end{pmatrix}; \quad \begin{pmatrix} ksp \\ (k+1)sp \end{pmatrix}, \begin{pmatrix} sp \\ p \end{pmatrix}, \begin{pmatrix} p \\ 1 \end{pmatrix}.$$

Since $(k+1)(sp+1)-p-(sp+1) \geqslant (s-1)p+1>0$, the multiplicity of $\tilde{\lambda}$ is $sp+1$. Also $\rho = \deg(G)-(sp+1) = ksp$. If $\deg(g) \geqslant 1$,

$$\big(U \cdot \{x=0\}\big)_S \leqslant (sp+1)\deg(g) \leqslant (k-1)(sp+1)-k = (k-1)sp-1 < \rho.$$

If $\deg(g)=0$, again $(U \cdot \{x=0\})_S = p-k < \rho$. Hence $\tilde{\lambda}$ is asymptotic with asymptote $\{x=0\}$.

g is uniquely determined by the condition

$$y^k \big| \big(y+g^s\big)^p - g^{sp+1}.$$

6.9.2. In the **(gga−)**-case 6.3.2 we have, with $s, p \geqslant 1$, $sp \geqslant 2$,

$$G(x,y) = \frac{(y+(xy^k+g(y))^s)^p - (xy^k+g(y))^{sp-1}}{y^k}$$

and

$$\deg(G) = (k+1)sp-k.$$

$\tilde{\lambda}$, λ have centers on $\{x=0\}$, $\{y=0\}$ and characteristic pairs

$$\begin{pmatrix} sp \\ ((k+1)s-1)p \end{pmatrix}, \begin{pmatrix} p \\ 1 \end{pmatrix}; \quad \begin{pmatrix} k(sp-1) \\ (k+1)(sp-1) \end{pmatrix}, \begin{pmatrix} sp-1 \\ p \end{pmatrix}.$$

We argue as above that $\tilde{\lambda}$ is asymptotic with asymptote $\{x=0\}$.

g is uniquely determined by the condition

$$y^k \big| \big(y+g^s\big)^p - g^{sp-1}.$$

6.9.3. In the **sq**-case 6.3.3 we have

$$G(x,y) = \frac{y-16y^2+4y(xy^k+g(y))-8y(xy^k+g(y))^2+(xy^k+g(y))^3-(xy^k+g(y))^4}{y^k}$$

and

$$\deg(G) = 3k+4.$$

$\tilde{\lambda}$, λ have centers on $\{x=0\}$, $\{y=0\}$ and characteristic pairs

$$\begin{pmatrix} 4 \\ 4k+2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}; \quad \begin{pmatrix} 3k \\ 3(k+1) \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

We argue as above that $\tilde{\lambda}$ is asymptotic with asymptote $\{x=0\}$.

g is uniquely determined by the condition

$$y^k \big| y-16y^2+4yg-8yg^2+g^3-g^4.$$

6.10. We consider the **bga**-cases from 6.4. We again write $g = yg^\star$. Now $g^\star$ is of degree at most $k-2$ and uniquely determined. By construction, λ is asymptotic with $\{y=0\}$ as a good asymptote.

6.10.1. In the **(bga+)**-case 6.4.2 we have

$$G(x,y) = \frac{(1+y(xy^{k-1}+g^\star(y))^{s+1})^p(xy^{k-1}+g^\star(y)) - 1}{y^{k-1}}$$

and

$$\deg(G) = k(s+1)p + p + 1.$$

$\tilde{\lambda}$, λ have centers on $\{x=0\}$, $\{y=0\}$ and characteristic pairs

$$\begin{pmatrix} (s+1)p+1 \\ k\big((s+1)p+1\big)+p \end{pmatrix}; \quad \begin{pmatrix} \big((k-1)(s+1)+1\big)p \\ \big(k(s+1)+1\big)p \end{pmatrix}, \begin{pmatrix} p \\ 1 \end{pmatrix}.$$

Again $\tilde{\lambda}$ is asymptotic with asymptote $\{x=0\}$.

We have $g^\star = 0$ if $k=1$. Otherwise $g^\star$ is uniquely determined by the condition

$$y^{k-1} \big| \big(1+g^{\star s+1}\big)^p g^\star - 1.$$

6.10.2. In the **(bga−)**-case 6.4.1 we have

$$G(x,y) = \frac{(1+y(xy^{k-1}+g^\star(y))^{s+1})^p - (xy^{k-1}+g^\star(y))}{y^{k-1}}$$

and

$$\deg(G) = k(s+1)p + p + 1.$$

$\tilde{\lambda}$, λ have centers on $\{x=0\}$, $\{y=0\}$ and characteristic pairs

$$\begin{pmatrix} (s+1)p \\ (k(s+1)+1)p \end{pmatrix}, \begin{pmatrix} p \\ 1 \end{pmatrix}; \quad \begin{pmatrix} (k-1)((s+1)p-1)+p \\ k((s+1)p-1)+p \end{pmatrix}.$$

Again $\tilde{\lambda}$ is asymptotic with asymptote $\{x=0\}$.

We have $g^\star = 0$ if $k=1$. Otherwise $g^\star$ is uniquely determined by the condition

$$y^{k-1} \big| \big(1+g^{\star s+1}\big)^p - g^\star.$$

We record the following.

6.11. Proposition. *In all cases of* 6.9 *and* 6.10*, U has two asymptotes.*

7. Classification of embeddings up to isomorphism of $\mathbb{C}^2$

A curve $U = \mathbb{C}^* \subset \mathbb{C}^2 = \mathit{Spec}\,\mathbb{C}[x, y] = S$ as constructed in the last section "in general" uniquely determines the family it comes from and the discrete parameters that define it within the family, as we will see below. We will also list the exceptions.

The following is a simple key observation.

7.1. Lemma. *Let λ be a branch at infinity w.r.t. S and suppose λ is asymptotic w.r.t. the $\{x, y\}$-coordinate system with asymptote $\{y = 0\}$. Then λ is also asymptotic w.r.t. the $\{x', y'\}$-coordinate system with asymptote $\{y' = 0\}$ if and only if $x' = \alpha x + \phi(y)$, $y' = \beta y$, $\alpha, \beta \neq 0$.*

Proof. Suppose λ is asymptotic with asymptote y'. If x', y' are linear in $\{x, y\}$, then $y = \beta y$. Suppose y' is not linear. Consider a sequence of minimal length of elementary automorphisms that linearizes y', the first one non-linear. After the first automorphism, λ and y have the same center at infinity, but the centers of y' and y are different. By induction on the length of the sequence we conclude that $\{y' = 0\}$ is not an asymptote for λ in any $\{x', y'\}$-coordinate system. □

By the above result, y is determined up to a constant multiple in all cases of 6.8, 6.9 and 6.10. In all cases where there is a second asymptote, its equation is similarly determined. In the exceptional cases 6.8.1.2(b) and 6.8.1.3, the degree condition on g determines x up to a constant multiple. We obtain the following.

7.2. Proposition. *In all cases of* 6.8–6.10 *the coordinates x, y (x', y in the cases of* 6.8.1.1, 6.8.2(b)) *are determined up to constant multiples unless $\{x = 0\}$, $\{y = 0\}$ ($\{x' = 0\}$, $\{y = 0\}$) are both very good asymptotes or both good asymptotes. In that case an interchange of x and y (x' and y) can occur.*

In 6.8, a, b and k are discrete parameters and the coefficients of the polynomials $g(y)$ or $g'(y)$ (resp. $g^\star(y)$) of degree at most $k - 1$ (resp. $k - 2$) give continuous parameters. We recall that $g(0) = 1$, $g'(0) = 1$. It is straightforward to deduce from 7.1 the following result regarding the continuous parameters.

7.3. Proposition.

(i) *Let g, $\tilde{g}$ be polynomials of degree at most $k - 1$ with $g(0) = \tilde{g}(0) = 0$. Let a, b, k be given and let $G(x, y)$ and correspondingly $\tilde{G}(x, y)$ be defined as in* 6.8.1.2 *and* 6.8.1.3. *Then G, $\tilde{G}$ define equivalent embeddings of $\mathbb{C}^*$ if and only if*

$$\tilde{g}(y) = g(\beta y), \qquad \beta^a = 1,$$

the equivalence sending x to $\beta^{-k}x$, y to βy.
In case 6.8.1.1, *an analogous result holds for $G'(x', y)$, $\tilde{G}'(x', y)$ if and only if*

$$\tilde{g}'(y) = g'(\beta y),$$

the equivalence again sending x' to $\beta^{-k}x'$, y to βy.

(ii) *Let $g^\star, \tilde{g}^\star$ be polynomials of degree at most $k-2$. Let a, b, k be given and let $G(x, y)$ and correspondingly $\tilde{G}(x, y)$ be defined as in* 6.8.1.2. *Then $G, \tilde{G}$ define equivalent embeddings of $\mathbb{C}^*$ if and only if*

$$\tilde{g}(y) = \alpha^{-1}\beta^{1-k}g(\beta y),$$

the equivalence sending x to αx, y to βy, with $\alpha^b = \beta^{kb-a}$.

In 6.8–6.10, U has two very good asymptotes precisely in the cases 6.8.1.1 and 6.8.2(b). Suppose then $b > 1$ in 6.8.1 and $k > 1$ in 6.8.2. The characteristic pairs for $\lambda, \tilde{\lambda}$ then uniquely determine b, a and k in each case (recall that $a < b$ in 6.8.2), and it follows from 7.2 that different choices lead to inequivalent embeddings in each case. An equivalence could possibly occur between a U from 6.8.1.2(a) and a U from 6.8.2(a). However, if also $k > 1$ in 6.8.1.2, λ cannot be described by one characteristic pair, as in 6.8.2(a). If $k = 1$, we can describe λ by the pair $\binom{b}{2b+a}$, and this can equal a pair in 6.8.2(a) of the form $\binom{\kappa\beta-\alpha}{(\kappa+1)\beta-\alpha}$ only if $\kappa = 1$. Using this and further information from 6.8 we obtain the following.

7.4. Proposition.

(i) *Suppose $b > 1$ (resp. $k > 1$). Then the equations*

$$G(x, y) = y^a - \left(xy^k + g(y)\right)^b$$

as in 6.8.1 (*resp.*

$$G(x, y) = 1 - y^{b-a}\left(xy^{k-1} + g^\star(y)\right)^b$$

as in 6.8.2) *define inequivalent embeddings of $\mathbb{C}^*$ for different choices of b, a and k. No equivalence occurs between the two types.*

(ii) *An embedding of $\mathbb{C}^*$ has two very good asymptotes precisely when it has, in suitably labelled coordinates, an equation of the form*

$$G(x, y) = x^\alpha y^\beta - 1, \quad \alpha, \beta \geqslant 1, \qquad GCD(\alpha, \beta) = 1.$$

(*This corresponds to the cases* 6.8.1.1, 6.8.2(a) *with $g^\star = 0$, and* 6.6.2(b).)

(iii) *In the situation of* 6.8, *if U has a very good and a good asymptote, then its equation is*

$$G(x, y) = y - \left(xy^k + 1\right)^b, \quad b > 1,\ k \geqslant 1,$$

or

$$G(x, y) = 1 - y\left(xy^{k-1} + 1\right)^b, \quad b \geqslant 1,\ k > 1.$$

No equivalence occurs between these two types. (*This corresponds to the last equation in* 6.8.1.1, *where we can assume $c = 1$ with suitable choice of x', y, and to the final equations in* 6.8.1.2(a) *and* 6.8.2(a).)

7.4.1. Remark. Among the embeddings described in 7.4, precisely those in (iii) will be equivalent to one of those in 6.9 or 6.10, with the roles of x and y reversed.

7.5. Proposition. *In the following cases of* 6.9 *and* 6.10*,* $\tilde{\lambda}$ *is asymptotic with* $\{x = 0\}$ *as a very good asymptote and hence the embedding is equivalent, with an interchange of* x *and* y*, to a case of* 7.4(iii).

(i) *The* **(gga+)***-case* 6.9.1.
  (i.1) $k = 1,\ p = 1,\ s \geqslant 1,\ g = 1,\ G = 1 - x(xy + 1)^s$;
  (i.2) $k = 2,\ p = 1,\ s = 1,\ g = y + 1,\ G = -(xy + 1)^2 - x$.
(ii) *The* **(gga−)***-case* 6.9.2.
  (ii.1) $k = 1,\ p = 1,\ s \geqslant 2,\ g = 1,\ G = 1 - x(xy + 1)^s$;
  (ii.2) $k = 2,\ s = 2,\ p = 1,\ g = -y + 1,\ G = x + (xy - 1)^2$.
(iii) *The* **(bga+)***-case* 6.10.1.
  (iii.1) $k = 1,\ p \geqslant 1,\ s \geqslant 1,\ g^\star = 0,\ G = (1 + yx^{s+1})^p x - 1$;
  (iii.2) $k = 2,\ p = 1,\ s \geqslant 1,\ 4g^\star = 1,\ G = x + (xy + 1)^{s+1}$.
(iv) *The* **(bga−)***-case* 6.10.2.
  (iv.1) $k = 1,\ p \geqslant 1,\ s \geqslant 1,\ sp \geqslant 2,\ g^\star = 0,\ G = (1 + yx^{s+1})^p - x$;
  (iv.2) $k = 2,\ p = 1,\ s \geqslant 2,\ g^\star = 1,\ G = (xy + 1)^{s+1} - x$.

Proof. These cases represent the solutions to $\deg(G(0, y) = 0$ under the assumption $\deg(g(y)) = k - 1$. We will see later that no other equivalences occur. See 7.11. □

7.6. Proposition. *In the following cases of* 6.9 *and* 6.10*,* $\tilde{\lambda}$ *is asymptotic with* $\{x = 0\}$ *as a good asymptote and hence the embedding is equivalent, with an interchange of* x *and* y*, to another case of* 6.9 *or* 6.10.

(i) *The* **(gga+)***-case* 6.9.1.
  (i.1) $k = 1,\ p = 2,\ s \geqslant 1,\ g = 1,$ *with characteristic pairs*

$$\binom{2s+1}{4s};\quad \binom{2s}{2(2s+1)},\ \binom{2}{1}.$$

  (i.2) $k = 2,\ p = 1,\ s = 2,\ g = y + 1,$ *with characteristic pairs*

$$\binom{3}{8};\quad \binom{4}{6},\ \binom{2}{1}.$$

  (i.3) $k = 2,\ p = 2,\ s = 1,\ g = 2y + 1,$ *with characteristic pairs*

$$\binom{3}{7};\quad \binom{4}{6},\ \binom{2}{3}.$$

  (i.4) $k = 3,\ p = 1,\ s = 1,\ g = -y^2 + y + 1,$ *with characteristic pairs*

$$\binom{2}{7};\quad \binom{3}{4}.$$

(ii) *The* **(gga−)**-*case* 6.9.2.
 (ii.1) $k=1,\ p=2,\ s\geqslant 2,\ g=1$, *with characteristic pairs*

$$\begin{pmatrix}2s\\2(2s-1)\end{pmatrix},\ \begin{pmatrix}2\\1\end{pmatrix};\quad \begin{pmatrix}2s-1\\4s\end{pmatrix}.$$

 (ii.2) $k=2,\ p=1,\ s=3,\ g=-y+1$, *with characteristic pairs*

$$\begin{pmatrix}3\\8\end{pmatrix};\quad \begin{pmatrix}4\\6\end{pmatrix},\ \begin{pmatrix}2\\1\end{pmatrix}.$$

 (ii.3) $k=3,\ p=1,\ s=2,\ g=-y^2-y+1$, *with characteristic pairs*

$$\begin{pmatrix}2\\7\end{pmatrix};\quad \begin{pmatrix}3\\4\end{pmatrix}.$$

(iii) *The* **(sq)**-*case* 6.9.3. $k=1,\ g=1$, *with characteristic pairs*

$$\begin{pmatrix}4\\6\end{pmatrix},\ \begin{pmatrix}2\\3\end{pmatrix};\quad \begin{pmatrix}3\\7\end{pmatrix}.$$

(iv) *The* **(bga+)**-*case* 6.10.1.
 $k=2,\ p=2,\ s\geqslant 1,\ g^\star=1$, *with characteristic pairs*

$$\begin{pmatrix}2s+3\\4s+8\end{pmatrix};\quad \begin{pmatrix}2(s+2)\\2(2s+3)\end{pmatrix},\ \begin{pmatrix}2\\1\end{pmatrix}.$$

(v) *The* **(bga−)**-*case* 6.10.2.
 (v.1) $k=2,\ p=2,\ s\geqslant 1,\ g^\star=1$, *with characteristic pairs*

$$\begin{pmatrix}2(s+1)\\2(2s+3)\end{pmatrix},\ \begin{pmatrix}2\\1\end{pmatrix};\quad \begin{pmatrix}2s+3\\4s+4\end{pmatrix}.$$

 (v.2) $k=3,\ p=1,\ s=1$ *with characteristic pairs*

$$\begin{pmatrix}2\\7\end{pmatrix};\quad \begin{pmatrix}3\\4\end{pmatrix}.$$

Proof. These cases represent the solutions to $\deg(G(0,y)=1$ under the assumption $\deg(g(y))=k-1$. We will see later that no other cases with two good asymptotes occur. See 7.11. □

Comparing characteristic pairs we deduce the following from 7.6.

7.6.1. Corollary. *With an interchange of x and y equivalent embeddings are provided by*

(i) *the* **(gga+)**-*case with* $k=1,\ p=2,\ s=\sigma\geqslant 2$ *and the* **(bga−)**-*case with* $k=2,\ p=2,\ s=\sigma-1$,
(ii) *the* **(gga+)**-*case with* $k=1,\ p=2,\ s=1$ *and the* **(gga+)**-*case with* $k=3,\ p=1,\ s=1$, *the* **(gga−)**-*case with* $k=2,\ p=1,\ s=2$ *and the* **(bga−)**-*case with* $k=3,\ p=1,\ s=1$,

(iii) *the* **(gga−)**-*case with* $k = 1$, $p = 2$, $s = \sigma \geqslant 3$ *and the* **(bga+)**-*case with* $k = 2$, $p = 2$, $s = \sigma - 2$,
(iv) *the* **(gga−)**-*case with* $k = 1$, $p = 2$, $s = 2$ *and the* **(gga+)**-*case with* $k = 2$, $p = 1$, $s = 2$ *and the* **(gga−)**-*case with* $k = 2$, $p = 1$, $s = 3$,
(v) *the* **sq**-*case with* $k = 1$ *and the* **(gga+)**-*case with* $k = 2$, $p = 2$, $s = 1$.

7.7. We study further equivalences between embeddings with two asymptotes. In view of 7.2, such an equivalence produces a direct or a cross match between the characteristic pairs of $\tilde{\lambda}$ and λ. We will determine the possibilities for this. To avoid a number of trivial repetitions, we exclude the following cases, easily settled separately:

$b = 1$ in case **gvga**,
$b - a = 1$, $k = 1$ in case **bvga**,
$k = 1$, $p = 1$, $s = 1$ in case **gga+**,
$k = 1$, $p = 2$, $s = 1$ in case **gga−**,
$k = 1$, $p = 1$ in cases **bga+**, **bga−**.

We note that in these cases U has a simple branch at infinity.

7.8. We will say an embedding U has *pair type* $(\tilde{h}, h)$ if the minimal number of characteristic pairs needed to describe $\tilde{\lambda}$ and λ is $\tilde{h}$ and h respectively.

It is to be noted that the values of $\tilde{h}$ and h depend on the parameters involved. We illustrate this by an example and then provide the complete list of possibilities.

7.8.1. Consider the **(gga+)**-case 6.9.1 with characteristic pairs

$$\begin{pmatrix} sp+1 \\ (k+1)(sp+1)-p \end{pmatrix}; \quad \begin{pmatrix} ksp \\ (k+1)sp \end{pmatrix}, \begin{pmatrix} sp \\ p \end{pmatrix}, \begin{pmatrix} p \\ 1 \end{pmatrix}.$$

If $k > 1$, $p > 1$, $s > 1$, the pair type is $(1, 3)$.
If $k > 1$, $p > 1$, $s = 1$, the pairs reduce to

$$\begin{pmatrix} p+1 \\ (k+1)(p+1)-p \end{pmatrix}; \quad \begin{pmatrix} kp \\ (k+1)sp \end{pmatrix}, \begin{pmatrix} p \\ p+1 \end{pmatrix}$$

and the pair type is (1,2).
If $k = 1$, $p > 1$, $s = 1$, he pairs reduce to

$$\begin{pmatrix} p+1 \\ p+2 \end{pmatrix}; \quad \begin{pmatrix} p \\ 3p+1 \end{pmatrix}$$

and the pair type is $(1, 1)$.

7.9. The following is the list of embeddings with two asymptotes grouped by pair type. Here **(type)**(κ, α, β) means that we substitute κ for k, α for a, β for b in the cases **gvga**, **bvga** and κ for k, α for p, β for s in the cases **gga±**, **bga±**.

7.9.1. Pair type (1,3):

$$(\mathbf{gga+})(>1,>1,>1),\ \begin{pmatrix} sp+1 \\ (k+1)(sp+1)-p \end{pmatrix};\quad \begin{pmatrix} ksp \\ (k+1)sp \end{pmatrix},\ \begin{pmatrix} sp \\ p \end{pmatrix},\ \begin{pmatrix} p \\ 1 \end{pmatrix}.$$

7.9.2. Pair type (2,2):

$$(1)\qquad (\mathbf{gga-})(>1,>1,>1),\ \begin{pmatrix} sp \\ ((k+1)s-1)p \end{pmatrix},\ \begin{pmatrix} p \\ 1 \end{pmatrix};\quad \begin{pmatrix} k(sp-1) \\ (k+1)(sp-1) \end{pmatrix},\ \begin{pmatrix} sp-1 \\ p \end{pmatrix}.$$

$$(2)\qquad (\mathbf{sq})(>1),\ \begin{pmatrix} 4 \\ 4k+2 \end{pmatrix},\ \begin{pmatrix} 2 \\ 3 \end{pmatrix};\quad \begin{pmatrix} 3k \\ 3(k+1) \end{pmatrix},\ \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

7.9.3. Pair type (1,2):

$$(1)\qquad (\mathbf{gvga})(>1,\geqslant 1,>1),\ \begin{pmatrix} b \\ (k+1)b-a \end{pmatrix};\quad \begin{pmatrix} kb \\ (k+1)b \end{pmatrix},\ \begin{pmatrix} b \\ a \end{pmatrix}.$$

$$(2)\qquad (\mathbf{gga+})(>1,1,>1),\ \begin{pmatrix} s+1 \\ (k+1)(s+1)-1 \end{pmatrix};\quad \begin{pmatrix} ks \\ (k+1)s \end{pmatrix},\ \begin{pmatrix} s \\ 1 \end{pmatrix}.$$

$$(3)\qquad (\mathbf{gga+})(>1,>1,1),\ \begin{pmatrix} p+1 \\ (k+1)(p+1)-p \end{pmatrix};\quad \begin{pmatrix} kp \\ (k+1)p \end{pmatrix},\ \begin{pmatrix} p \\ p+1 \end{pmatrix}.$$

$$(4)\qquad (\mathbf{gga+})(1,>1,>1),\ \begin{pmatrix} sp+1 \\ 2(sp+1)-p \end{pmatrix};\quad \begin{pmatrix} sp \\ (2s+1)p \end{pmatrix},\ \begin{pmatrix} p \\ 1 \end{pmatrix}.$$

$$(5)\qquad (\mathbf{gga-})(>1,1,>2),\ \begin{pmatrix} s \\ (k+1)(s-1 \end{pmatrix};\quad \begin{pmatrix} k(s-1) \\ (k+1)(s-1) \end{pmatrix},\ \begin{pmatrix} s-1 \\ 1 \end{pmatrix}.$$

$$(6)\qquad (\mathbf{bga+})(>1,>1,\geqslant 1),\ \begin{pmatrix} (s+1)p+1 \\ k((s+1)p+1)+p \end{pmatrix};\quad \begin{pmatrix} ((k-1)(s+1)+1)p \\ (k(s+1)+1)p \end{pmatrix},\ \begin{pmatrix} p \\ 1 \end{pmatrix}.$$

7.9.4. Pair type (2,1):

$$(1)\qquad (\mathbf{gga-})(1,>1,\geqslant 2),\ \begin{pmatrix} sp \\ (2s-1)p \end{pmatrix},\ \begin{pmatrix} p \\ 1 \end{pmatrix};\quad \begin{pmatrix} sp-1 \\ 2(sp-1)+p \end{pmatrix}.$$

$$(2)\qquad (\mathbf{bga-})(\geqslant 1,>1,\geqslant 1),\ \begin{pmatrix} (s+1)p \\ \big(k(s+1)+1\big)p \end{pmatrix},\ \begin{pmatrix} p \\ 1 \end{pmatrix};\quad \begin{pmatrix} (k-1)\big((s+1)p-1\big)+p \\ k\big((s+1)p-1\big)+p \end{pmatrix}.$$

$$(3)\qquad (\mathbf{sq})(1),\ \begin{pmatrix} 4 \\ 6 \end{pmatrix},\ \begin{pmatrix} 2 \\ 3 \end{pmatrix};\quad \begin{pmatrix} 3 \\ 7 \end{pmatrix}.$$

7.9.5. Pair type (1,1):

$$\textbf{(gvga)}(1, \geqslant 1, > 1),\ \begin{pmatrix} b \\ 2b-a \end{pmatrix}; \quad \begin{pmatrix} b \\ 2b+a \end{pmatrix}. \tag{1}$$

$$\textbf{(bvga)}(> 1, \geqslant 1, > 1),\ \begin{pmatrix} b \\ (k+1)b-a \end{pmatrix}; \quad \begin{pmatrix} kb-a \\ (k+1)b-a \end{pmatrix}. \tag{2}$$

$$\textbf{(gga+)}(1, 1, > 1),\ \begin{pmatrix} s+1 \\ 2s+1 \end{pmatrix}; \quad \begin{pmatrix} s \\ 2s+1 \end{pmatrix}. \tag{3}$$

$$\textbf{(gga+)}(> 1, 1, 1),\ \begin{pmatrix} 2 \\ 2k+1 \end{pmatrix}; \quad \begin{pmatrix} k \\ k+1 \end{pmatrix}. \tag{4}$$

$$\textbf{(gga+)}(1, > 1, 1),\ \begin{pmatrix} p+1 \\ p+2 \end{pmatrix}; \quad \begin{pmatrix} p \\ 3p+1 \end{pmatrix}. \tag{5}$$

$$\textbf{(gga}-\textbf{)}(> 1, 1, 2),\ \begin{pmatrix} 2 \\ 2k+1 \end{pmatrix}; \quad \begin{pmatrix} k \\ k+1 \end{pmatrix}. \tag{6}$$

$$\textbf{(gga}-\textbf{)}(1, 1, \geqslant 3),\ \begin{pmatrix} s \\ 2s-1 \end{pmatrix}; \quad \begin{pmatrix} s-1 \\ 2s-1 \end{pmatrix}. \tag{7}$$

$$\textbf{(bga+)}(> 1, 1, \geqslant 1),\ \begin{pmatrix} s+2 \\ k(s+2)+1 \end{pmatrix}; \quad \begin{pmatrix} (k-1)(s+1)+1 \\ k(s+1)+1 \end{pmatrix}. \tag{8}$$

$$\textbf{(bga+)}(1, > 1, \geqslant 1),\ \begin{pmatrix} (s+1)p+1 \\ (s+2)p+1 \end{pmatrix}; \quad \begin{pmatrix} p \\ (s+2)p+1 \end{pmatrix}. \tag{9}$$

$$\textbf{(bga}-\textbf{)}(> 1, 1, \geqslant 1),\ \begin{pmatrix} s+1 \\ k(s+1)+1 \end{pmatrix}; \quad \begin{pmatrix} (k-1)s+1 \\ ks+1 \end{pmatrix}. \tag{10}$$

7.10. Under the restrictions of 7.7, the following is the list of cross-matches between the characteristic pairs of $\tilde{\lambda}$ and λ.

(1) **(gvga)**$(1, 1, 2)$ and **(gga+)**$(1, 1, 2)$.
(2) **(gvga)**$(1, 1, 2)$ and **(gga−)**$(2, 1, 2)$.
(3) **(gvga)**$(1, 1, s+2)$ and **(bga+)**$(2, 1, s)$.
(4) **(bvga)**$(2, 1, 2)$ and **(gga+)**$(1, 1, 2)$.
(5) **(bvga)**$(2, s, s+1)$ and **(gga−)**$(1, 1, s+2)$.
(6) **(bvga)**$(s+2, p-1, p)$ and **(bga+)**$(1, p, s)$.
(7) **(bvga)**$(s+2, 1, p)$ and **(bga−)**$(1, p, s)$.
(8) **(gga+)**$(3, 1, 1)$ and **(gga+)**$(1, 2, 1)$.
(9) **(gga+)**$(1, 2, 1)$ and **(gga−)**$(3, 1, 2)$.
(10) **(gga+)**$(1, 2, 1)$ and **(bga−)**$(3, 1, 1)$.
(11) **(gga+)**$(2, 1, 2)$ and **(gga−)**$(1, 2, 2)$.

(12) (**gga+**)$(2, 2, 1)$ and (**sq**)(1).
(13) (**gga+**)$(2, 1, 3)$ and (**gga−**)$(1, 2, 2)$.
(14) (**gga+**)$(1, 2, s + 1)$ and (**bga−**)$(2, 2, s)$.
(15) (**bga+**)$(2, 2, s)$ and (**gga−**)$(1, 2, s + 2)$.

7.11. Proposition. *The lists in* 7.5 (*resp.* 7.6) *of embeddings with a very good and a good* (*resp. two good*) *asymptotes are complete.*

Proof. Cases (1) to (7) of 7.10 show this for 7.5 and cases (8) to (15) for 7.6. □

7.12. Under the restrictions of 7.7, the following is the list of direct matches between the characteristic pairs of $\tilde{\lambda}$ and λ.

(1) (**gga+**)$(1, 1, s)$ and (**gga−**)$(k, 1, s + 1)$.
(2) (**gga+**)$(k, 1, 1)$ and (**gga−**)$(k, 1, 2)$ and (**bga−**)$(k, 1, 1)$.
(3) (**gga−**)$(k, 1, 2)$ and (**bga−**)$(k, 1, 1)$.
(4) (**bga+**)$(k, 1, s)$ and (**bga−**)$(k, 1, s + 1)$.

It is straightforward to verify that there are equivalences to produce these matches. As an example we treat (2). In case (**gga+**)$(k, 1, 1)$ we have

$$G(x, y) = \frac{y + H - H^2}{y^k}$$

with $H(x, y) = xy^k + g(y)$. If we put $x = (-1)^k\xi$, $y = -\eta$, we have

$$G(x, y) = \tilde{G}(\xi, \eta) = (-1)^{k-1}\frac{\eta + \tilde{H}^2 - \tilde{H}}{\eta^k},$$

and this is an instance of (**gga−**)$(k, 1, 2)$ with $\tilde{H}(\xi, \eta) = \xi\eta^k + g(-\eta)$. Writing $G = yH^* + 1$, we find

$$G(x, y) = \frac{1 - H^* - yH^{*2}}{y^{k-1}}.$$

Here the substitution $x = (-1)^{k-1}\xi$, $y = -\eta$ shows that this is an instance of (**bga−**)$(k, 1, 1)$.

8. Summary

We summarize our results as follows. Equations are to be understood as "up to a suitable choice of coordinates."

8.1. Theorem. *The special one place curves introduced in* 1.4 *have one of the following equations in* $X = Spec\,\mathbb{C}[u, v]$. *See* 6.1.

(i) $v^a - u^b = 0$, $GCD(a, b) = 1$. *See* 6.2.3.
(ii) $(v + u^s)^p - u^{sp+1} = 0$, $s, p \geqslant 1$. *See* 6.3.1.

(iii) $(v+u^s)^p - u^{sp-1} = 0$, $s, p \geqslant 1$, $s \geqslant 2$. *See* 6.3.2.
(iv) $v - 16v^2 + 4uv - 8u^2v + u^3 - u^4 = 0$. *See* 6.3.3.

8.2. Theorem. *Equations here are w.r.t.* $S = Spec\,\mathbb{C}[x, y]$. *See* 6.6.1.

(i) *All embeddings of* $\mathbb{C}^*$ *with a very good asymptote are listed in* 6.8. *All equivalences between these embeddings are described in* 7.4. *The equations are*

(i.1) $$G(x, y) = y^a - \bigl(xy^k + g(y)\bigr)^b,$$
with $a, b \geqslant 1$, $GCD(a, b) = 1$, $k \geqslant 1$, $g(0) = 1$ *and* g *otherwise arbitrary of degree at most* $k - 1$,

(i.2) $$G(x, y) = 1 - y^{b-a}\bigl(xy^{k-1} + g^\star(y)\bigr)^b,$$
with $b > a \geqslant 1$, $GCD(a, b) = 1$, $k \geqslant 1$, $g^\star$ *arbitrary of degree at most* $k - 2$.

(ii) *All embeddings of* $\mathbb{C}^*$ *with a good asymptote are listed in* 6.9 *and* 6.10. *These embeddings have two asymptotes. The equations are*

(ii.1) $$G(x, y) = \frac{(y + (xy^k + g(y))^s)^p - (xy^k + g(y))^{sp+1}}{y^k}$$
with $s, p \geqslant 1$, $k \geqslant 1$, $g(0) = 1$ *and* g *of degree* $\leqslant k - 1$ *uniquely determined by the fact that* G *is a polynomial,*

(ii.2) $$G(x, y) = \frac{(y + (xy^k + g(y))^s)^p - (xy^k + g(y))^{sp-1}}{y^k}$$
with $s, p \geqslant 1$, $sp \geqslant 2$, $k \geqslant 1$, $g(0) = 1$ *and* g *of degree* $\leqslant k - 1$ *uniquely determined by the fact that* G *is a polynomial,*

(ii.3) $$G(x, y) = \frac{y - 16y^2 + 4y(xy^k + g(y)) - 8y(xy^k + g(y))^2 + (xy^k + g(y))^3 - (xy^k + g(y))^4}{y^k}$$
with $k \geqslant 1$, $g(0) = 1$ *and* g *of degree* $\leqslant k - 1$ *uniquely determined by the fact that* G *is a polynomial,*

(ii.4) $$G(x, y) = \frac{(1 + y(xy^{k-1} + g^\star(y))^{s+1})^p(xy^{k-1} + g^\star(y)) - 1}{y^{k-1}}$$
with $s, p \geqslant 1$, g^* *of degree* $\leqslant k - 2$ *uniquely determined by the fact that* G *is a polynomial,*

(ii.5) $$G(x, y) = \frac{(1 + y(xy^{k-1} + g^\star(y))^{s+1})^p - (xy^{k-1} + g^\star(y))}{y^{k-1}}$$
with $s, p \geqslant 1$, g^* *of degree* $\leqslant k - 2$ *uniquely determined by the fact that* G *is a polynomial.*

(iii) *All equivalences between an embedding from* (i) *and one from* (ii) *are listed in* 7.5. *These embeddings have a very good and a good asymptote.*

(iv) *Equivalences between two embeddings from* (ii) *are of two kinds*:
(iv.1) *Equivalences that interchange the two asymptotes. These are listed in* 7.6.1. *The embeddings then have two good asymptotes.*
(iv.2) *Equivalences that preserve the two asymptotes. These are listed in* 7.12.

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