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Separatrices and Solitary Periodic Solutions

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1. INTRODUCTION

A separatrix trajectory of a general solution to an ordinary differential equation is one which differs topologically from near by trajectories. Maximal regions of parallel flow are separated by a union of separatrices. The structure of these solutions has been a useful tool in the qualitative theory, especially in the plane (see, for example, [2, 6, 7]). We will consider the separatrix structure of a flow near a solitary periodic solution in 3-space (cf. [5]). A periodic orbit γ is solitary if it has a compact neighborhood (neighborhood of solitude) U such that any negative (respectively, positive) semitrajectory contained in U has its α-limit (resp., ω-limit) at γ.

Within a neighborhood of solitude, trajectories are distinguished by their eventual behavior in time. For those sets of trajectories which are elliptic, that is, are contained in U and hence have α- and ω-limit at γ, an analysis of separatrix structure in a slightly different situation has already been set forth [4]. Our interest here will be primarily in the set A_+ of positively attracted trajectories, i.e., those which have ω-limit at γ, but leave U in the negative time direction.

In contrast with the situation in two-dimensional settings, where each separatrix trajectory is thought of as separating two canonical regions, our analysis of separatrix structure must be concerned with connected components of the union of all separatrices. Thus, whereas a study of a planar flow is concerned with the geometry of individual trajectories, we must concern ourselves with the geometry of "surfaces" of separatrices. In general, these surfaces may be quite different from manifolds, even for C^∞ flows. Our approach here is to restrict attention to those flows whose separatrix sets satisfy some kind of "manifold hypothesis." We will also demand that no positive semitrajectory with initial state in the boundary of A_+ be internally tangent to the boundary of the neighborhood of solitude. Under a strict version of these hypotheses, a classification of regions of A_+ is given by boundary type.

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We find that in every case but one, the separatrix structure is trivial and trajectories are uniformly asymptotic to γ. In the remaining case, where the region of A_+ is a fan, the separatrix structure is closely associated with nearby elliptic regions.

To increase understanding by means of comparison, we weaken the manifold hypothesis in the final sections. A few properties will be discussed, and some examples will be presented which emphasize the differences between the weak and strict version of the manifold hypothesis.

There are many basic properties of solitary periodic solutions and of separatrices which do not depend on any additional hypotheses. Some of these are set forth in the next section.

2. Definitions, Basic Properties

All our definitions are given for flows on compact spaces. The case of a flow on $\mathbf{R}^n$ generated by a vector field η is easily handled by forming the one point compactification of $\mathbf{R}^n$ and declaring that the point ∞ will be a zero of the "extension" of η. Let $\phi: X \times \mathbf{R} \to \mathbf{R}$ be a continuous flow on a compact metric space X; let $\alpha(x, \phi)$, $\omega(x, \phi)$, respectively, denote the α- and ω-limits of the ϕ-trajectory through x. We will use $\alpha(x)$, $\omega(x)$ when the context is clear. Let $d(\cdot, \cdot)$ denote the distance on X and let $d_{\mathrm{H}}(\cdot, \cdot)$ denote the induced Hausdorff distance between compact subsets of X. Continuity of set valued maps will always be with respect to the Hausdorff metric topology.

We say that ϕ is *positively parallel* near its trajectory through x and write $x \in \mathscr{P}^+(\phi)$ (or $x \in \mathscr{P}^+$ when context is clear) if ω is continuous at x and if there is a neighborhood N of x and a function $T(\epsilon)$ such that $d(\phi(y, t), \omega(x)) < \epsilon$ whenever $y \in N$, $t > T(\epsilon)$, and $\epsilon > 0$. Using $\alpha(x)$ and negative times, one similarly defines ϕ being negatively parallel near the trajectory through x, $x \in \mathscr{P}^-(\phi)$. If $x \in \mathscr{P} = \mathscr{P}^+ \cap \mathscr{P}^-$, then ϕ is *parallel* near x. The *separatrix set* $\mathscr{S} = \mathscr{S}(\phi) = X - \mathscr{P}$ is partitioned into three parts: the *boundary separatrix set*, $\mathscr{S}_0 = \mathscr{S}_0(\phi) = \{x \mid \alpha \text{ or } \omega \text{ is discontinuous at } x\}$, the *primary separatrix set*, $\mathscr{S}_1 = \mathscr{S}_1(\phi) = \{x \in \mathscr{S} - \mathscr{S}_0 \mid \text{there exists } x_n \to x,\ t_n \to \infty \text{ such that either } \lim \phi(x_n, -t_n) \notin \alpha(x) \text{ or } \lim \phi(x_n, t_n) \notin \omega(x)\}$, and the *secondary separatrix* set, $\mathscr{S}_2 = \mathscr{S}_2(\phi) = \mathscr{S} - (\mathscr{S}_0 \cup \mathscr{S}_1)$. Notice that $\mathscr{S}_1$ consists of trajectories which have a nontrivial prolongation (cf. [1]): the α-prolongation of the ϕ-trajectory through x is given by $\Lambda^-(x, \phi) = \Lambda^-(x) = \{y \mid y \notin \alpha(x) \text{ and there exists } x_n \to x,\ t_n \to -\infty, \text{ such that } \phi(x_n, t_n) \to y \text{ as } n \to \infty\}$; the ω-prolongation is defined similarly using $t \to +\infty$.

Remark 2.1. Where α and ω are continuous, $\mathscr{P}^+$, $\mathscr{P}^-$, and $\mathscr{P}$ are open.

Remark 2.2. $\mathscr{S}_2 \subset \mathscr{C}\ell(\mathscr{S} - \mathscr{S}_2)$ ($\mathscr{C}\ell$ = closure).

THEOREM 2.3. *Let V be a ϕ-invariant open region of X contained in $X - (\mathscr{S}_0 \cup \alpha(X) \cup \omega(X))$. Then there is a global cross section to ϕ on $\mathscr{P} \cap V$, that is, there is a continuous function $g: \mathscr{P} \cap V \to \mathbf{R}$, strictly monotone on trajectories and such that $g^{-1}(0)$ intersects every trajectory in $\mathscr{P} \cap V$. Moreover, if $\mathscr{P} \cap V$ has a smooth (C^∞) manifold structure (compatible with d) and if ϕ is generated by a continuous vector field, then g may be chosen smooth with nonzero derivative along trajectories.*

Proof. The essence of this proof is due to Wilson, [9, Sect. 1]. Let $\lambda = \alpha(X) \cup \omega(X)$. Although Wilson was concerned only when λ is a periodic solution, his argument remains valid when λ is compact as in the case at hand. For the sake of completeness, we summarize the main ideas of his proof. Define the function $g_+, g_-: \mathscr{P} \cap V \to \mathbf{R}$ by

$$
\begin{aligned}
g_+(x) &= \sup\{d(\phi(x, t), \lambda)((1 + 2t)/(1 + t)) \mid t \geqslant 0\}, \\
g_-(x) &= \sup\{d(\phi(x, t), \lambda)((1 - 2t)/(1 - t)) \mid t \leqslant 0\}.
\end{aligned}
\tag{2.4}
$$

Using the uniformity condition from the definition of $\mathscr{P}$, one establishes the continuity of g_+, and shows that the derivative $\lim_{t \to 0^+} \sup((g_+\phi(x, t) - g_+(x))/t)$ is negative and uniformly bounded away from 0 on compact subsets of $\mathscr{P} \cap V$. The same properties hold for $g = g_+ - g_-$. If $\mathscr{P} \cap V$ is smooth, then using [10, 2.5] one may assume g is a smooth function. Finally, a consideration of the limiting values of g_+ and g_- on each trajectory as $t \to \pm\infty$ shows that g is somewhere zero.

THEOREM 2.5. *Let V and λ be as in 2.3 and let $x \in V$. Then a necessary and sufficient condition that $x \in \mathscr{P}$ is the existence of local surface of section W to ϕ through x intersecting each trajectory at most once, such that $\phi(\mathscr{C}\ell(W), \mathbf{R})$ is closed in $M - \lambda$.*

This was proved in [4] in a straightforward fashion, using our Theorem 2.3 to establish the necessity.

In regions where α and ω are continuous, it might be expected that $\mathscr{P}$ is not only open, but also dense in $M - \lambda$. However, Beck [2] has constructed an example of a flow in the plane which, when extended to the 2-sphere S^2, has $\alpha(S^2) = \omega(S^2) = \{\infty\}$ and has a dense set of primary separatrices. Hence every trajectory is a separatrix, by Remarks 2.1 and 2.2. Such examples motivate the following theorem, first proved in [4] for different definitions.

THEOREM 2.6. *The set $\mathscr{S}_1$ is of first category in X.*

Proof. We need some notation and a lemma. For each $\epsilon > 0$ let $\mathscr{S}_+(\epsilon)$ be the set of all $x \in \mathscr{S}_1$ such that there exist sequences $x_n \to x$, $t_n \to +\infty$ satisfying $\phi(x_n, t_n) \to y$ with $d(y, \omega(x)) > \epsilon$. Define $\mathscr{S}_-(\epsilon)$ similarly using α and $t_n \to -\infty$.

Noting that $\mathscr{S}_+(\epsilon_1) \subset \mathscr{S}_+(\epsilon_2)$ whenever $\epsilon_1 \geqslant \epsilon_2$, and similarly for $\mathscr{S}_-$, we can write $\mathscr{S}_1$ as the countable union $\bigcup(\mathscr{S}_+(1/n) \cup \mathscr{S}_-(1/n))$ taken over all positive integers n. In that which follows we will concern ourselves only with $\mathscr{S}_+(\epsilon)$ since analogous statements can be established for $\mathscr{S}_-(\epsilon)$ by considering ϕ with time orientation reversed. We complete the proof of the theorem with a lemma.

LEMMA 2.7. *$\mathscr{S}_+(\epsilon)$ is nowhere dense.*

Proof. Let $t(x, y) = \sup\{t \in \mathbf{R} \mid d(\phi(y, t), \omega(x)) > \epsilon\}$. If $x \in \mathscr{S}_+(\epsilon)$, then ω is continuous at x and $t(x, y)$ is finite for y close enough to x, but because $\Lambda^+(x)$ is not contained in the ϵ-neighborhood of $\omega(x)$, the function $t(x, \cdot)$ must be unbounded in every neighborhood of x.

Suppose the lemma is false. Then there is a nonempty open set V_0 contained in $\mathscr{C}\ell(\mathscr{S}_+(\epsilon))$ and a point $x_1 \in V_0 \cap \mathscr{S}_+(\epsilon)$. Since ω is continuous at points of $\mathscr{S}_1$, we can find an open neighborhood V_1 of x_1 with $V_1 \subset V_0$ and such that $d_H(\omega(x_1), \omega(x')) < \epsilon/4$ if $x' \in V_1$. Pick $T > 0$. Since $x_1 \in \mathscr{S}_+(\epsilon)$, $t(x_1, \cdot)$ is unbounded on V_1, so there is a point u_1 in V_1 and a corresponding $t_1 \geqslant T$ such that $d(\phi(u_1, t_1), \omega(x_1)) > \epsilon$, and hence $\phi(u_1, t_1)$ is in the open set

$$W = \{y \mid d(y, \omega(x_1)) > \tfrac{3}{4}\epsilon\}.$$

Choose an open set V_2 containing u_1, of diameter less and $\frac{1}{2}$, and with its closure contained in $V_1 \cap \phi(W, -t_1)$. Since $\mathscr{S}_+(\epsilon)$ is dense in V_0 and so also in V_2, we can find $x_2 \in V_2 \cap \mathscr{S}_+(\epsilon)$. Then $t(x_2, \cdot)$ is unbounded in V_2, so there is a point $u_2 \in V_2$ and a corresponding $t_2 \geqslant 2T$ such that $d(\phi(u_2, t_2), \omega(x_2)) > \epsilon$. Hence $d(\phi(u_2, t_2), \omega(x_1)) \geqslant d(\phi(u_2, t_2), \omega(x_2)) - d_H(\omega(x_2), \omega(x_1)) > \frac{3}{4}\epsilon$ and so $\phi(u_2, t_2) \in W$. Proceeding recursively, given $u_n \in V_n$ with $\operatorname{diam}(V_n) \leqslant 1/n$ and $\mathscr{C}\ell(V_n) \subset V_{n-1}$, and given $t_n \geqslant nT$ satisfying $\phi(u_n, t_n) \in W$, we find an open set V_{n+1} with diameter less than $1/(n+1)$ containing u_n and with $\mathscr{C}\ell(V_{n+1}) \subset V_n \cap \phi(W, -t_n)$. Since $\mathscr{S}_+(\epsilon)$ is dense in V_0, so is it in V_{n+1} and we can pick $x_{n+1} \in V_{n+1} \cap \mathscr{S}_+(\epsilon)$. This means that $t(x_{n+1}, \cdot)$ is unbounded in V_{n+1} so there is a point $u_{n+1} \in V_{n+1}$ and a corresponding $t_{n+1} \geqslant (n+1)T$ satisfying $d(\phi(u_{n+1}, t_{n+1}), \omega(x_{n+1})) > \epsilon$. So

$$d(\phi(u_{n+1}, t_{n+1}), \omega(x_1)) \geqslant d(\phi(u_{n+1}, t_{n+1}), \omega(x_{n+1})) - d_H(\omega(x_{n+1}), \omega(x_1)) > \tfrac{3}{4}\epsilon,$$

that is, $\phi(u_{n+1}, t_{n+1}) \in W$.

Note that $V_n \subset V_k$ whenever $n \geqslant k$, and that $\phi(V_n, t_k) \subset W$ whenever $n > k$. Since the diameter of V_n goes to 0 as n becomes large, we have by the finite intersection property of compact sets that the intersection $\bigcap \mathscr{C}\ell(V_n)$ taken over all n is a singleton, which we denote by $\{y\}$. Then y is in V_n and $d(\phi(y, t_n), \omega(x_1)) > \frac{3}{4}\epsilon$, so that $d_H(\omega(x_1), \omega(y)) \geqslant \frac{3}{4}\epsilon$. But $y \in V_1$, which is in contradiction with the defining property of V_1. Hence the lemma is proved, as is Theorem 2.6.

The following lemma supports a very basic part of our intuition.

LEMMA 2.8. *Suppose X is a manifold (C^0 is sufficient). Then $\omega(x)$ meets the closure of each connected component of $A^+(x)$.*

Proof. Let $\{W_n\}$ be a fundamental sequence of neighborhoods at x with each W_n homomorphic to an open ball in Euclidean space. Then

$$y \in \bigcap_n \mathscr{C}\ell(\phi(W_n, [n, \infty))).$$

Pick $x_n \in W_n$ and $t_n \geqslant n$ such that $x_n \to x$ and $\phi(x_n, t_n) \to y$. Choose an arc a_n in $\mathscr{C}\ell(\phi(W_n, [n, \infty)))$ which originates at $\phi(x_n, t_n)$ and terminates at $\phi(x, t_n)$. If m is an integer such that $1/m < d(y, \omega(x))$, let $B_m = \{u \in X \mid d(u, \omega(x)) \leqslant 1/m\}$. For n large enough, a_n intersects B_m, and for such n we let $\bar{a}_n$ be the subarc of a_n originating at $\phi(x_n, t_n)$ and terminating at the first (in the parameterization of a_n) point where a_n meets B_m. In the Hausdorff metric topology of compact subsets of M, the sequence $\{\bar{a}_n\}$ has a subsequence converging to a continuum α_m. Note that $y \in \alpha_m$ and $\alpha_m \subset A^+(x)$, so that α_m must be contained in the same component of $A^+(x)$ as y. Furthermore, $\alpha_m \cap B_m \neq \varnothing$, and since X is compact lim sup$\{\alpha_m \cap B_m\}$ is nonempty. But such a limit is simultaneously contained in $\omega(x)$ and in the closure of the component of $A^+(x)$ under consideration.

Remark 2.9. An easy modification of the above proof maintains the result when X is locally connected. Obviously, the statement dual to 2.8 regarding A^- and α is valid. Furthermore, if $\mathscr{S}_1 = A^-(X) \cup A^+(X)$ (in particular whenever α and ω are continuous), each component of $\mathscr{S}_1$ is connected $\alpha(X) \cup \omega(X)$. Finally, if one defines the analogs of α, ω, A^-, and A^+ for actions of the integers on X, then a similar lemma holds. This point of view is particularly convenient when a Poincaré map is induced on a surface of section, and we will adopt it in the next section.

3. Solitary Periodic Solutions

We now suppose that ϕ is generated by a smooth (C^∞) vector field η on 3-space. We furthermore assume that ϕ has a solitary periodic solution γ, which is not a rest point.

LEMMA 3.1. *There is a neighborhood of solitude U for γ satisfying:*

(i) *U is diffeomorphic with $D^2 \times S^1$ such that $\gamma = \{0\} \times S^1$*

($D^2 = \{z \in \mathbf{C} \mid |z| \leqslant 1\}$;

$S^1 = \partial D^2$; ∂ = boundary of manifold with boundary);

(ii) *There is a Riemannian metric on U such that if η is represented in $D^2 \times S^1$ coordinates as $\eta(z, \theta) = (\dot{z}, \dot{\theta})$, then $\dot{\theta} \equiv 2\pi$.*

Proof. We may choose a tubular neighborhood $f\colon \mathbf{R}^2 \times S^1 \to \mathbf{R}^3$ of γ such that $\gamma = f(\{0\} \times S^1)$ and so that $\phi(x, \cdot)\colon \mathbf{R} \to \gamma$ is a covering of $f \mid \{0\} \times S'$ for some $x \in \gamma$. In the coordinates given by f, $\dot{\theta} \equiv 1$ on γ. The continuity of $\dot{\theta}$ assures that a disk D, in $\mathbf{R}^2$ centered at the origin, may be chosen small enough that $\dot{\theta}$ is 0 nowhere on $U = f(D^2 \times S^1)$. On the tangent space of U, we multiply the given coordinates by $2\pi/\dot{\theta}$ to obtain new coordinates with $\dot{\theta} \equiv 2\pi$, as desired.

Given solitude U around γ satisfying 3.1, we would like to perturb the boundary of U slightly to obtain nice tangency with η. This is done in 3.3 after "nice" is defined in 3.2. By "circle" we mean a homeomorphic image of S^1.

DEFINITION 3.2. The vector field η has *generic contact* with the submanifold ∂U in $\mathbf{R}^3$ if the following conditions are satisfied: the subset of ∂U where η is tangent to ∂U is either empty or a finite set of circles collectively denoted by τ. Each circle separates a region of egress from a region of ingress of η relative to U. Everywhere on τ the component of η normal to ∂U (it is 0 precisely on τ) has nonzero derivative in a direction in ∂U transverse to τ in ∂U. Furthermore the subset of τ where f is tangent to τ is either empty or a finite set of points, collectively denoted χ. Each point of χ separates an open subarc of τ where η points toward a region of ingress from an open subarc of τ where η points toward a region of egress. Finally, at each point of χ the component of η tangent to τ (is 0 precisely on χ by definition and) has nonzero derivative in a direction of τ.

The following lemma is a special case of [5, Theorem 1]. Its proof is given essentially by Percell [8, Proof of Theorem 2.5].

LEMMA 3.3. *Suppose U satisfies* 3.1 *and V is an open set containing U with $\eta \neq 0$ on V. Let $\mathscr{A}^r$ denote the space of C^r embeddings of $S^1 \times S^1$ in V with the C^r topology, and let $\mathscr{A}_0^r \subset \mathscr{A}^r$ consist of all embeddings f such that $f(S^1 \times S^1)$ has generic contact with η. If $r > 3$, then $\mathscr{A}_0^r$ is open and dense in $\mathscr{A}^r$.*

We will say that U is a generic neighborhood of solitude if ∂U has generic contact with η. *Henceforth we use U to denote a fixed generic neighborhood of solitude for γ, satisfying* 3.1. Notice that if x is a point of the tangency set $\tau \subset \partial U$ and $\eta(x)$ points from ingress to egress then the ϕ-trajectory through x is externally tangent to ∂U at x. Dually, if $\eta(x)$ points from egress to ingress, an internal tangency results. If $\eta(x)$ is tangent to τ, i.e., $x \in \chi$, then the ϕ-trajectory through x crosses from inside U to the outside, or vice versa, at χ. Hence χ is called the set of crossing tangencies.

Basic regions of U are defined as follows:

$$
\begin{aligned}
U_\pm &= \{x \in U \mid \phi(x, \pm t) \in U \text{ for all } t \geqslant 0\};\\
E &= U_+ \cap U_- - \gamma && \text{(elliptic set);}\\
A_+ &= U_+ - (E \cup \gamma) && \text{(positively attracted set);}\\
A_- &= U_- - (E \cup \gamma) && \text{(negatively attracted set);}\\
H &= U - (U_+ \cup U_-) && \text{(hyperbolic set).}
\end{aligned}
$$

Remark 3.4. U is an isolating neighborhood for γ in the sense of Conley and Easton [3] if and only if $E = \varnothing$.

Remark 3.5. It is convenient to notice that $U_{\pm}$ are closed in U, as is E in $U - \gamma$, whereas H is open in U and $U - \gamma$. However, $A_{\pm}$ may be open or closed or neither in $U - \gamma$.

In the local analysis we are developing, we would like the boundaries of E, H, $A_{\pm}$ to be composed of separatrices. However, due to the global nature of the definition of separatrix, this is usually not so, since E, H, $A_{\pm}$ are relative to our choice of U. Hence, a modification of definition is necessary for the local situation. Our naive viewpoint is that U should be like a point at infinity, and any trajectory which leaves U is lost from our view. Therefore, let g be a smooth nonnegative valued function on $\mathbf{R}^3$ with $g^{-1}(0) = \mathbf{R}^3 - \mathrm{Int}(U)$ (Int $=$ topological interior), and let ψ be the solution flow to the vector field $g \cdot \eta$. Use σ generically to denote a maximum trajectory segment of $\phi(\phi(x, I)$ where I is the maximum closed, not necessarily bounded, interval such that $\phi(x, I) \subset U$), and note that each σ is a union of ψ-trajectories. Say σ is a separatrix of ϕ relative to U if it contains a ψ-separatrix. Our notation:

$$
\begin{aligned}
\mathscr{S}(\phi, U) &= \{x \in \sigma \mid \sigma \cap \mathscr{S}(\psi) \neq \varnothing\};\\
\mathscr{S}_0(\phi, U) &= \{x \in \sigma \mid \sigma \cap \mathscr{S}_0(\psi) \neq \varnothing\};\\
\mathscr{S}_1(\phi, U) &= \{x \in \sigma \mid \sigma \not\subset \mathscr{S}_0(\phi, U) \text{ and } \sigma \cap \mathscr{S}_1(\psi) \neq \varnothing\};\\
\mathscr{S}_2(\phi, U) &= \mathscr{S}(\phi, U)(\mathscr{S}_0(\phi, U) \cup \mathscr{S}_1(\phi, U));\\
\mathscr{P}(\phi, U) &= U - \mathscr{S}(\phi, U).
\end{aligned}
\tag{3.6}
$$

Henceforth in the paper, we will be concerned only with separatrices of ϕ relative to U. Therefore we write $\mathscr{S}$, $\mathscr{S}_0$, $\mathscr{S}_1$, $\mathscr{S}_2$, and $\mathscr{P}$ to denote the respective sets above.

In order to clarify some relations between the tangency subset of U and the set of separatrices, we need some notation. Let τ_i, τ_e, χ, respectively, denote the sets of internal, external, and crossing tangencies. Let $\mathscr{T}$ denote the union of all ϕ-trajectory segments σ such that $\sigma \subset U$ and $\sigma \cap \tau_i \neq \varnothing$. Let Γ_+ (respectively, Γ_-) be the set of $x \in U$ where $\omega(x, \psi) = \gamma$ but $\omega(\cdot, \psi)$ is discontinuous at x (respectively, using α). Notice that if $\omega(x, \psi) \neq \gamma$, then it is a singleton subset of ∂U, in which case we slightly abuse notation to write $\omega(x, \psi) \in \partial U$, and similarly for α.

LEMMA 3.7. *$\omega(\cdot, \psi)$ is continuous at x if $\omega(x, \psi) \in \partial U - \tau_i$.*

Proof. Let $y = \omega(x, \psi) \in \partial U - \tau_i$. We will proceed, considering various cases. In each case, we choose local coordinates (u, v, w) in a neighborhood V of y in $\mathbf{R}^3$ so that the vector field η is given by $\dot{w} = 1$ and $\dot{u} = \dot{v} = 0$ and, in addition, τ and ∂U are "nicely" related to η in the local coordinates. In every case, the choice of such coordinates is based on a theorem of Percell [8, 2.2]. Since

$y = \phi(x, t)$ for some $t \geqslant 0$, there is no loss of generality in assuming $x \in V$. Our first case is $y \in \partial U - \tau$. Here (u, v, w) may be chosen so that, in addition, ∂U is given near y by $w = 0$. The resulting continuity of $\omega(\cdot, \psi)$ is now obvious. Next, suppose y is an external tangency of ϕ (so $x = y$). In this case the coordinates may be chosen to straighten η as above and so that $U \cap V$ is given by $v \geqslant w^2$ with $\tau_e \cap V$ given by $w = v = 0$. Again, the local coordinates make continuity of $\omega(\cdot, \psi)$ obvious. Finally, if $y \in \chi$ and $\eta(y)$ points from τ_i to τ_e, we may obtain:

$$U \cap V = \{(u, v, w) \mid v \geqslant w^3 - uw\},$$
$$\tau \cap V = \{(u, v, w) \mid v = w^3 - uw \text{ and } \partial v/\partial w = 0\},$$

and $y = (0, 0, 0)$. (If $\eta(y)$ points from τ_e to τ_i, we alter only the characterization of $U \cap V$ to $v \leqslant w^3 - uw$.) This viewpoint not only allows an easy verification of the continuity of $\omega(\cdot, \psi)$, but it also provides an interesting look at discontinuities along τ_i accumulating at χ.

COROLLARY 3.8. $\mathscr{S}_0 \subset \mathscr{T} \cup \Gamma_+ \cup \Gamma_-$.

We also have the following partial converse to 3.7.

THEOREM 3.9. *$\omega(\cdot, \psi)$ is discontinuous at x if $\omega(x, \psi) \in \tau_i$ and $x \notin \Lambda^+(x, \phi)$.*

Proof. Let $y = \omega(x, \psi) \in \tau_i$. Letting $t_0 \geqslant 0$ be determined by $\phi(x, t_0) = y$, we see that $\phi(x, t) \in U$ for some $t > t_0$ because $\phi(x, t_0) = y \in \tau_i$. Consider first the case where $\phi(x, t') \notin U$ for some $t' > t_0$. Then there are times t_1 and t_2 such that $0 \leqslant t_0 < t_1 < t_2$, $\phi(x, [0, t_1]) \subset U$, and $\phi(x, [t_1, t_2]) \cap U = \phi$. We may choose a neighborhood V_0 of y with local coordinates (u, v, w) satisfying the following: (i) η is given by $\dot{u} = \dot{v} = 0$, $\dot{w} = 1$; (ii) $U \cap V_0$, $\tau_i \cap V_0$, y are given, respectively, by $v \leqslant w^2$, $v = w = 0$, $(0, 0, 0)$; (iii) the intersection of $\phi(x, [0, t_2])$ with $\mathscr{C}\ell(V_0)$ is the trajectory segment $u = v = 0$. For $\epsilon > 0$ let B_ϵ be that subset of V_0 given by $u^2 + v^2 < \epsilon$ and $w = 0$. Then ϵ may be chosen small enough, along with a $\delta > 0$ so that $\phi(B_\epsilon, (-\delta, \delta)) \subset V_0$, so that $\phi(B_\epsilon, (-t_0 - \delta, t_2 - t_0 + \delta))$ is an open neighborhood of $\phi(x, [0, t_2])$ which intersects V_0 only in the cylinder $u^2 + v^2 < \epsilon$, and so that $\phi(B_\epsilon, t_2 - t_0 + \delta) \cap U \neq \varnothing$. It is clear from the last condition that $\omega(z, \psi) \in \partial U$ for all z in the neighborhood $\phi(B_\epsilon, (-t_0 - \delta, -t_0 + \delta))$ of x. But because of the properties of V_0, it is also clear that for an open set of such z, $\omega(z, \psi)$ is not in the neighborhood $\phi(B_\epsilon, (-\delta, \delta))$ of $y = \omega(x, \psi)$. Hence $\omega(\cdot, \psi)$ is not continuous at x in this case.

Finally consider the case where $\phi(x, t) \in U$ for all $t \geqslant 0$, but $x \in \Lambda^+(x, \psi)$. Let V_0 and its local coordinates be chosen as above. Choose a sequence $\{y_n\}$ in the region of V_0 given by $v < 0$ and such that $\lim y_n = y$ as $n \to \infty$. Letting $x_n = \phi(y_n, -t_0)$, we have $x_n \to x$. Suppose $\omega(\cdot, \psi)$ were continuous at x. Then $\omega(x_n, \psi) \to y$, so for each n there is a $t_n \geqslant 0$ such that $\omega(x_n, \psi) = \phi(x_n, t_n)$ and

$\phi(x_n, [0, t_n]) \subset U$. Since $\omega(x_n, \psi) \to y$, we also have $\phi(x_n, t_n - t_0) \to x$. But since $\omega(x, \phi) = \gamma$, we deduce $t_n \to \infty$. But then $x \in A^+(x, \phi)$, contradicting the lemma's hypothesis, so $\omega(\cdot, \psi)$ cannot be continuous at x.

COROLLARY 3.10. $\mathscr{T} \subset \mathscr{S}_0 \cup \mathscr{S}_1$.

Proof. We have just seen that if $x \in \mathscr{T} - \mathscr{S}_0$ then $x \in A^+(x, \phi) \cup A^-(x, \phi)$. Moreover, the actual procedure used also shows that $x \in A^+(x, \psi) \cup A^-(x, \psi)$, which shows that $x \in \mathscr{S}_1$.

THEOREM 3.11. $\mathscr{S}_0 \cup \mathscr{S}_1 = \mathscr{T} \cup \Gamma_+ \cup \Gamma_- \cup \mathscr{S}_1(\psi)$.

Proof. Inclusion of the right side in the left follows immediately from 3.6 and 3.10. For the converse question, 3.8 shows that it will suffice to show $\mathscr{S}_1 \subset \mathscr{T} \cup \mathscr{S}_1(\psi)$. Let σ be a maximum ϕ-trajectory segment in $\mathscr{S}_1$. Then σ is a union of ψ-trajectories and contains a primary ψ-separatrix. But σ contains more than one noncritical ψ-trajectory iff $\sigma \subset \mathscr{T}$, and rest points of ψ in σ are easily accounted for.

LEMMA 3.12. *$\mathscr{T}$ has Lebesgue measure zero and is nowhere dense.*

Proof. Since $\mathscr{T} \subset \mathscr{C}\ell(\times) \subset \phi(\tau \times \mathbf{R}) \cup \gamma$, the lemma follows from an application of Sard's theorem to the restriction of ϕ to $\tau \times \mathbf{R}$.

LEMMA 3.13. *Γ_+ is nowhere dense in U.*

Proof. Let V be any open set in U. If $x \in \Gamma_+ \cap V$, then for $\omega(\cdot, \psi)$ to be discontinuous at x, there must be a point $y \in V$ such that $\omega(y, \psi) \in U$. There are two possibilities: either $\omega(y, \psi) \in \partial U - \tau$ or $\omega(y, \psi) \in \tau$. In the former case, there is an open set V_1 containing y so that $\omega(z, \psi) \in \partial U - \tau$ for every $z \in V_1$. Hence the open subset $V_1 \cap V$ is excluded from $V \cap \Gamma_+$, and Γ_+ cannot be dense in V. In case $\omega(y, \psi) \in \tau$, the generic contect makes it easy to find a point z near y ($z \in V$) such that $\omega(z, \psi) \in \partial U - \tau$. This puts us back in the former case, with Γ_+ not dense in V. Since V was arbitrary, Γ_+ is nowhere dense.

THEOREM 3.14. *$\mathscr{S}_0 \cup \mathscr{S}_1$ is of first category in U.*

Proof. This is a direct consequence of 3.11–3.13 together with 2.6.

4. THE REGIONS $A_\pm$

Because of the duality between A_+ and A_-, we restrict our attention to A_+. Our first concern is $b(A_+)$ (b = topological boundary in U), especially in its relation to internal tangencies. Consider a point $x \in b(A_+) - \gamma$, and let

$L = L(x) = \sup\{t \mid \phi(x, [0, t]) \subset b(A_+)\}$. It is easily seen that L is infinite when $x \in b(A_+) \cap E$. Furthermore, if $x \in b(A_+) - E$, then $x \in b(H)$.

THEOREM 4.1. *If $x \in b(A_+) \cap b(H) - E$, then L may be finite or infinite. In the finite case $\phi(x, L)$ is an internal tangency with $\phi(x, [0, L]) \subset b(H)$ and $\phi(x, [L, \infty)) \subset \mathrm{Int}(A_+)$. If L is infinite, then $\phi(x, \mathbf{R}^+) \subset b(A_+) \cap b(H)$ and there is a $T \geqslant 0$ such that $\phi(x, (T, \infty))$ contains no internal tangencies. Furthermore if $x_n \to \phi(x, t)$ with $t > T$ and $x_n \in H$ (all n), then there is a divergent sequence of positive times, $\{t_n\}$, such that $\phi(x_n, [0, t_n]) \subset H$ (all n).*

Proof. To verify that L may be finite consider the vector field $\dot{z} = -z$, $\dot{\theta} \equiv 1$ on $\mathbf{R}^2 \times \mathscr{S}^1$ and let $\partial U \cap (\mathbf{R}^2 \cap \{\theta\})$ be as shown in Fig. 1. That L may be infinite is clear from considering a simple vector field $\dot{u} = u^2 + v^2$, $\dot{v} \equiv 0$, $\dot{\theta} \equiv 1$ for $(u, v, \theta) \in \mathbf{R}^2 \times S^1$, with $U = \{(u, v) \mid u^2 + v^2 \leqslant 1\} \times S^1$.

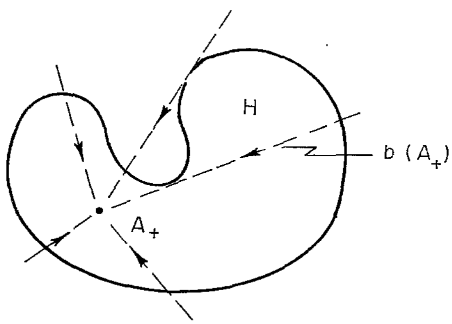

FIGURE 1

Suppose $x \in b(H)$ and L is finite. Now if $\phi(x, t) \in b(A_+) \cap \mathrm{Int}(U)$, then $\phi(x, t + \epsilon) \in b(A_+)$ for sufficiently small $\epsilon > 0$. Therefore, since L is finite, $\phi(x, L) \in \partial U$, $x \in B(A_+) \subset U_+$, $\phi(x, L)$ must be an internal tangency. Since x is in $\mathscr{C}\ell(A_+)$ which is positively ϕ-invariant, for $t > L$ we must have $\phi(x, t) \in \mathscr{C}\ell(A_+) - (b(A_+) \cup \gamma) = \mathrm{Int}(A_+)$, that is, $\phi(x, (L, \infty)) \subset \mathrm{Int}(A_+)$. This clearly precludes $\phi(x, s) \in E$ for $s \leqslant L$; therefore $\phi(x, [0, L]) \subset b(H)$.

Finally suppose $x \in b(H) - E$ and L is infinite. Then $\phi(x, \mathbf{R}^+) \subset b(A_+) - E = b(A_+) \cap b(H)$. Since $\omega(x) = \gamma$, we may find $T = \inf\{t \geqslant 0 \mid \phi(x, (t, \infty)) \subset \mathrm{Int}\, U\}$. If $x_n \to x$ with $x_n \in H$, ley $t_n = \sup\{t \mid \phi(x_n, [0, t]) \subset U\}$. Since $x_n \in H$, t_n is finite, and certainly $\phi(x_n[0, t_n]) \in H$. To show that $\{t_n\}$ diverges observe that for each $\epsilon > 0$ there is a neighborhood V_ϵ of x such that for some $t > 1/\epsilon$ $\phi(V_\epsilon, t)$ is contained in the ϵ-neighborhood of γ. Since $\{x_n\}$ is eventually in V_ϵ, we also eventually have $t_n > 1/\epsilon$. The proof is complete.

Any analysis of A_+ and its boundary is complicated by the possibility of $b(A_+)$ not being positively ϕ-invariant. A reasonable question is: can U be altered, preferably by a shrinking, to effect the positive invariance? Certainly it can in the simple example of Fig. 1. That situation is typical to the extent that in general, one would need to excise a portion of U bounded by ∂U and a "surface" of trajectory segments emanating from an arc of internal tangency. There ends the generality of our example. A first "excision" theorem of this sort is given in [5]. The scope of the entire problem is beyond our present efforts. However, on the basis of our implied conjecture and Theorem 4.1, we henceforth assume

(A) $\phi(b(A_+), \mathbf{R}^+)$ contains no internal tangencies ($\mathbf{R}^+ = \{t \in \mathbf{R} \mid t > 0\}$).

Remark 4.2. As an immediate consequence of Assumption A and 4.1, we have $\phi(b(A_+), \mathbf{R}^+) \subset b(A_+)$.

For purposes of intuition, we often think of separatrices as lying on smooth surfaces. This is not generally the case: examples are given in [4] of smooth flows where $\mathscr{S} = \mathscr{S}_1$ and $\mathscr{S}$ is not a manifold. The remainder of this paper will be concerned with the effect of a strict hypothesis of manifold structure for $\mathscr{S}$, with some consideration of a weakening of that hypothesis. The strict manifold hypothesis is

H. $(\mathscr{S}_0 \cup \mathscr{S}_1) \cap \mathscr{Cl}(A_+) - \gamma$ is an embedded submanifold of U.

By "submanifold of U" we mean that for each point x of $\mathscr{S}_0 \cup \mathscr{S}_1 - \gamma$ either (a) $x \in \operatorname{Int}(U)$ and the usual condition holds, or (b) $x \in \partial U$ and either the usual condition holds in U, or else there is a submanifold chart in the double of U for $\mathscr{S}_0 \cup \mathscr{S}_1 - \gamma$ at x. This weakened definition allows $\mathscr{S}_0$ or $\mathscr{S}_1$ to be internally tangent to ∂U.

Henceforth, let $M = (\mathscr{S}_0 \cup \mathscr{S}_1) \cap \mathscr{Cl}(A_+) - \gamma$.

THEOREM 4.3. *M is two dimensional.*

Proof. By 3.14, $\dim(M) \leqslant 2$, but since M is positively ϕ-invariant $\dim(M) \geqslant 1$. The ϕ-invariance of M, together with the transversality of ϕ with the coordinate disks $D_\theta = D^2 \times \{\theta\}$, implies that $M \cap D_\theta$ is an embedded submanifold of D_θ. It will suffice to show that no point of that submanifold is isolated in D_θ.

Let x be in $M \cap D_\theta$. We consider cases: $x \in \operatorname{Int}(A_+)$ or $x \in b(A_+)$. Assume the former. Then $x \notin \Gamma_+$. But $x \notin \mathscr{T}$ by 3.2, so that $x \notin \Gamma_-$. Hence $x \in \mathscr{S}_1(\psi)$, and in fact, one may check that $A^-(x, \psi) = \varnothing$ so that we may pick $y \in A^+(x, \psi) \cap D_\theta$. Then $x \in A^-(y, \psi)$ and consideration of 2.8 and 2.9 shows that x is connected to y within $D_\theta \cap A^-(y, \psi)$. But $A^-(y, \psi) \subset \mathscr{S}_1(\psi)$ and so x is not isolated in $M \cap D_\theta$.

If $x \in b(A_+)$ we consider two subcases: $x \in b(A_+) \cap b(H)$ or $x \in b(A_+) - b(H)$. In the former case, we may use 4.1 and 4.2 to find $y \in A^+(x, \psi)$ and proceed as above. In the latter case, we must have $x \in b(A_+) \cap (b(A_-) \cap b(E)) \subset U_+ \cap U_- = E \cup \gamma$, that is, $x \in E$. It is easy to obtain points of $b(A_+)$ if $\phi(x, \mathbf{R}^-)$ contains an internal tangency, so assume to the contrary that $\phi(x, \mathbf{R}) \subset \operatorname{Int}(U)$. Then one finds, by considering the accumulation of A_+ at x, that $A^-(x, \phi)$ is not empty. Since $\phi(x, \mathbf{R}) \subset \mathrm{Int}(U)$, we have $A^-(x, \psi) = A^-(x, \epsilon)$, and hence by 2.8 and 3.11 $\mathscr{S}_1(\psi) \cap \mathscr{C}\ell(A_+)$ accumulates near x. Hence $\dim(M \cap D_\theta) \geqslant 1$, and finally $\dim(M) = 2$.

Since M is two dimensional and transverse to D_θ, the components of $M \cap D_\theta$ are embedded one-dimensional submanifolds of D_θ, and hence must be homeomorphic to an interval (0, 1), (0, 1], or [0, 1], or the circle S^1. If $f\colon (0, 1] \to D_\theta$ is an embedding onto a component of $M \cap D_\theta$, then $f(1) \in \partial U$ and the limit as $s \to 0$ of $f(s)$ is $D_\theta \cap \gamma$. When confusion will not arise, we will identify (0, 1] with its image under f, and write $1 \in \partial D_\theta$, $0 \in \gamma$, and etc. In the case that a component of $M \cap D_\theta$ is S^1, then the region bounded by S^1 is contained in A_+. Since points of A_+ must exit from U in negative time, there must be a $t \leqslant 0$ such that $\phi(S^1, t)$ meets ∂U. It is precisely because of this that we use a weak definition of submanifold for the manifold hypothesis.

We need some notation. Let $\theta\colon U = D^2 \times S^1 \to S^1$ also be the projection. Let A be a connected component of A_+; fix x in A; let $\tilde{A}_t$ denote the component of $A \cap \mathrm{Int}(D^2 \times \{\theta(\phi(x, t))\})$ which contains $\phi(x, t)$; and finally let $A_t = \mathscr{C}\ell(\tilde{A}_t) - \gamma$.

THEOREM 4.4. *Assume H. Then there is a $T \geqslant 0$ such that one of the following holds.*

(a) *There is a homeomorphism $F\colon \bigcup_{t \geqslant T} A_t \to D^2 \times [0, \infty)$ such that $F(A_t) = D^2 \times \{t - T\}$. In this case* lim sup $A_t = \gamma$ *as $t \to \infty$ in the Hausdorff metric topology, and* $\mathrm{Int}(A) \subset \mathscr{P}$.

(b) *There is a homeomorphism $F\colon A \to (0, 1] \times S^1$. Here $F(A_t) \subset (0, 1] \times e^{\pi i t/m}$ where m is a positive integer such that $\phi(A_t, mn) \subset A_t$ for every positive integer n. F extends to a homeomorphism of $A \cup \gamma$ onto a quotient space* $[0, 1] \times S^1/R$ *where R is the relation $(s_1, \beta_1)\, R(s_2, \beta_2)$ if $s_1 = s_2 = 0$ and $\beta_2 = \beta_1 e^{\pi i n/m}$ for some integer n and with m as before. γ is mapped onto $\{0\} \times S^1/R$. Finally $A \subset \mathscr{S}_0$.*

(c) *Let (r, ζ) be polar coordinates for the plane. There is a sequence (possibly void or finite) of nonoverlapping subintervals $\{(a_i, b_i)\}$ of $(0, \pi)$ and continuous functions $r_i\colon (a_i, b_i) \to (0, 1)$ with $r_i(\zeta) \to 0$ as either $\zeta \to a_i$ or $\zeta \to b_i$, such that for each $t \geqslant T$ there is a homeomorphism*

$$f_t\colon A_t \to \{(r, \zeta) \mid 0 < r \leqslant 1, 0 \leqslant \zeta \leqslant \pi \text{ and } r \geqslant r_i(\zeta) \text{ when } a_i < \zeta < b_i\}.$$

Here $f_t(\partial U \cap A_t) = \{(r, \zeta) \mid r = 1, 0 \leqslant \zeta \leqslant \pi\}$ and if $x_n \to \gamma$ in A_t and $(r_n, \zeta_n) = f_t(x_n)$, then $r_n \to 0$. For each i, the set

$$E_i = \phi(f_T^{-1}(\mathit{graph}\ r_i), \mathbf{R}) \cup \gamma$$

is a tube, wrapping around γ and bounding a component of the elliptic set. Precisely, E_i is an embedded copy of one of the following quotient spaces:

(c1) $(S^1 \times \mathbf{R})/R$ *where* R *is the relation determined by a preferred point* $w_0 \in S^1$ *with* $(\alpha_1, \beta_1) R(\alpha_2, \beta_2)$ *if* $\alpha_1 = \alpha_2 = w_0$ *and* $\beta_2 = \beta_1 + n$ *for some integer* n.

(c2) $(S^1 \times S^1)/R$ *where* R *is the relation determined by a preferred point* $w_0 \in S^1$ *and a positive integer* m *with* $(\alpha_1, \beta_1) R(\alpha_2, \beta_2)$ *if* $\alpha_1 = \alpha_2 = w_0$ *and* $\beta_2 = \beta_1 e^{2\pi i n/m}$ *for some integer* n. *This integer* m *also satisfies* $\phi(A_t, mn) \subset A_t$ *for all positive integers* n.

In either case, (c1) *or* (c2), A *is not necessarily homeomorphic to* $A_t \times S^1$.

(d) *This may be viewed as a degenerate case of* (c). *With the same notation, the subintervals* (a_i, b_i) *are in* $(0, 2\pi]$ *and* $f_t\colon A_t \to \{(r, \zeta) \mid 0 < r \leqslant 1\}$ *with* $f_t(\partial U \cap A_t) = \{(r, \zeta) \mid r = 1\}$. *After these modifications, we use the description in* (c) *for the tubes* E_i. *Corresponding to* (c1) *and* (c2) *are respective cases which we call* (d1) *and* (d2).

Proof. Consider the topological type of a component of $b(A_t) - \gamma$. If it is S^1, then since $\phi(b(A_t), s) \cap \partial U = \varnothing$ for $s \geqslant 0$, it must be true that $A_{t+s} = \phi(A_t, s)$, and ϕ gives a homeomorphism of $D^2 \times [0, \infty)$ onto $\cup_{s \geqslant 0} A_{t+s}$. We also need to show $\limsup A_{t+s} = \gamma$ as $s \to \infty$. Since $A_t \subset U_+$, $\gamma \subset \limsup A_{t+s}$. Furthermore, given a neighborhood V of γ, if A_{t+s} were not eventually in V, one would obtain a contradiction to the embedding aspect of H. If A is the component of A_+ which contains A_t, then ϕ is parallel for positive time in $\mathrm{Int}(A)$ since $\limsup A_{t+s} = \gamma$, and likewise for negative time because of 4.1. Hence we have established conclusion (a) for any $x \in A_+$ where $b(A_T)$ is a circle for some $T \geqslant 0$.

Consider the possibility that a component of $b(A_t) - \gamma$ is homeomorphic to $(0, 1]$. Abusing notation, $1 \in \partial U$ and $0 \in \gamma$. Since points of such component are arbitrarily close to $A_{t+s} \neq \varnothing$ for each $s < 0$ and so for all real s. Since $\dot{\theta} = 2\pi$, A_{t+n} is in the same disk D_θ as A_t, for all integers n. But only finitely many A_{t+n} can be distinct if H holds. Furthermore, the distinct ones may be listed as $A_{t+1}, A_{t+2}, \ldots, A_{t+m} = A_t$.

There are two cases: that where $A_t = b(A_t) - \gamma$, and that where $b(A_t) \subset \mathscr{C}\ell(A_t - b(A_t))$. In the former case, part (b) of the theorem follows from what we have established above. In the case where $(0, 1] \subset \mathscr{C}\ell(A_t - b(A_t))$, one may see that there is a distinct component of A_t, also homeomorphic to $(0, 1]$, with $1 \in \partial D_\theta$, $0 \in \gamma$. Let these two components be denoted b_1 and b_2, respectively We may assume they have been chosen so that one of the two sectors of D_θ between them contains points of A and yet contains no other boundary component of type $(0, 1]$. Let d be the open arc of ∂D_θ between b_1 and b_2 in that sector. We claim that $d \subset A_+$. Indeed if $y \in d \cap b(A_+)$, then y must be a point of ingress by A, so that $y \in U - U_-$ and consequently $y \in b(H)$. But then one may use 2.8 and 2.9 to obtain a component of $A^+(y, \phi) \cap D_\theta \subset b(A_+)$ which extends from γ_θ to ∂D_θ, in contradiction of the choice of b_1 and b_2. Since $b_1 \cup b_2 \cup d \cup \gamma_\theta$ is contained in U_+, so is the closed sector R of D_θ bounded by that union. If $A_t = R - \gamma$, then an argument, similar to that given above when $A_t = (0, 1]$, establishes conclusion (c) with a void set of functions C_i. On the other hand, if A_t is properly contained in $R - \gamma$, we have $R - (A_t \cup \gamma) \subset U_+ - (A_t \cup \gamma) \subset E - \gamma$. Furthermore, if $y \in R \cap E$, then y is connected to ∂D_θ or γ within $D_\theta \cap U_-$ (by 2.8 and 2.9). Since $d \subset A_+$, each component of $E \cap R$ contains γ_θ in its closure, and hence the mutual boundary of such a component E_i and A_t is given by $(0, 1) \cup \gamma_\theta$, with E_i bounded by $(0, 1) \cup \gamma_\theta$ in D_θ. There can be only a countable number of the regions E_i. If $A_{t+m} = A_t$, it may be the case that $\phi(E_i, m) = E_i$. But it is easy to construct examples and index the E_i in such a way that $\phi(E_i, m) = E_{i+1}$ for every integer i. In any case, conclusion (c) of the theorem is satisfied.

We have just seen one way in which the boundary type $(0, 1)$ can occur. The region R bounded by such a curve and γ must be contained in U_+. Furthermore $R \subset U_-$, since if a point in R were to exit U in negative time then the boundary type $(0, 1)$ would be broken up to contain boundary components of type $(0, 1]$, which cannot happen in view of our previous treatment of $(0, 1]$. Hence the component A must be exterior to $(0, 1) \cup \gamma_\theta$. Now either $A \cap D_\theta$ has boundary components of type $(0, 1]$ as above, and conclusion (c) holds, or $\partial U \subset A$ and conclusion (d) holds.

It remains to be shown that boundary type $[0, 1]$ cannot occur as an eventuality. Suppose $[0, 1]$ were a component of $b(A_t)$ and denote it b_0. Let the component of $b(A_{t+s})$ be denoted by b_s. Since $b_0 \subset U_+$, it is impossible that $\phi(b_0, n) \cup b_0$ for any positive integer n, and furthermore $\gamma_\theta \in \limsup b_n$ as $n \to \infty$. Under H, we must have $\gamma_\theta = \limsup b_n$, or $\lim b_n = \gamma_\theta$ as $n \to \infty$. The only way this can occur is for the endpoints of b_s to come together as s increases, so that for sufficiently large n, b_n is a circle and conclusion (a) applies. This completes the proof of Theorem 4.4.

THEOREM 4.5. *If C is a component of $A \cap \mathscr{S}_1$, then A is described by* (c) *or* (d) *of* 4.4, *and description* (b) *of* 4.4 *applies to C. If A is described by* (c2) *or* (d2), *the corresponding integer m may be different than the one used to describe C. For each $x \in C$, the prolongation $\Lambda^+(x)$ is a nontrivial union of tubes E_i.*

Proof. If C is a component of $A \cap \mathscr{S}_1$, then C is a component of A, then $C \subset \text{Int}(A)$ and $\mathscr{C}\ell(C) \cap b(A) = \varnothing$ because of H. Hence C is a component of $A_+ \cap (\mathscr{S}_0 \cup \mathscr{S}_1)$ and is a submanifold of $U - \gamma$, invariant under ϕ in positive time. Let x be a fixed point in C and let $\theta = \theta(x)$. Now $\Lambda^+(x) \subset \mathscr{C}\ell(A) \subset U_+$. But $\Lambda^+(x) \subset U_-$ so that $\Lambda^+(x) \subset E \cap b(A)$. Using the methods of (2.8), one may show that each component of $C \cap D_\theta$ intersects ∂D_θ and contains γ_θ in its closure. Hence C is described by 4.4b. Since $\Lambda^+(x)$ must be nonempty and contained in $E \cap b(A)$, it must be a union of tubes, by 4.4c, d.

5. Weakening the Hypothesis H

For contrast with the above theorems, it is instructive to consider the following weaker hypothesis: Assume $\mathscr{S}_0 \cup \mathscr{S}_1 - \gamma$ an immersed submanifold of U, such that each component of $\mathscr{S}_0 \cap (D^2 \times \{\theta\}) - \gamma$ or $\mathscr{S}_1 \cap (D^2 \times \{\theta\}) - \gamma$ is an embedded submanifold. A relevent example is constructed by suspending the composition of two diffeomorphisms, the first of which maps the region X_i onto X_{i+1} (Fig. 2) with no lateral effect, and the second of which moves every point strictly to its right, excepting γ and the right endpoints of the X_i, which shall be fixed, and having only enough vertical displacement so that the X_i are invariant sets. Then $A_+ = \mathscr{C}\ell(\bigcup_i X_i)$, and the remainder is in H, save the ray emanating from γ to the right, which is in A_-.

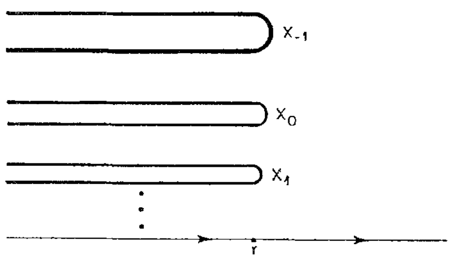

Figure 2

6. Other Examples

Define a vector field $(\dot{x}, \dot{\theta}) = \eta(x, \theta)$ for $(x, \theta) \in \mathbf{R}^2 \times S'$ as follows: $\dot{\theta} \equiv 2\pi$ and $\dot{x} = f(x)$ so that the solution flow to $\dot{x} = f(x)$ in the plane has trajectories as indicated by the solid curves in Fig. 3. Call the rest point γ_0. If V is the closed region of $\mathbf{R}^2$ bounded by the closed (broken) curve of uniform dashes in Fig. 3, then $U_1 = V \times S^1$ is a neighborhood of solitude for the periodic solution $\gamma_0 \times S^1$ in $\mathbf{R}^2 \times S^1$. Furthermore, U_1 satisfies conditions A and H of Section 4. There is a region of H on the left, one of E at the top, and one of A_+ on the right of U_1.

We now alter U_1 slightly to obtain neighborhoods of solitude U_2 and U_3. Fix θ_0 in S^1. We will let the respective intersections of U_1, U_2, U_3 with $\mathbf{R}^2 \times \{\theta\}$ be all the same if θ is not near θ_0 in S', so that the alterations of U_1 will occur only in $\mathbf{R}^2 \times \{\theta\}$ for θ near θ_0. Let $U_2 \cap (\mathbf{R}^2 \times \{\theta_0\})$ be given by that alteration of V in Fig. 3 indicated by the dotted curve. Let $U_3 \cap (\mathbf{R}^2 \times \{\theta_2\})$ be similarly indicated by the dahes of alternating size. Finally, let U_2 and U_3 be smooth, using an isotopy over θ in a neighborhood of θ_0 in S' to patch together the required shapes.

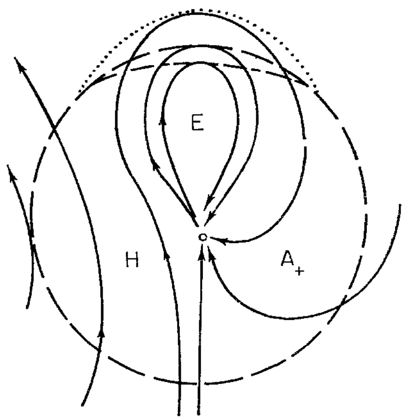

FIGURE 3

The qualitative analysis of U_3 is the easier. We have cut into the elliptic region of U_1, so that trajectories, which remained in U_1 for all time, briefly exit from U_3. Hence in U_3 there is a tube of A_- (type (4.4a) for A_-) spiraling down the hyperbolic side of the elliptic set of U_3.

In U_2 notice that the elliptic region is the same as in U_1, but some of the trajectories which are hyperbolic in U_1 become part of A_+ in U_2. To get a complete idea of the change, let α be an arc in $(U_2 - U_1) \cap (\mathbf{R}^2 \times \{\theta_0\})$ contained in $A_+(U_2)$ and abutting on E. For $t \leqslant 0$ let α_t be the image of α moved by the flow for time t. Notice that there is a $T < 0$ such that $\alpha_t \subset U_2$ if and only if $t \geqslant T$. However, each α_t abuts on E. Hence the union of the α_t over $t \leqslant 0$ is a fin attached to E, spiraling around to have limit as $t \to -\infty$ in the closure of the separatrix surface which comes up from the bottom to meet γ. If one fattens α out to include the entire part of $A_+(U_2)$ which is outside U_1, a satisfactory vision of U_2 may be obtained. Notice that neither condition A nor H of Section 4 is satisfied.

7. ON THE POSSIBLE LACK OF SEMICONTINUITY OF α AND ω

Although we have concerned ourselves only in whether or not α and ω are continuous, a brief note on the semicontinuity of these functions may be of interest. Recall the definitions of semicontinuity: for example, ω is lower s.c. at x if for every open set V meeting $\omega(x)$, there is a neighborhood N of x such that $y \in N$ implies $\omega(y)$ meets V; and ω is upper s.c. at x if for every open set V containing $\omega(x)$, there is a neighborhood N of x such that $y \in N$ implies $\omega(y) \subset V$.

We can construct smooth vector fields such that the resulting ω-limit function is (a) neither upper s.c. nor lower s.c., (b) lower s.c. near some point but not upper s.c. there, or (c) upper s.c. near some point but not lower s.c. there.

For (a) and (b) let (r, θ) be polar coordinates for the plane. If the vector field $(\dot{r}, \dot{\theta}) = \eta(r, \theta)$ is given by $\dot{r} = r(1 - r^2)$, $\dot{\theta} \equiv 1$, then the resulting omega function satisfies (a) at the origin. If we multiply η by a nonnegative valued function which is zero exactly at $(r, \theta) = (1, 0)$ the resulting ω function satisfies (b) at this new rest point.

Finally, for (c) we construct an example in the Cartesian (x, y) plane as follows. Set $\dot{x} = -x \exp(-1/x^2)$ and in the closure of quadrants two through four set $\dot{y} \equiv 0$. For x, y positive, let u be given by $y = u(1 + x + \sin(1/x))$ and set $\dot{u} = -u(u - 1)^2 \exp(-1/x^2$. This determines $\dot{y}$ in the open first quadrant, and one may verify that $\dot{x}, \dot{y}$ are C^∞ in the entire plane. Furthermore it is possible to verify that

$$\begin{aligned}\omega(x_0, y_0) &= \{(0, y_0)\} \text{ if either } x_0 \leqslant 0 \text{ or } y_0 \leqslant 0;\\ &= \{(0, 0)\} \text{ if } x_0 > 0 \text{ and } 0 \leqslant y_0 < 1 + x_0 + \sin(1/x_0);\\ &= \{(0, y) \mid 0 \leqslant y \leqslant 2\} \text{ if } x_0 > 0 \text{ and } y_0 \geqslant 1 + x_0 + \sin(1/x_0).\end{aligned}$$

Hence (c) is fulfilled for (x_0, y_0) on the curve $y = 1 + x + \sin(1/x)$ $(x > 0)$.

References

1. J. Auslander, N. P. Bhatia, and P. Seibert, Attractors in dynamical systems, *Bol. Soc. Mat. Mexicana* **9** (1964), 55–66.
2. A. Beck, "Continuous Flows in the Plane," Vol. 201, Grundlehren der Mathematische Wissenschaften, Springer-Verlag, Berlin, 1974.
3. C. C. Conley and R. W. Easton, Isolated invariant sets and isolating blocks, *Trans. Amer. Math. Soc.* **158** (1971), 35–61.
4. R. H. Elderkin, Separatrix structure for elliptic flows, *Amer. J. Math.* **97** (1975), 221–247.
5. R. H. Elderkin and F. W. Wilson, Solitary invariant sets, *in* "Dynamical Systems, An International Symposium" (J. P. LaSalle, Ed.), Vol. 2, pp. 13–18, Academic Press, New York, 1976.
6. W. Kaplan, Regular curve families filling the plane, I and II, *Duke J. Math.* **7** (1940), 154–185; **8** (1941), 11–46.
7. L. Markus, Global structure of ordinary differential equations in the plane, *Trans. Amer. Math. Soc.* **26** (1954), 127–148.
8. P. B. Percell, Structural stability on manifolds with boundary, *Topology* **12** (1973), 123–144.
9. F. W. Wilson, Elliptic flows are trajectory equivalent, *Amer. J. Math.* **92** (1970), 779–792.
10. F. W. Wilson, Smoothing derivatives of functions and applications, *Trans. Amer. Math. Soc.* **139** (1969), 413–428.