TRIANGULATED \( n \)-MANIFOLDS ARE DETERMINED BY THEIR \([n/2]+1\)-SKELETONS

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We will reconstruct compact, triangulated \( n \)-manifolds—without-boundaries from just their \([n/2]+1\) skeletons. Therefore, two such manifolds are isomorphic if they have simplicially isomorphic \([n/2]+1\)-skeletons. Furthermore, when \( n \) is even, and the \( n/2 \)-homology group is zero, then the \( n/2 \)-skeleton is sufficient.


manifold
triangulation
skeleton

The purpose of this paper is to establish this result:

**Theorem 1.** Let \( M \) and \( N \) be compact, triangulated homology \( n \)-manifolds. Let \( m \geq n/2 + 1 \). Given a simplicial isomorphism \( f: \text{Skel}_m(M) \rightarrow \text{Skel}_m(N) \), there is a simplicial isomorphism \( f_* \) of \( M \) onto \( N \) which is an extension of \( f \).

Later, we will show that compact triangulated topological manifolds with different dimensions must have different high-dimensional skeletons. This is Theorem 14. It is in contrast to Caratheodory's result (Counterexample 18) that triangulated spheres of different dimensions can have the same low-dimensional skeletons.

A homology \( n \)-manifold is a (locally-finite) simplicial complex with the property that the link of every vertex has the homology of an \((n-1)\)-sphere.

Sometimes the dimension of the necessary skeleton may be reduced by 1.

**Theorem 2.** Let \( M \) and \( N \) be two compact triangulated homology \( 2m \)-manifolds. Suppose that either

(a) \( H_m(M, \mathbb{Z}_2) = 0 = H_m(N, \mathbb{Z}_2) \), or
(b) both \( M \) and \( N \) are orientable manifolds and both \( H_m(M, \mathbb{Z}) \) and \( H_m(N, \mathbb{Z}) \) are finite groups.
Then
\[ \text{Skel}_m(M) \equiv \text{Skel}_m(N) \Rightarrow M \cong N. \]

The author is grateful to Micha Perles for introducing him to this subject. Dr. Perles ([1]) had previously established the special cases of Theorems 1 and 2 when \( M \) and \( N \) are PL \( n \)-spheres which are the boundaries of convex linear cells.

Perles answered Question 3 for these \( n \)-spheres, \( M = S^n \), when \( 2k \leq n \). He showed that \( \text{Skel}_kC(S^k, S^n) \) when \( S^k = \partial \Delta^{k+1}, \Delta^{k+1} \subseteq S^n \) and that \( \text{Skel}_kC(S^k, S^n) \) is homotopically equivalent to \( S^{n-k-1} \) otherwise.

We shall establish Theorem 1 by first showing that \( n \)-manifolds may be reconstructed from just their \( m \)-skeleta, \( m > n/2 + \frac{1}{2} \); this is Theorem 11. The crux of this reconstruction is finding an answer to this question:

**Question 3.** Which \( k \)-spheres in \( \text{Skel}_k M \) (which are simplicially isomorphic to the boundary of a \( (k+1) \)-simplex) bound a \( (k+1) \)-simplex in \( M \)?

The useful answer is Main Lemma 10.

We shall now lay the necessary groundwork to establish Theorem 1.

**Definitions and notations.** Let \( L \) be a subcomplex of a simplicial complex \( K \). Suppose that each simplex of \( K \), which has all its vertices in \( L \), is also a simplex of \( L \). Then \( L \) is full in \( K \). This is denoted by \( L \triangleleft K \).

Let \( L \) be any subcomplex of a simplicial complex \( K \). The complementary skeleton \( C(L, K) \) of \( L \) in \( K \) is
\[ C(L, K) = \{ \sigma \in K | \sigma \cap |L| = \emptyset \}. \]

We use \( \text{Skel}_r K \) to denote the \( r \)-skeleton of the simplicial complex \( K \).

**Note.** From this point, all homology is with \( \mathbb{Z} \) coefficients, unless otherwise noted.

**Lemma 4.** If \( L \) is a full subcomplex of a complex \( K \) then \( |K| - |L| \) deformation retracts to \( C(L, K) \) and hence the induced maps:
\[ i_*: H_j(C(L, K)) \rightarrow H_j(|K| - |L|) \]
are isomorphisms for each integer \( j \).

**Lemma 5.** Let \( K \) be a complex, \( L \) a subcomplex of \( K \) and \( L_F \) the subcomplex of \( K \) consisting of the simplices of \( K \), all of whose vertices are in \( L \). Then \( C(L, K) \equiv C(L_F, K) \).

**Lemma 6.** Let \( M \) be a triangulated homology \( n \)-manifold. Let \( S^k \) be a subcomplex of \( M \) which is simplicially isomorphic to the boundary of a \( (k+1) \)-simplex.
(a) If \( S^k = \partial \Delta^{k+1} \), then \( \Delta^{k+1} \) is a simplex in \( M \) if and only if \( S^k \not\subseteq M \).
(b) If \( S^k = \partial \Delta^{k+1} \) and \( \Delta^{k+1} \) is a simplex in \( M \), then
\[ H_j(C(S^k, M)) \equiv H_j(M), \text{ when } j \leq n - 2. \]
(c) If $S^k \neq \partial \Delta^{k+1}$, for any $(k + 1)$-simplex $\Delta^{k+1}$ in $M$, then

$$H_j(C(S^k, M)) \cong H_j(|M| - |S^k|) \quad \text{for all } j.$$ 

**Proof.** (a) Since $S^k$ is simplicially isomorphic to the boundary of a $(k + 1)$ simplex, $S^k$ must contain precisely $(k + 2)$ vertices. Also, these $(k + 2)$ vertices may contain the vertices of possibly only one additional simplex in $M$, namely some $(k + 1)$-simplex $\Delta^{k+1}$, but then $\partial \Delta^{k+1} = S^k$. Thus $S^k \not\subset M$ if and only if there is a $(k + 1)$-simplex $\Delta^{k+1} \in M$ such that $\partial \Delta^{k+1} = S^k$.

(b) When $S^k = \partial \Delta^{k+1}$, $\Delta^{k+1}$ will be full in $M$ since $\Delta^{k+1}$ is a simplex of $M$. Therefore (using first Lemma 5 and then Lemma 4)

$$H_j(C(S^k, M)) \cong H_j(C(\Delta^{k+1}, M)) \cong H_j(|M| - |\Delta^{k+1}|) \cong H_j(M), \quad j \leq n - 2.$$ 

(c) When $S^k \neq \partial \Delta^{k+1}$, then $S^k \not\subset M$ (part (a)). Therefore (by Lemma 4):

$$H_j(C(S^k, M)) \cong H_j(|M| - |S^k|) \quad \text{for all } j.$$ 

**Lemma 7.** Let $M$ be a triangulated homology $n$-manifold and let $k \geq 1$. Then:

$$H_{n-j}(M, M - S^k) \cong H^j(S^k) \cong \begin{cases} Z_2, & \text{if } j = k \text{ or } j = 0, \\ 0, & \text{otherwise}. \end{cases}$$

**Remark.** Lemma 7 is a corollary of Poincaré Duality.

The standard long exact sequence for $S^k \subset M$ is

$$\cdots \to H_{n-j+1}(M, M - S^k) \to H_{n-j}(M - S^k) \to H_{n-j}(M) \to H_{n-j}(M, M - S^k) \to \cdots$$

Putting Lemma 7 into this long exact sequence, we obtain these short exact sequences:

$$0 \to H_j(M - S^k) \to H_j(M) \to 0, \quad j \neq n - k, n - k - 1, n, n - 1 \quad \text{(1)}$$

and

$$0 \to H_{n-k}(M - S^k) \to H_{n-k}(M) \to Z_2 \to H_{n-k-1}(M - S^k) \to H_{n-k-1}(M) \to 0. \quad \text{(2)}$$

This last sequence is the main result behind Part (ii) of the next lemma. In turn, Lemma 8 Part (ii), together with Lemma 6 Parts (a) and (b) will be used to determine which $k$-spheres in $M$ (or in $\text{Skel}_k M$) bound a $(k + 1)$-simplex in $M$.

**Lemma 8.** Suppose $S^k \not\subset M$, $M$ a compact homology $n$-manifold, then:

(i) $H_j(\text{Skel}_k C(S^k, M)) \cong H_j(\text{Skel}_k(M)), \quad j \neq n - k, n - k - 1, n, n - 1; j < r$;

(ii) when $n < 2k$, either

$$H_{n-k}(\text{Skel}_k C(S^k, M)) \neq H_{n-k}(\text{Skel}_k M).$$
or
\[ H_{n-k-1}(\text{Skel}_k C(S^k, M)) \neq H_{n-k-1}(\text{Skel}_k M). \]

**Sublemma 9.** The inclusion induced homomorphism \( H_j(\text{Skel}_k (K)) \to H_j(K) \) is an isomorphism whenever \( j \leq n - 2 \).

**Proof of Lemma 8.** (i) Using (1) first and then Lemma 6 parts (a) and (c) we see that
\[ H_j(M) \cong H_j(\lvert M \rvert - \lvert S^k \rvert) \cong H_j(C(S^k, M)). \]
This and Sublemma 9 will establish part (i).

(ii) Putting the results of Lemma 6 parts (a) and (c) into (2) yields the exact sequence
\[ 0 \to H_{n-k}(C(S^k, M)) \to H_{n-k}(M) \to \mathbb{Z} \to H_{n-k-1}(C(S^k, M)) \to H_{n-k-1}(M) \to 0. \]
(3)

Since this sequence is exact, either
\[ H_{n-k}(C(S^k, M)) \cong H_{n-k}(M) \]
or
\[ H_{n-k-1}(C(S^k, M)) \cong H_{n-k-1}(M). \]
This and Sublemma 9 will establish part (ii).

**Main Lemma 10.** Let \( M \) be a compact triangulated homology \( n \)-manifold. Let \( S^k \) be a subcomplex of \( M \) which is simplicially isomorphic to the boundary of a \((k + 1)\)-simplex and let \( 2k > n \). Then \( S^k \) does not bound a \((k + 1)\)-simplex in the triangulation of \( M \) if and only if condition (ii) of Lemma 8 is valid.

**Proof.** When \( S^k = \partial \Delta^{k+1} \), for some simplex \( \Delta^{k+1} \in M \), then
\[ H_i(\text{Skel}_k C(S^k, M)) \cong H_i(\text{Skel}_k M) \quad \text{for both} \quad i = n - k \quad \text{and} \quad i = n - k - 1, \]
by Lemma 6(b) and Sublemma 9.

When \( S^k \neq \partial \Delta^{k+1} \) for any simplex \( \Delta^{k+1} \in M \), then \( S^k \lhd M \) (Lemma 6(a)) and hence condition (ii) of Lemma 8 is valid.

**Theorem 11.** Let \( M \) be a compact, triangulated homology \( n \)-manifold. Let \( m \geq n/2 + \frac{1}{2} \). Then a simplicial triangulation of \( M \) may be reconstructed from just its \( m \)-skeleton.

**Proof.** Suppose that we know \( \text{Skel}_k(M) \). To construct the \((k + 1)\)-skeleton we need only locate the positions of the \((k + 1)\)-simplices of \( M \). For each \( \Delta^{k+1} \in M \), \( \partial \Delta^{k+1} \) is a subcomplex of \( \text{Skel}_k(M) \). Therefore locating the positions of the \((k + 1)\)-simplices
in \( \text{Skel}_{k+1}(M) \) is equivalent to Question 3. Question 3 is answered by the Main Lemma 10. Thus the \( \text{Skel}_{k+1}(M) \) is obtained from \( \text{Skel}_k(M) \) by attaching \((k + 1)\)-simplices to precisely those subcomplexes \( S^k \) of \( \text{Skel}_k(M) \) which are simplicially isomorphic to the boundary of a \((k + 1)\)-simplex and for which

\[
H_i(\text{Skel}_k C(S^k, M)) \cong H_i(\text{Skel}_k M) \quad \text{for both } i = n - k \text{ and } i = n + k.
\]

**Proof of Theorem 1.** We may assume inductively that a simplicial isomorphism, \( f_k : \text{Skel}_k(M) \rightarrow \text{Skel}_k(N) \), has been constructed for \( k \geq m \geq n/2 + \frac{1}{2} \). For each \((k + 1)\)-simplex \( \Delta^{k+1} \) in \( M \), the commutative diagram:

\[
\begin{array}{ccc}
C(\partial \Delta^{k+1}, \text{Skel}_k M) & \xrightarrow{(f_k)_*} & \text{Skel}_k(M) \\
\downarrow & & \downarrow \leftarrow \text{Skel}_k(N) \\
C(f_k(\partial \Delta^{k+1}), \text{Skel}_k N) & \xrightarrow{(f_k)_*} & \text{Skel}_k(N)
\end{array}
\]

will induce the commutative diagram

\[
\begin{array}{ccc}
H_i(C(\partial \Delta^{k+1}, \text{Skel}_k M)) = H_i(\text{Skel}_k(C(\partial \Delta^{k+1}, M))) & \xrightarrow{\ast} & H_i(\text{Skel}_k M) \\
\downarrow & & \downarrow \leftarrow \text{Skel}_k(N) \\
H_i(C(f_k(\partial \Delta^{k+1}), \text{Skel}_k N)) = H_i(\text{Skel}_k(C(f(\partial \Delta^{k+1}), N))) & \xrightarrow{\ast} & H_i(\text{Skel}_k N).
\end{array}
\]

The vertical arrows are isomorphisms since \( f_k \) is a simplicial isomorphism. Since \( k \geq m \geq n/2 + \frac{1}{2} \) implies \( 2k > n \), it follows, from the Main Lemma 10, that the horizontal top arrow is an isomorphism. Therefore the bottom horizontal arrow is also an isomorphism. Therefore it follows from the Main Lemma 10 that there exists a \((k + 1)\)-simplex \( \delta^{k+1} \) in \( N \) such that \( f_k(\partial \Delta^{k+1}) = \partial \delta^{k+1} \). Thus \( f_k \) can be extended to a simplicial isomorphism \( \text{Skel}_k(M) \cup \Delta^{k+1} \rightarrow \text{Skel}_k(N) \cup \delta^{k+1} \). By doing this construction over all the \((k + 1)\)-simplices of \( \text{Skel}_{k+1}(M) \), \( f_k \) can be extended to a one-to-one simplicial map \( f_{k+1} : \text{Skel}_{k+1}(M) \rightarrow \text{Skel}_{k+1}(N) \). In the same way, one may check that each \( k + 1 \) simplex of \( N \) is the image under \( f_{k+1} \) of a \( k + 1 \) simplex of \( M \). Therefore \( f_{k+1} \) will be a simplicial isomorphism. Thus the induction is established. The last isomorphism, \( f_n : M \rightarrow N \), establishes Theorem 1.

We now proceed to 'extend' Theorem 1 to Theorem 2. All that really remains is to construct the \((m + 1)\)-skeleton of \( M \) from the \( m \)-skeleton. For this we need the next lemma instead of Lemma 8.

**Lemma 12.** Suppose \( S^k \triangleleft M \), \( M \) a compact homology \( 2m \)-manifold.

(i) If \( H_m(M, Z_2) \cong 0 \), then

\[
H_{m-1}(\text{Skel}_m C(S^m, M), Z_2) \neq H_{m-1}(\text{Skel}_m M, Z_2).
\]

(ii) If \( M \) is orientable and \( H_m(M, Z) \) is a finite group, then

\[
H_{m-1}(\text{Skel}_m C(S^m, M), Z) \neq H_{m-1}(\text{Skel}_m M, Z).
\]
Proof. (i) Setting \( k = m \) (and hence, \( n - k = m \)) and \( H_m(M) = 0 \) in (3) yields the exact sequence

\[
0 = H_m(M) \to \mathbb{Z} \to H_{m-1}(C(S^m, M)) \to H_{m-1}(M) \to 0.
\]

Since this sequence is exact,

\[
H_{m-1}(C(S^m, M)) \neq H_{m-1}(M).
\]

This and Sublemma 9 will establish part (i).

(ii) For orientable manifolds, Poincaré Duality is valid for homology groups with the integers \( \mathbb{Z} \) as the ring of coefficients. Therefore, for orientable manifolds, (2) with \( \mathbb{Z} \) instead of \( \mathbb{Z}_2 \) is valid for these homology groups with \( \mathbb{Z} \) coefficients.

Again setting \( k = m \) (and \( n - k = m \)) in (3) yields the exact sequence

\[
H_m(M, \mathbb{Z}) \to \mathbb{Z} \to H_{m-1}(C(S^m, M), \mathbb{Z}) \to H_{m-1}(M, \mathbb{Z}) \to 0.
\]

When \( H_m(M, \mathbb{Z}) \) is a finite group, the image of \( H_m(M, \mathbb{Z}) \) is zero. This implies that

\[
H_{m-1}(C(S^m, M), \mathbb{Z}) \neq H_{m-1}(M, \mathbb{Z}).
\]

This and Sublemma 9 will establish part (ii).

Theorem 13. Let \( M \) be a compact triangulated homology \( 2m \)-manifold. Suppose either that \( H_{m+1}(M, \mathbb{Z}) = 0 \) or that \( M \) is an orientable manifold and \( H_m(M, \mathbb{Z}) \) is a finite group. Then \( M \) may be reconstructed from just its \( m \)-skeleton.

Proof. First, we want to construct the \((m+1)\)-skeleton from the \( m \)-skeleton. Again, the basic problem is to locate the positions of the \((m+1)\)-simplices of \( M \). Here, this problem is equivalent to Question 3 when \( k = m \). Here the answer is implied by Lemma 6 together with Lemma 12 part (i) or (ii). Therefore \( \text{Skel}_{m+1}(M) \) is obtained from \( \text{Skel}_m(M) \) by attaching \((m+1)\)-simplices to precisely those subcomplexes \( S^m \) of \( \text{Skel}_m(M) \), which are simplicially isomorphic to the boundary of an \((m+1)\)-simplex and for which

(a) \( H_{m-1}(\text{Skel}_m C(S^m, M), \mathbb{Z}_2) \neq H_{m-1}(\text{Skel}_m(M, \mathbb{Z}_2)) \), when \( M \) is non-orientable or

(b) \( H_{m-1}(\text{Skel}_m C(S^m, M), \mathbb{Z}) \neq H_{m-1}(\text{Skel}_m(M, \mathbb{Z})) \), when \( M \) is orientable and \( H_m(M, \mathbb{Z}) \) is a finite group.

Having constructed the \((m+1)\)-skeleton, Theorem 11 ‘will construct’ the rest of the manifold \( M \).

Proof of Theorem 2. In the same manner, as in the proofs of Theorems 1 and 13, the isomorphism, between the \( m \)-skeletons of \( M \) and \( N \), may be extended to an isomorphism between the \((m+1)\)-skeletons. Theorem 1 will now complete the proof of Theorem 2.

The next result says that compact triangulated manifolds with different dimensions must have different (high dimensional) skeletons.
Theorem 14. Let $M$ and $N$ be compact triangulated topological manifolds; let $n = \text{Dim. } N$ and $r = \text{Dim. } M$. Suppose that $n > r$ and that either

(i) $m \geq n/2 + \frac{1}{2}$ or 
(ii) $m = n/2$ and $H_m(N; \mathbb{Z}_2) \cong 0$ and $m \neq r - 1$.

Then $\text{Skel}_m M$ is not simplicially isomorphic to $\text{Skel}_m N$.

Remark. That the condition $2m \geq n + 1$ cannot be relaxed to $2m = n - 2$ was demonstrated by Carathéodory (Counterexample 18).

We begin by establishing a simple case of Theorem 14.

Lemma 15. Let $M$ and $N$ be triangulated topological manifolds, and let $r = \text{Dim. } M < \text{Dim. } N$. Then $M$ is not the $r$-skeleton of $N$.

Proof. When $M \equiv S'$, then the only possibility for $\text{Skel}_{r+1}(N)$ is a single $(r+1)$-simplex. There are no possible $(r+2)$-skeletons for $N$.

Suppose $M \not\equiv S'$. By invariance of domain, $M$ contains no subcomplex which is isomorphic to $\partial \Delta^{r+1}$. Therefore there is no possible $(r+1)$-skeleton for $N$. We see that $N$ does not exist. This establishes Lemma 15.

The key to the proof of Theorem 14 is the next lemma.

Lemma 16. Let $M$ be a compact triangulated homology $r$-manifold. Then

$$H_j(\text{Skel}_m C(S^k, M)) \cong H_j(\text{Skel}_m M), \quad j \neq r - k, r - k - 1, r, r - 1 \text{ and } j < m$$

where $S^k$ is a subcomplex of $M$, which is simplicially isomorphic to $\partial \Delta^{k+1}$.

Proof. When $S^k \not\subset M$, this lemma is the same as Lemma 8(i). When $S^k \subset M$ then $S^k = \partial \Delta^{k+1}$, for some $\Delta^{k+1} \in M$ and therefore this lemma follows from Lemma 6(b).

Lemma 17. Let $M$ and $N$ be compact, triangulated homology manifolds, and let $n = \text{Dim. } N$ and $r = \text{Dim. } M$, where $r \neq n \neq r - 1$. Suppose that $\text{Skel}_m M \equiv \text{Skel}_m N$, that $r \neq n - m$ and $r \neq n - m + 1$ and that either

(i) $m \geq n/2 + \frac{1}{2}$ or 
(ii) $m = n/2$ and $H_m(N; \mathbb{Z}_2) \cong 0$.

Then each $S^m$ in $N$ bounds an $(m+1)$-simplex in $N$ (where $S^m$ is a subcomplex of $N$ which is simplicially isomorphic to $\partial \Delta^{m+1}$).

Proof. Case (i): $|n - r| \geq 2$. Lemma 16 says that:

$$H_j(\text{Skel}_m C(S^m, M)) \cong H_j(\text{Skel}_m M) \quad \text{for } j = n - m \text{ and } j = n - m - 1.$$  (4)
Therefore (since Skel\textsubscript{m} N \equiv Skel\textsubscript{m} M)
\begin{equation}
H_j(\text{Skel}_m C(S^m, N)) \equiv H_j(\text{Skel}_m N) \quad \text{for } j = n - m, n - m - 1.
\end{equation}

When \( m \geq n/2 + 1 \), the Main Lemma 10 and (5) imply that \( S^m = \partial \Delta^{m+1} \) for some \( \Delta^{m+1} \in N \).

When \( m = n/2 \) and \( H_m(M; Z_2) \equiv 0 \), Lemma 12(i) and (5) imply that \( S^m \not< N \) and hence \( S^m = \partial \Delta^{m+1} \), for some \( \Delta^{m+1} \in N \).

Case (iia): \( |n - r| = 1 \) and \( S^m = \partial \Delta^{m+1}, \Delta^{m+1} \in M \). Here Lemma 6(b) implies that
\begin{equation}
H_j(C(S^m, M)) \equiv H_j(M) \quad \text{for } j = n - m \text{ and } j = n - m - 1.
\end{equation}
This implies (4). Then the rest of the proof of case (i) will show that \( S^m = \partial \Delta^{m+1}, \Delta^{m+1} \in M \).

Case (iib): \( n = r + 1 \) and \( S^m \not< M \). We shall show that this case (iib) is not possible.

To begin Lemma 8 part (i) implies that:
\begin{equation}
H_{n-m-2}(\text{Skel}_m C(S^m, N)) \equiv H_{n-m-2}(\text{Skel}_m N).
\end{equation}

Since \( M \) and \( N \) have the same \( m \)-skeletons and since the \( n - m - 2 \) homology groups live in these skeletons, (since \( m \geq n/2 \Rightarrow n - m - 2 < m \))
\begin{equation}
H_{n-m-2}(C(S^m, M)) \equiv H_{n-m-2}(M).
\end{equation}

This isomorphism reduces (3) to the exact sequence
\[ 0 \to H_{n-m-1}(C(S^m, M)) \to H_{n-m-1}(M) \to \mathbb{Z}_2 \to 0. \]

Therefore, the induced homomorphism,
\[ \varphi_M : H_{n-m-1}(C(S^m, M)) \to H_{n-m-1}(M), \]
is one-one but not onto.

Since \( H_{n-m-1}(N) \) also lives in \( \text{Skel}_{n-m-1}(N) \), the induced homomorphism \( \varphi_N : H_{n-m-1}(C(S^m, N)) \to H_{n-m-1}(N) \) must also be one-one and not onto. But (3) implies that \( \varphi_N \) is onto, when \( S^m \not< N \) and Lemma 6(b) implies that \( \varphi_N \) is onto when \( S^m \not< N \). This contradiction on \( \varphi_N \) implies that this case (iib) does not occur.

**Proof of Theorem 14.** We will be using Lemma 17. Therefore we shall first check the special cases \( r = n - m \) and \( r = n - m + 1 \). The hypothesis (i) \( m > n/2 \), together with these two equations for \( r \), implies that \( n - r \geq m - 1 > n/2 - 1 \). Therefore \( n/2 \geq r - 1 \). Combining this with the hypothesis (i) \( m > n/2 \), yields \( m > r - 1 \) or \( m \geq r \). Since a manifold does not have a skeleton of higher dimension, \( m \) must equal \( r \), and hence \( M = \text{Skel}_m(M) \equiv \text{Skel}_m(N) \). In this situation Lemma 15 establishes Theorem 14. Similarly the hypothesis (ii) \( m = n/2 \) and \( m \neq r - 1 \) will also imply that \( m \geq r \). Again \( M = \text{Skel}_m(M) \) and Lemma 15 is applicable. Thus Theorem 14 is established for the two special cases: \( r = n - m \) and \( r = n - m + 1 \).

The rest of the proof of Theorem 14 will be a proof by contradiction. Let us assume that \( r \neq n - m, n - m + 1 \) and that \( \text{Skel}_m(M) \equiv \text{Skel}_m(N) \).
Claim. Each $S^n$, in $M$ and $N$, bounds an $(m+1)$-simplex in $M$ and $N$. ($S^n$ is a subcomplex which is simplicially isomorphic to $\Delta^{m+1}$.) Also, $\text{Skel}_{m+1}(M) \cong \text{Skel}_{m+1}(N)$.

Proof. When $|n-r| \geq 2$, by using Lemma 17 twice, once with the interchanging of the roles of $M$ and $N$, it will be established that each $S^n$ in $M$ and $N$ bounds an $(m+1)$-simplex. When $|n-r| = 1$, Lemma 17 together with the fact that case (iib) of its proof does not occur, will establish the fact that each $S^n$ in $M$ and $N$ bounds an $(m+1)$ simplex.

Since each $S^n$ in both $M$ and $N$ bounds an $(m+1)$-simplex, the isomorphism on the $m$-skeletons extends to the $(m+1)$-skeletons. This establishes the claim.

Proof of Theorem 14 (continued). Repeated use of this Claim will establish that $\text{Skel}_m(N) = \text{Skel}_m(M)$. But Lemma 15 says this is not permitted. Therefore $\text{Skel}_m(M)$ could not have been isomorphic to $\text{Skel}_m(N)$. Therefore Theorem 14 is established.

We thank Professor Amos Altshuler for calling our attention to the following counterexample which complements Theorem 14.

Counterexample 18. (Carathéodory [2]). For each $p > n$, there is a triangulation $S^n_\kappa$ of the $n$-sphere $S^n$ such that:

$$\text{Skel}_m(S^n_\kappa) \cong \text{Skel}_m(\partial \Delta^p),$$

when $m = [(n-1)/2]$ (and $\Delta^p$ is a $p$-simplex).

This result was rediscovered by Professors Gale [3], [4] and Motzkin [5]. This is discussed in [7]. Dr. Walkup has shown that each connected $P1$ 3-manifold has a triangulation whose skeleton is isomorphic to the 1 skeleton of some $n$-sphere (Theorem 4 of [6]).

This paper leaves the following question unanswered. Is there a triangulation of $M = S^3 \times S^3$ such that every 3 vertices ‘span’ a 2-simplex? If so, such a triangulation would have a 2-skeleton which is the same as the 2-skeleton for the $(p-1)$-sphere ($\partial \Delta^p$) and hence, by Carathéodory’s result (Counterexample 18) would also be the same as the 2-skeleton for some triangulation of the 6-sphere.

References