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Uniformly convex subsets of the Hilbert space with modulus of convexity of the second order

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ABSTRACT

We prove that in the Hilbert space every uniformly convex set with modulus of convexity of the second order at zero is an intersection of closed balls of fixed radius. We also obtain an estimate of this radius.

1. Introduction

We begin by some definitions for a Banach space E over $\mathbb{R}$. Let $B_r(a) = \{x \in E \mid \|x - a\| \leqslant r\}$. Let $\operatorname{cl} A$ denote the *closure* and $\operatorname{int} A$ the *interior* of a subset $A \subset E$. The *diameter* of a subset $A \subset E$ is defined as $\operatorname{diam} A = \sup_{x,y\in A} \|x - y\|$. The *Minkowski sum* of two sets $A, B \subset E$ is the set

$$A + B = \{a + b \mid a \in A,\ b \in B\}.$$

We denote the *convex hull* of a set $A \subset E$ by $\operatorname{co} A$. The *supporting function* of a subset $A \subset E$ is defined as follows

$$s(p, A) = \sup_{x\in A}(p, x), \quad \forall p \in E^*. \tag{1.1}$$

The supporting function of any set A is always lower semicontinuous, positively homogeneous and convex. If a set A is bounded then the supporting function is Lipschitz continuous [1,13]. If a subset $A \subset E$ of a reflexive Banach space E is closed convex and bounded then for any vector $p \in E^*$ the set $A(p) = \{x \in A \mid (p, x) \geqslant s(p, A)\}$ is the subdifferential of the supporting function $s(\cdot, A)$ at the point p. In this case the set $A(p)$ is nonempty, weakly compact and convex (cf. [1,13]), $A(0) = A$.

We denote the inner product in the Hilbert space $\mathcal{H}$ by $(\cdot,\cdot)$.

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Definition 1.1. (See [14].) Let E be a Banach space and let a subset $A \subset E$ be convex and closed. *The modulus of convexity* $\delta_A : [0, \operatorname{diam} A) \to [0, +\infty)$ is the function defined by

$$\delta_A(\varepsilon) = \sup\left\{\delta \geqslant 0 \;\middle|\; B_\delta\left(\frac{x_1 + x_2}{2}\right) \subset A,\ \forall x_1, x_2 \in A\colon\ \|x_1 - x_2\| = \varepsilon\right\}.$$

Definition 1.2. (See [14].) Let E be a Banach space and let a subset $A \subset E$ be convex and closed, $A \neq E$. If the modulus of convexity $\delta_A(\varepsilon)$ is strictly positive for all $\varepsilon \in (0, \operatorname{diam} A)$, then we call the set A *uniformly convex* (*with the modulus* $\delta_A(\cdot)$).

We proved in [3] that every uniformly convex set is bounded and if the Banach space E contains a nonsingleton uniformly convex set then it admits a uniformly convex equivalent norm. We also proved that the function $\varepsilon \to \delta_A(\varepsilon)/\varepsilon$ is increasing (see also [6, Lemma 1.e.8]), and for any uniformly convex set A there exists a constant $C > 0$ such that $\delta_A(\varepsilon) \leqslant C\varepsilon^2$.

Definition 1.3. (See [6].) Let E be a Banach space. We call the space E *uniformly convex* with the modulus $\delta_E(\varepsilon)$, $\varepsilon \in [0, 2)$, if the closed unit ball in E is uniformly convex set with the modulus δ_E.

In a Banach space E consider a set

$$A = \bigcap_{x \in X} B_R(x) \neq \emptyset,$$

where $X \subset E$ is an arbitrary subset. Such sets have been considered by several authors (see [2,9,10,13] for details), they are called *R-convex*, or *strongly convex of radius R*. In particular, strongly convex sets of radius R are closely related to the classical notions of diametrically maximal sets and constant width sets, see [4,5,7,8,12,13]. It is obvious that if the space E is uniformly convex with the modulus δ_E then any strongly convex of radius R set A is uniformly convex with the modulus

$$\delta_A(\varepsilon) \geqslant R\delta_E\left(\frac{\varepsilon}{R}\right), \quad \forall \varepsilon \in [0, \operatorname{diam} A).$$

We want to consider the converse question. Suppose that in a Banach space E a subset $A \subset E$ is uniformly convex with the modulus δ_A. What can we say about geometric properties of the set A? In particular, is the set A an intersection of balls of fixed radius?

2. The main result

We give an affirmative answer in the Hilbert space $\mathcal{H}$. Our main result is given in the following theorem.

Theorem 2.1. *Let $\mathcal{H}$ be the Hilbert space. Suppose that a nonempty closed convex subset $A \subset \mathcal{H}$ is uniformly convex with the modulus of convexity of the second order at zero*: *there exists $C > 0$ such that*

$$\delta_A(\varepsilon) = C\varepsilon^2 + o\big(\varepsilon^2\big), \quad \varepsilon \to +0.$$

Then there exists a subset $X \subset \mathcal{H}$ such that

$$A = \bigcap_{x \in X}\left(x + \frac{1}{8C}B_1(0)\right),$$

and $\frac{1}{8C}$ is sharp in the sense that for any $r < \frac{1}{8C}$ and any subset $Y \subset \mathcal{H}$,

$$A \neq \bigcap_{x \in Y} B_r(x).$$

3. Preliminary lemmas

The key idea of the proof of Theorem 2.1 is to use the definition of the *generating set*.

Definition 3.1. (See [2].) Let E be a Banach space. A closed convex bounded subset $M \subset E$ is called a *generating set*, if for any nonempty subset $A \subset E$ such that

$$A = \bigcap_{x \in X}(M + x),$$

where $X \subset E$, there exists another closed convex subset $B \subset E$ with $A + B = M$. The set A above is said to be *M-strongly convex*.

Proposition 3.1. *(See [2], [13, Theorem 4.1.3].) Let M be a closed convex bounded set in a reflexive Banach space E. Then $M \subset E$ is a generating set if and only if for any nonempty set $A = \bigcap_{x\in X}(M + x)$ and any unit vector $p \in E^*$ and any $x_p^A \in A(p)$ there exists a point $x_p^M \in M(p)$ with*

$$A \subset M + x_p^A - x_p^M. \tag{3.1}$$

The inclusion (3.1) is a special supporting principle. Indeed, each closed convex set coincides with the intersection of supporting half-spaces (see [1,13]). Proposition 3.1 says that if a set M is generating then each M-strongly convex set A coincides with the intersection of *supporting shifts* of the set M.

Proposition 3.2. *(See [2], [13, Theorem 4.2.7].) A closed ball in the Hilbert space is a generating set.*

We wish to point out that for $\mathcal{H} = \mathbb{R}^n$ the result of Proposition 3.2 was in fact proved in [10].

By Propositions 3.1 and 3.2 we obtain that for any strongly convex set $A \subset \mathcal{H}$ of radius R and for any unit vector $p \in \mathcal{H}$ the following inclusion holds:

$$A \subset B_R\big(x_p^A - Rp\big), \quad \big\{x_p^A\big\} = A(p). \tag{3.2}$$

Lemma 3.1. *A bounded closed convex set $A \subset \mathcal{H}$ is strongly convex with the radius R if and only if the function $f(p) = R\|p\| - s(p, A)$ is convex.*

Proof. We conclude from Definition 3.1 and Proposition 3.2 that if a closed set $A \subset \mathcal{H}$ is strongly convex of radius $R > 0$ then there exists another convex set B such that $A + B = B_R(0)$. Taking the supporting functions, we get $f(p) = R\|p\| - s(p, A) = s(p, B)$ which is a convex function.

If the function $f(p) = R\|p\| - s(p, A)$ is convex then, keeping in mind that $f(p)$ is also continuous and positively homogeneous, we obtain that $f(p)$ is the supporting function for the set $B = \{x \in \mathcal{H} \mid (p, x) \leqslant f(p),\ \forall p \in \mathcal{H}\}$, i.e. $f(p) = s(p, B)$ [13, Corollary 1.11.2]. Hence $s(p, A) + s(p, B) = s(p, A + B) = R\|p\|$ and by the convexity and closedness of the set A we have $A + B = B_R(0)$. Thus $A = \bigcap_{b\in B} B_R(-b)$. □

Definition 3.2. (See [11].) For a set $A \subset \mathcal{H}$, $A \subset B_R(a)$, a *strongly convex hull of radius $R > 0$* is defined to be the intersection of all closed balls of radius R each of which contains A. We denote the strongly convex hull of radius R of a set A by $\operatorname{strco}_R A$.

Let $\|a-b\| < 2R$. Any intersection of the set $\operatorname{strco}_R\{a, b\} \subset \mathcal{H}$ by a 2-dimensional plane L, $\{a, b\} \subset L$, represents the planar convex set between two smaller arcs of the circles of radius R which pass through the points a and b. Also if $0 < r < R$ and $\operatorname{strco}_r A \neq \emptyset$, then $\operatorname{strco}_R A \subset \operatorname{strco}_r A$ (see [2], [13, Theorem 4.4.2]). We define the smaller arc of a circle of radius R, the center $z \in \mathcal{H}$ and the endpoints $x, y \in \mathcal{H}$ by $D_R(z)(x, y)$.

Lemma 3.2. *Let $R > 0$. Let a subset $A \subset \mathcal{H}$ be closed, convex and bounded. Suppose that*

$$\exists \varepsilon_0 > 0\ \forall a, b \in A: \quad \|a - b\| \leqslant \varepsilon_0 \quad \Rightarrow \quad \operatorname{strco}_R\{a, b\} \subset A.$$

Then the set A is strongly convex of radius R.

Proof. The boundary of the set A contains no nondegenerate line segments. By the inclusion $\operatorname{strco}_R\{a, b\} \subset A$, $\forall a, b \in A$ and $\|a - b\| \leqslant \varepsilon_0$, and by the property of strongly convex hull of two points we obtain that the set A is uniformly convex with the modulus

$$\delta_A(t) \geqslant R - \sqrt{R^2 - \frac{t^2}{4}}, \quad \forall t \in (0, \varepsilon_0).$$

By Corollary 2.2 of [3], the function $\partial B_1(0) \ni p \to a(p) = \arg\max_{x\in A}(p, x)$ is uniformly continuous. It is easy to see that $a(p)$ is also uniformly continuous on each set of the form $\{p \in \mathcal{H} \mid \|p\| > r > 0\}$.

Fix any pair of linear independent vectors $p_1, p_2 \in \mathcal{H}$ (i.e. $0 \notin [p_1, p_2]$). The condition of uniform continuity on the set $[p_1, p_2]$ is

$$\exists \delta_0 > 0\ \forall q_1, q_2 \in [p_1, p_2]: \quad \|q_1 - q_2\| < \delta_0, \qquad \big\|a(q_1) - a(q_2)\big\| \leqslant \varepsilon_0. \tag{3.3}$$

Consider $f(p) = R\|p\| - s(p, A)$, $p \in [p_1, p_2]$. Fix $q_1, q_2 \in [p_1, p_2]$ such that $\|q_1 - q_2\| < \delta_0$. By formula (3.3) we obtain for points $a_i = a(q_i)$, $i = 1, 2$, that $\|a_1 - a_2\| \leqslant \varepsilon_0$ and using the condition of lemma we have $\operatorname{strco}_R\{a_1, a_2\} \subset A$. Using the convexity of the function $R\|p\| - s(p, \operatorname{strco}_R\{a_1, a_2\})$ (Lemma 3.1) we obtain that

$$f\left(\frac{1}{2}(q_1+q_2)\right) = R\left\|\frac{1}{2}(q_1+q_2)\right\| - s\left(\frac{1}{2}(q_1+q_2), A\right) \leqslant R\left\|\frac{1}{2}(q_1+q_2)\right\| - s\left(\frac{1}{2}(q_1+q_2), \operatorname{strco}_R\{a_1,a_2\}\right)$$
$$\leqslant \frac{1}{2}\big(R\|q_1\| - s\big(q_1, \operatorname{strco}_R\{a_1,a_2\}\big)\big) + \frac{1}{2}\big(R\|q_2\| - s\big(q_2, \operatorname{strco}_R\{a_1,a_2\}\big)\big)$$
$$= \frac{1}{2}\big(R\|q_1\| - (q_1,a_1)\big) + \frac{1}{2}\big(R\|q_2\| - (q_2,a_2)\big) = \frac{1}{2}f(q_1) + \frac{1}{2}f(q_2).$$

Thus for any $q_1, q_2 \in [p_1, p_2]$ with $\|q_1 - q_2\| < \delta_0$,

$$f\left(\frac{q_1+q_2}{2}\right) \leqslant \frac{1}{2}f(q_1) + \frac{1}{2}f(q_2).$$

Let us show that the above condition of convexity holds if we replace δ_0 by $2\delta_0$, i.e.

$$\forall q_1, q_2 \in [p_1, p_2]: \quad \|q_1 - q_2\| < 2\delta_0, \qquad f\left(\frac{q_1+q_2}{2}\right) \leqslant \frac{1}{2}f(q_1) + \frac{1}{2}f(q_2).$$

Let $q_3, q_4 \in [p_1, p_2]$: $\|q_3 - q_4\| < 2\delta_0$, $p_0 = \frac{1}{2}(q_3+q_4)$. Let $q_1 = \frac{1}{2}(q_3+p_0)$, $q_2 = \frac{1}{2}(q_4+p_0)$, $\|q_1 - q_2\| < \delta_0$. We have $f(p_0) \leqslant \frac{1}{2}(f(q_1)+f(q_2))$, $f(q_1) \leqslant \frac{1}{2}(f(q_3)+f(p_0))$, $f(q_2) \leqslant \frac{1}{2}(f(q_4)+f(p_0))$. We obtain from the last three inequalities that $f(p_0) \leqslant \frac{1}{4}f(q_3) + \frac{1}{4}f(q_4) + \frac{1}{2}f(p_0)$, i.e. $f(p_0) \leqslant \frac{1}{2}f(q_3) + \frac{1}{2}f(q_4)$.

By induction we obtain that for all $q_1, q_2 \in [p_1, p_2]$,

$$f\left(\frac{1}{2}(q_1+q_2)\right) \leqslant \frac{1}{2}f(q_1) + \frac{1}{2}f(q_2).$$

If p_1 and p_2 are parallel then the latter inequality holds due to the positive homogeneity of the function f. Finally, by continuity of the function f we conclude that f is convex. Hence, by Lemma 3.1 the set A is strongly convex of radius R. □

Lemma 3.3. *Let a subset $A \subset \mathcal{H}$ be uniformly convex with the modulus of convexity $\delta_A(\varepsilon)$, $C > 0$, and $\delta_A(\varepsilon) = C\varepsilon^2 + o(\varepsilon^2)$, $\varepsilon \to +0$. Let $0 < K < C$. Then the set A is strongly convex of radius $\frac{1}{4K}$.*

Proof. By [3, Theorem 2.1] the set A is bounded. Fix any $K \in (0, C)$. From the asymptotic equality $\delta_A(\varepsilon) \sim C\varepsilon^2$, $\varepsilon \to +0$, we obtain that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ we have $\delta_A(\varepsilon) > K\varepsilon^2$ and $\delta_A(\varepsilon) < \frac{\varepsilon}{2}$. (See Fig. 1.)

Fix an arbitrary pair of points $a, b \in A$ with $\|a - b\| = \varepsilon \leqslant \varepsilon_0$. Then

$$B = \operatorname{co}\left(\{a\} \cup B_{\delta_A(\varepsilon)}\left(\frac{a+b}{2}\right) \cup \{b\}\right) \subset A.$$

Consider an arbitrary 2-dimensional affine plane L such that $\{a, b\} \subset L$. Let $w = \frac{1}{2}(a+b)$. Let $l \subset L$ be a line such that $a \in l$ and l is a tangent line to the circle $L \cap \partial B_{\delta_A(\varepsilon)}(w)$ (at the point z). Note that the segment $[a, z]$ is a part of the boundary ∂B. Let $m \subset L$ be a line such that $w \in m$ and m is orthogonal to the line $\operatorname{aff}\{a, b\}$. Let the point $s \in m$ be such that the line $\operatorname{aff}\{s, a\}$ is orthogonal to the line l. Then the circle $L \cap \partial B_{\|a-s\|}(s)$ is tangent to the line l at the point a.

Let the line $l_1 \subset L$ be symmetric to the line l with respect to the line m. Let the point $z_1 = l_1 \cap (L \cap \partial B_{\delta_A(\varepsilon)}(w))$ be symmetric to the point z with respect to the line m. Let $R = \|s - a\|$; $\|s - a\| \geqslant \|a - w\| = \frac{\varepsilon}{2} > \delta_A(\varepsilon)$. Let $x = l \cap m \cap l_1$. The arc $D_R(s)(a, b)$ is the homothetic image of the arc $D_{\delta_A(\varepsilon)}(w)(z, z_1)$ under the homothety with the center x and the coefficient $k = \frac{\|a-s\|}{\|z-w\|} = \frac{R}{\delta_A(\varepsilon)}$. So we see that $D_R(s)(a, b) \subset L \cap B$.

By the similarity of the triangles saw and awz we have $\frac{\|z-w\|}{\|a-w\|} = \frac{\|a-w\|}{\|a-s\|}$, or

$$\frac{2\delta_A(\varepsilon)}{\varepsilon} = \frac{\varepsilon}{2R}.$$

Hence, using inequality $\delta_A(\varepsilon) > K\varepsilon^2$, we obtain that

$$R \leqslant \frac{1}{4K}.$$

By the symmetry of the set B with respect to the line $\operatorname{aff}\{a, b\}$ and the arbitrary choice of L we have

$$\operatorname{strco}_{\frac{1}{4K}}\{a, b\} \subset \operatorname{strco}_R\{a, b\} \subset \operatorname{co}\left(\{a\} \cup B_{\delta_A(\varepsilon)}\left(\frac{a+b}{2}\right) \cup \{b\}\right) \subset A.$$

By Lemma 3.2 we obtain that the set A is strongly convex of radius $\frac{1}{4K}$. □

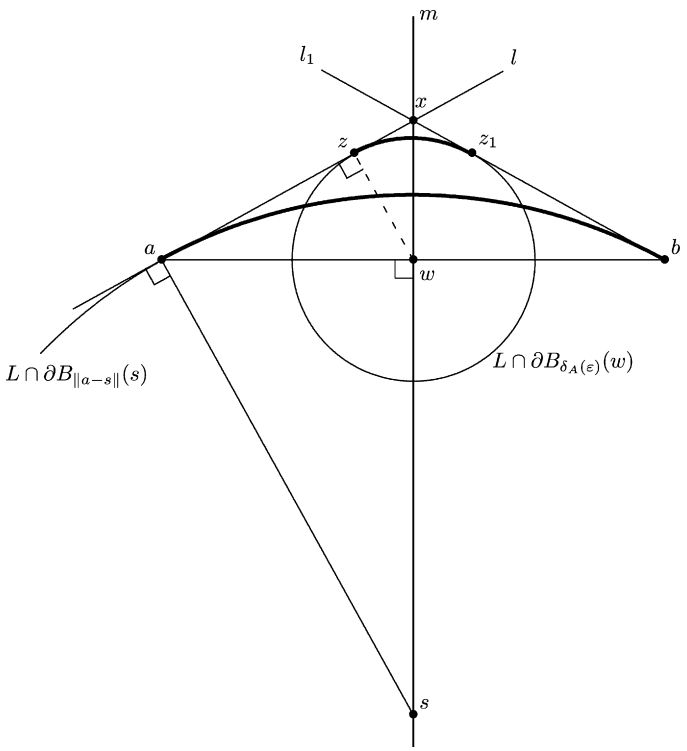

Fig. 1.

Lemma 3.4. *Let a subset $A \subset \mathcal{H}$ be uniformly convex with the modulus of convexity $\delta_A(\varepsilon)$, $C > 0$, and $\delta_A(\varepsilon) = C\varepsilon^2 + o(\varepsilon^2)$, $\varepsilon \to +0$. Let $0 < K < C$. Let the set A be strongly convex of radius $R > \frac{1}{8K}$. Then the set A is strongly convex of radius*

$$R_1 = \frac{2R}{8RK + 1}.$$

Proof. See Fig. 2, where w, l, m, etc., are defined as in the proof of Lemma 3.4.

Fix any $K \in (0, C)$. From the asymptotic equality $\delta_A(\varepsilon) \sim C\varepsilon^2$, $\varepsilon \to +0$, we obtain that there exists such $\varepsilon_0 \in (0, \frac{1}{100K})$ that for all $\varepsilon \in (0, \varepsilon_0]$ we have $\delta_A(\varepsilon) < \frac{\varepsilon}{2}$ and

$$\frac{\delta_A(\varepsilon)}{\varepsilon^2}\left(1 - \frac{\delta_A(\varepsilon)}{R}\right) > K. \tag{3.4}$$

Choose any pair of points $a, b \in A$, $\|a - b\| = \varepsilon \leqslant \varepsilon_0$. Then

$$B = \operatorname{strco}_R\big\{\{a\} \cup B_{\delta_A(\varepsilon)}(w) \cup \{b\}\big\} \subset A,$$

where $w = \frac{a+b}{2}$.

Let L be any 2-dimensional affine plane, $\{a, b\} \subset L$. Let $m \subset L$ be a line such that $w \in m$ and $m \perp \operatorname{aff}\{a, b\}$. Consider a circle of radius R with a center $s \in L$ such that $L \cap B_R(s) \supset L \cap B_{\delta_A(\varepsilon)}(w)$, it passes through the point a, tangents to $L \cap \partial B_{\delta_A(\varepsilon)}(w)$ and define

$$\big[L \cap \partial B_R(s)\big] \cap \big[L \cap \partial B_{\delta_A(\varepsilon)}(w)\big] = \{z\}.$$

Such circle exists because $R > \varepsilon/2$.

Let $D_1 = D_R(s)(a, z)$, D_2 is symmetric to the D_1 with respect to the line m, D_3 and D_4 are symmetric to the D_1 and D_2 with respect to the line $\operatorname{aff}\{a, b\}$, respectively. We have

$$L \cap B \supset \operatorname{co}\big\{D_1 \cup D_2 \cup D_3 \cup D_4 \cup \big(L \cap B_{\delta_A(\varepsilon)}(w)\big)\big\}.$$

Let l be the tangent line to the circle $L \cap \partial B_R(s)$ at the point a. Let φ be the angle between the lines l and $\operatorname{aff}\{a, b\}$; α be the angle between the lines $\operatorname{aff}\{a, b\}$ and $\operatorname{aff}\{a, s\}$; $\varphi + \alpha = \pi/2$. Consider the triangle aws: $\|a - w\| = \frac{\varepsilon}{2}$, $\|a - s\| = R$, $\|w - s\| = R - \delta_A(\varepsilon)$. Hence by the Cosine theorem we get

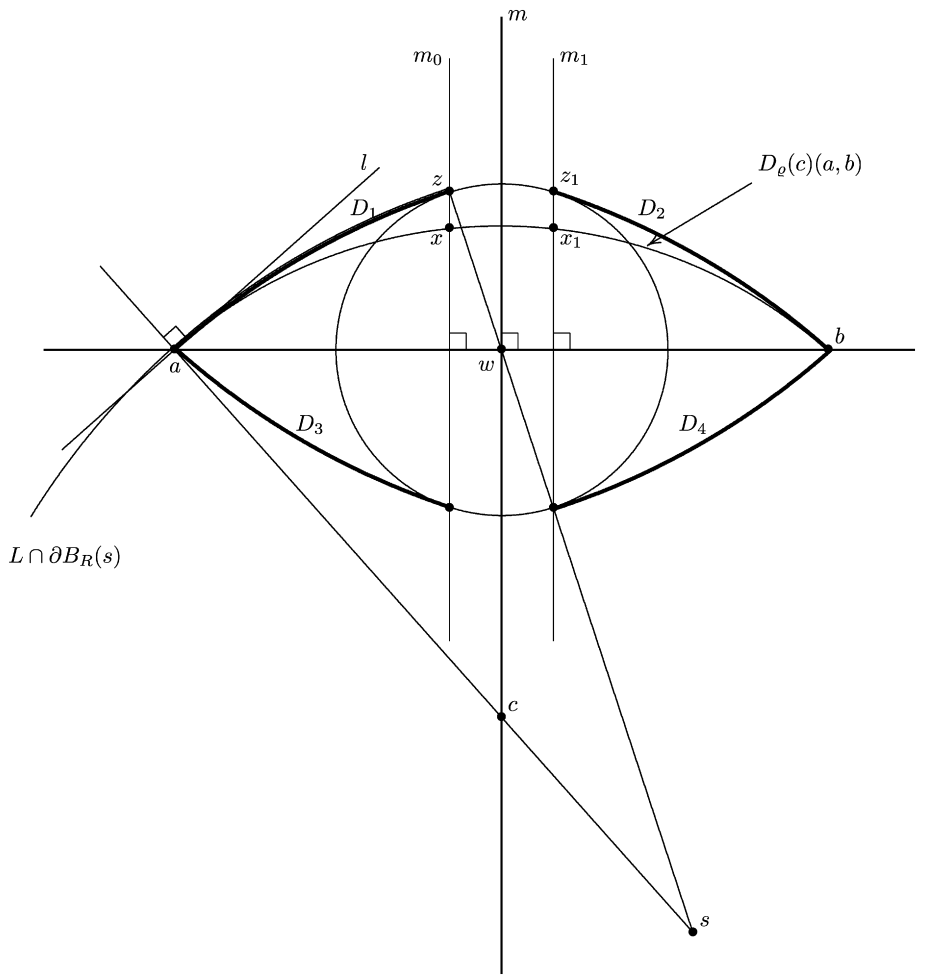

Fig. 2.

$$\sin\varphi = \cos\alpha = \frac{\|a-s\|^2 + \|a-w\|^2 - \|w-s\|^2}{2\|a-s\|\cdot\|a-w\|} = \frac{\varepsilon}{4R} + \frac{2\delta_A(\varepsilon)}{\varepsilon} - \frac{\delta_A^2(\varepsilon)}{R\varepsilon} = \frac{\varepsilon}{4R} + \frac{2\delta_A(\varepsilon)}{\varepsilon}\left(1 - \frac{\delta_A(\varepsilon)}{R}\right),$$

and using (3.4) we obtain that

$$\sin\varphi = \cos\alpha \geqslant \frac{\varepsilon}{4R} + 2K\varepsilon. \tag{3.5}$$

Let the point $z_1 = D_2 \cap (L \cap \partial B_{\delta_A(\varepsilon)}(w))$ be symmetric to the point z with respect to the line m. Let the lines $m_0 \subset L$ and $m_1 \subset L$ be parallel to the line m, $z \in m_0$, $z_1 \in m_1$. Let $\varrho = \frac{\varepsilon}{2\sin\varphi}$ and $c = m \cap \mathrm{aff}\{a, s\}$. Note that $\|a - c\| = \varrho$. Let $x = D_\varrho(c)(a,b) \cap m_0$, $x_1 = D_\varrho(c)(a,b) \cap m_1$.

The circle $L \cap \partial B_\varrho(c)$ touches the line l at the point a. Taking into account that $K > 1/8R$, we get by the formula (3.5)

$$\varrho = \frac{\varepsilon}{2\sin\varphi} \leqslant \frac{1}{\frac{1}{2R} + 4K} < \frac{1}{\frac{1}{2R} + \frac{1}{2R}} = R.$$

By the last inequality the points a and s are separated by the line m and the points a, z (and b, z_1) are situated in the same half-plane with respect to the line m. Hence the points x and x_1 belong to the disk $L \cap B_{\delta_A(\varepsilon)}(w)$. The arc $D_\varrho(c)(a,x)$ lies between the arc $D_R(s)(a,z)$ and the line $\mathrm{aff}\{a,b\}$, by the symmetry of the arc $D_\varrho(c)(a,b)$ with respect to the line m we have

$$D_\varrho(c)(a,x) \cup D_\varrho(c)(b,x_1) \subset L \cap B.$$

By the inequality $\varrho \geqslant \frac{\varepsilon}{2} > \delta_A(\varepsilon)$ we have

$$D_\varrho(c)(x,x_1) \subset L \cap B_{\delta_A(\varepsilon)}(w).$$

Thus $D_\varrho(c)(a,b) \subset L \cap B$ and $\varrho \leqslant \frac{1}{\frac{1}{2R}+4K} = R_1$. By the symmetry of the set $L \cap B$ with respect to the line $\operatorname{aff}\{a,b\}$ and the arbitrary choice of L we have

$$\operatorname{strco}_{R_1}\{a,b\} \subset \operatorname{strco}_\varrho\{a,b\} \subset B \subset A.$$

By Lemma 3.2 we obtain that the set A is strongly convex with the radius R_1. □

Lemma 3.5. *Under the assumptions of Lemma* 3.4 *the set A is strongly convex of radius* $r = \frac{1}{8K}$.

Proof. Define inductively the sequence $\{R_n\}_{n=1}^\infty$ as follows: $R_1 = R$, and, for $n \geqslant 1$,

$$R_{n+1} = \frac{2R_n}{8R_nK + 1}.$$

It is not difficult to show that $R_{n+1} \leqslant R_n$. After having a look to the function $f(x) = \frac{2x}{8xK+1}$ (which is increasing for $x \geqslant 0$), and using $R > \frac{1}{8K}$, it is not difficult to show that also $R_n > \frac{1}{8K}$ for every $n \in \mathbb{N}$. Hence by the Weierstrass theorem we get $R_n \to r = \frac{2r}{8rK+1}$, that is, $r = \frac{1}{8K}$.

Using Lemma 3.4 we know that the set A is strongly convex of radius R_n for every $n \in \mathbb{N}$. By Lemma 3.2, the latter assertion means that the functions $f_n(p) = R_n\|p\| - s(p,A)$ are convex for all $n \in \mathbb{N}$. Taking the limit of the sequence of the functions we get that

$$f(p) = \frac{1}{8K}\|p\| - s(p,A)$$

is a convex function as well. This shows, again by Lemma 3.2, that A is strongly convex of radius $\frac{1}{8K}$. □

4. Proof of Theorem 2.1

Let $K_n = C - \frac{1}{n}$. By Lemma 3.3 the set A is strongly convex of radius $\frac{1}{4K_n}$. By Lemma 3.5 the set A is strongly convex of radius $\frac{1}{8K_n}$. By Lemma 3.1 this is equivalent to the convexity of the function $\frac{1}{8K_n}\|p\| - s(p,A)$ for all natural n. Taking the limit $n \to \infty$, we obtain the convexity of the function $\frac{1}{8C}\|p\| - s(p,A)$. Hence by Lemma 3.1 the set A is strongly convex of radius $\frac{1}{8C}$.

Suppose that there exist a number $r \in (0, \frac{1}{8C})$ and a subset $Y \subset \mathcal{H}$ such that

$$A = \bigcap_{x \in Y} B_r(x).$$

For the ball $B_r(0)$ in the Hilbert space we have (cf. [6])

$$\delta_{B_r(0)}(\varepsilon) = r\delta_{\mathcal{H}}\left(\frac{\varepsilon}{r}\right) = r - \sqrt{r^2 - \frac{\varepsilon^2}{4}} = \frac{\varepsilon^2}{8r} + o\big(\varepsilon^2\big), \quad \varepsilon \to +0.$$

Due to the fact that the set A is the intersection of closed balls of the radius r we conclude that $C\varepsilon^2 + o(\varepsilon^2) = \delta_A(\varepsilon) \geqslant r\delta_{\mathcal{H}}(\frac{\varepsilon}{r}) = \frac{\varepsilon^2}{8r} + o(\varepsilon^2)$, for all $\varepsilon > 0$. Hence $C \geqslant \frac{1}{8r}$. This contradicts the inequality $r < \frac{1}{8C}$. □

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