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Soft substructures of rings, fields and modules

Akın Osman Atagün [a,*], Aslıhan Sezgin [b]

[a] Department of Mathematics, Bozok University, 66100 Yozgat, Turkey

[b] Department of Mathematics, Amasya University, 05100 Amasya, Turkey

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ABSTRACT

Soft set theory, proposed by Molodtsov, has been regarded as an effective mathematical tool to deal with uncertainties. In this paper, we introduce and study soft subrings and soft ideals of a ring by using Molodtsov's definition of the soft sets. Moreover, we introduce soft subfields of a field and soft submodule of a left R-module. Some related properties about soft substructures of rings, fields and modules are investigated and illustrated by many examples.

1. Introduction

The complexities of modeling uncertain data in economics, engineering, environmental science, sociology, medical science and many other fields cannot be successfully dealt with by classical methods. While probability theory, fuzzy set theory [1,2], rough set theory [3,4], vague set theory [5] and the interval mathematics [6] are useful approaches to describing uncertainty, each of these theories has its inherent difficulties. Consequently, Molodtsov [7] proposed a completely new approach for modeling vagueness and uncertainty, which is called soft set theory. Now, works on soft set theory are progressing rapidly. Maji et al. [8] described the applications of soft set theory and have published a detailed theoretical study on soft sets [9]. Molodtsov [7] demonstrated a lot of potential applications of soft sets in different fields including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory. Aktaş and Çağman [10,11] studied the basic concepts of soft set theory and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also defined and studied soft group and derived their basic properties by using Molodtsov's definition of the soft sets. Ali et al. [12] introduced some new notions such as the restricted intersection, the restricted union, the restricted difference and the extended intersection of two soft sets. Feng et al. [13] introduced and investigated soft semirings, soft subsemirings, soft ideals, idealistic soft semirings and soft semiring homomorphisms. In [14], Çağman and Enginoğlu defined soft matrices and their operations to construct a soft max–min decision making method which can be successfully applied to the problems that contain uncertainties. Acar et al. [15] introduced initial concepts of soft rings. Atagün and Sezgin [16] introduced the notions of soft near-rings, soft subnear-rings, soft (left, right) ideals, (left, right) idealistic soft near-rings and soft near-ring homomorphisms and investigated them with many corresponding examples. Sezgin et al. [17] extended the study of soft near-rings especially with respect to the idealistic soft near-rings as well. The algebraic structure of set theories dealing with uncertainties has also been studied by some authors. Rosenfeld [18] proposed the concept of fuzzy groups in order to establish the algebraic structures of fuzzy

* Corresponding author. Tel.: +90 3542421021; fax: +90 3542421022.
E-mail addresses: aosman.atagun@bozok.edu.tr, aoatagun@erciyes.edu.tr (A.O. Atagün), sezgin.nearring@hotmail.com, aslihan.sezgin@amasya.edu.tr (A. Sezgin).

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sets. Abou-Zaid [19] introduced the notion of a fuzzy subnear-ring and studied fuzzy ideals of a near-ring. This concept is also discussed by many authors (e.g., [20–23]). Atagün [24] defined the notions of soft subnear-rings, soft ideals and soft N-subgroups of near-rings. He also established the bi-intersection and product operation of soft subnear-rings, soft ideals and soft N-groups of near-rings. Moreover, he showed that for all soft subnear-rings (resp. soft ideals, soft N-groups) of a near-ring N, there exists at least one subnear-ring (resp. ideal, N-subgroup) of N. Rough groups were defined by Biswas et al. [25] and some other authors (e.g., [26,27]) have studied the algebraic properties of rough sets as well.

In this paper, using soft set theory, we deal with the algebraic soft substructures of rings, fields and modules. We define the notions of soft subring and soft ideal of a ring, soft subfield of a field and soft submodule of a module with several illustrating examples. We also establish the restricted intersection and the product operations of these soft substructures and sum operations for soft ideals of a ring and soft submodules of a module. Moreover, we show that for all soft subrings (resp. soft ideals) of a ring R, there exists at least one subring (resp. ideal) of R and that for all soft subfields (resp. soft submodules) of a field F (resp., module M), there exists at least one subfield (resp. submodule) of F (resp., M).

2. Preliminaries

By a ring, we shall mean an algebraic system $(R, +, .)$, where

(i) $(R, +)$ forms an abelian group,
(ii) $(R, .)$ forms a semi-group and
(iii) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$ (i.e., right and left distributive laws hold).

Throughout this paper, R will always denote a ring. A subgroup S of $(R, +)$ with $SS \subseteq S$ is called a *subring* of R and denoted by $S < R$. A subgroup I of $(R, +)$ is called a *left ideal* (resp., *right ideal*) of R if $ri \in I$ (resp., $ir \in I$) for all $r \in R$ and $i \in I$ and denoted by $I \lhd_l R$ (resp., $I \lhd_r R$). If I is both left and right ideals of R, then it is called an *ideal* of R and denoted by $I \lhd R$.

A ring $(F, +, .)$ is called a *field* if $(F - \{0_F\}, \cdot)$ is an abelian group. A *subfield* S of a field F is a subset containing 0_F and 1_F, closed under the operations $+, -, \cdot$ and multiplicative inverses and with its own operations defined by restriction. Hence the subset S of a field F is a *subfield* if and only if the conditions

(i) $0_F \in S$,
(ii) $x - y \in S$ for all $x, y \in S$,
(iii) $1_F \in S$ and
(iv) $xy^{-1} \in S$ for all $x, y \in S$ $(y \neq 0_F)$

hold.

A *left R-module* over a ring R consists of an abelian group $(M, +)$ and an operation $R \times M \longrightarrow M$ such that for all $r, s \in R, x, y \in M$, we have

(i) $r(x + y) = rx + ry$
(ii) $(r + s)x = rx + sx$
(iii) $(rs)x = r(sx)$.

It is denoted by R^M. Clearly R itself is a (left) R-module by natural operation. Suppose M is a left R-module and N is a subgroup of M. Then N is called a *submodule* (or *R-submodule*, to be more explicit) if, for any $n \in N$ and any $r \in R$, the product rn is in N.

Molodtsov [7] defined the soft set in the following manner: Let U be an initial universe set, E be a set of parameters, $P(U)$ be the power set of U and $A \subseteq E$.

Definition 1 (*[7]*)**.** A pair (F, A) is called a soft set over U, where F is a mapping given by

$$F : A \to P(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-elements of the soft set (F, A), or as the set of ε-approximate elements of the soft set. To illustrate this idea, Molodtsov considered several examples in [7].

In fact, there exists a mutual correspondence between soft sets and binary relations as shown in [28,29]. That is, let A and B be nonempty sets and assume that α refers to an arbitrary binary relation between an element of A and an element of B. A set-valued function $F : A \to P(B)$ can be defined as $F(x) = \{y \in B \mid (x, y) \in \alpha\}$ for all $x \in A$. Then, the pair (F, A) is a soft set over B, which is derived from the relation α.

Definition 2 (*[12]*)**.** Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$. The *restricted intersection* of (F, A) and (G, B) is denoted by $(F, A) \Cap (G, B)$, and is defined as $(F, A) \Cap (G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$.

3. Soft substructures of rings

Throughout this section, we denote a ring by R and a subring (resp. ideal) S of R by $S < R$ (resp. $S \lhd R$).

Definition 3. Let S be a subring of R and let (F, S) be a soft set over R. If for all $x, y \in S$,

(s1) $F(x-y) \supseteq F(x) \cap F(y)$ and
(s2) $F(xy) \supseteq F(x) \cap F(y)$,

then the soft set (F, S) is called a *soft subring* of R and denoted by $(F, S)\tilde{<}R$ or simply $F_S\tilde{<}R$.

Example 1. Given the ring $R = (\mathbb{Z}_6, +, .)$, $S_1 = \{0, 3\} < R$ and the soft set (F, S_1) over R, where $F : S_1 \to P(R)$ is a set-valued function defined by $F(0) = \{0, 1, 4, 5\}$ and $F(3) = \{0, 4, 5\}$. Then one can easily show that $F_{S_1}\tilde{<}R$.

Given $S_2 = \{0, 2, 4\} < R$ and the soft set (G, S_2) over R, where $G : S_2 \to P(R)$ is a set-valued function defined by $G(0) = \{0, 1, 3, 4, 5\}$, $G(2) = \{1, 3\}$ and $G(4) = \{0, 1, 3, 4\}$. Then one can easily show that $G_{S_2}\tilde{<}R$. However if we define the soft set (T, S_2) over R such that $T : S_2 \to P(R)$ is a set-valued function defined by $T(0) = \{0, 1, 3, 4, 5\}$, $T(2) = \{1, 3\}$ and $T(4) = \{1, 2\}$, then $T(2 \cdot 2) = T(4) = \{1, 2\} \not\supseteq T(2) \cap T(2) = T(2) = \{1, 3\}$. It follows that (T, S_2) is not a soft subring of R.

Example 2. Given the ring $R = M_2(\mathbb{Z}_4)$, i.e. 2×2 matrices with $\mathbb{Z}_4$ terms, with the operations addition and multiplication of matrices.

Let $P = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right\}$. It is obvious that P is a subring of R.

Let the soft set (J, P) over R, where $J : P \to P(R)$ is a set-valued function defined by $J\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}\right\}$ and $J\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = \left\{\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}\right\}$. One can easily show that $J_P\tilde{<}R$.

However, if we define the soft set (W, P) over R such that $W\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\right\}$ and $W\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = \left\{\begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}\right\}$, then $W\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = W\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \not\supseteq W\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) \cap W\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = W\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)$. Then, (W, P) is not a soft subring of R.

Theorem 1. *If $F_{S_1}\tilde{<}R$ and $G_{S_2}\tilde{<}R$, then $F_{S_1} \Cap G_{S_2}\tilde{<}R$.*

Proof. Since S_1 and S_2 are subrings of R, then $S_1 \cap S_2$ is a subring of R. By Definition 2, let $F_{S_1} \Cap G_{S_2} = (F, S_1) \Cap (G, S_2) = (H, S_1 \cap S_2)$, where $H(x) = F(x) \cap G(x)$ for all $x \in S_1 \cap S_2 \neq \emptyset$. Then for all $x, y \in S_1 \cap S_2$,

$$\begin{aligned} H(x-y) &= F(x-y) \cap G(x-y) \\ &\supseteq (F(x) \cap F(y)) \cap (G(x) \cap G(y)) \\ &= (F(x) \cap G(x)) \cap (F(y) \cap G(y)) \\ &= H(x) \cap H(y), \end{aligned}$$

$$\begin{aligned} H(xy) &= F(xy) \cap G(xy) \\ &\supseteq (F(x) \cap F(y)) \cap (G(x) \cap G(y)) \\ &= (F(x) \cap G(x)) \cap (F(y) \cap G(y)) \\ &= H(x) \cap H(y). \end{aligned}$$

Therefore $F_{S_1} \Cap G_{S_2} = H_{S_1 \cap S_2}\tilde{<}R$. □

Definition 4. Let R_1 and R_2 be rings and let (F, S_1) and (G, S_2) be two soft subrings of R_1 and R_2, respectively. The product of soft subrings (F, S_1) and (G, S_2) is defined as $(F, S_1) \times (G, S_2) = (Q, S_1 \times S_2)$, where $Q(x, y) = F(x) \times G(y)$ for all $(x, y) \in S_1 \times S_2$.

Theorem 2. *If $F_{S_1}\tilde{<}R_1$ and $G_{S_2}\tilde{<}R_2$, then $F_{S_1} \times G_{S_2}\tilde{<}R_1 \times R_2$.*

Proof. Since S_1 and S_2 are subrings of R_1 and R_2, respectively, then $S_1 \times S_2$ is a subring of $R_1 \times R_2$. By Definition 4, let $F_{S_1} \times G_{S_2} = (F, S_1) \times (G, S_2) = (Q, S_1 \times S_2)$, where $Q(x, y) = F(x) \times G(y)$ for all $(x, y) \in S_1 \times S_2$. Then for all $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$,

$$\begin{aligned} Q((x_1, y_1) - (x_2, y_2)) &= Q(x_1 - x_2, y_1 - y_2) \\ &= F(x_1 - x_2) \times G(y_1 - y_2) \\ &\supseteq (F(x_1) \cap F(x_2)) \times (G(y_1) \cap G(y_2)) \\ &= (F(x_1) \times G(y_1)) \cap (F(x_2) \times G(y_2)) \\ &= Q(x_1, y_1) \cap Q(x_2, y_2), \end{aligned}$$

$$\begin{aligned} Q((x_1, y_1)(x_2, y_2)) &= Q(x_1x_2, y_1y_2) \\ &= F(x_1x_2) \times G(y_1y_2) \\ &\supseteq (F(x_1) \cap F(x_2)) \times (G(y_1) \cap G(y_2)) \\ &= (F(x_1) \times G(y_1)) \cap (F(x_2) \times G(y_2)) \\ &= Q(x_1, y_1) \cap Q(x_2, y_2). \end{aligned}$$

Hence $F_{S_1} \times G_{S_2} = Q_{S_1 \times S_2}\tilde{<}R_1 \times R_2$. □

Proposition 1. *If $F_S \tilde{<} R$, then $F(0) \supseteq F(x)$ for all $x \in S$.*

Proof. Since (F, S) is a soft subring of R, then $F(0) = F(x - x) \supseteq F(x) \cap F(x) = F(x)$ for all $x \in S$. $\square$

Proposition 2. *If $F_S \tilde{<} R$, then $S_F = \{x \in S \mid F(x) = F(0)\}$ is a subring of S.*

Proof. We need to show that $x - y \in S_F$ and $xy \in S_F$ for all $x, y \in S_F$, which means that $F(x - y) = F(0)$ and $F(xy) = F(0)$ have to be satisfied. Since $x, y \in S_F$, then $F(x) = F(y) = F(0)$. By Proposition 1, $F(0) \supseteq F(x - y)$ and $F(0) \supseteq F(xy)$ for all $x, y \in S_F$. Since (F, S) is a soft subring of R, then $F(x - y) \supseteq F(x) \cap F(y) = F(0)$ and $F(xy) \supseteq F(x) \cap F(y) = F(0)$ for all $x, y \in S_F$. Therefore S_F is a subring of S. $\square$

To illustrate Theorems 1 and 2, we have the following example:

Example 3. We take $(F, S_1)\tilde{<}\mathbb{Z}_6$ and $(G, S_2)\tilde{<}\mathbb{Z}_6$ in Example 1. By Definition 2, $F_{S_1} \Cap G_{S_2} = (F, S_1) \Cap (G, S_2) = (W, S_1 \cap S_2)$, where $W(x) = F(x) \cap G(x)$ for all $x \in S_1 \cap S_2 = \{0\}$. Then $Q(0) = \{0, 1, 4, 5\}$. It is obvious that $W_{S_1 \cap S_2} \tilde{<} R$.

By Definition 4, $F_{S_1} \times G_{S_2} = (F, S_1) \times (G, S_2) = (Q, S_1 \times S_2)$, where $Q(x, y) = F(x) \times G(y)$ for all $(x, y) \in S_1 \times S_2 = \{(0, 0), (0, 2), (0, 4), (3, 0), (3, 2), (3, 4)\}$. Then it can be easily seen that $Q_{S_1 \times S_2} \tilde{<} R \times R$. We show the operations for some elements of $S_1 \times S_2$:

$$\begin{aligned} Q((0, 2) - (3, 4)) = Q(0 - 3, 2 - 4) &= Q(3, 4) \\ &= F(3) \times G(4) = \{0, 4, 5\} \times \{0, 1, 3, 4\} \end{aligned}$$

$$\begin{aligned} Q(0, 2) \cap Q(3, 4) &= (F(0) \times G(2)) \cap (F(3) \times G(4)), \\ &= (\{0, 1, 4, 5\} \times \{1, 3\}) \cap (\{0, 4, 5\} \times \{0, 1, 3, 4\}) \\ &= \{(0, 1), (0, 3), (4, 1), (4, 3), (5, 1), (5, 3)\}, \end{aligned}$$

$$\begin{aligned} Q((0, 2)(3, 4)) = Q(0 \cdot 3, 2 \cdot 4) &= Q(0, 2) \\ &= F(0) \times G(2) = (\{0, 1, 4, 5\} \times \{1, 3\}). \end{aligned}$$

It is seen that $Q((0, 2) - (3, 4)) \supseteq Q(0, 2) \cap Q(3, 4)$ and $Q((0, 2)(3, 4)) \supseteq Q(0, 2) \cap Q(3, 4)$.

Definition 5. Let I be an ideal of R and let (F, I) be a soft set over R. If for all $x, y \in I$ and $r \in R$,

(i_1) $F(x - y) \supseteq F(x) \cap F(y)$ and
(i_2) $F(rx) \supseteq F(x)$,
(i_3) $F(xr) \supseteq F(x)$,

then (F, I) is called a *soft ideal* of R and denoted by $(F, I)\tilde{\lhd} G$ or simply $F_I \tilde{\lhd} R$.

Example 4. Let $R = (\mathbb{Z}_{12}, +, .)$, $I_1 = \{0, 6\} \lhd R$ and the soft set (F, I_1) over R, where $F : I_1 \to P(R)$ is a set-valued function defined by $F(0) = \mathbb{Z}_{12}$ and $F(6) = \{1, 7\}$. It can be easily illustrated that $F_{I_1} \tilde{\lhd} R$.

Let $I_2 = \{0, 4, 8\} \lhd R$ and the soft set (G, I_2) over R, where $G : I_2 \to P(R)$ is a set-valued function defined by $G(0) = \mathbb{Z}_{12}$, $G(4) = G(8) = \{3, 9\}$. It can be easily illustrated that $G_{I_2} \tilde{\lhd} R$. However if we define the soft set (H, I_2) over R such that the soft set $H : I_2 \to P(R)$ is a set-valued function defined by $H(0) = \mathbb{Z}_{12}$, $H(4) = \{1, 3\}$ and $H(8) = \{1, 2\}$, then $H(5 \cdot 4) = H(8) = \{1, 2\} \not\supseteq H(4) = \{1, 3\}$. It follows that (H, I_2) is not a soft ideal of R.

Theorem 3. *If $F_{I_1} \tilde{\lhd} R$ and $G_{I_2} \tilde{\lhd} R$, then $F_{I_1} \Cap G_{I_2} \tilde{\lhd} R$.*

Proof. Since $I_1, I_2 \lhd R$, then $I_1 \cap I_2 \lhd R$. By Definition 2, $F_{I_1} \Cap G_{I_2} = (F, I_1) \Cap (G, I_2) = (H, I_1 \cap I_2)$, where $H(x) = F(x) \cap G(x)$ for all $x \in I_1 \cap I_2 \neq \emptyset$. Then for all $x, y \in I_1 \cap I_2$ and for all $r \in R$,

$$\begin{aligned} H(x - y) &= F(x - y) \cap G(x - y) \\ &\supseteq (F(x) \cap F(y)) \cap (G(x) \cap G(y)) \\ &= (F(x) \cap G(x)) \cap (F(y) \cap G(y)) \\ &= H(x) \cap H(y), \end{aligned}$$

$$\begin{aligned} H(rx) &= F(rx) \cap G(rx) \\ &\supseteq F(x) \cap G(x) \\ &= H(x), \end{aligned}$$

$$\begin{aligned} H(xr) &= F(xr) \cap G(xr) \\ &\supseteq F(x) \cap G(x) \\ &= H(x). \end{aligned}$$

Therefore $F_{I_1} \Cap G_{I_2} = H_{I_1 \cap I_2} \tilde{\lhd} R$. $\square$

Definition 6. Let R_1 and R_2 be rings and let (F, I_1) and (G, I_2) be two soft ideals of R_1 and R_2, respectively. The product of soft ideals (F, I_1) and (G, I_2) is defined as $(F, I_1) \times (G, I_2) = (Q, I_1 \times I_2)$, where $Q(x, y) = F(x) \times G(y)$ for all $(x, y) \in I_1 \times I_2$.

Theorem 4. *If $F_{I_1} \tilde{\triangleleft} R_1$ and $G_{I_2} \tilde{\triangleleft} R_2$, then $F_{I_1} \times G_{I_2} \tilde{\triangleleft} R_1 \times R_2$.*

Proof. Since I_1 and I_2 are ideals of R_1 and R_2, respectively, then $I_1 \times I_2$ is an ideal of $R_1 \times R_2$. By Definition 6, $F_{I_1} \times G_{I_2} = (F, I_1) \times (G, I_2) = (Q, I_1 \times I_2)$, where $Q(x, y) = F(x) \times G(y)$ for all $(x, y) \in I_1 \times I_2$. Then for all $(x_1, y_1), (x_2, y_2) \in I_1 \times I_2$ and $(r_1, r_2) \in R_1 \times R_2$,

$$\begin{aligned} Q((x_1, y_1) - (x_2, y_2)) &= Q(x_1 - x_2, y_1 - y_2) \\ &= F(x_1 - x_2) \times G(y_1 - y_2) \\ &\supseteq (F(x_1) \cap F(x_2)) \times (G(y_1) \cap G(y_2)) \\ &= (F(x_1) \times G(y_1)) \cap (F(x_2) \times G(y_2)) \\ &= Q(x_1, y_1) \cap Q(x_2, y_2), \end{aligned}$$

$$\begin{aligned} Q((r_1, r_2)(x_1, y_1)) &= Q(r_1x_1, r_2y_1) \\ &= F(r_1x_1) \times G(r_2y_1) \\ &\supseteq F(x_1) \times G(y_1) \\ &= Q(x_1, y_1), \end{aligned}$$

$$\begin{aligned} Q((x_1, y_1)(r_1, r_2)) &= Q(x_1r_1, y_1r_2) \\ &= F(x_1r_1) \times G(y_1r_2) \\ &\supseteq F(x_1) \times G(y_1) \\ &= Q(x_1, y_1). \end{aligned}$$

Therefore $F_{I_1} \times G_{I_2} = Q_{I_1 \times I_2} \tilde{\triangleleft} R_1 \times R_2$. □

It is worth noting that if I_1 and I_2 are two ideals of a ring $(R, +, .)$, then the sum of these two ideals is defined by $I_1 + I_2 = \{i_1 + i_2 \mid i_1 \in I_1 \wedge i_2 \in I_2\}$.

Definition 7. Let (F, I_1) and (G, I_2) be two soft ideals of R. If $I_1 \cap I_2 = \{0\}$, then the sum of soft ideals (F, I_1) and (G, I_2) is defined by $(F, I_1) + (G, I_2) = (H, I_1 + I_2)$, where $H(x + y) = F(x) + G(y)$ for all $x + y \in I_1 + I_2$.

Theorem 5. *If $F_{I_1} \tilde{\triangleleft} R$ and $G_{I_2} \tilde{\triangleleft} R$, where $I_1 \cap I_2 = \{0\}$, then $F_{I_1} + G_{I_2} \tilde{\triangleleft} R$.*

Proof. Since I_1 and I_2 are ideals of R, then $I_1 + I_2$ is an ideal of R. By Definition 7, let $F_{I_1} + G_{I_2} = (F, I_1) + (G, I_2) = (H, I_1 + I_2)$, where $H(x + y) = F(x) + G(y)$ for all $x + y \in I_1 + I_2$. It is seen that H is well defined because $I_1 \cap I_2 = \{0\}$. Then for all $x_1 + y_1, x_2 + y_2 \in I_1 + I_2$ and $r \in R$,

$$\begin{aligned} H((x_1 + y_1) - (x_2 + y_2)) &= H((x_1 - x_2) + (y_1 - y_2)) \\ &= F(x_1 - x_2) + G(y_1 - y_2) \\ &\supseteq (F(x_1) \cap F(x_2)) + (G(y_1) \cap G(y_2)) \\ &= (F(x_1) + G(y_1)) \cap (F(x_2) + G(y_2)) \\ &= H(x_1 + y_1) \cap H(x_2 + y_2), \end{aligned}$$

$$\begin{aligned} H(r(x_1 + y_1)) &= H(rx_1 + ry_1) \\ &= F(rx_1) + G(ry_1) \\ &\supseteq F(x_1) + G(y_1) \\ &= H(x_1 + y_1), \end{aligned}$$

$$\begin{aligned} H((x_1 + y_1)r) &= H(x_1r + y_1r) \\ &= F(x_1r) + G(y_1r) \\ &\supseteq F(x_1) + G(y_1) \\ &= H(x_1 + y_1). \end{aligned}$$

Therefore $F_{I_1} + G_{I_2} = H_{I_1 + I_2} \tilde{\triangleleft} R$. □

To illustrate Theorem 5, we have the following example:

Example 5. We take $(F, I_1) \tilde{\triangleleft} \mathbb{Z}_{12}$ and $(G, I_2) \tilde{\triangleleft} \mathbb{Z}_{12}$ in Example 4. By Definition 7, $F_{I_1} + G_{I_2} = (F, I_1) + (G, I_2) = (Q, I_1 + I_2)$, where $Q(x + y) = F(x) + G(y)$ for all $x + y \in I_1 + I_2 = \{0, 2, 4, 6, 8, 10\}$. It can be easily seen that $Q_{I_1 + I_2} \tilde{\triangleleft} R$. We show the operations for some elements of $I_1 + I_2$:

$$\begin{aligned} Q((6 + 4) - (6 + 8)) &= Q((6 - 6) + (4 - 8)) \\ &= Q(0 + 8) = F(0) + G(8) \\ &= \mathbb{Z}_{12} \end{aligned}$$

$$Q(3 \cdot (6 + 4)) = Q(6 + 0) = F(6) + G(0) = \mathbb{Z}_{12}$$

$$Q((6 + 4) \cdot 3) = Q(6 + 0) = F(6) + G(0) = \mathbb{Z}_{12}$$

and $Q(6 + 4) = F(6) + G(4) = \{4, 10\}$, $Q(6 + 8) = F(6) + G(8) = \{4, 10\}$. Thus, $Q(6 + 4) \cap Q(6 + 8) = \{4, 10\}$. It is obvious that, $Q((6+4)-(6+8)) \supseteq Q(6+4) \cap Q(6+8)$, $Q(3 \cdot (6+4)) \supseteq Q(6+4)$, $Q((6+4) \cdot 3) \supseteq Q(6+4)$. Similarly the other elements of $I_1 + I_2$ and $r \in R$ can be easily illustrated.

Definition 8. Let (F, I) be a soft subring (soft ideal) of R. Then,

(i) (F, I) is said to be *trivial* if $F(x) = \{0_R\}$ for all $x \in I$.
(ii) (F, I) is said to be *whole* if $F(x) = R$ for all $x \in I$.

Proposition 3. *Let (F, I_1) and (G, I_2) be soft subrings (resp. soft ideals) of R. Then,*

(i) *If (F, I_1) and (G, I_2) are trivial soft subrings (resp. soft ideals) of R, then $(F, I_1) \Cap (G, I_2)$ is a trivial soft subring (resp. soft ideal) of R.*
(ii) *If (F, I_1) and (G, I_2) are whole soft subrings (resp. soft ideals) of R, then $(F, I_1) \Cap (G, I_2)$ is a whole soft subring (resp. soft ideal) of R.*
(iii) *If (F, I_1) is a trivial soft subring (resp. soft ideal) of R and (G, I_2) is a whole soft subring (resp. soft ideal) of R, then $(F, I_1) \Cap (G, I_2)$ is a trivial soft subring (resp. soft ideal) of R.*
(iv) *If (F, I_1) and (G, I_2) are trivial soft ideals of R, where $I_1 \cap I_2 = \{0\}$, then $(F, I_1) + (G, I_2)$ is a trivial soft ideal of R.*
(v) *If (F, I_1) and (G, I_2) are whole soft ideals of R, where $I_1 \cap I_2 = \{0\}$, then $(F, I_1) + (G, I_2)$ is a whole soft ideal of R.*
(vi) *If (F, I_1) is a trivial soft ideal of R and (G, I_2) is a whole soft ideal of R, where $I_1 \cap I_2 = \{0\}$, then $(F, I_1) + (G, I_2)$ is a whole soft ideal of R.*

Proof. The proof is easily seen by Definitions 2, 7 and 8, Theorems 1, 3 and 5. □

Proposition 4. *Let (F, I_1) and (G, I_2) be two soft subrings (resp. soft ideals) of R_1 and R_2, respectively. Then,*

(i) *If (F, I_1) and (G, I_2) are trivial soft subrings (resp. soft ideals) of R_1 and R_2, respectively, then $(F, I_1) \times (G, I_2)$ is a trivial soft subring (resp. soft ideal) of $R_1 \times R_2$.*
(ii) *If (F, I_1) and (G, I_2) are whole soft subrings (resp. soft ideals) of R_1 and R_2, respectively, then $(F, I_1) \times (G, I_2)$ is a whole soft subring (resp. soft ideal) of $R_1 \times R_2$.*

Proof. The proof is easily seen by Definitions 4, 6 and 7, Theorems 2 and 4. □

Proposition 5. *If $F_I \tilde{\triangleleft} G$, then $I_F = \{x \in I \mid F(x) = F(0)\}$ is an ideal of R.*

Proof. We need to show that (i) $x - y \in I_F$, (ii) $rx \in I_F$ and (iii) $xr \in I_F$ for all $x, y \in I_F$ and $r \in R$. If $x, y \in I_F$, then $F(x) = F(y) = F(0)$. In view of Proposition 1, $F(0) \supseteq F(x - y)$, $F(0) \supseteq F(rx)$ and $F(0) \supseteq F(xr)$ for all $r \in R$ and $x, y \in I_F$. Since (F, I) is a soft ideal of R, then for all $x, y \in I_F$ and $r \in R$, (i) $F(x - y) \supseteq F(x) \cap F(y) = F(0)$, (ii) $F(rx) \supseteq F(x) = F(0)$ and (iii) $F(xr) \supseteq F(x) = F(0)$. Hence $F(x - y) = F(0)$, $F(rx) = F(0)$ and $F(xr) = F(0)$ for all $r \in R$ and $x, y \in I_F$. Therefore I_F is an ideal of R. □

Theorem 6. *Let R_1 and R_2 be two rings and $(F_1, S_1) \tilde{<} R_1$, $(F_2, S_2) \tilde{<} R_2$. If $f : S_1 \to S_2$ is a ring homomorphism, then*

(a) *If f is an epimorphism, then $(F_1, f^{-1}(S_2)) \tilde{<} R_1$,*
(b) *$(F_2, f(S_1)) \tilde{<} R_2$,*
(c) *$(F_1, \mathrm{Ker}f) \tilde{<} R_1$.*

Proof. (a) Since $S_1 < R_1$, $S_2 < R_2$ and $f : S_1 \to S_2$ is a ring epimorphism, then it is clear that $f^{-1}(S_2) < R_1$. Since $(F_1, S_1) \tilde{<} R_1$ and $f^{-1}(S_2) \subseteq S_1$, $F_1(x - y) \supseteq F_1(x) \cap F_1(y)$ and $F_1(xy) \supseteq F_1(x) \cap F_1(y)$ for all $x, y \in f^{-1}(S_2)$. Hence $(F_1, f^{-1}(S_2)) \tilde{<} R_1$.
(b) Since $S_1 < R_1$, $S_2 < R_2$ and $f : S_1 \to S_2$ is a ring homomorphism, then $f(S_1) < R_2$. Since $f(S_1) \subseteq S_2$, the result is obvious by Definition 3.
(c) Since $\mathrm{Ker}f < R_1$ and $\mathrm{Ker}f \subseteq S_1$, the rest of the proof is clear by Definition 3. □

Corollary 1. *Let $(F_1, S_1) \tilde{<} R_1$, $(F_2, S_2) \tilde{<} R_2$ and $f : S_1 \to S_2$ is a ring homomorphism, then $(F_2, \{0_{S_2}\}) \tilde{<} R_2$.*

Proof. By Theorem 6(c), $(F_1, \mathrm{Ker}f) \tilde{<} R_1$. Then $(F_2, f(\mathrm{Ker}f)) = (F_2, \{0_{S_2}\}) \tilde{<} R_2$ by Theorem 6(b). □

4. Soft substructures of fields

Throughout this section, we denote a field by F and a subfield S of F by $S < F$.

Definition 9. Let S be a subfield of F and let (G, S) be a soft set over F. If for all $x, y \in S$,

(s1) $G(x - y) \supseteq G(x) \cap G(y)$ and
(s2) $G(xy^{-1}) \supseteq G(x) \cap G(y)$ $(y \neq 0_F)$,

then the soft set (G, S) is called a *soft subfield* of F and denoted by $(G, S) \tilde{<} F$ or simply $G_S \tilde{<} F$.

Example 6. Let $F = (\mathbb{Z}_3, +, \cdot)$, $S = \mathbb{Z}_3 < \mathbb{Z}_3$ and the soft set (G, S) over F, where $G : S \to P(F)$ is a set-valued function by $G(0) = \mathbb{Z}_3$, $G(1) = G(2) = \{1, 2\}$. Then it can be easily seen that $(G, S) \tilde{<} F$. However if we define the soft set (H, S) over F such that $H : S \to P(F)$ is a set-valued function defined by $H(0) = \mathbb{Z}_3$, $H(1) = \{1, 2\}$ and $H(2) = \{0, 1\}$, then $H(2 \cdot 2^{-1}) = H(2 \cdot 2) = H(1) = \{1, 2\} \not\supseteq H(2) \cap H(2) = H(2) = \{0, 1\}$. It follows that (H, S) is not a soft subfield of F.

Theorem 7. *If $G_{S_1} \tilde{<} F$ and $H_{S_2} \tilde{<} F$, then $G_{S_1} \Cap H_{S_2} \tilde{<} F$.*

Proof. Since S_1 and S_2 are subfields of F, then $S_1 \cap S_2$ is a subfield of F. By Definition 2, let $G_{S_1} \Cap H_{S_2} = (G, S_1) \Cap (H, S_2) = (T, S_1 \cap S_2)$, where $T(x) = G(x) \cap H(x)$ for all $x \in S_1 \cap S_2 \neq \emptyset$. Then for all $x, y \in S_1 \cap S_2$,

(s1) $T(x-y) = G(x-y) \cap H(x-y) \supseteq (G(x) \cap G(y)) \cap (H(x) \cap H(y)) = (G(x) \cap H(x)) \cap (G(y) \cap H(y)) = T(x) \cap T(y)$,
(s2) $T(xy^{-1}) = G(xy^{-1}) \cap H(xy^{-1}) \supseteq (G(x) \cap G(y)) \cap (H(x) \cap H(y)) = (G(x) \cap H(x)) \cap (G(y) \cap H(y)) = T(x) \cap T(y)\ (y \neq 0_F)$.

Therefore $G_{S_1} \Cap H_{S_2} = T_{S_1 \cap S_2} \tilde{<} F$. □

Definition 10. Let F_1 and F_2 be fields and let (G, S_1) and (H, S_2) be two soft subfields of F_1 and F_2, respectively. The product of soft subfields (G, S_1) and (H, S_2) is defined as $(G, S_1) \times (H, S_2) = (Q, S_1 \times S_2)$, where $Q(x, y) = F(x) \times G(y)$ for all $(x, y) \in S_1 \times S_2$.

Theorem 8. *If $G_{S_1} \tilde{<} F_1$ and $H_{S_2} \tilde{<} F_2$, then $G_{S_1} \times H_{S_2} \tilde{<} F_1 \times F_2$.*

Proof. Since S_1 and S_2 are subfields of F_1 and F_2, respectively, then $S_1 \times S_2$ is a subfield of $F_1 \times F_2$. By Definition 10, let $G_{S_1} \times H_{S_2} = (G, S_1) \times (H, S_2) = (Q, S_1 \times S_2)$, where $Q(x, y) = F(x) \times G(y)$ for all $(x, y) \in S_1 \times S_2$. Then for all $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$,

(s1) $Q((x_1, y_1) - (x_2, y_2)) = Q(x_1 - x_2, y_1 - y_2) = G(x_1 - x_2) \times H(y_1 - y_2) \supseteq (G(x_1) \cap G(x_2)) \times (H(y_1) \cap H(y_2)) = (G(x_1) \times H(y_1)) \cap (G(x_2) \times H(y_2)) = Q(x_1, y_1) \cap Q(x_2, y_2)$,
(s2) $Q((x_1, y_1)(x_2, y_2)^{-1}) = Q(x_1 x_2^{-1}, y_1 y_2^{-1}) = G(x_1 x_2^{-1}) \times H(y_1 y_2^{-1}) \supseteq (G(x_1) \cap G(x_2)) \times (H(y_1) \cap H(y_2)) = (G(x_1) \times H(y_1)) \cap (G(x_2) \times H(y_2)) = Q(x_1, y_1) \cap Q(x_2, y_2)$ (here $(x_2, y_2) \neq (0_{F_1}, 0_{F_2})$).

Hence $G_{S_1} \times H_{S_2} = Q_{S_1 \times S_2} \tilde{<} F_1 \times F_2$. □

Proposition 6. *If $G_S \tilde{<} F$, then $G(0_F) \supseteq G(x)$ for all $x \in S$.*

Proof. Since (G, S) is a soft subfield of F, then for all $x \in S$, $G(0_F) = G(x - x) \supseteq G(x) \cap G(x) = G(x)$ for all $x \in S$. □

Proposition 7. *If $G_S \tilde{<} F$ and $G(1_F) = G(0_F)$, then $S_G = \{x \in S \mid G(x) = G(0_F)\}$ is a subfield of S.*

Proof. We need to show that $0_F \in S_G$, $1_F \in S_G$, $x - y \in S_G$ and $xy^{-1} \in S_G\ (y \neq 0_F)$ for all $x, y \in S_G$, which means that (i) $G(0_F) = G(0_F)$, (ii) $G(1_F) = G(0_F)$, (iii) $G(x - y) = G(0_F)$ and (iv) $G(xy^{-1}) = G(0_F)$ have to be satisfied. (i) is obvious and (ii) comes from the assumption. Since $x, y \in S_G$, then $G(x) = G(y) = G(0_F)$. Since (G, S) is a soft subfield of F, then $G(x - y) \supseteq G(x) \cap G(y) = G(0_F)$ and $G(xy^{-1}) \supseteq G(x) \cap G(y) = G(0_F)$ for all $x, y \in S_G\ (y \neq 0_F)$. Moreover, by Proposition 6, $G(0_F) \supseteq G(x - y)$ and $G(0_F) \supseteq G(xy^{-1})$. Therefore S_G is a subfield of S. □

Definition 11. Let (G, S) be a soft subfield of F. Then,

(i) (G, S) is said to be *trivial* if $G(x) = \{0_F\}$ for all $x \in S$.
(ii) (G, S) is said to be *whole* if $G(x) = F$ for all $x \in S$.

Proposition 8. *Let (G, S_1) and (H, S_2) be soft subfields of F. Then,*

(i) *If (G, S_1) and (H, S_2) are trivial soft subfields of F, then $(G, S_1) \Cap (H, S_2)$ is a trivial soft subfield of F.*
(ii) *If (G, S_1) and (H, S_2) are whole soft subfields of F, then $(G, S_1) \Cap (H, S_2)$ is a whole soft subfield of F.*
(iii) *If (G, S_1) is a trivial soft subfield of F and (H, S_2) is a whole soft subfield of F, then $(G, S_1) \Cap (H, S_2)$ is a trivial soft subfield of F.*

Proof. The proof is easily seen by Definitions 2 and 11 and Theorem 7. □

Proposition 9. *Let (G, S_1) and (H, S_2) be two soft subfields of F_1 and F_2, respectively. Then,*

(i) *If (G, S_1) and (H, S_2) are trivial soft subfields of F_1 and F_2, respectively, then $(G, S_1) \times (H, S_2)$ is a trivial soft subfield of $F_1 \times F_2$.*
(ii) *If (G, S_1) and (H, S_2) are whole soft subfields of F_1 and F_2, respectively, then $(G, S_1) \times (H, S_2)$ is a whole soft subfield of $F_1 \times F_2$.*

Proof. The proof is easily seen by Definitions 10 and 11 and Theorem 8. □

Theorem 9. *Let F_1 and F_2 be fields and $(G_1, S_1) \tilde{<} F_1$, $(G_2, S_2) \tilde{<} F_2$. If $f : S_1 \to S_2$ is a field homomorphism, then*

(a) *If f is an epimorphism, then $(G_1, f^{-1}(S_2)) \tilde{<} F_1$,*
(b) *$(G_2, f(S_1)) \tilde{<} F_2$,*
(c) *$(G_1, \mathrm{Kerf}) \tilde{<} F_1$.*

Proof. (a) Since $S_1 < F_1, S_2 < F_2$ and $f : F_1 \to F_2$ is a field epimorphism, then it is obvious that $f^{-1}(S_2) < F_1$. Since $(G_1, S_1)\tilde{<}F_1$ and $f^{-1}(S_2) \subseteq S_1$, $G_1(x-y) \supseteq G_1(x) \cap G_1(y)$ for all $x, y \in f^{-1}(S_2)$ and $G_1(xy^{-1}) \supseteq G_1(x) \cap G_1(y)$ $(y \neq 0_{F_1})$. Hence $(G_1, f^{-1}(S_2))\tilde{<}F_1$.

(b) Since $S_1 < F_1, S_2 < F_2$ and $f : S_1 \to S_2$ is a field homomorphism, then $f(S_1) < S_2$. Since $f(S_1) \subseteq S_2$, the result is obvious by Definition 9.

(c) Since Ker$f < F_1$ and Ker$f \subseteq S_1$, the rest of the proof is clear by Definition 9. □

Corollary 2. *Let $(G_1, S_1)\tilde{<}F_1$, $(G_2, S_2)\tilde{<}F_2$ and $f : S_1 \to S_2$ is a field homomorphism, then $(G_2, \{0_{S_2})\}\tilde{<}F_2$.*

Proof. By Theorem 9(c), $(G_1, \text{Ker}f)\tilde{<}F_1$. Then $(G_2, f(\text{Ker}f)) = (G_2, \{0_{S_2}\})\tilde{<}F_2$ by Theorem 9(b). □

5. Soft substructures of modules

Throughout this section, we denote a module by M and a submodule (resp. ideal) N of M by $N < M$.

Definition 12. Let N be a submodule of M and let (F, N) be a soft set over M. If for all $x, y \in N$ and for all $r \in R$,

(s1) $F(x-y) \supseteq F(x) \cap F(y)$ and
(s2) $F(rx) \supseteq F(x)$,

then the soft set (F, N) is called a *soft submodule* of M and denoted by $(F, N)\tilde{<}M$ or simply $F_N\tilde{<}M$.

Example 7. Let $R = (\mathbb{Z}_{10}, +, .)$, $M = (\mathbb{Z}_{10}, +)$ be a left R-module with natural operation and $N_1 = \{0, 5\}$ be a submodule of M. Let the soft set (F, N_1) over M, where $F : N_1 \to P(M)$ is a set-valued function defined by $F(0) = \{0, 3, 4, 9\}$ and $F(5) = \{0, 9\}$. Then it can be easily seen that $(F, N_1)\tilde{<}M$.

Let $N_2 = \{0, 2, 4, 6, 8\} < M$ and the soft set (G, N_2) over M, where $G : N_2 \to P(M)$ is a set-valued function defined by $G(0) = \{0, 2, 5, 7, 9\}$ and $G(2) = G(4) = G(6) = G(8) = \{2, 9\}$. Then $(G, N_2)\tilde{<}M$, too. However if we define the soft set (H, N_2) over M such that $H(0) = \mathbb{Z}_{10}$, $H(2) = \{1, 7\}$, $H(4) = \{3, 5, 7\}$, $H(6) = \{1, 2, 8\}$, $H(8) = \{2, 4, 7\}$, then $H(7 \cdot 6) = H(2) = \{1, 7\} \not\supseteq H(6) = \{1, 2, 8\}$. Therefore, (H, N_2) is not a soft submodule over M.

Theorem 10. *If $F_{N_1}\tilde{<}M$ and $G_{N_2}\tilde{<}M$, then $F_{N_1} \Cap G_{N_2}\tilde{<}M$.*

Proof. Since N_1 and N_2 are submodules of M, then $N_1 \cap N_2$ is a submodule of M. By Definition 2, let $F_{N_1} \Cap G_{N_2} = (F, N_1) \Cap (G, N_2) = (H, N_1 \cap N_2)$, where $H(x) = F(x) \cap G(x)$ for all $x \in N_1 \cap N_2 \neq \emptyset$. Then for all $x, y \in N_1 \cap N_2$ and $r \in R$,

(s1) $H(x-y) = F(x-y) \cap G(x-y) \supseteq (F(x) \cap F(y)) \cap (G(x) \cap G(y)) = (F(x) \cap G(x)) \cap (F(y) \cap G(y)) = H(x) \cap H(y)$,
(s2) $H(rx) = F(rx) \cap G(rx) \supseteq F(x) \cap G(x) = H(x)$.

Therefore $F_{N_1} \Cap G_{N_2} = H_{N_1 \cap N_2}\tilde{<}M$. □

Definition 13. Let M_1 and M_2 be left R-modules and let (F, N_1) and (G, N_2) be two soft submodules of M_1 and M_2, respectively. The product of soft submodules (F, N_1) and (G, N_2) is defined as $(F, N_1) \times (G, N_2) = (Q, N_1 \times N_2)$, where $Q(x, y) = F(x) \times G(y)$ for all $(x, y) \in N_1 \times N_2$.

Theorem 11. *If $F_{N_1}\tilde{<}M_1$ and $G_{N_2}\tilde{<}M_2$, then $F_{N_1} \times G_{N_2}\tilde{<}M_1 \times M_2$.*

Proof. Since N_1 and N_2 are submodules of M_1 and M_2, respectively, then $N_1 \times N_2$ is a submodule of $M_1 \times M_2$. By Definition 13, let $F_{N_1} \times G_{N_2} = (F, N_1) \times (G, N_2) = (Q, N_1 \times N_2)$, where $Q(x, y) = F(x) \times G(y)$ for all $(x, y) \in M_1 \times M_2$. Then for all $(x_1, y_1), (x_2, y_2) \in M_1 \times M_2$ and $(r_1, r_2) \in R \times R$,

(s1) $Q((x_1, y_1) - (x_2, y_2)) = Q(x_1 - x_2, y_1 - y_2) = F(x_1 - x_2) \times G(y_1 - y_2) \supseteq (F(x_1) \cap F(x_2)) \times (G(y_1) \cap G(y_2)) = (F(x_1) \times G(y_1)) \cap (F(x_2) \times G(y_2)) = Q(x_1, y_1) \cap Q(x_2, y_2)$,
(s2) $Q((r_1, r_2)(x_1, y_1)) = Q(r_1x_1, r_2y_1) = F(r_1x_1) \times G(r_2y_1) \supseteq F(x_1) \times G(y_1) = Q(x_1, y_1)$.

Hence $F_{N_1} \times G_{N_2} = Q_{N_1 \times N_2}\tilde{<}M_1 \times M_2$. □

To illustrate Theorems 10 and 11, we have the following example:

Example 8. Let $(F, N_1)\tilde{<}\mathbb{Z}_{10}$ and $(G, N_2)\tilde{<}\mathbb{Z}_{10}$ in Example 7. By Definition 2, $(F, N_1) \Cap (G, N_2) = (T, N_1 \cap N_2)$, where $T(x) = F(x) \cap G(x)$ for all $x \in N_1 \cap N_2 = \{0\}$. Then $T(0) = F(0) \cap G(0) = \{0, 9\}$. It is obvious that $(T, N_1 \cap N_2)\tilde{<}M$.

By Definition 4, $F_{N_1} \times G_{N_2} = (F, N_1) \times (G, N_2) = (Q, N_1 \times N_2)$, where $Q(x, y) = F(x) \times G(y)$ for all $(x, y) \in N_1 \times N_2 = \{(0, 0), (0, 2), (0, 4), (0, 6), (0, 8), (5, 0), (5, 2), (5, 4), (5, 6), (5, 8)\}$. Then it can be easily seen that $Q_{N_1 \times N_2}\tilde{<}\mathbb{Z}_{10} \times \mathbb{Z}_{10}$. We show the operations for some elements of $N_1 \times N_2$:

$$\begin{aligned} Q((5,2)-(0,8)) &= Q(5-0,2-8) = Q(5,4) \\ &= F(5) \times G(4) = \{0,9\} \times \{2,9\} \\ &= \{(0,2),(0,9),(9,2),(9,9)\} \end{aligned}$$

$$\begin{aligned} Q(5,2) \cap Q(0,8) &= (F(5) \times G(2)) \cap (F(0) \times G(8)) \\ &= (\{0,9\} \times \{2,9\}) \cap (\{0,3,4,9\} \times \{2,9\}) \\ &= \{(0,2),(0,9),(9,2),(9,9)\} \end{aligned}$$

$$\begin{aligned} Q((7,9)(5,2)) &= Q(7 \cdot 5, 9 \cdot 2) = Q(5,8) \\ &= F(5) \times G(8) = (\{0,9\} \times \{2,9\}) \\ &= \{(0,2),(0,9),(9,2),(9,9)\}. \end{aligned}$$

It is seen that $Q((5,2)-(0,8)) \supseteq Q(5,2) \cap Q(0,8)$ and $Q((7,9)(5,2)) \supseteq Q(5,2) = F(5) \times G(2) = \{(0,2),(0,9),(9,2),(9,9)\}$.

It is worth noting that if N and K are two submodules of a left R-module M, then the sum of these two submodules is defined by $N+K = \{n+k \mid n \in N \wedge k \in K\}$.

Definition 14. Let (F,N) and (G,K) be two soft submodules of M. If $N \cap K = \{0\}$, then the sum of soft submodules (F,N) and (G,K) is defined as $(F,N)+(G,K) = (T,N+K)$, where $T(x+y) = F(x)+G(y)$ for all $x+y \in N+K$.

Theorem 12. *If $F_N \tilde{\lesssim} M$ and $G_K \tilde{\lesssim} M$, where $N \cap K = \{0\}$, then $F_N + G_K \tilde{\lesssim} M$.*

Proof. Since N and K are submodules of M, then $N+K$ is a submodule of M. By Definition 14, let $F_N+G_K = (F,N)+(G,K) = (T,N+K)$, where $T(x+y) = F(x)+G(y)$ for all $x+y \in N+K$. Since $N \cap K = \{0\}$, T is well defined. Then for all $x_1+y_1, x_2+y_2 \in N+K$ and $r \in R$,

$$\begin{aligned} T((x_1+y_1)-(x_2+y_2)) &= T((x_1-x_2)+(y_1-y_2)) \\ &= F(x_1-x_2)+G(y_1-y_2) \\ &\supseteq (F(x_1) \cap F(x_2)) + (G(y_1) \cap G(y_2)) \\ &= (F(x_1)+G(y_1)) \cap (F(x_2)+G(y_2)) \\ &= T(x_1+y_1) \cap T(x_2+y_2), \end{aligned}$$

$$\begin{aligned} T(r(x_1+y_1)) &= T(rx_1+ry_1) \\ &= F(rx_1)+G(ry_1) \\ &\supseteq F(x_1)+G(y_1) \\ &= T(x_1+y_1). \end{aligned}$$

Therefore $F_N + G_K = T_{N+K} \tilde{\lesssim} M$. □

Proposition 10. *If $F_N \tilde{\lesssim} M$, then $F(0) \supseteq F(x)$ for all $x \in N$.*

Proof. Since (F,N) is a soft submodule of M, then for all $x \in N$, $F(x-x) = F(0) \supseteq F(x) \cap F(x) = F(x)$ for all $x \in N$. □

Proposition 11. *If $F_N \tilde{\lesssim} M$, then $N_F = \{x \in N \mid F(x) = F(0)\}$ is a submodule of N.*

Proof. We need to show that $x-y \in N_F$ and $rx \in N_F$ for all $x,y \in N_F$ and $r \in R$, which means that $F(x-y) = F(0)$ and $F(rx) = F(0)$ have to be satisfied. Since $x,y \in N_F$, then $F(x) = F(y) = F(0)$. Since (F,N) is a soft submodule of M, then $F(x-y) \supseteq F(x) \cap F(y) = F(0)$ and $F(rx) \supseteq F(x) = F(0)$ for all $x,y \in N_F$ and $r \in R$. Moreover, by Proposition 10, $F(0) \supseteq F(x-y)$ and $F(0) \supseteq F(rx)$. Therefore N_F is a submodule of N. □

Definition 15. Let (F,N) be a soft submodule of M. Then,

(i) (F,N) is said to be *trivial* if $F(x) = \{0_M\}$ for all $x \in N$.
(ii) (F,N) is said to be *whole* if $F(x) = M$ for all $x \in N$.

Proposition 12. *Let (F,N_1) and (G,N_2) be soft submodules of M. Then,*

(i) *If (F,N_1) and (G,N_2) are trivial soft submodules of M, then $(F,N_1) \Cap (G,N_2)$ is a trivial soft submodule of M.*
(ii) *If (F,N_1) and (G,N_2) are whole soft submodules of M, then $(F,N_1) \Cap (G,N_2)$ is a whole soft submodule of M.*
(iii) *If (F,N_1) is a trivial soft submodule of M and (G,N_2) is a whole soft submodule of M, then $(F,N_1) \Cap (G,N_2)$ is a trivial soft submodule of M.*
(iv) *If (F,N_1) and (G,N_2) are trivial soft submodules of M, where $N_1 \cap N_2 = \{0\}$, then $(F,N_1)+(G,N_2)$ is a trivial soft submodule of M.*
(v) *If (F,N_1) and (G,N_2) are whole soft submodules of M, where $N_1 \cap N_2 = \{0\}$, then $(F,N_1)+(G,N_2)$ is a whole soft submodule of M.*
(vi) *If (F,N_1) is a trivial soft submodule of M and (G,N_2) is a whole soft submodule of M, where $N_1 \cap N_2 = \{0\}$, then $(F,N_1)+(G,N_2)$ is a whole soft submodule of M.*

Proof. The proof is easily seen by Definitions 2, 14 and 15, Theorems 10 and 12. □

Proposition 13. *Let (F, N_1) and (G, N_2) be two soft submodules of M_1 and M_2, respectively. Then,*

(i) *If (F, N_1) and (G, N_2) are trivial soft submodules of M_1 and M_2, respectively, then $(F, N_1) \times (G, N_2)$ is a trivial soft submodule of $M_1 \times M_2$.*
(ii) *If (F, N_1) and (G, N_2) are whole soft submodules of M_1 and M_2, respectively, then $(F, N_1) \times (G, N_2)$ is a whole soft submodule of $M_1 \times M_2$.*

Proof. The proof is easily seen by Definitions 13 and 15 and Theorem 11. □

Theorem 13. *Let M_1 and M_2 be two R-modules and $(F_1, N_1)\tilde{<}M_1$, $(F_2, N_2)\tilde{<}M_2$. If $f : N_1 \to N_2$ is a module homomorphism, then*

(a) *If f is an epimorphism, then $(F_1, f^{-1}(N_2))\tilde{<}M_1$,*
(b) *$(F_2, f(N_1))\tilde{<}M_2$,*
(c) *$(F_1, \mathrm{Kerf})\tilde{<}M_1$.*

Proof. (a) Since $N_1 < M_1$, $N_2 < M_2$ and $f : N_1 \to N_2$ is a module epimorphism, then it is clear that $f^{-1}(N_2) < M_1$. Since $(F_1, N_1)\tilde{<}M_1$ and $f^{-1}(N_2) \subseteq N_1$, $F_1(x - y) \supseteq F_1(x) \cap F_1(y)$ and $F_1(rx) \supseteq F_1(x)$ for all $x, y \in f^{-1}(N_2)$ and $r \in R$. Hence $(F_1, f^{-1}(N_2))\tilde{<}M_1$.
(b) Since $N_1 < M_1$, $N_2 < M_2$ and $f : N_1 \to N_2$ is a module homomorphism, then $f(N_1) < M_2$. Since $f(N_1) \subseteq N_2$, the result is obvious by Definition 12.
(c) Since $\mathrm{Kerf} < M_1$ and $\mathrm{Kerf} \subseteq N_1$, the rest of the proof is clear by Definition 12. □

Corollary 3. *Let $(F_1, N_1)\tilde{<}M_1$, $(F_2, N_2)\tilde{<}M_2$ and $f : N_1 \to N_2$ is a module homomorphism, then $(F_2, \{0_{N_2}\})\tilde{<}M_2$.*

Proof. By Theorem 13(c), it is seen that $(F_1, \mathrm{Kerf})\tilde{<}M_1$. Then $(F_2, f(\mathrm{Kerf})) = (F_2, \{0_{N_2}\})\tilde{<}M_2$ by Theorem 13(b). □

6. Conclusion

Throughout this paper, we deal with the algebraic soft substructures of rings, fields and modules. We have introduced soft subrings and soft ideals of rings. By theoretical aspect we have applied some of the operations defined on soft sets to our soft substructures. Furthermore, we introduce the notion of soft subfields of fields and soft submodules of modules and study their related properties with several examples. To extend this work, one could study the soft substructures of other algebraic structures such as vector spaces and algebras.

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