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Solvability of infinite systems of singular integral equations in Fréchet space of continuous functions

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ABSTRACT

The aim of this paper is to show how some measures of noncompactness in the Fréchet space of continuous functions defined on an unbounded interval can be applied to an infinite system of singular integral equations. The results obtained generalize and improve several ones.

1. Introduction

The theory of infinite systems of integral or differential equations creates an important branch of nonlinear analysis. It is connected naturally with a large number of problems considered in mechanics, engineering, in the theory of branching processes, the theory of neutral nets and so on.

The infinite system of equations can be considered as a particular case of equations in Banach spaces [1–10]. A considerable number of those results were formulated in terms of measures of noncompactness.

It seems that a more effective approach consists in applying suitable regular measures of noncompactness for some Fréchet spaces of continuous functions defined on a bounded or an unbounded interval (see [11]).

The aim of this paper is to show how the measure $\overline{\omega}_0$ introduced in [11] can be applied to infinite systems of functional singular integral equations. The results of this paper improve and generalize those obtained in paper [8].

2. Notation and auxiliary facts

In this section, we gather definitions and auxiliary facts which will be needed further on. If X is a subset of a linear topological space, then $\overline{X}$ and $\mathrm{Conv}\, X$ denote closure and convex closure of X, respectively.

Let $C[0, T]$ denote the Banach space consisting of all real functions, defined and continuous on $[0, T]$. The space $C[0, T]$ is furnished with the standard norm

$$\|x\| = \max\{|x(t)| : t \in [0, T]\}.$$

Now, we recollect the definitions of some quantities which will be used further on. These quantities were introduced in [1].

To this end let us fix a nonempty bounded subset X of $C[0, T]$. For $x \in X$ and $\varepsilon > 0$ let us denote by $\omega^T(x, \varepsilon)$ *the modulus of continuity* of the function x on the interval $[0, T]$, i.e.

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \le \varepsilon\}.$$

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Further, let us put

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : x \in X\},$$
$$\omega_0^T(X) = \lim_{\varepsilon\to 0+} \omega^T(X,\varepsilon).$$

The function ω_0^T is an example of a regular measure of noncompactness in the space $C[0,T]$ (see [1]).

Next, let us denote by $\mathbb{R}^\infty$ the linear space of all real sequences equipped with the distance

$$d_{\mathbb{R}^\infty}(a,b) = \sup\{2^{-i}|a_i - b_i|/(1+|a_i-b_i|) : i \in \mathbb{N}\}$$

for $a=(a_i), b=(b_i) \in \mathbb{R}^\infty$.

The space $\mathbb{R}^\infty$ is a Fréchet space (i.e. a linear, metrizable and complete space).

For any sequence $a=(a_i)\in\mathbb{R}^\infty$ we put $\pi_i(a) = a_i$.

Further, denote by $C([0,T],\mathbb{R}^\infty)$ the space consisting of all functions defined and continuous on $[0,T]$ with values in the space $\mathbb{R}^\infty$.

For $x=(x_i(t)) \in C([0,T],\mathbb{R}^\infty)$ we put $\pi_i(x) = x_i$. Obviously $\pi_i(x)\in C[0,T]$.

If $X \subset C([0,T],\mathbb{R}^\infty)$ then for a fixed $i\in\mathbb{N}$ we denote by $\pi_i(X)$ the following set situated in $C[0,T]$:

$$\pi_i(X) = \{\pi_i(x) : x \in X\}.$$

The space $C([0,T],\mathbb{R}^\infty)$ will be equipped with the distance

$$d_{C_T}(x,y) = \sup\{d_{\mathbb{R}^\infty}(x(t),y(t)) : t\in[0,T]\} \quad \text{for } x,y \in C([0,T],\mathbb{R}^\infty).$$

Now we introduce the next function space.

Let $C(\mathbb{R}_+,\mathbb{R}^\infty)$ be the space of all continuous functions defined on $\mathbb{R}_+ = [0,\infty)$ with values in $\mathbb{R}^\infty$. This space equipped with the family of seminorms

$$|x|_n = \sup\{|\pi_i(x)(t)| : i \le n, t \in [0,n]\}$$

and the distance $d_C(x,y) = \sup\{2^{-n}|x-y|_n/(1+|x-y|_n) : n\in\mathbb{N}\}$ for $x,y\in C(\mathbb{R}_+,\mathbb{R}^\infty)$ becomes a Fréchet space.

The convergence and the compactness in $C(\mathbb{R}_+,\mathbb{R}^\infty)$ are characterized by the following conditions [11]:

(A) A sequence (x_n) is convergent to x in $C(\mathbb{R}_+,\mathbb{R}^\infty)$ if and only if $\pi_i(x_n)$ is uniformly convergent to $\pi_i(x)$ on $[0,T]$ for each $i\in\mathbb{N}$ and $T>0$.

(B) A subset $X \subset C(\mathbb{R}_+,\mathbb{R}^\infty)$ is relatively compact if and only if the functions of the set $\pi_i(X)$ are equicontinuous and uniformly bounded on every interval $[0,T]$ for each $i\in\mathbb{N}$ (or, equivalently, if $\pi_i(X)$ are relatively compact in $C[0,T]$ for each $i\in\mathbb{N}$ and $T>0$).

Obviously, $\lim_{n\to\infty} d_C(x,x_n) = 0$ iff $\lim_{n\to\infty} d_{C_T}(x,x_n)=0$, for every $T>0$.

Definition 2.1. A nonempty subset $Z\subset\mathbb{R}^\infty$ is said to be *bounded* if

$$\sup\{|\pi_i(z)| : z\in Z\} < \infty \quad \text{for } i = 1,2,\ldots.$$

A subset Y of $\mathbb{R}_+\times\mathbb{R}^\infty$ is called *bounded* if Y is contained in a set of the form $[0,T]\times Z$ where $T>0$ and Z is bounded in $\mathbb{R}^\infty$.

A nonempty subset $X \subset C(\mathbb{R}_+,\mathbb{R}^\infty)$ is said to be *bounded* if the functions of the set $\pi_i(X)$ are uniformly bounded on $[0,T]$ for each $i\in\mathbb{N}$ and $T>0$ i.e.

$$\sup\{|\pi_i(x)(t)| : t\in[0,T], x\in X\} < \infty \quad \text{for } i=1,2,\ldots \text{ and } T>0.$$

Next, let us define

$$\mathfrak{M}_{C(\mathbb{R}_+,\mathbb{R}^\infty)} = \{X \subset C(\mathbb{R}_+,\mathbb{R}^\infty) : X \neq \emptyset \text{ and } X \text{ is bounded}\}$$

while $\mathfrak{N}_{C(\mathbb{R}_+,\mathbb{R}^\infty)}$ stands for the family of all nonempty and relatively compact subsets of $C(\mathbb{R}_+,\mathbb{R}^\infty)$.

Now we will define the regular measure of noncompactness $\overline{\omega}_0$ in the space $C(\mathbb{R}_+,\mathbb{R}^\infty)$ (see [11]). To this end assume that $p_i : \mathbb{R}_+ \to (0,\infty)$ $(i=1,2,\ldots)$ is a sequence of functions.

Next, for $X\in\mathfrak{M}_{C(\mathbb{R}_+,\mathbb{R}^\infty)}$ let us put

$$\overline{\omega}_0^T(X) = \sup\{p_i(T)\omega_0^T(\pi_i(X)) : i\in\mathbb{N}\},$$
$$\overline{\omega}_0(X) = \sup\{\overline{\omega}_0^T(X) : T>0\}. \tag{1}$$

The following theorem presents the basic properties of the measure $\overline{\omega}_0$ [11].

Theorem 2.2. *The mapping $\overline{\omega}_0 : \mathfrak{M}_{C(\mathbb{R}_+,\mathbb{R}^\infty)} \to [0,\infty]$ satisfies the following conditions:*

1° $\overline{\omega}_0(X) = 0$ *if and only if X is a relatively compact subset of $C(\mathbb{R}_+,\mathbb{R}^\infty)$.*

2° $X \subset Y \Rightarrow \overline{\omega}_0(X) \le \overline{\omega}_0(Y)$.
3° $\overline{\omega}_0(\mathrm{Conv}\, X) = \overline{\omega}_0(X)$.
4° $\overline{\omega}_0(\lambda X + (1-\lambda)Y) \le \lambda\overline{\omega}_0(X) + (1-\lambda)\overline{\omega}_0(Y)$ *for* $\lambda \in [0, 1]$.
5° *If* (X_n) *is a sequence of closed sets from* $\mathfrak{M}_{C(\mathbb{R}_+,\mathbb{R}^\infty)}$ *such that* $X_{n+1} \subset X_n$ $(n = 1, 2, \ldots)$ *and if* $\lim_{n\to\infty} \overline{\omega}_0(X_n) = 0$, *then the intersection* $X_\infty = \bigcap_{n=1}^{\infty} X_n$ *is nonempty.*
6° $\overline{\omega}_0(\lambda X) = |\lambda|\overline{\omega}_0(X)$ *for* $\lambda \in \mathbb{R}$.
7° $\overline{\omega}_0(X + Y) \le \overline{\omega}_0(X) + \overline{\omega}_0(Y)$.
8° $\overline{\omega}_0(X \cup Y) \le \max\{\overline{\omega}_0(X), \overline{\omega}_0(Y)\}$.

Remark 2.3. Observe that in contrast to the definition of the concept of a measure of noncompactness given in [1], our measures of noncompactness may take the value ∞. This fact is very essential in our considerations in the setting of Fréchet spaces.

Other facts concerning measures of noncompactness in Fréchet spaces and their properties may be found in [11–14]. For our purposes we will only need the following fixed point theorem (see [1,12]).

Theorem 2.4. *Let* Q *be a nonempty bounded closed convex subset of the space* $C(\mathbb{R}_+, \mathbb{R}^\infty)$ *such that* $\overline{\omega}_0(Q) < \infty$ *and let* $F : Q \to Q$ *be a continuous transformation such that* $\overline{\omega}_0(FX) \le q\overline{\omega}_0(X)$ *for any nonempty subset* X *of* Q, *where* q *is a constant,* $q \in [0, 1)$. *Then* F *has a fixed point in the set* Q.

3. Main result

In this section we are going to show how the measure $\overline{\omega}_0$, defined in the previous section, can be applied to an infinite systems of nonlinear integral equations.

Let us consider the following system of singular integral equations of the form

$$x_i(t) = g_i(t, x_1(t), x_2(t), \ldots) + \int_0^t \frac{f_i(\tau, x_1(\tau), x_2(\tau), \ldots)\mathrm{d}\tau}{(t-\tau)^\alpha} \tag{2}$$

where $i = 1, 2, \ldots$ and $t \ge 0$. Apart from this we assume that α is a fixed number in the interval $(0, 1)$. For simplicity, we will write $f_i(\tau, x(\tau))$ instead of $f_i(\tau, x_1(\tau), x_2(\tau), \ldots)$.

We will consider the system (2) under the following assumptions:

(H1) The functions $f_i : \mathbb{R}_+ \times \mathbb{R}^\infty \to \mathbb{R}$ are continuous $(i = 1, 2, \ldots)$ and for each $i \in \mathbb{N}$, the family of functions $\{f_i(t, x)\}_{t\in[0,T]}$ is equicontinuous on bounded subsets of $\mathbb{R}^\infty$ for every $T > 0$.

Moreover, there are continuous functions $a_i, b_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f_i(t, x_1, x_2, \ldots)| \le a_i(t) + b_i(t)|x_i| \tag{3}$$

for $i = 1, 2, \ldots$ and $x = (x_i) \in \mathbb{R}^\infty$.

Apart from this, the functions b_i are uniformly bounded on compact intervals of $\mathbb{R}_+$.

(H2) The functions $g_i : \mathbb{R}_+ \times \mathbb{R}^\infty \to \mathbb{R}$ are continuous and there exist the constant $k_{ij} \ge 0$ such that

$$|g_i(t, x_1, x_2, \ldots) - g_i(t, y_1, y_2, \ldots)| \le \sum_{j=1}^{\infty} k_{ij}|x_j - y_j|$$

for $i = 1, 2, \ldots$ and $x = (x_i), y = (y_i) \in \mathbb{R}^\infty$.

Moreover, for each $i \in \mathbb{N}$, the family of functions $\{g_i(t, x)\}_{x\in Z}$ is equicontinuous on compact intervals of $\mathbb{R}_+$ for every bounded subset $Z \subset \mathbb{R}^\infty$.

(H3) There exists a constant $q \in [0, 1)$ and there are nondecreasing functions $m_i : \mathbb{R}_+ \to (0, \infty)$ such that

$$|g_i(t, 0, 0, \ldots)| + \int_0^t \frac{a_i(\tau)\mathrm{d}\tau}{(t-\tau)^\alpha} \le m_i(t)$$

and

$$\sum_{j=1}^{\infty} k_{ij}\frac{m_j(t)}{m_i(t)} \le q$$

for $i = 1, 2, \ldots$ and $t \ge 0$.

Remark 3.1. Let us notice that assumption (H1) on the equicontinuity of the family functions $\{f_i(t, x)\}_{t\in[0,T]}$ on bounded subsets of $\mathbb{R}^\infty$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sup\{|f_i(t, x) - f_i(t, y)| : t \in [0, T]\} \le \varepsilon$ for each $x, y \in Z$ such that $d_{\mathbb{R}^\infty}(x, y) \le \delta$.

Similarly, the equicontinuity of the family functions $\{g_i(t, x)\}_{x\in Z}$ on the compact intervals of $\mathbb{R}_+$ means that for every $\varepsilon > 0, T > 0$ there exists $\delta > 0$ such that $\sup\{|g_i(t_2, x) - g_i(t_1, x)| : x \in Z\} \le \varepsilon$ for each $t_2, t_1 \in [0, T]$ such that $|t_2 - t_1| \le \delta$.

Remark 3.2. Let us observe that uniform continuity on bounded subsets of $\mathbb{R}_+ \times \mathbb{R}^\infty$ each of the functions f_i and g_i, $(i = 1, 2, \ldots)$ is sufficient for equicontinuity of f_i and g_i.

Remark 3.3. In next section we give weaker but convenient and handy version of assumption (H3).

Now we can formulate our main result.

Theorem 3.4. *Under assumptions* (H1)–(H3), *the infinite system* (2) *has at least one solution* $x(t) = (x_i(t))$ *such that* $x(t) \in C(\mathbb{R}_+, \mathbb{R}^\infty)$.

Proof. Let us consider the operator F defined on the space $C(\mathbb{R}_+, \mathbb{R}^\infty)$ as follows:

$$(Fx)(t) = (\pi_i(Fx)(t)) = \left(g_i(t, x_1(t), x_2(t), \ldots) + \int_0^t \frac{f_i(\tau, x_1(\tau), x_2(\tau), \ldots)\mathrm{d}\tau}{(t-\tau)^\alpha} \right).$$

Obviously, the function Fx is continuous on the interval $\mathbb{R}_+$.

Firstly we show that there are functions $r_i : \mathbb{R}_+ \to (0, \infty)$ $(i = 1, 2, \ldots)$ such that

$$\text{if } x \in C(\mathbb{R}_+, \mathbb{R}^\infty) \text{ and } |\pi_i(x)(t)| \le r_i(t) \text{ then } |\pi_i(Fx)(t)| \le r_i(t) \quad \text{for } t \ge 0 \text{ and } i = 1, 2, \ldots. \tag{4}$$

Observe that in view of assumptions (H1) and (H2) we have

$$|\pi_i(Fx)(t)| \le |g_i(t, 0, 0, \ldots)| + \sum_{j=1}^\infty k_{ij}|x_j(t)| + \int_0^t \frac{a_i(\tau)\mathrm{d}\tau}{(t-\tau)^\alpha} + \int_0^t \frac{b_i(\tau)|x_i(\tau)|\mathrm{d}\tau}{(t-\tau)^\alpha}. \tag{5}$$

Let us observe, that

$$\begin{aligned}&\text{if } \beta : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is a nondecreasing function}\\ &\text{then the function } t \to \int_0^t \frac{\beta(\tau)\mathrm{d}\tau}{(t-\tau)^\alpha} \text{ is also nondecreasing on } \mathbb{R}_+.\end{aligned} \tag{6}$$

The standard proof will be omitted (see [8]).

Note from (H1) that there is a function $b : \mathbb{R}_+ \to (0, \infty)$ that $b_i(t) \le b(t)$ for $t \ge 0$, $i = 1, 2, \ldots$.

Next, let us consider the following second kind singular Volterra integral equation

$$\frac{1}{1-q} + \int_0^t \frac{\overline{b}(\tau)\phi(\tau)\mathrm{d}\tau}{(1-q)(t-\tau)^\alpha} = \phi(t), \quad t \ge 0 \tag{7}$$

where $\overline{b}(\tau) = \sup\{b(\eta) : \eta \le \tau\}$.

The solution $\phi(t)$ of Eq. (7) can be expressed as Liouville–Neumann series and, in virtue of (6), we derive that $\phi(t)$ is nondecreasing on $\mathbb{R}_+$.

Eq. (7) can be rewritten as follows

$$1 + q\phi(t) + \int_0^t \frac{\overline{b}(\tau)\phi(\tau)\mathrm{d}\tau}{(t-\tau)^\alpha} = \phi(t), \quad t \ge 0.$$

Keeping in mind (H3) we get

$$1 + \sum_{j=1}^\infty k_{ij}\frac{m_j(t)}{m_i(t)}\phi(t) + \int_0^t \frac{\overline{b}(\tau)\phi(\tau)\mathrm{d}\tau}{(t-\tau)^\alpha} \le \phi(t). \tag{8}$$

Let us denote

$$r_i(t) = m_i(t)\phi(t) \quad i = 1, 2, \ldots. \tag{9}$$

Obviously, the functions r_i are nondecreasing.

Putting $\phi(t) = \frac{r_j(t)}{m_j(t)}$, $\phi(\tau) = \frac{r_i(\tau)}{m_i(\tau)}$, $\phi(t) = \frac{r_i(t)}{m_i(t)}$ into (8) we get

$$m_i(t) + \sum_{j=1}^\infty k_{ij}r_j(t) + \int_0^t \frac{\overline{b}(\tau)r_i(\tau)}{(t-\tau)^\alpha}\frac{m_i(t)}{m_i(\tau)}\mathrm{d}\tau \le r_i(t)$$

and, by the definition of $\overline{b}(t)$ and the monotonicity of $m_i(t)$ we obtain

$$m_i(t) + \sum_{j=1}^\infty k_{ij}r_j(t) + \int_0^t \frac{b_i(\tau)r_i(\tau)\mathrm{d}\tau}{(t-\tau)^\alpha} \le r_i(t). \tag{10}$$

Now, we take $x \in C(\mathbb{R}_+, \mathbb{R}^\infty)$ such that $|\pi_i(x)(t)| \le r_i(t)$ for $t \ge 0, i = 1, 2, \dots$. Using (5), assumption (H3) and (10) we get $|\pi_i(Fx)(t)| \le r_i(t)$ what confirms (4).

Next, let Q be the subset of the space $C(\mathbb{R}_+, \mathbb{R}^\infty)$ consisting of all functions $x(t) = (x_i(t))$ such that $|x_i(t)| \le r_i(t)$ for $t \ge 0$ and $i = 1, 2 \dots$. Obviously, Q is closed, convex and nonempty subset of $C(\mathbb{R}_+, \mathbb{R}^\infty)$. Condition (4) implies that $F : Q \to Q$.

Now, we will estimate the modulus of continuity $\omega^T(\pi_i(Fx), \varepsilon)$ of the function $\pi_i(Fx)$. Let us take a nonempty set $X \subset Q$. Next, fix arbitrarily $T > 0$ and $\varepsilon > 0$. Choose a function $x = (x_i) \in X$ and take $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| \le \varepsilon$. Without loss of generality, we may assume $t_1 \le t_2$. Then by virtue of the imposed assumptions we have

$$\begin{aligned} |\pi_i(Fx)(t_2) - \pi_i(Fx)(t_1)| &\le |g_i(t_2, x(t_2)) - g_i(t_2, x(t_1))| + |g_i(t_2, x(t_1)) - g_i(t_1, x(t_1))| \\ &\quad + \int_0^{t_1} \left(\frac{f_i(\tau, x(\tau))}{(t_2-\tau)^\alpha} - \frac{f_i(\tau, x(\tau))}{(t_1-\tau)^\alpha} \right) d\tau + \int_{t_1}^{t_2} \frac{f_i(\tau, x(\tau))}{(t_2-\tau)^\alpha} d\tau \\ &\le \sum_{j=1}^\infty k_{ij}|x_j(t_2) - x_j(t_1)| + \nu^T(g_i, \varepsilon) + \left(\overline{a}_i(T) + r_i(T)\overline{b}_i(T)\right) \varepsilon^{2-\alpha}/(1-\alpha), \end{aligned}$$

where

$$\nu^T(g_i, \varepsilon) = \sup\{|g_i(t_2, x) - g_i(t_1, x)| : t_2, t_1 \in [0, T], |t_2 - t_1| \le \varepsilon, x = (x_i) \in \mathbb{R}^\infty, |x_i| \le r_i(T)\},$$
$$\overline{a}_i(T) = \sup_{t \le T} a_i(t), \qquad \overline{b}_i(T) = \sup_{t \le T} b_i(t).$$

Hence

$$\omega^T(\pi_i(Fx), \varepsilon) \le \sum_{j=1}^\infty k_{ij}\omega^T(\pi_j(x), \varepsilon) + \nu^T(g_i, \varepsilon) + \left(\overline{a}_i(T) + r_i(T)\overline{b}_i(T)\right) \varepsilon^{2-\alpha}/(1-\alpha). \tag{11}$$

Let us observe that, by virtue of (H2), the functions $g_i(t, x)$ are equicontinuous on $[0, T]$ so $\omega^T(g_i, \varepsilon) \to 0$ as $\varepsilon \to 0+$.

Thus, in view of (11) we obtain

$$\omega_0^T(\pi_i(FX)) \le \sum_{j=1}^\infty k_{ij}\omega_0^T(\pi_j(X)). \tag{12}$$

In what follows we will work with the measure of noncompactness $\overline{\omega}_0$ defined in the Fréchet space $C(\mathbb{R}_+, \mathbb{R}^\infty)$ by the formula (1), where

$$p_i(T) = r_i^{-1}(T).$$

Taking into account (12), (9) and assumption (H3), we get

$$\begin{aligned} r_i^{-1}(T)\omega_0^T(\pi_i(FX)) &\le \sum_{j=1}^\infty k_{ij} \frac{\omega_0^T(\pi_j(X))}{r_j(T)} \frac{r_j(T)}{r_i(T)} \\ &\le \sum_{j=1}^\infty k_{ij}\overline{\omega}_0^T(X) \frac{m_j(T)}{m_i(T)} \le q\overline{\omega}_0^T(X) \end{aligned}$$

and consequently

$$\overline{\omega}_0^T(FX) \le q\overline{\omega}_0^T(X),$$
$$\overline{\omega}_0(FX) \le q\overline{\omega}_0(X). \tag{13}$$

Moreover, $\omega^T(\pi_i(x), \varepsilon) \le 2r_i(T)$ for $x \in Q$ and therefore

$$\overline{\omega}_0(Q) \le 2. \tag{14}$$

In the sequel we show that the operator F is continuous on the set Q. To do this let us fix $x \in Q$ and take a sequence $(x_n) \in Q$ such that $x_n \to x$ in $C(\mathbb{R}_+, \mathbb{R}^\infty)$. In virtue of (A), this is equivalent to

$$\lim_{n\to\infty} \sup_{t \le T} |\pi_i(x)(t) - \pi_i(x_n)(t)| = 0, \quad \text{for } T > 0,\ i = 1, 2, \dots.$$

Let us fix $T > 0$, $i \in \mathbb{N}$ and take $t \in [0, T]$. Then, applying (H2) we get

$$\begin{aligned} |\pi_i(Fx)(t) - \pi_i(Fx_n)(t)| &\le |g_i(t, x(t)) - g_i(t, x_n(t))| + \int_0^t \frac{f_i(\tau, x(\tau)) - f_i(\tau, x_n(\tau))}{(t-\tau)^\alpha} d\tau \\ &\le \sum_{j=1}^\infty k_{ij}|\pi_j(x)(t) - \pi_j(x_n)(t)| + \nu^T(f_i, d_{C_T}(x, x_n)) \frac{T^{1-\alpha}}{1-\alpha} \end{aligned} \tag{15}$$

where

$$\nu^T(f_i,\varepsilon) = \sup\{|f_i(\tau,x)-f_i(\tau,y)| : \tau\in[0,T], x,y\in\mathbb{R}^\infty, |\pi_j(x)|, |\pi_j(y)| \le r_j(T) \text{ for } j=1,2,\ldots, d_{\mathbb{R}^\infty}(x,y)\le\varepsilon\}.$$

Observe that $\nu^T(f_i,\varepsilon)\to 0$ as $\varepsilon\to 0+$, which is a simple consequence of equicontinuity of the family function $\{f_i(t,x)\}_{t\in[0,T]}$ on the set $Z=\{x\in\mathbb{R}^\infty : |\pi_j(x)|\le r_j(T) \text{ for } j=1,2,\ldots\}$.

We will show that the series in (15) is convergent. Keeping in mind (9) and (H2) we get

$$\begin{aligned}\sup_{t\le T}\sum_{j=1}^{\infty}k_{ij}|\pi_j(x)(t)-\pi_j(x_n)(t)| &\le 2\sup_{t\le T}\sum_{j=1}^{\infty}k_{ij}r_j(t)\\ &\le 2\sup_{t\le T}\sum_{j=1}^{\infty}k_{ij}\frac{m_j(t)}{m_i(t)}\phi(t)m_i(t)\le 2qr_i(T)<\infty.\end{aligned}$$

This implies that there is sufficiently big number $j_0\in\mathbb{N}$ such that

$$\sup_{t\le T}\sum_{j>j_0}^{\infty}k_{ij}|\pi_j(x)(t)-\pi_j(x_n)(t)|$$

is sufficiently small. Moreover, for sufficiently big numbers $n\in\mathbb{N}$, the numbers

$$\sup_{t\le T}\sum_{j=1}^{j_0}k_{ij}|\pi_j(x)(t)-\pi_j(x_n)(t)|$$

are sufficiently small. Linking all above obtained facts we infer that the right side of the inequality (15) is arbitrarily small for sufficiently big numbers $n\in\mathbb{N}$. This confirm continuity of F on Q.

Finally, taking into account (13), (14), the properties of the set Q and the operator $F : Q\to Q$ established above and applying Theorem 2.4, we infer that the operator F has at least one fixed point x in the set Q. Obviously the function $x(t)=(x_i(t))$ is a solution of the system (2). This completes the proof. □

4. Final remarks and an example

The first part of this section is devoted to discussing a few facts concerning assumptions of Theorem 3.4.

Let us consider the following assumption:

(H′3) The functions $|g_i(t,0,0,\ldots)|$ and $\alpha_i(t)$ $(i=1,2,\ldots)$ are uniformly bounded on compact intervals of $\mathbb{R}_+$ and there exists a constant $q\in[0,1)$ such that

$$\sum_{j=1}^{\infty}k_{ij}\le q$$

for $i=1,2,\ldots$.

Notice that assumption (H′3) implies (H3). Indeed, putting

$$m(t)=\sup\left\{|g_i(t,0,0,\ldots)|+\int_0^t\frac{\alpha_i(\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau : i=1,2,\ldots\right\}$$

we derive that the functions

$$m_i(t)=\sup\{m(\tau):\tau\le t\}\quad\text{for } i=1,2,\ldots \tag{16}$$

are nondecreasing and satisfy the inequalities of assumption (H3). Therefore, we have the following theorem.

Theorem 4.1. *Under assumptions* (H1), (H2) *and* (H′3), *the infinite system* (2) *has at least one solution* $x(t)=(x_i(t))$ *such that* $x(t)\in C(\mathbb{R}_+,\mathbb{R}^\infty)$.

Next we denote by (H′1) assumption (H1) modified in such a way that we replace (3) by the following inequality

$$|f_i(t,x_1,x_2,\ldots)|\le a_i(t)+b_i(t)\sup\{|x_k|:k\in\mathbb{N}\}$$

for $i=1,2,\ldots$ and $x=(x_i)\in\mathbb{R}^\infty$ such that $\sup\{|x_k|:k\in\mathbb{N}\}<\infty$.

Using (16) and a reasoning similar to that from the proof of Theorem 3.4 we can prove the another existence result:

Theorem 4.2. *Under assumptions* (H′1), (H2) *and* (H′3), *the infinite system* (2) *has at least one solution* $x(t)=(x_i(t))$ *such that* $x(t)\in C(\mathbb{R}_+,\mathbb{R}^\infty)$.

Observe, that above theorem improves and generalizes Theorem 2 [8].

Example 4.3. In order to illustrate our investigations, let us consider the following infinite systems of integral equations of the form

$$x_i(t) = \mathrm{e}^{it} + \sum_{j=1}^{i-1} jx_j(t) + \int_0^t \frac{x_i^3(\tau) + \mathrm{e}^{(i-1)\tau}\sqrt{\tau}/2}{1 + \tau + \sum_{j=1}^{i} x_j^2}(t-\tau)^{-1/2}\mathrm{d}\tau, \tag{17}$$

where $i = 1, 2, \ldots$ and $t \geq 0$.

Notice that this equation is a particular case of the infinite system of Eq. (2), where

$$g_i(t, x_1, x_2, \ldots) = \mathrm{e}^{it} + \sum_{j=1}^{i-1} jx_j, \qquad \alpha = 1/2,$$

$$f_i(t, x_1, x_2, \ldots) = \frac{x_i^3 + \mathrm{e}^{(i-1)t}\sqrt{t}/2}{1 + t + \sum_{j=1}^{i} x_j^2}.$$

We show that (17) satisfies Theorem 3.4 with

$$a_i(t) = \mathrm{e}^{(i-1)t}\sqrt{t}/2, \qquad b_i(t) = 1, \qquad k_{ij} = \begin{cases} j & \text{for } j < i \\ 0 & \text{for } j \geq i. \end{cases}$$

It is clear that assumptions (H1) and (H2) are satisfied. We show that assumption (H3) is satisfied with

$$q = 1/2 \quad \text{and} \quad m_i(t) = 2\mathrm{e}^{it}\prod_{j=1}^{i}(2j-1).$$

In fact, we have

$$|g_i(t, 0, 0, \ldots)| + \int_0^t \frac{a_i(\tau)\mathrm{d}\tau}{\sqrt{t-\tau}} \leq \mathrm{e}^{it} + \mathrm{e}^{(i-1)t}\frac{\sqrt{t}}{2}\int_0^t \frac{\mathrm{d}\tau}{\sqrt{t-\tau}} \leq 2\mathrm{e}^{it} \leq m_i(t).$$

Moreover

$$\sum_{j=1}^{\infty} k_{ij}\frac{m_j(t)}{m_i(t)} = \sum_{j=1}^{i-1} j\frac{m_j(t)}{m_i(t)} = \frac{\sum_{j=1}^{i-1} \mathrm{e}^{jt} j \prod_{k=1}^{j}(2k-1)}{\mathrm{e}^{it}\prod_{j=1}^{i}(2j-1)} \leq \frac{\sum_{j=1}^{i-1} j \prod_{k=1}^{j}(2k-1)}{\prod_{j=1}^{i}(2j-1)}.$$

By the method of mathematical induction, we can prove that

$$\sum_{j=1}^{i-1} j\prod_{k=1}^{j}(2k-1) = \frac{1}{2}\left(\prod_{j=1}^{i}(2j-1) - 1\right) \quad \text{for } i = 2, 3, \ldots.$$

We omit the standard proof.

Therefore, in view of the above inequality we get

$$\sum_{j=1}^{\infty} k_{ij}\frac{m_j(t)}{m_i(t)} \leq 1/2.$$

Hence, on the basis of Theorem 3.4 we deduce that system (17) has at least one solution $x(t) = (x_i(t)) \in C(\mathbb{R}_+, \mathbb{R}^\infty)$.

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