Differential Geometry and its Applications 30 (2012) 428–437

Contents lists available at SciVerse ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

An integral transform on a cylinder and the twistor correspondence

Fuminori Nakata *, 1

Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, Noda, Chiba 278-8510, Japan

A R T I C L E   I N F O

Article history:
Received 1 February 2012
Available online 2 July 2012
Communicated by M.G. Eastwood

MSC:
53C28
35L05
53C50
32G10

Keywords:
Twistor method
Holomorphic disks
Indefinite metric
Wave equation
Monopole equation
Radon transform

A B S T R A C T

Twistor correspondences for $\mathbb{R}$-invariant indefinite self-dual conformal structures on $\mathbb{R}^4$ are established explicitly. These correspondences are written down by using a natural integral transform from functions on a two dimensional cylinder to functions on the flat Lorentz space $\mathbb{R}^{1,2}$ which is related to the wave equation and the Radon transform. A general method on the twistor construction of indefinite self-dual 4-spaces and indefinite Einstein–Weyl 3-spaces are also summarized.

1. Introduction

A twistor correspondence for self-dual Zollfrei conformal structure of the neutral signature $(--++)$ was established by C. LeBrun and L.J. Mason [9]. Here an indefinite conformal structure is called *Zollfrei* if and only if all the maximal null geodesics are closed. In this theory, the twistor space is the pair $(\mathbb{CP}^3, P)$ where P is an embedded $\mathbb{RP}^3$ in $\mathbb{CP}^3$, and the self-dual space is recovered as the space of holomorphic disks on $\mathbb{CP}^3$ with boundaries lying on P. LeBrun and Mason also showed that any neutral self-dual Zollfrei 4-manifold is compact and is homeomorphic to $S^2 \times S^2$ or $(S^2 \times S^2)/\mathbb{Z}_2$.

LeBrun and Mason also established twistor correspondence for indefinite Einstein–Weyl structure on $\mathbb{R} \times S^2$ in [10] (see also [11]). These two types of twistor correspondences are related by the Jones–Tod reduction theory [4], which say that an Einstein–Weyl 3-space is obtained as the orbit space of 1-dimensional group action on a self-dual 4-space, and conversely that self-dual 4-spaces are obtained from solutions of the generalized monopole equation on an Einstein–Weyl 3-space.

Following LeBrun and Mason, the author wrote down the twistor correspondence for Tod–Kamada metrics explicitly in [12]. Tod–Kamada metrics are S^1-invariant indefinite self-dual metrics on $S^2 \times S^2$ which are first constructed by K.P. Tod [13] and are also rediscovered by H. Kamada in the investigation of indefinite Kähler surfaces with Hamiltonian S^1-symmetry [6]. As a consequence of this work, we find that all the Tod–Kamada metrics are Zollfrei.

The construction of Tod–Kamada metrics are based on the Jones–Tod reduction, and is obtained from solutions of the monopole equation on the de Sitter 3-space S^3_1. In the study of twistor correspondence for the Tod–Kamada metric, we

* Current address: Faculty of Human Development and Culture, Fukushima University, Kanayagawa, Fukushima-city 960-1296, Japan.
E-mail address: fnakata@educ.fukushima-u.ac.jp.

1 Supported by Grant-in-Aid for Scientific Research of the Japan Society for the Promotion of Science.

0926-2245/$ – see front matter
http://dx.doi.org/10.1016/j.difgeo.2012.06.004

obtain simple expressions of solutions of the monopole equation or the wave equation on S^3_1 in terms of Radon-type integral transforms from functions on S^2 to functions on S^3_1. This result may give a new insight for the theory of hyperbolic PDEs.

In this article we relax the Zollfrei condition, and study the twistor correspondence for $\mathbb{R}$-invariant indefinite self-dual metrics on $\mathbb{R}^4$ by a similar method as the Tod–Kamada metric case. We start from the twistor correspondence for the flat metric and deform it respecting an $\mathbb{R}$-action. This construction can also be considered as the indefinite analogue of the taub-NUT metric (see [1] for the taub-NUT metric and its twistor space).

Similarly to the Tod–Kamada case, our twistor correspondence is related to the theory of hyperbolic PDEs and integral transforms. In this article, we need some results for the wave equation on the Lorentz space $\mathbb{R}^{1,2} = \{(t, x_1, x_2)\}$:

$$-\Box u = -\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0. \tag{1.1}$$

We introduce a natural integral transform from functions on a two dimensional cylinder to functions on $\mathbb{R}^{1,2}$ of which the image solves the wave equation (1.1). One of the two ways of the twistor correspondence, from a twistor space to a self-dual space, is written down by using this integral transform. The converse correspondence is obtained by using a general formula for the solutions to the wave equation (1.1) which is written in terms of Radon transform (Theorem 2.1).

In the LeBrun–Mason theory, the Zollfrei condition gives a nice restriction on the space of self-dual metrics. Instead of assuming the Zollfrei condition, in this article we assume the rapidly decreasing condition for the solutions of wave equation. Under this rapidly decreasing condition, we can reasonably establish the correspondence from self-dual metric to the twistor space.

The organization of the paper is the following. In Section 2, we introduce an integral transform of which the image solves the wave equation (1.1). Its inverse correspondence is given in Theorem 2.1. In Section 3, we summarize the general method of twistor-type construction of indefinite self-dual 4-spaces and indefinite Einstein–Weyl 3-spaces as the space of holomorphic disks. In Section 4, we will write down the twistor correspondence for the flat indefinite metric on $\mathbb{R}^4$ and its $\mathbb{R}$-quotient explicitly. Finally we study the deformation of this standard case in Sections 5 and 6. In Section 5, we deform the twistor space and determine the corresponding self-dual spaces, and the converse is studied in Section 6.

Notations. We denote the complex unit disk by $\mathbb{D} = \{\omega \in \mathbb{C} \mid |\omega| \leqslant 1\}$. For the pair (Z, P) of a complex manifold Z and a totally real submanifold $P \subset Z$, a *holomorphic disk on* (Z, P) means the image of a continuous map $(\mathbb{D}, \partial\mathbb{D}) \to (Z, P)$ which is holomorphic on the interior of $\mathbb{D}$.

2. Planar circles on a cylinder

We first study the geometry of circles on a two dimensional cylinder. We introduce an integral transform and explain the relation with the wave equation on the flat Lorentz space $\mathbb{R}^{1,2}$.

The two dimensional cylinder and planar circles. Let $\mathscr{C} = \{(\omega, v) \in \mathbb{R}^2 \times \mathbb{R} \mid |\omega| = 1\} \simeq S^1 \times \mathbb{R}$ be a two dimensional cylinder embedded in $\mathbb{R}^3$. Each plane on $\mathbb{R}^3$ cuts out a circle on $\mathscr{C}$ if the plane is not parallel to the v-axis. We call such circles *planar*, and let M be the set of planar circles on $\mathscr{C}$. Then M is an affine subset of the set of planes on $\mathbb{R}^3$, and is coordinated by $(t, x) \in \mathbb{R} \times \mathbb{R}^2$ so that the corresponding planar circle $C_{(t,x)}$ is cut out by the plane $\{(\omega, v) \in \mathbb{R}^2 \times \mathbb{R} \mid v = t + \langle \omega, x\rangle\}$, that is,

$$C_{(t,x)} = \bigl\{(\omega, v) \in \mathscr{C} \bigm| v = t + \langle \omega, x\rangle\bigr\}. \tag{2.1}$$

Now we equip M a flat Lorentz metric

$$g = -dt^2 + dx_1^2 + dx_2^2 \quad \bigl(x = (x_1, x_2)\bigr). \tag{2.2}$$

This metric naturally arises from the twistor correspondence which we will see later (Section 4). With respect to this Lorentz metric, the geometry on M is nicely related to the geometry on $\mathscr{C}$ in the following way.

First, for each point $p \in \mathscr{C}$ let $\Pi_p \subset M$ be the set of circles passing through p, i.e.

$$\Pi_p = \bigl\{(t, x) \in M \bigm| p \in C_{(t,x)}\bigr\} = \bigl\{(t, x) \in M \bigm| v = t + \langle \omega, x\rangle\bigr\} \quad \bigl(p = (\omega, v)\bigr).$$

Then Π_p is a *null plane* on (M, g), that is the metric g degenerates on Π_p. Since every null plane on M is written in this way by a unique $p \in \mathscr{C}$, the cylinder $\mathscr{C}$ is identified with the set of null planes on M.

Next notice that each geodesic (i.e. straight line) on M corresponds to a 1-parameter family of planes on $\mathbb{R}^3$ with a common axis. If this axis intersects with $\mathscr{C}$ at two distinct points p and q, then the planes of this family cut out planar circles passing through common points p and q. In this case, the geodesic is written as $\Pi_p \cap \Pi_q$ and is a *space-like geodesic*. If the axis is tangent to $\mathscr{C}$ at p, then we obtain a family of planar circles which are mutually tangent at $p \in \mathscr{C}$. In this case, the geodesic is a *null geodesic* contained in the null plane Π_p. If the axis is apart from $\mathscr{C}$, then we obtain a family of planar circles which folliate the cylinder $\mathscr{C}$. This family corresponds to a *time-like geodesic*.

Finally, let us notice the two domains $\Omega^{\pm}_{(t,x)}$ on $\mathscr{C}$ which are divided by $C_{(t,x)}$:

$$\Omega^{\pm}_{(t,x)} = \left\{(\omega, v) \in \mathscr{C} \;\middle|\; \pm\big(t + \langle \omega, x\rangle - v\big) \geqslant 0\right\}. \tag{2.3}$$

Then the sets $\{(t', x') \in M \mid C_{(t',x')} \subset \Omega^{\pm}_{(t,x)}\}$ are the *future* and the *past cone* with vertex at $(t, x) \in M$.

Later, we also use a complex parameter $z = x_1 + ix_2$ and we write $\omega = (\cos\theta, \sin\theta) = e^{i\theta}$ identifying $S^1 \cong U(1)$. Then (2.1) is also written as

$$C_{(t,z)} = \left\{(\omega, v) \in \mathscr{C} \;\middle|\; v = t + \operatorname{Re}\big(ze^{-i\theta}\big)\right\}. \tag{2.4}$$

Integral transform R and the wave equation. Now we introduce an integral transform $R : C^\infty(\mathscr{C}) \to C^\infty(M)$ so that for given function $h(\omega, v) \in C^\infty(\mathscr{C})$ the function $Rh(t, x) \in C^\infty(M)$ is

$$Rh(t,x) = \frac{1}{2\pi}\int_{C_{(t,x)}} h\,d\theta = \frac{1}{2\pi}\int_{|\omega|=1} h\big(\omega, t + \langle\omega, x\rangle\big)\,d\theta \quad \big(\omega = e^{i\theta}\big). \tag{2.5}$$

A significant property of the transform R is that its image $u = Rh$ satisfies the wave equation (1.1). Here notice that R has a non-trivial kernel. Actually, if the function $h(\omega, v)$ is independent of v, then $Rh(t, x)$ is constant and is equal to the constant term of the Fourier expansion of $h = h(\omega)$ which can be vanish for non-trivial h. One natural way to avoid such obvious kernel is to replace the function space $C^\infty(\mathscr{C})$ with the rapidly decreasing functions $\mathcal{S}(\mathscr{C})$, where $\mathcal{S}(\mathscr{C})$ is the space of smooth functions h on $\mathscr{C}$ which for any integers $k, l \geqslant 0$ and any differential operator D on S^1 satisfy

$$\sup_{(\omega,v)\in\mathscr{C}} \left| \big(1 + |v|^k\big) \frac{\partial^l}{\partial v^l}(Dh)(\omega, v) \right| < \infty.$$

Radon transform and the inverse of R. The inverse transform of R is obtained using the *Radon transform*, which we summarize here (see [2] for the detail). Let $M_0 = \{(0, x) \in M\} \simeq \mathbb{R}^2$ be the *initial plane*. Notice that each null surface $\Pi_{(\omega,v)} \subset M$ corresponding to $(\omega, v) \in \mathscr{C}$ intersects with M_0 by a straight line

$$l_{(\omega,v)} = \Pi_{(\omega,v)} \cap M_0 = \left\{x \in M_0 \;\middle|\; \langle\omega, x\rangle = v\right\}.$$

Since $l_{(\omega,v)} = l_{(-\omega,-v)}$, the cylinder $\mathscr{C}$ is identified with the double cover of the set of straight lines on M_0.

Let $\mathcal{S}(M_0)$ be the space of rapidly decreasing functions on M_0. Here a smooth function $f(x)$ on M_0 is called rapidly decreasing if and only if, for each integer $k, l, m \geqslant 0$, f satisfies

$$\left| |x|^k \partial^l_{x_1} \partial^m_{x_2} f(x) \right| < \infty.$$

For given function $f(x) \in \mathcal{S}(M_0)$, its Radon transform $\hat{f}(\omega, v) \in \mathcal{S}(\mathscr{C})$ is defined by

$$\hat{f}(\omega, v) = \int_{l_{(\omega,v)}} f\,dm \tag{2.6}$$

where dm is the Euclid measure on the line $l_{(\omega,v)}$. On the other hand, for a function $h(\omega, v) \in C^\infty(\mathscr{C})$, its *dual Radon transform* $\check{h}(x) \in C^\infty(M_0)$ is defined by

$$\check{h}(x) = \frac{1}{2\pi}\int_{|\omega|=1} h\big(\omega, \langle\omega, x\rangle\big)\,d\theta = Rh(0, x).$$

The inversion formula of the Radon transform for $f(x) \in \mathcal{S}(M_0)$ is

$$f = \frac{1}{2i}\big(\mathcal{H}_v \partial_v \hat{f}\big)^\vee \quad \text{where } \mathcal{H}_v h(\omega, v) = \frac{i}{\pi}\,\mathbf{pv}.\int_{-\infty}^{\infty} \frac{h(\omega, \nu)}{\nu - v}\,d\nu. \tag{2.7}$$

Recall that $\mathcal{H}_v$ is called the *Hilbert transform*.

Now the inverse of R is given in the following way.

Theorem 2.1. *Suppose $u(t, x) \in C^\infty(M)$ satisfies the following conditions*:

- *u solves the wave equation, i.e. $\Box u = 0$,*
- *$f_0(x) = u(0, x)$ and $f_1(x) = u_t(0, x)$ are rapidly decreasing functions on $\mathbb{R}^2$.*

Then we can write $u = Rh$ where $h \in \mathcal{S}(\mathscr{C})$ is defined by

$$h = \frac{1}{4\pi i}\mathcal{H}_v(\partial_v \hat{f}_0 + \hat{f}_1). \tag{2.8}$$

This theorem is easily proved by the inversion formula (2.7) and the uniqueness theorem for the solutions to a hyperbolic PDE (see [2]).

3. General method on the twistor correspondence

In this section, we summarize the general method to construct indefinite self-dual 4-spaces and indefinite Einstein–Weyl 3-spaces from a family of holomorphic disks. The detail is found in [9] for the self-dual case, and in [11] for the Einstein–Weyl case.

Partial indices of a holomorphic disk. First, we recall the notion of the *partial indices* of a holomorphic disk (see [8]). Let (Z, P) be the pair of a complex n-manifold Z and a totally real submanifold P, and $(D, \partial D)$ be a holomorphic disk on (Z, P). We write $\mathcal{N}$ the complex normal bundle of D in Z and $\mathcal{N}_{\mathbb{R}}$ the real normal bundle of ∂D in P. Notice that we have $\mathcal{N}_{\mathbb{R}} \subset \mathcal{N}|_{\partial D}$. We can construct a virtual holomorphic vector bundle $\widehat{\mathcal{N}}$ on the double $\mathbb{CP}^1 = D \cup_{\partial D} \overline{D}$ by patching $\mathcal{N} \to D$ and $\overline{\mathcal{N}} \to \overline{D}$ so that $\mathcal{N}_{\mathbb{R}}$ coincides. Then we can write $\widehat{\mathcal{N}} = \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_{n-1})$, and we call the $(n-1)$-tuple of integers $(k_1, \ldots, k_{n-1})$ the *partial indices* of the holomorphic disk D. These indices is uniquely determined by D up to permutation.

Construction of self-dual conformal structures of signature $(--++)$. Let $\mathscr{T}$ be a complex 3-manifold and let $\mathscr{T}_{\mathbb{R}}$ be a totally real submanifold on $\mathscr{T}$. Suppose that there exists a family of holomorphic disks on $(\mathscr{T}, \mathscr{T}_{\mathbb{R}})$ smoothly parametrized by a real 4-manifold M. Such family is described by the following diagram:

$$\begin{array}{ccc} & (\mathcal{Z}, \mathcal{Z}_{\mathbb{R}}) & \\ {}^{\mathfrak{p}}\swarrow & & \searrow^{(\mathfrak{f}, \mathfrak{f}_{\mathbb{R}})} \\ M & & (\mathscr{T}, \mathscr{T}_{\mathbb{R}}) \end{array} \tag{3.1}$$

where $(\mathcal{Z}, \mathcal{Z}_{\mathbb{R}})$ is a smooth $(\mathbb{D}, \partial\mathbb{D})$-bundle on M, $\mathfrak{f}$ is a smooth map which is holomorphic along each fiber of $\mathfrak{p}$, and $\mathfrak{f}_{\mathbb{R}}$ is the restriction of $\mathfrak{f}$. For each $x \in M$ let us write $D_x = \mathfrak{f}(\mathfrak{p}^{-1}(x))$ which is the holomorphic disk corresponding to x. Now we assume the following conditions:

A1) the differential $\mathfrak{f}_*$ is of full-rank on $\mathcal{Z} \setminus \mathcal{Z}_{\mathbb{R}}$ (i.e. $\mathfrak{f}$ is locally isomorphic on $\mathcal{Z} \setminus \mathcal{Z}_{\mathbb{R}}$),
A2) the differential $(\mathfrak{f}_{\mathbb{R}})_*$ is of full-rank, and
A3) for each $x \in M$, the partial indices of the corresponding disk D_x is $(1, 1)$.

In this setting, the following holds (see Section 10 of [9]).

Proposition 3.1. *There exists a unique smooth self-dual conformal structure $[g]$ of signature $(--++)$ on M such that for each $p \in \mathscr{T}_{\mathbb{R}}$ the set $\mathfrak{S}_p = \{x \in M \mid p \in \partial D_x\}$ gives a β-surface for $[g]$.*

Here a *β-surface* is a surface on which the conformal structure vanishes and of which the tangent bivector is anti-self-dual every where.

Construction of Einstein–Weyl structure of signature $(-++)$. There is a similar construction for three dimensional Einstein–Weyl structure. Recall that an Einstein–Weyl structure is the pair $([g], \nabla)$ of a conformal structure $[g]$ and an affine connection ∇ satisfying the compatibility condition (Weyl condition) $\nabla g \propto g$ and the Einstein–Weyl condition $R_{(ij)} \propto g_{ij}$ where $R_{(ij)}$ is the symmetrized Ricci tensor of ∇ (see [3,11]).

Let $\mathscr{S}$ be a complex surface and $\mathscr{S}_{\mathbb{R}}$ be a totally real submanifold on $\mathscr{S}$. Suppose that there exists a family of holomorphic disks on $(\mathscr{S}, \mathscr{S}_{\mathbb{R}})$ smoothly parametrized by a real 3-manifold $\underline{M}$, which is described by the following diagram:

$$\begin{array}{ccc} & (\mathcal{W}, \mathcal{W}_{\mathbb{R}}) & \\ {}^{\underline{\mathfrak{p}}}\swarrow & & \searrow^{(\underline{\mathfrak{f}}, \underline{\mathfrak{f}}_{\mathbb{R}})} \\ \underline{M} & & (\mathscr{S}, \mathscr{S}_{\mathbb{R}}) \end{array} \tag{3.2}$$

Let us assume the conditions:

B1) the differential $\underline{\mathfrak{f}}_*$ is of full-rank on $\mathcal{W} \setminus \mathcal{W}_{\mathbb{R}}$,
B2) the differential $(\underline{\mathfrak{f}}_{\mathbb{R}})_*$ is of full-rank, and
B3) for each $x \in \underline{M}$, the partial index of the corresponding disk $\underline{D}_x = \underline{\mathfrak{f}}(\underline{\mathfrak{p}}^{-1}(x))$ is 2.

Then the following holds (see [11]).

Proposition 3.2. *There exists a unique smooth Einstein–Weyl structure* $([g], \nabla)$ *of signature* $(-++)$ *on* $\underline{M}$ *so that for each* $p \in \mathscr{S}_{\mathbb{R}}$ *the set* $\underline{\mathfrak{S}}_p = \{x \in \underline{M} \mid p \in \partial \underline{D}_x\}$ *is a totally geodesic null surface on* $\underline{M}$.

Here recall that a surface on a conformal manifold $(\underline{M}, [g])$ of signature $(-++)$ is called *null* if and only if $[g]$ degenerate on it.

The Einstein–Weyl structure on $\underline{M}$ defined by Proposition 3.2 satisfies the following properties (cf. [11]):

- for each distinguished $p, q \in \mathscr{S}_{\mathbb{R}}$, the set $\underline{\mathfrak{S}}_p \cap \underline{\mathfrak{S}}_q$ is a *space-like geodesic*,
- for each p and non-zero $v \in T_p\mathscr{S}_{\mathbb{R}}$, the set $\underline{\mathfrak{C}}_{p,v} = \{x \in \underline{M} \mid p \in \partial \underline{D}_x,\ v \in T_p(\partial \underline{D}_x)\}$ is a *null geodesic*, and
- for each $\zeta \in \mathscr{S} \setminus \mathscr{S}_{\mathbb{R}}$, the set $\underline{\mathfrak{C}}_\zeta = \{x \in \underline{M} \mid \zeta \in \underline{D}_x\}$ is a *time-like geodesic*.

4. Standard model

In this section, we describe the twistor correspondence for $\mathbb{R}^{2,2}$ and $\mathbb{R}^{1,2}$ explicitly.

Twistor correspondence for $\mathbb{R}^{2,2}$. Let H be the degree 1 holomorphic line bundle over $\mathbb{CP}^1$, and we put $\mathscr{T} = H \oplus H$. The total space of $\mathscr{T}$ can be embedded in $\mathbb{CP}^3$ as

$$\mathscr{T} = \left\{[y_0 : y_1 : y_2 : y_3] \in \mathbb{CP}^3 \;\middle|\; (y_0, y_1) \neq (0,0)\right\}$$

with the projection $[y_i] \mapsto [y_0 : y_1]$. We introduce an antiholomorphic involution σ on $\mathscr{T}$ by $\sigma : [y_0 : y_1 : y_2 : y_3] \longmapsto [\bar{y}_1 : \bar{y}_0 : \bar{y}_3 : \bar{y}_2]$, and let $\mathscr{T}_{\mathbb{R}}$ be the fixed point set of σ. Then

$$\mathscr{T}_{\mathbb{R}} = \left\{\left[e^{-\frac{i\theta}{2}} : e^{\frac{i\theta}{2}} : \eta : \bar{\eta}\right] \in \mathbb{CP}^3 \;\middle|\; \left(e^{i\theta}, \eta\right) \in S^1 \times \mathbb{C}\right\}.$$

The space of σ-invariant holomorphic sections on $\mathscr{T} = H \oplus H$ is parametrized by $\mathbb{C}^2$ so that $(a,b) \in \mathbb{C}^2$ correspondes to the section

$$L_{(a,b)} = \left\{[1 : \omega : \bar{a}\omega + b : \bar{b}\omega + a] \in \mathscr{T} \;\middle|\; \omega \in \mathbb{C} \cup \{\infty\}\right\}.$$

Each section $L_{(a,b)}$ intersect with the set $\mathscr{T}_{\mathbb{R}}$ by S^1, and divided into two disks

$$D_{(a,b)} = \left\{[1 : \omega : \bar{a}\omega + b : \bar{b}\omega + a] \in \mathscr{T} \;\middle|\; \omega \in \mathbb{D}\right\} \quad \text{and}$$
$$D'_{(a,b)} = \left\{[\omega : 1 : \bar{a} + b\omega : \bar{b} + a\omega] \in \mathscr{T} \;\middle|\; \omega \in \mathbb{D}\right\}. \tag{4.1}$$

Hence we obtain two $\mathbb{C}^2$-families of holomorphic disks $\{D_{(a,b)}\}$ and $\{D'_{(a,b)}\}$ on $(\mathscr{T}, \mathscr{T}_{\mathbb{R}})$. In the rest of this article, we only deal with the family $\{D_{(a,b)}\}$.

The double fibration associated with the holomorphic disks $\{D_{(a,b)}\}$ is given as follows. In our case, the parameter space M is $\mathbb{C}^2$. Let $\mathfrak{p} : (\mathcal{Z}, \mathcal{Z}_{\mathbb{R}}) \to \mathbb{C}^2$ be the trivial $(\mathbb{D}, \mathbb{D}_{\mathbb{R}})$ bundle, and we define a map $\mathfrak{f} : (\mathcal{Z}, \mathcal{Z}_{\mathbb{R}}) \to (\mathscr{T}, \mathscr{T}_{\mathbb{R}})$ by $\mathfrak{f}(a,b;\omega) = [1 : \omega : \bar{a}\omega + b : \bar{b}\omega + a]$. It is easy to verify that the conditions A1 to A3 hold (see the final part of this section), so we obtain a self-dual conformal structure on $\mathbb{C}^2$ by Proposition 3.1. This conformal structure is characterized by the condition: for each $\omega \in U(1)$ and each constant $c \in \mathbb{C}$ the set $\{(a,b) \in \mathbb{C}^2 \mid \bar{a}\omega + b = c\}$ is a β-surface. Since the flat metric

$$g = -|da|^2 + |db|^2 \tag{4.2}$$

satisfies this condition, the induced self-dual conformal structure on $\mathbb{C}^2$ is represented by this metric. We write $\mathbb{R}^{2,2}$ for $\mathbb{C}^2$ equipped with this metric g.

Twistor correspondence for $\mathbb{R}^{1,2}$ as a quotient. In this part, we establish the twistor correspondence for $\mathbb{R}^{1,2}$ as the quotient of $\mathbb{R}^{2,2}$. Let $(\mathscr{T}, \mathscr{T}_{\mathbb{R}})$ be as above, and let us introduce a $(\mathbb{C}, \mathbb{R})$-action on $(\mathscr{T}, \mathscr{T}_{\mathbb{R}})$ by

$$\nu \cdot [y_0 : y_1 : y_2 : y_3] = [y_0 : y_1 : y_2 - i\nu y_1 : y_3 + i\nu y_0] \quad (\nu \in \mathbb{C}). \tag{4.3}$$

Here "$(\mathbb{C}, \mathbb{R})$-action on $(\mathscr{T}, \mathscr{T}_{\mathbb{R}})$" is a $\mathbb{C}$-action on $\mathscr{T}$ of which the restriction to $\mathbb{R} \subset \mathbb{C}$ preserves $\mathscr{T}_{\mathbb{R}}$.

The quotient space of this $(\mathbb{C}, \mathbb{R})$-action is described as follows. Let $\mathscr{S} = H^2$ be the degree 2 holomorphic line bundle on $\mathbb{CP}^1$. We use the weighted homogeneous coordinate $[y_0 : y_1; v]$ on $\mathscr{S}$ where $[y_0 : y_1]$ is the coordinate of the base $\mathbb{CP}^1$

and $[y_0 : y_1; v] = [\lambda y_0 : \lambda y_1; \lambda^2 v]$ for $\lambda \in \mathbb{C}^*$. We introduce an involution σ on $\mathscr{S}$ by $[y_0 : y_1; v] \mapsto [\bar{y}_1 : \bar{y}_0; \bar{v}]$. The fixed point set of σ is identified with the real cylinder $\mathscr{C} = S^1 \times \mathbb{R}$ so that

$$\mathscr{C} = \left\{ \left[e^{-\frac{i\theta}{2}} : e^{\frac{i\theta}{2}}; v \right] \in \mathscr{S} \;\middle|\; \left(e^{i\theta}, v\right) \in S^1 \times \mathbb{R} \right\}.$$

Let us define a map $\pi : (\mathscr{T}, \mathscr{T}_{\mathbb{R}}) \to (\mathscr{S}, \mathscr{C})$ by:

$$\pi\left([y_0 : y_1 : y_2 : y_3]\right) = \left[y_0 : y_1; \frac{y_0 y_2 + y_1 y_3}{2} \right]. \tag{4.4}$$

Notice that each fiber of this map π is a $\mathbb{C}$-orbit on $\mathscr{T}$, and the real submanifold $\mathscr{T}_{\mathbb{R}}$ is mapped onto $\mathscr{C}$. Hence π is the quotient map of the $(\mathbb{C}, \mathbb{R})$-action on $(\mathscr{T}, \mathscr{T}_{\mathbb{R}})$.

Corresponding to this $(\mathbb{C}, \mathbb{R})$-action on the twistor space $(\mathscr{T}, \mathscr{T}_{\mathbb{R}})$, we can introduce an $\mathbb{R}$-action on the parameter space $\mathbb{R}^{2,2}$ so that $D_{\nu \cdot (a,b)} = \nu \cdot D_{(a,b)}$ for $\nu \in \mathbb{R}$ where $D_{(a,b)}$ is the holomorphic disk defined in (4.1). This $\mathbb{R}$-action is written as $\nu \cdot (a, b) = (a + i\nu, b)$, so its quotient map is given by $\varpi : \mathbb{R}^{2,2} \to \mathbb{R} \times \mathbb{C}$ where $\varpi(a, b) = (\mathrm{Re}\, a, b)$. The quotient space $\mathbb{R} \times \mathbb{C} = \{(t, z)\}$ has a natural indefinite metric

$$\underline{g} = -dt^2 + |dz|^2 \tag{4.5}$$

as a quotient of the metric g in (4.2). We also write the quotient space $\mathbb{R} \times \mathbb{C}$ as $\mathbb{R}^{1,2}$.

Let ∇ be the Levi-Civita connection of $\underline{g}$. Then Jones–Tod theory [4] asserts that the Einstein–Weyl structure $([\underline{g}], \nabla)$ is a structure which corresponds to the twistor space $(\mathscr{S}, \mathscr{C})$ in the sense of Proposition 3.2. Actually, suppose $\varpi(a, b) = (t, z)$ (i.e. $(t, z) = (\mathrm{Re}\, a, b)$), then π maps the holomorphic disk $D_{(a,b)}$ to the holomorphic disk

$$\underline{D}_{(t,z)} = \left\{ \left[1 : \omega; \frac{z}{2} + t\omega + \frac{\bar{z}}{2}\omega^2 \right] \in \mathscr{S} \;\middle|\; \omega \in \mathbb{D} \right\} \tag{4.6}$$

on $(\mathscr{S}, \mathscr{C})$. So the space $\mathbb{R}^{1,2}$ is considered as the parameter space of holomorphic disks $\{\underline{D}_{(t,z)}\}$ on $(\mathscr{S}, \mathscr{C})$. We can construct a double fibration with respect to this family of holomorphic disks as follows. Let $\underline{\mathfrak{p}} : (\mathcal{W}, \mathcal{W}_{\mathbb{R}}) \to \mathbb{R}^{1,2}$ be a trivial $\mathbb{D}$ bundle, and let us define a map $\underline{\mathfrak{f}} : (\mathcal{W}, \mathcal{W}_{\mathbb{R}}) \to (\mathscr{S}, \mathscr{C})$ by $\underline{\mathfrak{f}}(t, z, \omega) = [1 : \omega; \frac{z}{2} + t\omega + \frac{\bar{z}}{2}\omega^2]$. Then, we obtain the following diagram:

$$\begin{array}{ccccc}
 & & (\mathcal{Z}, \mathcal{Z}_{\mathbb{R}}) & & \\
 & \swarrow^{\mathfrak{p}} & \downarrow{\scriptstyle \Pi} & \searrow^{\mathfrak{f}} & \\
\mathbb{R}^{2,2} & & (\mathcal{W}, \mathcal{W}_{\mathbb{R}}) & & (\mathscr{T}, \mathscr{T}_{\mathbb{R}}) \\
\downarrow{\scriptstyle \varpi} & \swarrow^{\underline{\mathfrak{p}}} & & \searrow^{\underline{\mathfrak{f}}} & \downarrow{\scriptstyle \pi} \\
\mathbb{R}^{1,2} & & & & (\mathscr{S}, \mathscr{C})
\end{array} \tag{4.7}$$

where $\Pi : (\mathcal{Z}, \mathcal{Z}_{\mathbb{R}}) \to (\mathcal{W}, \mathcal{W}_{\mathbb{R}})$ is given by $(a, b, \omega) \mapsto (\mathrm{Re}\, a, b, \omega)$. We can check that the family $\{\underline{D}_{(t,z)}\}$ satisfies the conditions B1 to B3. Since each β-plane on $\mathbb{R}^{2,2}$ is mapped to a totally geodesic null surface on $\mathbb{R}^{1,2}$, we see that the induced Einstein–Weyl structure coincides to $([\underline{g}], \nabla)$.

Now the observations in Section 2 for planar circles on $\mathscr{C}$ are explained as follows. Under the identification $z = x_1 + ix_2$, we find that each planar circle $C_{(t,x)}$ on $\mathscr{C}$ given by (2.1) is obtained as the boundary circle of the holomorphic disk $D_{(t,z)}$ on $(\mathscr{S}, \mathscr{C})$. The observations for null planes and geodesics are derived from the properties of the Einstein–Weyl structure which is explained in the last part of Section 3.

Distributions. In the rest of this section, we go into more detail of the above construction. We notice the distributions on $(\mathcal{Z}, \mathcal{Z}_{\mathbb{R}})$ and $(\mathcal{W}, \mathcal{W}_{\mathbb{R}})$ introduced from the diagram (4.7) and give an explicit description of them.

First, we define a rank 3 complex distribution $\mathcal{E}$ on $\mathcal{Z} \setminus \mathcal{Z}_{\mathbb{R}}$ by $\mathcal{E} = \ker\{\mathfrak{f}_*^{1,0} : T_{\mathbb{C}}\mathcal{Z} \to T^{1,0}\mathcal{T}\}$ where $\mathfrak{f}_*^{1,0}$ is the composition of the differential $\mathfrak{f}_* : T_{\mathbb{C}}\mathcal{Z} \to T_{\mathbb{C}}\mathcal{T}$ and the projection $T_{\mathbb{C}}\mathcal{T} \to T^{1,0}\mathcal{T}$. If we define complex tangent vector fields $\mathfrak{m}_1$ and $\mathfrak{m}_2$ on $\mathcal{Z} = \{(a, b; \omega) \in \mathbb{C}^2 \times \mathbb{D}\}$ by

$$\mathfrak{m}_1 = -2\frac{\partial}{\partial \bar{a}} + 2\omega\frac{\partial}{\partial b}, \qquad \mathfrak{m}_2 = -2\omega\frac{\partial}{\partial a} + 2\frac{\partial}{\partial \bar{b}}, \tag{4.8}$$

then we obtain $\mathcal{E} = \langle \mathfrak{m}_1, \mathfrak{m}_2, \frac{\partial}{\partial \bar{\omega}} \rangle$ on $\mathcal{Z} \setminus \mathcal{Z}_{\mathbb{R}}$. Notice that on $\mathcal{Z}_{\mathbb{R}}$ we have $\mathfrak{m}_2 = \omega \bar{\mathfrak{m}}_1$.

We also define a rank 2 real distribution $\mathcal{D}$ on $\mathcal{Z}_{\mathbb{R}}$ by $\mathcal{D} = \ker\{(\mathfrak{f}_{\mathbb{R}})_* : T\mathcal{Z}_{\mathbb{R}} \to T\mathscr{T}_{\mathbb{R}}\}$. Then $\mathcal{D} \otimes \mathbb{C} = \langle \mathfrak{m}_1, \mathfrak{m}_2 \rangle$. Notice that the self-dual indefinite metric $[g]$ on $\mathbb{C}^2 = \mathbb{R}^{2,2}$ is defined so that for each $u \in \mathcal{Z}_{\mathbb{R}}$ the tangent plane $\mathfrak{p}_*(\mathcal{D}_u)$ in $T_{\mathfrak{p}(u)}\mathbb{C}^2$ gives a β-plane for $[g]$. Such conformal structure must be represented by the flat metric (4.2) as we already explained.

Similarly, let us define a rank 3 complex distribution $\underline{\mathcal{E}}$ on $\mathcal{W} \setminus \mathcal{W}_{\mathbb{R}}$ by $\underline{\mathcal{E}} = \ker\{\underline{\mathfrak{f}}_*^{1,0} : T_{\mathbb{C}}\mathcal{W} \to T^{1,0}\mathcal{S}\}$ and a rank 2 real distribution $\underline{\mathcal{D}}$ on $\mathcal{W}_{\mathbb{R}}$ by $\underline{\mathcal{D}} = \ker\{\underline{\mathfrak{f}}_* : T\mathcal{W}_{\mathbb{R}} \to T\mathcal{C}\}$. If we define complex tangent vector fields on $\mathcal{W} = \{(t, z; \omega)\}$ by

$$\underline{\mathfrak{m}}_1 = -\frac{\partial}{\partial t} + 2\omega\frac{\partial}{\partial z}, \qquad \underline{\mathfrak{m}}_2 = -\omega\frac{\partial}{\partial t} + 2\frac{\partial}{\partial \bar{z}}, \tag{4.9}$$

then we obtain

$$\underline{\mathcal{E}} = \left\langle \underline{\mathfrak{m}}_1, \underline{\mathfrak{m}}_2, \frac{\partial}{\partial\bar{\omega}} \right\rangle \quad \text{on } \mathcal{W} \setminus \mathcal{W}_{\mathbb{R}},$$
$$\underline{\mathcal{D}} \otimes \mathbb{C} = \langle \underline{\mathfrak{m}}_1, \underline{\mathfrak{m}}_2 \rangle \quad \text{on } \mathcal{W}_{\mathbb{R}}.$$

The Einstein–Weyl structure on $\mathbb{R} \times \mathbb{C} = \mathbb{R}^{1,2}$ is defined so that the image of each integral surface of $\underline{\mathcal{D}}$ by the projection $\underline{\mathfrak{p}}$ gives a totally geodesic null surface.

Notice that $\Pi_*(\mathfrak{m}_i) = \underline{\mathfrak{m}}_i$ for $i = 1, 2$. More precisely, if we consider $(\mathcal{Z}, \mathcal{Z}_{\mathbb{R}})$ as the trivial $\mathbb{R}$-bundle over $(\mathcal{W}, \mathcal{W}_{\mathbb{R}})$ with a fiber coordinate $s = \operatorname{Im} a$, then we obtain

$$\mathfrak{m}_1 = \underline{\mathfrak{m}}_1 - i\frac{\partial}{\partial s}, \qquad \mathfrak{m}_2 = \underline{\mathfrak{m}}_2 + i\omega\frac{\partial}{\partial s}. \tag{4.10}$$

5. Deformation: from twistors to space–times

In this section, we deform the standard twistor space $(\mathcal{T}, \mathcal{T}_{\mathbb{R}})$ and determine the corresponding self-dual conformal structures explicitly. In this article, we only consider $\mathbb{R}$-invariant deformation of the real twistor space $\mathcal{T}_{\mathbb{R}}$ by which the quotient space $(\mathcal{S}, \mathcal{C})$ is not deformed.

Deformation of the twistor space. Recall that the map $\pi : \mathcal{T} \to \mathcal{S}$ is considered as a trivial $\mathbb{C}$-bundle, and its restriction $\mathcal{T}|_{\mathcal{C}}$ is trivialized as $\mathcal{C} \times \mathbb{C} \xrightarrow{\sim} \mathcal{T}|_{\mathcal{C}}$ by

$$\left(e^{i\theta}, v; \nu\right) \longmapsto \left[e^{-\frac{i\theta}{2}} : e^{\frac{i\theta}{2}} : e^{\frac{i\theta}{2}}(v - i\nu) : e^{-\frac{i\theta}{2}}(v + i\nu)\right]. \tag{5.1}$$

By this notation, the real twistor space $\mathcal{T}_{\mathbb{R}}$ is written as

$$\mathcal{T}_{\mathbb{R}} = \left\{\left(e^{i\theta}, v; \nu\right) \in \mathcal{C} \times \mathbb{R}\right\} = \left\{\left(e^{i\theta}, v; \nu\right) \in \mathcal{T}|_{\mathcal{C}} \;\middle|\; \operatorname{Im}(\nu) = 0\right\}.$$

Now, for each smooth function $h(e^{i\theta}, v) \in C^\infty(\mathcal{C})$, we define a deformation $\mathcal{P}_h$ of $\mathcal{T}_{\mathbb{R}}$ by

$$\mathcal{P}_h = \left\{\left(e^{i\theta}, v; \nu\right) \in \mathcal{T}|_{\mathcal{C}} \;\middle|\; \operatorname{Im}(\nu) = h\left(e^{i\theta}, v\right)\right\}. \tag{5.2}$$

Notice that $\mathcal{P}_0 = \mathcal{T}_{\mathbb{R}}$ and that all the $\mathbb{R}$-invariant deformations of the twistor space fixing the quotient space $(\mathcal{S}, \mathcal{C})$ are written in this way.

Recall that we defined the integral transform $R : C^\infty(\mathcal{C}) \to C^\infty(\mathbb{R}^{1,2})$ in (2.5). Let $\check{d}$ and $\check{*}$ be the exterior derivative and Hodge's operator along the $\mathbb{R}^2$-direction of $(t, x) \in \mathbb{R} \times \mathbb{R}^2 = \mathbb{R}^{1,2}$, that is,

$$\check{d}u = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2,$$
$$\check{*}dx_1 = dx_2, \qquad \check{*}dx_2 = -dx_1 \quad \text{and so on.}$$

In this section, we will show the following.

Theorem 5.1. *For each $h \in C^\infty(\mathcal{C})$, there is a smooth $\mathbb{C}^2$-family of holomorphic disks on $(\mathcal{T}, \mathcal{P}_h)$. Moreover if $\partial_t Rh < 1$, a natural self-dual conformal structure on the parameter space $\mathbb{C}^2 = \mathbb{R} \times \mathbb{R}^{1,2}$ is induced and is represented by the metric*

$$g_{(V,A)} = -V^{-1}(ds + A)^2 + V\underline{g} \tag{5.3}$$

where (V, A) is the pair of the function $V \in C^\infty(\mathbb{R}^{1,2})$ and the 1-form $A \in \Omega^1(\mathbb{R}^{1,2})$ defined by

$$V = 1 - \partial_t Rh \quad \textit{and} \quad A = \check{*}\check{d}Rh. \tag{5.4}$$

Though the self-duality of $g_{(V,A)}$ defined above is deduced from the twistor construction, we can also check it directly in the following way. First notice that the metric of the form (5.3) is well studied, and the following proposition holds (see [5–7,12]).

Proposition 5.2. *The metric $g_{(V,A)}$ defined by (5.3) is self-dual if and only if (V, A) satisfies the non-degeneracy condition $V > 0$ and the monopole equation $*dV = dA$.*

On the other hand, the following proposition is easily checked.

Proposition 5.3. *For a function $u \in C^\infty(\mathbb{R}^{1,2})$, let us define*

$$V = 1 - \partial_t u \quad \text{and} \quad A = \check{*}\check{d}u. \tag{5.5}$$

Then (V, A) solves the monopole equation $*dV = dA$ if and only if u solves the wave equation $\square u = 0$.

Since $u = Rh$ solves the wave equation $\square u = 0$, we obtain the following.

Corollary 5.4. *The metric $g_{(V,A)}$ on $\mathbb{R}^4$ induced from a function $h \in C^\infty(\mathscr{C})$ by (5.3) and (5.4) is self-dual.*

Deformation of the holomorphic disks. In this part, we construct a family of holomorphic disks on $(\mathscr{T}, \mathscr{P}_h)$. Recall that we have a family of holomorphic disks $\{\underline{D}_{(t,z)}\}_{(t,z)\in\mathbb{R}^{1,2}}$ on $(\mathscr{S}, \mathscr{C})$ as in (4.6).

Proposition 5.5. *For each $(t, z) \in \mathbb{R}^{1,2}$, there is an $\mathbb{R}$-family of holomorphic disks $\{\mathcal{D}_{(s,t,z)}\}_{s\in\mathbb{R}}$ on $(\mathscr{T}, \mathscr{P}_h)$ satisfying $\pi(\mathcal{D}_{(s,t,z)}) = \underline{D}_{(t,z)}$.*

Proof. First, we lift the boundary circle $C_{(t,z)} = \partial\underline{D}_{(t,z)}$ to the real twistor space $\mathscr{P}_h$. Recall that $C_{(t,z)}$ can be written as (2.4). By the description (5.2) of $\mathscr{P}_h$, any lift of $C_{(t,z)}$ on $\mathscr{P}_h$ can be written as

$$\left\{\left(e^{i\theta}, t + \operatorname{Re}\left(ze^{-i\theta}\right), \kappa\left(e^{i\theta}\right) + iH\left(t, z; e^{i\theta}\right)\right) \in \mathscr{P}_h \;\middle|\; e^{i\theta} \in S^1\right\} \tag{5.6}$$

by using unknown real-valued function $\kappa(e^{i\theta}) \in C^\infty(S^1)$ and the function

$$H\left(t, z; e^{i\theta}\right) = h\left(e^{i\theta}, t + \operatorname{Re}\left(ze^{-i\theta}\right)\right).$$

Now we determine $\kappa(e^{i\theta})$ so that the circle (5.6) extends holomorphically to $|\omega| < 1$ by putting $\omega = e^{i\theta}$. By using the trivialization (5.1), this condition is satisfied if and only if the following two functions on $S^1 = \{|\omega| = 1\}$ extend holomorphically to $|\omega| \leqslant 1$:

$$\omega\left(\operatorname{Re}\left(z\omega^{-1}\right) - i\kappa(\omega) + H(t, z; \omega)\right) \quad \text{and} \quad \operatorname{Re}\left(z\omega^{-1}\right) + i\kappa(\omega) - H(t, z; \omega).$$

If we put $\tilde{\kappa}(\omega) = \kappa(\omega) + \operatorname{Im}(z\omega^{-1})$, these functions are written as

$$z - \omega\left(i\tilde{\kappa}(\omega) + H(t, z; \omega)\right) \quad \text{and} \quad \bar{z}\omega + i\tilde{\kappa}(\omega) - iH(t, z; \omega). \tag{5.7}$$

Now let us take the Fourier expansion

$$H(t, z; \omega) = \sum_{k=-\infty}^{\infty} H_k(t, z)\omega^k. \tag{5.8}$$

Since H is real-valued, we have $\overline{H_k(t, z)} = H_{-k}(t, z)$ for each integer $k \in \mathbb{Z}$. Let us put

$$H_\pm(t, z; \omega) = \sum_{k=1}^{\infty} H_{\pm k}(t, z)\omega^{\pm k}, \qquad u(t, z) = H_0(t, z). \tag{5.9}$$

Then the functions (5.7) extend holomorphically to $|\omega| \leqslant 1$ if and only if $\tilde{\kappa}(\omega)$ can be written as

$$\tilde{\kappa}(\omega) = s + i\left(H_+(t, z; \omega) - H_-(t, z; \omega)\right)$$

for some real constant s. Hence we obtain $\mathbb{R}$-family of holomorphic disks $\{\mathcal{D}_{(s,t,z)}\}_{s\in\mathbb{R}}$ of which the boundary $\partial\mathcal{D}_{(s,t,z)}$ is written as

$$\left\{\left(e^{i\theta}, t + \operatorname{Re}\left(ze^{-i\theta}\right), s - \operatorname{Im}\left(ze^{-i\theta}\right) + i\eta\right) \in \mathscr{P}_h \;\middle|\; e^{i\theta} \in S^1\right\} \tag{5.10}$$

where $\eta = \eta(t, z; \omega) = u(t, z) + 2H_+(t, z; \omega)$. By the trivialization (5.1), we obtain

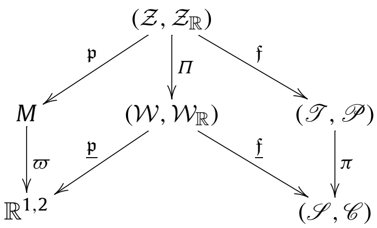
$$\mathcal{D}_{(s,t,z)} = \left\{\left[1 : \omega : (t - is + \eta)\omega + z : t + is - \eta + \bar{z}\omega\right] \in \mathscr{T} \;\middle|\; \omega \in \mathbb{D}\right\}. \qquad \square \tag{5.11}$$

Notice that the constant part u of the Fourier expansion of H is written as

$$u(t, z) = H_0(t, z) = \frac{1}{2\pi}\int_0^{2\pi} H\left(t, z; e^{i\theta}\right) d\theta = Rh(t, z)$$

by using the integral transform R defined in (2.5).

By the above proposition and the proof, we obtain a family of holomorphic disks $\{\mathcal{D}_{(s,t,z)}\}$ on $(\mathscr{T}, \mathscr{P}_h)$ parametrized by $(a,b) = (t+is, z) \in \mathbb{C}^2$. We write the parameter space by $M = \{(a,b) \in \mathbb{C}^2\}$ and we write $\mathcal{D}_{(a,b)} = \mathcal{D}_{(s,t,z)}$ when $(a,b) = (t+is, z)$. This family $\{\mathcal{D}_{(a,b)}\}$ is described by the following diagram:



Here $\mathfrak{p} : (\mathcal{Z}, \mathcal{Z}_{\mathbb{R}}) \to M$ is a trivial $(\mathbb{D}, \partial\mathbb{D})$-bundle, and the map $\mathfrak{f}$ is defined by

$$\mathfrak{f}(a,b;\omega) = \big[1 : \omega : \bar{a}\omega + b + \eta(\operatorname{Re} a, b; \omega) : \bar{b}\omega + a - \eta(\operatorname{Re} a, b; \omega)\big].$$

Notice that $\Pi : (\mathcal{Z}, \mathcal{Z}_{\mathbb{R}}) \to (\mathcal{W}, \mathcal{W}_{\mathbb{R}})$ is the trivial $\mathbb{R}$-bundle with fiber coordinate $s \in \mathbb{R}$. The holomorphic disks are given by $\mathcal{D}_{(a,b)} = \mathfrak{f}(\mathfrak{p}^{-1}(a,b))$.

Self-dual metrics and the monopoles. In this part, we determine the induced self-dual conformal structure on M in the sense of Proposition 3.1. First, the distribution $\mathcal{D} = \ker\{\mathfrak{f}_* : T\mathcal{Z}_{\mathbb{R}} \to T\mathscr{P}_h\}$ on $\mathcal{Z}_{\mathbb{R}}$ is determined in the following way. Here $\underline{\mathfrak{m}}_1$ and $\underline{\mathfrak{m}}_2$ are as in (4.9).

Proposition 5.6. *The distribution $\mathcal{D} \otimes \mathbb{C}$ is spanned by complex tangent vector fields*

$$\mathfrak{m}_1 = \underline{\mathfrak{m}}_1 - i(1 - u_t - 2\omega u_z)\frac{\partial}{\partial s} \quad \textit{and} \quad \mathfrak{m}_2 = \underline{\mathfrak{m}}_2 + i\big(\omega(1-u_t) - 2u_{\bar{z}}\big)\frac{\partial}{\partial s}. \tag{5.12}$$

Notice that the relation (4.10) of the standard case is recovered when $h = 0$, that is, when $u = Rh = 0$.

Proof. By construction, we have $\Pi_*(\mathcal{D}) = \underline{\mathcal{D}}$. Hence the complex distribution $\mathcal{D} \otimes \mathbb{C}$ is spanned by the vectors of the form

$$\mathfrak{m}_j = \underline{\mathfrak{m}}_j + \mathfrak{a}_j \frac{\partial}{\partial s} \quad (j = 1, 2).$$

Since the function $H(t, z; e^{i\theta}) = h(e^{i\theta}, t + \operatorname{Re}(ze^{-i\theta}))$ is constant along $\mathfrak{m}_1$ and $\mathfrak{m}_2$, we obtain $\mathfrak{m}_i H \equiv 0$ for $i = 1, 2$. By the expansion (5.8), we obtain

$$-\frac{\partial H_k}{\partial t} + 2\frac{\partial H_{k-1}}{\partial z} = 0, \qquad -\frac{\partial H_k}{\partial t} + 2\frac{\partial H_{k+1}}{\partial \bar{z}} = 0.$$

Using these relation, we obtain

$$\underline{\mathfrak{m}}_1 \eta = -u_t - 2\omega u_z, \qquad \underline{\mathfrak{m}}_2 \eta = \omega u_t + 2u_{\bar{z}}.$$

Then, by $\mathfrak{m}_j((t - is + \eta)\omega + z) = 0$ we obtain $\mathfrak{a}_1 = -i(1 + \underline{\mathfrak{m}}_1 \eta)$ and $\mathfrak{a}_2 = i(\omega - \underline{\mathfrak{m}}_2 \eta)$. □

Lemma 5.7. *The conditions* A1, A2, *and* A3 *are satisfied if and only if $u_t \neq 1$.*

Proof. By the description (5.12) of $\mathfrak{m}_j$, we can easily check that the conditions A1 and A2 hold if and only if $u_t \neq 1$. On the other hand, A3 always holds since $\mathcal{E}$ is obtained as a continuous deformation from the standard model. □

Lemma 5.8. *Suppose $1 - u_t > 0$ on M, then the induced self-dual conformal structure on M is represented by the indefinite metric $g_{(V,A)}$ as defined in (5.3) with $V = 1 - u_t$ and $A = \check{*}\check{d}u$.*

Proof. As explained in Section 3, there exists unique self-dual conformal structure on M defined by $\mathcal{D}$. Hence it is enough to check that the metric $g = g_{(V,A)}$ satisfies $g(\mathfrak{m}_j, \mathfrak{m}_k) = 0$ for every j and k, which is directly checked. □

Thus the proof of Theorem 5.1 is completed.

6. From space–times to twistors

In this section, we establish the converse correspondence, that is, we start from self-dual metrics on $\mathbb{R}^4$ and determine the corresponding twistor spaces. This is directly deduced from Theorem 2.1 under a rapidly decreasing assumption.

Self-dual metrics from monopoles. Let us start from the metric on $\mathbb{R}\times\mathbb{R}^{1,2}=\{(s,t,x)\}$ of the form

$$g_{(V,A)}=-V^{-1}(ds+A)^2+V\underline{g}$$

where $V\in C^\infty(\mathbb{R}^{1,2})$ is a function, $A\in\Omega^1(\mathbb{R}^{1,2})$ is a 1-form, and $\underline{g}$ is the flat Lorentz metric on $\mathbb{R}^{1,2}$ given by (4.5). As in Proposition 5.2, $g_{(V,A)}$ is self-dual if and only if $V>0$ and $*d\underline{V}=dA$. We call a solution to the monopole equation $*dV=dA$ just a monopole.

Two pairs (V,A) and (V',A') are called *gauge equivalent* if and only if $V'=V$ and $A'=A+d\phi$ for some function ϕ on $\mathbb{R}^{1,2}$. Notice that the metrics $g_{(V,A)}$ and $g_{(V',A')}$ are isometric if (V,A) and (V',A') are gauge equivalent. Now recall that we write $M_0=\{(0,x)\in\mathbb{R}^{1,2}\}$.

Proposition 6.1. *Let (V',A') is a monopole on $\mathbb{R}^{1,2}$. Suppose that the restrictions $(V'-1)|_{M_0}$ and $A'|_{M_0}$ are rapidly decreasing on M_0. Then there exists a pair of a unique function $u\in C^\infty(\mathbb{R}^{1,2})$ and a unique monopole (V,A) which is gauge equivalent to (V',A') such that*

- *$V=1-u_t$ and $A=\check{*}\check{d}u$,*
- *u solves the wave equation $\square u=0$, and*
- *the restrictions $u|_{M_0}$ and $u_t|_{M_0}$ are rapidly decreasing functions on M_0.*

Proof. This is proved in a completely similar way as the de Sitter case (Propositions 4.3 and 4.4 in [12]). Just one point which we should care is to determine a function $\check{\phi}\in C^\infty(M_0)$ satisfying $\Delta_{M_0}\check{\phi}=-\check{*}\check{d}\check{*}A$, which is cleared by the rapidly decreasing condition. $\square$

Summarizing all, we obtain the following.

Theorem 6.2. *Let (V,A) be a monopole on $\mathbb{R}^{1,2}$ such that the restrictions $(V-1)|_{M_0}$ and $A|_{M_0}$ are rapidly decreasing on M_0. Changing (V,A) by a uniquely determined gauge transform, we can write $V=1-u_t$ and $A=\check{*}\check{d}u$ by a unique solution $u\in C^\infty(\mathbb{R}^{1,2})$ to the wave equation $\square u=0$ satisfying the rapidly decreasing condition $u|_{M_0},u_t|_{M_0}\in\mathcal{S}(M_0)$. Further if $V>0$, then the metric $g_{(V,A)}$ on $\mathbb{R}\times\mathbb{R}^{1,2}$ is self-dual and is obtained from the twistor space $(\mathscr{T},\mathscr{P}_h)$ by the twistor construction where the function $h\in C^\infty(\mathscr{C})$ is determined from u by Theorem 2.1.*

Acknowledgements

The author would like to thank Claude LeBrun for providing helpful ideas, and Simons Center SUNY for its hospitality in the summer 2011. He would also like to thank Mitsuji Tamura for helpful conversations and comments for hyperbolic PDEs.

References

[1] L. Besse, Einstein Manifolds, Classics Math., Springer, 1987.
[2] S. Helgason, The Radon Transform, second edition, Progr. Math., vol. 5, Birkhäusar, 1999.
[3] N.J. Hitchin, Complex manifolds and Einstein's equations, in: Twistor Geometry and Non-Linear Systems, in: Lecture Notes in Math., vol. 970, 1982.
[4] P.E. Jones, K.P. Tod, Minitwistor spaces and Einstein–Weyl spaces, Class. Quantum Gravity 2 (1985) 565–577.
[5] H. Kamada, Self-dual Kähler Metrics of Neutral Signature on Complex Surfaces, Tohoku Math. Publ., vol. 24, Tohoku University, Sendai, 2002.
[6] H. Kamada, Compact scalar-flat indefinite Kähler surfaces with Hamiltonian S^1-symmetry, Comm. Math. Phys. 254 (2005) 23–44.
[7] C. LeBrun, Explicit self-dual metrics on $\mathbb{CP}_2\sharp\cdots\sharp\mathbb{CP}_2$, J. Differential Geom. 34 (1991) 223–253.
[8] C. LeBrun, Twistors, holomorphic disks, and Riemann surfaces with boundary, in: Perspectives in Riemannian Geometry, in: CRM Proc. Lecture Notes, vol. 40, Amer. Math. Soc., Providence, RI, 2006, pp. 209–221.
[9] C. LeBrun, L.J. Mason, Nonlinear gravitons, null geodesics, and holomorphic disks, Duke Math. J. 136 (2) (2007) 205–273.
[10] C. LeBrun, L.J. Mason, The Einstein–Weyl equations, scattering maps, and holomorphic disks, Math. Res. Lett. 16 (2009) 291–301.
[11] F. Nakata, A construction of Einstein–Weyl spaces via LeBrun–Mason type twistor correspondence, Comm. Math. Phys. 289 (2009) 663–699.
[12] F. Nakata, Wave equations and LeBrun–Mason correspondence, Trans. Amer. Math. Soc. 364 (2012) 4763–4800.
[13] K.P. Tod, Indefinite conformally-ASD metric on $S^2\times S^2$, in: Further Advances in Twistor Theory, vol. III, Chapman & Hall/CRC, 2001, pp. 61–63, reprinted from Twistor Newsletter 36 (1993).