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Discrete Delay, Distributed Delay and Stability Switches

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1. INTRODUCTION

In modelling in the biological, physical and social sciences, it is sometimes necessary to take account of time delays inherent in the phenomena. The inclusion of delays explicitly in the equations is often a simplification or idealization that is introduced because a detailed description of the underlying processes is too complicated to be modelled mathematically, or because some of the details are unknown. In these cases, it may be necessary to choose between a model with discrete or sharp delays and a model with distributed or continuous delay. A question of great importance is whether two models with parallel structure, one with discrete delay and one with distributed delay, will exhibit the same qualitative modes of behaviour. More generally, how does the qualitative behaviour depend on the form and magnitude of the delays? In this paper we shall examine certain aspects of this question.

The paper is divided into two parts. In the first part (Sections 2–6), we examine how the stability properties of certain models change when the delay is increased. It has frequently been observed that stability of an equilibrium may be lost when delays are increased. Less frequently, it has been seen that further increase in the delay may result in restabilization. In this paper, we examine the possibilities for several simple equations: (1) a first order linear differential-difference equation; (2) a second order delayed friction model; (3) a second order equation with delayed restoring force; and (4) a population growth model of J. Cushing. In (2) and (3) and in a general equation including both, we show that there may be arbitrarily many switches from stability to instability to stability as the delay is increased, but in (1) this is not possible. In (4), the equation has distributed delay, and

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much of the existing analysis concerns approximations by special kernels that make the equation equivalent to a set of ordinary differential equations. This is a distributed delay version of (1). We show that in contrast to (1), a switch from stability to instability must be followed by a single switch back to stability as the mean delay is increased.

In the second part of this paper (Sections 7–10), we relate the special kernels mentioned above to the transfer functions and frequency response associated with the corresponding set of linear equations. In particular we show that interpreting a linear system as generating frequency dependent delays can simplify the calculation of limit cycle solutions by Poincaré-type expansions. This is illustrated in the example of Goodwin's model.

2. First Order Equation with Discrete Delay

Consider the equation:

$$dx/dt = ax(t) + bx(t-\tau), \tag{1}$$

with constant delay $\tau > 0$. It is well-known (see Bellman and Cooke, [2]) that the stability of the zero equilibrium solution depends on the roots of the transcendental equation

$$\lambda - a - be^{-\tau\lambda} = 0. \tag{2}$$

If we let $z = \tau\lambda$, $p = a\tau$, $q = b\tau$, this equation is equivalent to

$$pe^{z} + q - ze^{z} = 0. \tag{3}$$

By [2, Theorem 13.8], a necessary and sufficient condition in order that all roots of Eq. (3) have negative real parts is that

(i) $p < 1$ and

(ii) $p < -q < (\theta^2 + p^2)^{1/2}$,

where θ is the unique root of $\theta = p \tan\theta$, $0 < \theta < \pi$, or $\theta = \pi/2$ if $p = 0$. By condition (i) we must have $a\tau < 1$ and by (ii) we must have $(a+b)\,\tau < 0$, or $a + b < 0$, and also

$$-b\tau < (\theta^2 + a^2\tau^2)^{1/2}.$$

If $b > 0$, these conditions reduce to $a\tau < 1$, $a + b < 0$; hence if $a \geqslant -b$ or if $a \geqslant 1/\tau$, there is instability. If there is stability when $\tau = 0$, that is, for the ordinary equation $dx/dt = (a + b)x$, then stability holds for all positive τ since $a < 0$.

If $b < 0$ (the trivial case $b = 0$ is omitted), the stability conditions are

$$a\tau < 1, \qquad a < |b| < [a^2 + (\theta/\tau)^2]^{1/2}.$$

If stability holds for $\tau = 0$, then it holds for larger τ until either $\tau = 1/a$ or until the first positive root of

$$|b| = [a^2 + (\theta/\tau)^2]^{1/2}. \tag{4}$$

Since $\tan\theta = \theta/(a\tau)$, we see that θ decreases as τ increases (for a either positive or negative), hence θ/τ decreases as τ increases. Consequently, when $a < |b|$ there is a unique root τ of Eq. (4), and for larger values of τ condition (ii) is violated.

We may therefore summarize as follows. *If Eq.* (1) *is stable for* $\tau = 0$ (*that is*, $a + b < 0$), *then either it is stable for all* $\tau \geqslant 0$, *or else there is a value* τ^* *such that it is stable for* $\tau < \tau^*$ *and unstable for* $\tau > \tau^*$. There is no possibility of restabilization for large τ.

This same conclusion can be reached in a different way. For a retarded equation, the supremum of the real parts of the roots of the transcendental equation varies continuously with τ (see Datko [7]). Therefore, if there is a transition from stability to instability, or the reverse, as τ varies, it must correspond to a purely imaginary root $\lambda = i\omega$. Any purely imaginary root must be simple, since at a multiple root one must have (2) and $1 + b\tau e^{-\tau\lambda} = 0$, which imply that $\lambda = a - 1/\tau$. It then follows from standard arguments, since $\lambda - a - be^{-\tau\lambda}$ is an analytic function of λ and τ, that a root $\lambda(\tau)$ is a differentiable function of τ near $i\omega$. From the characteristic equation (2), we have

$$\frac{d\lambda}{d\tau} = -\frac{be^{-\tau\lambda}\lambda}{1 + b\tau e^{-\tau\lambda}}.$$

At a value of τ (if any) for which $\lambda = i\omega$, we have

$$\frac{d\lambda}{d\tau} = -\frac{bi\omega e^{-i\omega\tau}}{1 + b\tau e^{-i\omega\tau}},$$

$$Re\left(\frac{d\lambda}{d\tau}\right) = \frac{d}{d\tau}(Re\lambda) = \frac{-b\omega\sin\omega\tau}{(1 + b\tau\cos\omega\tau)^2 + (b\tau\sin\omega\tau)^2}.$$

On the other hand, at a root $\lambda = i\omega$ we must have

$$i\omega - a - be^{-i\omega\tau} = 0.$$

Therefore, $b \sin \omega\tau = -\omega$, $b \cos \omega\tau = -a$, and

$$\frac{d}{d\tau}(Re\lambda) = \frac{\omega^2}{(1 - a\tau)^2 + (\omega\tau)^2}.$$

If $\omega \neq 0$, this derivative is positive, and so the root must pass from the negative to the positive half-plane as τ increases. On the other hand, $\omega = 0$ corresponds to a zero root $\lambda = 0$, which is impossible since we have assumed that $a + b \neq 0$. Thus, roots can cross the imaginary axis only from left to right as τ increases. If stability is lost at some critical value of τ (or does not exist for $\tau = 0$), it can never be regained.

Brauer [4] has also studied the dependence of stability for Eq. (1) on the delay τ. He defines the *characteristic return time* to the equilibrium $x = 0$ (when the equilibrium is stable) to be $-\tau/\sigma_{max}$, where σ_{max} is the real part of the characteristic root λ_{max} of largest real part. His principal results are as follows:

(i) If $b > 0$, $a + b < 0$, the trivial solution of (i) is asymptotically stable for all τ, and the characteristic return time is a monotone increasing function of τ.

(ii) If $b < 0$, the characteristic return time is a decreasing function of τ for $0 \leqslant \tau < \tau^*$, where τ^* is defined by

$$-b\tau^* e^{-a\tau^*} = e^{-1}.$$

It is an increasing function of τ for $\tau > \tau^*$, remaining finite for all τ for which the zero solution is asymptotically stable.

Brauer also considered nonlinear equations of the type

$$x'(t) = F[x(t - \tau)],$$

where $F(0) = 0$ and $r = -F'(0) > 0$. If $r\tau < \pi/2$, the zero solution is asymptotically stable, while if $r\tau > \pi/2$, $x = 0$ is unstable but there is a periodic solution. The linearization around $x = 0$ is of the form (1) with $a = 0$, $b = -r$. Brauer shows that if $0 < r\tau < 0.63336$, the characteristic return time is less than for $\tau = 0$, so that it may be said that a delay in this range tends to stabilize the system, even though a larger delay destroys stability of the equilibrium. Other extensions may be found in [3].

3. Delayed Friction

We consider the equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + b\frac{dx(t - \tau)}{dt} + cx(t) = 0, \tag{5}$$

where a, b are non-negative constants, and $c > 0$, $a + b > 0$, $\tau > 0$. This equation was studied by Minorsky [19, 20] in connection with studies of ship stabilization. Also, Minorsky suggested it as a model for the small vibrations of a pendulum of mass one, where x is the displacement from the equilibrium position, $cx(t)$ is the linear restoring force, $a\dot{x}(t)$ is the natural frictional force, and $b\dot{x}(t-\tau)$ is a frictional force introduced with a time delay τ. The stability of the same equation was subsequently studied by Pinney [24]. We shall show here that for any fixed a, b, c, with $a < b$, as τ is increased from 0 to ∞ the zero solution of the equation is alternately stable, unstable, and stable again. This cycle from stability to instability to stability can occur any finite number of times, but ultimately instability persists. This phenomenon was apparently detected by Ansoff and Krumhansl [1] by a method different from ours. It was observed by Mufti [22] for third order equations with delay in the zero-order term. These authors did not emphasize the notion of continuous variation of the delay and the resulting cycling of stability and instability. In the present paper, we want to direct attention to this phenomenon, which is not now widely known. In Section 6, we apply similar ideas to an integro-differential equation that arises from a population growth model.

In the discussion that follows, we shall use the following simple lemma.

LEMMA. *Let $f(\lambda, \tau) = \lambda^2 + a\lambda + b\lambda e^{-\tau\lambda} + c + de^{-\tau\lambda}$, where a, b, c, d, τ are real numbers and $\tau \geqslant 0$. Then, as τ varies, the sum of the multiplicities of zeros of f in the open right half-plane can change only if a zero appears on or crosses the imaginary axis.*

Proof. Let $\lambda = \lambda(\tau)$ be any root of $f(\lambda, \tau) = 0$. If we place a small disk around $\lambda(\tau)$, then for τ' sufficiently close to τ, the total multiplicity of roots in the disk equals the multiplicity of $\lambda(\tau)$. This follows from Rouché's theorem; see Dieudonné [8, Theorem 9.17.4]. In this sense, a root $\lambda(\tau)$ cannot suddenly disappear or appear or change its multiplicity at a finite point in the plane. Let $M(\tau)$ be the total (finite) multiplicity of zeros in the open right half-plane. Suppose that $M(\tau)$ changes but no roots appear on or cross the imaginary axis. This could only occur due to the appearance of a root at $\lambda = \infty$. That is, there would exist τ^* and a root $\lambda(\tau)$ such that $|\lambda(\tau)| \to \infty$ as $\tau \to \tau^* + 0$ (or $\tau \to \tau^* - 0$), with $\operatorname{Re} \lambda(\tau) \geqslant 0$. But then since $|e^{-\tau\lambda(\tau)}| \leqslant 1$, we deduce that

$$f(\lambda, \tau)/\lambda^2 = 1 + \lambda^{-2}(a\lambda + b\lambda e^{-\tau\lambda} + c + de^{-\tau\lambda})$$

tends to 1 as $|\lambda(\tau)| \to \infty$. This contradicts $f(\lambda(\tau), \tau) = 0$, and completes the proof.

The characteristic equation associated with Eq. (5) is

$$\lambda^2 + a\lambda + b\lambda e^{-\tau\lambda} + c = 0. \tag{6}$$

Suppose that $\lambda = i\omega$ is a purely imaginary root. It suffices to seek solutions with $\omega > 0$ since $\lambda = 0$ is not a root when $c > 0$ and since complex roots occur in conjugate pairs. We have

$$\omega^2 - c - b\omega \sin \tau\omega = 0, \qquad \omega(a + b \cos \tau\omega) = 0. \tag{7}$$

We shall now examine three cases.

Case 1. $a > b \geqslant 0$. The equation $a + b \cos \tau\omega = 0$ has no real solution ω in this case. Consequently, Eq. (6) has no roots on the imaginary axis. Following the line of argument used in the latter part of Section 2, we reason that the supremum of real parts of the roots varies continuously with τ. For $\tau = 0$, $a + b > 0$, $c > 0$, both roots lie in the left half-plane. Consequently, all roots satisfy $\text{Re}(\lambda) < 0$ for all $\tau \geqslant 0$, and the zero solution of Eq. (5) is stable for every positive τ.

Case 2. $0 \leqslant a < b$. From Eq. (7) we obtain

$$\tau\omega = \cos^{-1}(-a/b) + 2n\pi \qquad (n = 0, 1, 2,...).$$

If we define

$$\theta_1 = \cos^{-1}(-a/b), \qquad \pi/2 \leqslant \theta_1 < \pi,$$
$$\theta_2 = \cos^{-1}(-a/b), \qquad \pi < \theta_2 < 3\pi/2$$

then for the choice θ_1 we obtain

$$\sin \tau\omega = \sin(\theta_1 + 2n\pi) = [1 - (a/b)^2]^{1/2}.$$

Therefore,

$$\omega^2 - (b^2 - a^2)^{1/2}\, \omega - c = 0.$$

The only positive root is

$$\omega_1 = \tfrac{1}{2}(b^2 - a^2)^{1/2} + \tfrac{1}{2}(b^2 - a^2 + 4c)^{1/2}.$$

If we choose θ_2 then the sign of $\sin \tau\omega$ is reversed and we have

$$\omega^2 + (b^2 - a^2)^{1/2}\, \omega - c = 0.$$

The only positive root is

$$\omega_2 = -\tfrac{1}{2}(b^2 - a^2)^{1/2} + \tfrac{1}{2}(b^2 - a^2 + 4c)^{1/2}.$$

We see that we obtain two sequences of positive values of τ corresponding to pure imaginary roots. These are:

(a) $\tau_{n,1} = (\theta_1 + 2n\pi)/\omega_1 \quad (n = 0, 1, 2, \ldots)$,

(b) $\tau_{n,2} = (\theta_2 + 2n\pi)/\omega_2 \quad (n = 0, 1, 2, \ldots)$.

We can show that each of these roots is simple, for if $F(\lambda, \tau)$ is the function in Eq. (6) then $\partial F/\partial\lambda = 0$ takes the form

$$2\lambda + a + (b - b\tau\lambda)\, e^{-\tau\lambda} = 0.$$

Solving Eq. (6), for $\lambda \neq 0$, for $e^{-\tau\lambda}$, and substituting, we obtain

$$\tau\lambda^3 + (a\tau + 1)\,\lambda^2 + c\tau\lambda - c = 0.$$

For $\lambda = i\omega$, this implies $(a\tau + 1)\,\omega^2 + c = 0$, which is impossible. Thus, any root on the imaginary axis is simple. We may argue as in Section 2 that any root $\lambda(\tau)$ is differentiable with respect to τ at a value of τ for which $\lambda = i\omega$.

Continuing with Case 2, we compute $d\lambda/d\tau$ from Eq. (6), obtaining

$$\frac{d\lambda}{d\tau} = \frac{b\lambda^2 e^{-\tau\lambda}}{2\lambda + a + be^{-\tau\lambda} - b\tau\lambda e^{-\tau\lambda}} = \frac{b\lambda^2}{(2\lambda + a)\, e^{\tau\lambda} + b - b\tau\lambda}.$$

The real part when $\lambda = i\omega$ is

$$\frac{d}{d\tau}(\operatorname{Re}\lambda) = \operatorname{Re}\frac{d\lambda}{d\tau} = -b\omega^2(a\cos\tau\omega - 2\omega\sin\tau\omega + b)\, D^{-1},$$

where

$$D = (a\cos\tau\omega - 2\omega\sin\tau\omega + b)^2 + (2\omega\cos\tau\omega + a\sin\tau\omega - b\tau\omega)^2.$$

When $\tau = \tau_{n,1}$ and $\omega = \omega_1$, we have $\cos\tau\omega = -a/b$, $\sin\tau\omega = (1 - a^2/b^2)^{1/2}$ and so

$$\frac{d}{d\tau}(\operatorname{Re}\lambda) = -\omega_1^2[-a^2 - 2\omega_1(b^2 - a^2)^{1/2} + b^2]\, D^{-1}.$$

Since $D > 0$ and $2\omega_1(b^2 - a^2)^{1/2} > b^2 - a^2$, we see that $d(\operatorname{Re}\lambda)/d\tau > 0$. That is, at all those roots, the root crosses the imaginary axis from left to right as τ increases. On the other hand, when $\tau = \tau_{n,2}$ and $\omega = \omega_2$, we have $\cos\tau\omega = -a/b$, $\sin\tau\omega = -(1 - a^2/b^2)^{1/2}$ and

$$\frac{d}{d\tau}(\operatorname{Re}\lambda) = -\omega_2^2[-a^2 + 2\omega_2(b^2 - a^2)^{1/2} + b^2]\, D^{-1}.$$

Since $\omega_2 > 0$ and $b > a$, and again $D > 0$, we see that $d(\operatorname{Re}\lambda)/d\tau < 0$. Thus, all the roots that cross the imaginary axis at $i\omega_2$ cross from right to left as τ increases.

The total multiplicity of roots in the right half-plane can only change when a root crosses the imaginary axis, by the lemma. Observe that the multiplicity is zero for $0 \leqslant \tau < \tau_{0,1}$ and is two for $\tau_{0,1} < \tau < \tau_{0,2}$. Consequently, we have stability for $0 \leqslant \tau < \tau_{0,1}$ and instability for $\tau_{0,1} < \tau < \tau_{0,2}$. We shall now prove the following two propositions.

I. *For any fixed a, b and c with $a < b$ there exists an integer k such that*

$$\tau_{0,1} < \tau_{0,2} < \tau_{1,1} < \tau_{1,2} < \cdots < \tau_{k-1,1} < \tau_{k-1,2} < \tau_{k,1} \tag{8}$$

and consequently as τ varies from 0 *to $\tau_{k,1}$, we have, alternately, switching from stability to instability and back to stability k times, and moreover, the system is unstable for all $\tau > \tau_{k,1}$.*

II. *For any specified positive integer k there are parameter values a, b, c with $a < b$ for which* I. *holds. This can be achieved with $a = 0$.*

In order to prove these results, we must examine the conditions under which the two sequences $\{\tau_{n,1}\}$ and $\{\tau_{n,2}\}$ alternate as in I. We note immediately that since

$$\tau_{n+1,1} - \tau_{n,1} = 2\pi/\omega_1, \qquad \tau_{n+1,2} - \tau_{n,2} = 2\pi/\omega_2$$

and $\omega_2 < \omega_1$, this alternation cannot persist for the whole of the sequences. Eventually, therefore, there is an integer k such that

$$\tau_{k-1,1} < \tau_{k-1,2} < \tau_{k,1} < \tau_{k+1,1} < \tau_{k,2}. \tag{9}$$

Hence for $\tau > \tau_{k,1}$, the multiplicity of roots in the right half-plane is at least two and the system is unstable.

To prove II., we only need to show that $2\pi/\omega_1$ and $2\pi/\omega_2$ can be in any desired ratio. We shall, in fact, obtain specific inequalities on a, b, c sufficient to guarantee that (8) and (9) hold. We are looking for conditions under which the first k terms of the sequences $\{\tau_{n,1}\}$ and $\{\tau_{n,2}\}$ alternate, but $\tau_{k-1,2} < \tau_{k,1} < \tau_{k+1,1} < \tau_{k,2}$. Let $\theta_1 = \pi/2 + z$, $\theta_2 = 3\pi/2 - z$ $(0 \leqslant z < \pi/2)$. Since $\theta_1 < \theta_2$ and $\omega_1 < \omega_2$, we need only consider the inequalities

$$\frac{\theta_2}{\omega_2} < \frac{\theta_1 + 2\pi}{\omega_1},$$

$$\frac{\theta_2 + 2\pi}{\omega_2} < \frac{\theta_1 + 4\pi}{\omega_1}, \ldots, \frac{\theta_2 + 2(k-1)\pi}{\omega_2} < \frac{\theta_1 + 2k\pi}{\omega_1}$$

and

$$\frac{\theta_1 + 2(k+1)\pi}{\omega_1} < \frac{\theta_2 + 2k\pi}{\omega_2}.$$

Equivalently, consider

$$\frac{\omega_1}{\omega_2} < \frac{\frac{5}{2}\pi + z}{\frac{3}{2}\pi - z}, \qquad \frac{\omega_1}{\omega_2} < \frac{\frac{9}{2}\pi + z}{\frac{7}{2}\pi - z}, \ldots, \frac{\omega_1}{\omega_2} < \frac{(2k+\frac{1}{2})\pi + z}{(2k-\frac{1}{2})\pi - z}$$

and

$$\frac{\omega_1}{\omega_2} > \frac{(2k+\frac{5}{2})\pi + z}{(2k+\frac{3}{2})\pi - z}.$$

Since the function

$$\frac{(2n+\frac{1}{2})\pi + z}{(2n-\frac{1}{2})\pi - z}$$

is a strictly decreasing function of n for fixed $z \geqslant 0$ ($n \geqslant 1$), it is equivalent to require merely

$$\frac{\omega_1}{\omega_2} < \frac{(2k+\frac{1}{2})\pi + z}{(2k-\frac{1}{2})\pi - z} \quad \text{and} \quad \frac{\omega_1}{\omega_2} > \frac{(2k+\frac{5}{2})\pi + z}{(2k+\frac{3}{2})\pi - z}.$$

Substitution of the expression for ω_1 and ω_2 transforms these inequalities to

$$4k\pi(b^2 - a^2)^{1/2} < (\pi + 2z)(b^2 - a^2 + 4c)^{1/2}$$

and

$$4(k+1)\pi(b^2 - a^2)^{1/2} > (\pi + 2z)(b^2 - a^2 + 4c)^{1/2}.$$

These lead to the conditions

$$\frac{(4k\pi)^2 - (\pi + 2z)^2}{4(\pi + 2z)^2}(b^2 - a^2)$$
$$< c < \frac{4^2(k+1)^2\pi^2 - (\pi + 2z)^2}{4(\pi + 2z)^2}(b^2 - a^2).$$

Clearly, for any specified integer $k \geqslant 1$, and any real non-negative numbers a and b with $0 \leqslant a < b$, there is a range of values c for which these inequalities are satisfied. This proves II.

Case 3. $a = b$. Equation (7) becomes

$$\omega^2 - c - a\omega \sin \tau\omega = 0, \qquad \cos \tau\omega = -1.$$

Thus, $\sin \tau\omega = 0$, $\omega^2 = c$. The positive root is $\omega = \sqrt{c}$ and the critical values of τ are $\tau_n = (2n+1)\,\pi/\sqrt{c}$ $(n = 0, 1, 2, \ldots)$.

In this case, it can be shown that $\mathrm{Re}(d\lambda/d\tau) = 0$ at the critical values $\lambda = i\omega$. Therefore, it is necessary to compute $d^2\lambda/d\tau^2$ in order to determine whether the roots actually cross the axis. We forego this calculation here, since our main point has already been made in Case 2.

The following physical interpretation may be given to these results. First, if $a > b$, the larger part of the damping is not delayed, and the above results show that the zero solution remains stable, whatever be the delay in the delayed part of the damping. On the other hand, suppose that $a < b$, so that the delayed damping is larger than the undelayed damping. Then there are intervals of values of τ for which the zero solution is stable and intervals for which it is unstable. Instability occurs, speaking in engineering language, when the delay is of a size to cause a reversal of phase, so that the damping term in effect is reinforcing the vibrations instead of damping them. The larger is the restoring force c, the greater is the number of phase reversals permitting stability, but in all cases there is eventually instability for τ sufficiently large.

Finally, returning to Eq. (5), we consider the case in which the zero solution is unstable for $\tau = 0$. The interesting question is whether a switch to stability and back to instability, once or more, is possible as τ is being increased. We will show that this is indeed the case. Instability for $\tau = 0$ implies either $a + b < 0$ or $c < 0$. Consider the case with $a + b < 0$, $c > 0$. By Eq. (7), Eq. (6) has no imaginary root unless $|b| > |a|$. $a + b < 0$ is then fulfilled only if $b < 0$. Assuming $a > 0$, and defining $\theta_1 = \cos^{-1}(-a/b)$, $0 < \theta_1 < \pi/2$, and $\theta_2 = \cos^{-1}(-a/b)$, $3\pi/2 < \theta_2 < 2\pi$, we find from Eq. (7), $\omega^2 + (b^2 - a^2)^{1/2}\,\omega - c = 0$, or $\omega_1 = -\frac{1}{2}(b^2 - a^2)^{1/2} + \frac{1}{2}(b^2 - a^2 + 4c)^{1/2}$ for the choice θ_1; $\omega^2 - (b^2 - a^2)^{1/2}\,\omega - c = 0$, or $\omega_2 = \frac{1}{2}(b^2 - a^2)^{1/2} + \frac{1}{2}(b^2 - a^2 + 4c)^{1/2}$ for the choice θ_2. Corresponding to ω_1 and ω_2 there are sets $\tau_{n,1} = (\theta_1 + 2\pi n)/\omega_1$ and $\tau_{n,2} = (\theta_2 + 2\pi n)/\omega_2$, respectively. We note that $\theta_1/\theta_2 < \frac{1}{3}$. Although $\omega_1 < \omega_2$, for any given a and b satisfying the conditions specified above for this case one can choose $c > 0$ sufficiently large so that $1 > \omega_1/\omega_2 > \frac{1}{3}$ and $\tau_{0,1} = \theta_1/\omega_1 < \theta_2/\omega_2 = \tau_{0,2}$. As shown before, the total multiplicity of the roots for which $\mathrm{Re}\,\lambda > 0$ is increased by two as τ is increased and passes through any value of $\tau_{n,2}$, corresponding to the larger root ω_2, and is decreased by two at $\tau_{n,1}$. Since the multiplicity for $0 \leqslant \tau < \tau_{0,1}$ is 2, a switch to stability occurs at $\tau_{0,1}$ and a switch back to instability at $\tau_{0,2}$. Since the intervals between subsequent values of τ within the sets obey $2\pi/\omega_1 < 2\pi/\omega_2$, only a finite number of such switches can occur. For all sufficiently large τ, instability prevails.

The situation is similar if both $b < 0$ and $a < 0$. For $a + b > 0$ and $c < 0$, the condition $|b| > |a|$ implies $b > 0$ and then the larger ω is associated with the smaller θ, so that the first crossing of the imaginary axis increases the multiplicity of roots with $\operatorname{Re} \lambda > 0$. As τ is further increased, crossings with an increase in multiplicity occur more frequently than crossings in the other direction, so that the zero solution remains unstable.

4. Delayed Restoring Force

Consider the following equation, which models a process with instantaneous damping but delayed restoring force.

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx(t) + cx(t-\tau) = 0 \qquad (a, b, c > 0). \tag{10}$$

Let us see whether the switching of stability occurs as it did in the case of delayed friction. The characteristic equation is

$$\lambda^2 + a\lambda + b + ce^{-\tau\lambda} = 0.$$

If $\lambda = i\omega$ is a root, then $\omega \neq 0$ and $\omega^2 - b = c \cos \tau\omega$, $a\omega = c \sin \tau\omega$. Thus

$$\omega^4 + (a^2 - 2b)\,\omega^2 + b^2 - c^2 = 0.$$

If $(a^2 - 2b)^2 < 4(b^2 - c^2)$, there is no positive solution ω^2. If $\Delta = (a^2 - 2b)^2 - 4(b^2 - c^2) > 0$, or if $\Delta = 0$ and $2b > a^2$, then there may be positive solutions

$$\omega^2 = \tfrac{1}{2}[2b - a^2 \pm \sqrt{\Delta}].$$

There are several cases. If $2b - a^2 < 0$ and $b^2 > c^2$, then $\sqrt{\Delta} < |a^2 - 2b|$ and no positive solution exists. If $b^2 < c^2$, there is one positive solution ω^2 using the plus sign. If $2b > a^2$ and $b^2 > c^2$ there are two positive solutions. If $b^2 = c^2$ and $2b > a^2$, there is one positive solution. We now examine the cases with $\Delta > 0$.

Case 1. $b^2 < c^2$. We have one positive solution

$$\omega = \frac{1}{\sqrt{2}}[2b - a^2 + \sqrt{\Delta}]^{1/2}.$$

Then

$$\sin \tau\omega = a\omega/c > 0.$$

The equation

$$c \cos \tau\omega = \tfrac{1}{2}[-a^2 + \sqrt{\Delta}]$$

determines the sign of $\cos \tau\omega$, and thus the quadrant in which $\tau\omega$ must lie. Thus, τ must have one of the values

$$\tau_n = \frac{1}{\omega}\left[\sin^{-1}\left(\frac{a\omega}{c}\right) + 2n\pi\right], \qquad n = 0, \pm 1, \ldots,$$

where ω is as given above and where $\sin^{-1}(a\omega/c)$ is chosen in the first or second quadrant according to the sign of $\cos \tau\omega$. These roots $\lambda = i\omega$ are simple. Also we have

$$2\lambda \frac{d\lambda}{d\tau} + a\frac{d\lambda}{d\tau} - ce^{-\tau\lambda}\left(\tau\frac{d\lambda}{d\tau} + \lambda\right) = 0,$$

$$\frac{d\lambda}{d\tau} = \frac{c\lambda e^{-\tau\lambda}}{2\lambda + a - c\tau e^{-\tau\lambda}} = \frac{c\lambda}{(2\lambda + a)\,e^{\tau\lambda} - c\tau}.$$

At $\lambda = i\omega$, we have

$$\frac{d}{d\tau}(\operatorname{Re}\lambda) = \operatorname{Re}\frac{d\lambda}{d\tau}$$

$$= \frac{c\omega(2\omega \cos \tau\omega + a \sin \tau\omega)}{(a \cos \tau\omega - 2\omega \sin \tau\omega - c\tau)^2 + (2\omega \cos \tau\omega + a \sin \tau\omega)^2}.$$

Since $\sin \tau\omega = a\omega/c$, $\cos \tau\omega = (\omega^2 - b)/c$, at a root, we have

$$\operatorname{Re}\frac{d\lambda}{d\tau} = \frac{c^2\omega^2(2\omega^2 - 2b + a^2)}{(ab + a\omega^2 + c^2\tau)^2 + (2\omega^3 - 2b\omega + a^2\omega)^2}.$$

Since $2\omega^2 = 2b - a^2 + \sqrt{\Delta}$, $d(\operatorname{Re}\lambda)/d\tau$ is positive at every root. Therefore, every time a root crosses the imaginary axis with increasing τ, it crosses from left to right. Consequently, stability of the zero solution is lost at $\tau = \tau_0$, and instability persists for all larger τ.

Case 2. $b^2 > c^2$, $2b > a^2$. There are two positive values of ω^2,

$$\omega_1^2 = \tfrac{1}{2}[2b - a^2 + \sqrt{\Delta}],$$
$$\omega_2^2 = \tfrac{1}{2}[2b - a^2 - \sqrt{\Delta}],$$

and corresponding sequences $\{\tau_{n,1}\}$, $\{\tau_{n,2}\}$. The calculation of $\operatorname{Re} d\lambda/d\tau$ is unchanged. Since

$$2\omega_1^2 - 2b + a^2 = \sqrt{\Delta}, \qquad 2\omega_2^2 - 2b + a^2 = -\sqrt{\Delta},$$

we see that the sign of $d(\operatorname{Re}\lambda)/d\tau$ is positive for each root $i\omega_i$ and negative at each root $i\omega_2$. Next, observe that $\Delta < (a^2 - 2b)^2 \leqslant a^4$. Hence $2c\cos\tau\omega = -a^2 \pm\sqrt{\Delta} < 0$. Since $\sin\tau\omega$ is positive, the angle $\tau\omega$ must lie in the second quadrant. Let θ_1 and θ_2 be determined from

$$c\cos\theta_1 = \tfrac{1}{2}[-a^2 + \sqrt{\Delta}], \qquad \pi/2 < \theta_1 \leqslant \pi,$$
$$c\cos\theta_2 = \tfrac{1}{2}[-a^2 - \sqrt{\Delta}], \qquad \pi/2 < \theta_2 \leqslant \pi.$$

Then $\tau\omega = \theta + 2n\pi$, and the values of τ for which there are imaginary roots are

$$\tau_{n,1} = \frac{\theta_1 + 2n\pi}{\omega_1}, \qquad \tau_{n,2} = \frac{\theta_2 + 2n\pi}{\omega_2} \qquad (n = 0, 1, 2, \ldots).$$

Since $\omega_1 > \omega_2$ and $\theta_1 < \theta_2$, we have $\tau_{0,1} < \tau_{0,2}$ and also $2\pi/\omega_1 < 2\pi/\omega_2$. Therefore, there exists an integer k such that the first k terms in the two sequences alternate, but then two terms in the first sequence occur consecutively. At each value $\tau = \tau_{n,1}$, a pair of roots crosses the imaginary axis at $i\omega_i$ into the right half-plane, and at each $\tau = \tau_{n,2}$, a pair of roots crosses at $i\omega_2$ back into the left half-plane. Consequently, the same phenomenon of a finite number of stability switches occurs that was found in the case of delayed friction. There is instability for all τ greater than some value τ^*.

Case 3. $b^2 = c^2$, $2b > a^2$. There is one positive value $\omega^2 = 2b - a^2$, and one corresponding sequence $\{\tau_n\}$. At every root $i\omega$, $2\omega^2 - 2b + a^2 = 2b - a^2 > 0$, so that every root crosses the imaginary axis from left to right, and there is instability for all τ after the first crossing.

Observe that if the delayed restoring force is greater than or equal to the instantaneous restoring force ($c^2 \geqslant b^2$), as in Case 1 or 3, stability is lost at $\tau = \tau_0$ and never regained. However, if the delayed restoring force is less than the instantaneous restoring force ($c^2 < b^2$), finitely many stability switches can occur before final instability.

Finally, if Eq. (10) is unstable for $\tau = 0$, it remains unstable for every τ. This follows from the fact that the smallest value of τ for which there is an imaginary root is always associated with the larger frequency, which in turn implies an initial increase (from 2 to 4) of the total multiplicity of roots possessing positive real parts.

5. General Second Order Equation with One Delay

We consider now the following equation, with both delayed friction and delayed restoring force and the same delay.

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + b\frac{dx(t-\tau)}{dt} + cx(t) + dx(t-\tau) = 0$$

$$(a + b \neq 0, c + d \neq 0). \qquad (11)$$

This equation can arise from a nonlinear system with one delay, upon linearization. The characteristic equation is

$$\lambda^2 + a\lambda + b\lambda e^{-\lambda\tau} + c + de^{-\lambda\tau} = 0. \qquad (12)$$

If $\tau = i\omega$ is a root, then $\omega \neq 0$ and

$$c - \omega^2 + b\omega \sin \omega\tau + d \cos \omega\tau = 0,$$

$$a\omega + b\omega \cos \omega\tau - d \sin \omega\tau = 0. \qquad (13)$$

Thus,

$$(\omega^2 - c)^2 + a^2\omega^2 = b^2\omega^2 + d^2;$$

$$\omega^4 + (a^2 - b^2 - 2c)\,\omega^2 + c^2 - d^2 = 0. \qquad (14)$$

The roots are

$$\omega_\pm^2 = \tfrac{1}{2}(b^2 - a^2 + 2c) \pm \{\tfrac{1}{4}(b^2 - a^2 + 2c)^2 - (c^2 - d^2)\}^{1/2}.$$

There are two cases of interest:

1. $c^2 < d^2$. There is one imaginary solution, $\lambda = i\omega_+$, $\omega_+ > 0$.

2. $c^2 > d^2$. There are two imaginary solutions, $\lambda_\pm = i\omega_\pm$, with $\omega_+ > \omega_- > 0$, provided that (a) $b^2 - a^2 + 2c > 0$ and (b) $(b^2 - a^2 + 2c)^2 > 4(c^2 - d^2)$, and no such solutions otherwise.

The quantity of interest is again the sign of the derivative of $\operatorname{Re}\lambda$ with respect to τ at the points where λ is purely imaginary. From Eq. (12),

$$\{2\lambda + a + [b - \tau(b\lambda + d)]\, e^{-\lambda\tau}\}\frac{d\lambda}{d\tau} = \lambda(b\lambda + d)\, e^{-\lambda\tau}.$$

From this it may be seen that all purely imaginary roots are simple (unless $a = b = d = 0$). Also,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda + a)\, e^{\lambda\tau} + b}{\lambda(b\lambda + d)} - \frac{\tau}{\lambda}, \qquad e^{\lambda\tau} = \frac{-(b\lambda + d)}{\lambda^2 + a\lambda + c}.$$

Thus,

$$\begin{aligned}
\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\right\}_{\lambda=i\omega}
&= \operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega} \\
&= \operatorname{sign}\left\{\operatorname{Re}\left[\frac{-(2\lambda+a)}{\lambda(\lambda^2+a\lambda+c)}\right]_{\lambda=i\omega} + \operatorname{Re}\left[\frac{b}{\lambda(b\lambda+d)}\right]_{\lambda=i\omega}\right\} \\
&= \operatorname{sign}\left\{\frac{a^2-2(c-\omega^2)}{a^2\omega^2+(\omega^2-c)^2} - \frac{b^2}{b^2\omega^2+d^2}\right\} \\
&= \operatorname{sign}\{a^2-b^2-2c+2\omega^2\}.
\end{aligned}$$

Equation (14) was used in the last step. Inserting the expression for $\omega_\pm^2$, it is seen that the sign is positive for ω_+^2 and negative for ω_-^2. For case 1, only one imaginary root exists, $\lambda = i\omega_+$. Therefore, only crossing of the imaginary axis from left to right is possible as τ increases, and stability of the zero solution can only be lost but not regained. For case 2, crossing from left to right with increasing τ occurs whenever τ assumes a value corresponding to ω_+, and crossing from right to left occurs for values of the τ corresponding to ω_-. In this case, using Eq. (13), the two sets of values of τ for which there are imaginary roots are

$$\begin{aligned}
\tau_{n,1} &= \frac{1}{\omega_+}\cos^{-1}\left\{\frac{d(\omega_+^2-c)-\omega_+^2 ab}{b^2\omega_+^2+d^2}\right\} + \frac{2\pi n}{\omega_+} \qquad (n=0,1,...), \\
\tau_{n,2} &= \frac{1}{\omega_-}\cos^{-1}\left\{\frac{d(\omega_-^2-c)-\omega_-^2 ab}{b^2\omega_-^2+d^2}\right\} + \frac{2\pi n}{\omega_-}.
\end{aligned} \tag{15}$$

If Eq. (11) is stable for $\tau = 0$, then necessarily $\tau_{0,1} < \tau_{0,2}$ (since the multiplicity of roots with positive real parts cannot become negative). Since $\tau_{n+1,1} - \tau_{n,1} = 2\pi/\omega_+ < 2\pi/\omega_- = \tau_{n+1,2} - \tau_{n,2}$, there can be only a finite number of switches between stability and instability. That there exist sets of parameters realizing any number of such switches was demonstrated in the special cases of Sections 3 and 4, when either b or d was zero, and it is certainly true when both b and d are non-zero. If Eq. (11) is unstable for $\tau = 0$, examples similar to the case in Section 3 above can be chosen in order to illustrate that there is a range of parameters for which $\tau_{0,2} < \tau_{0,1}$, so that one or more switches from instability to stability to instability occur.

We can summarize the general results derived in this and in the previous sections as follows: *Consider any second order, linear, homogeneous differential-difference equation of the retarded type.* The number of different imaginary roots (roots different in sign only not considered different) of the characteristic equation can be zero, one or two. (I) If there are no such roots, the stability of the zero solution does not change as τ is increased from zero to infinity. (II) *If there is one imaginary root, an unstable zero solution never becomes stable.* If it is stable for $\tau = 0$, then it becomes unstable at the smallest value of τ for which an imaginary root exists and remains so as τ is increased. (III) *If there are two imaginary roots,* $i\omega_+$ *and* $i\omega_-$, *so that* $|\omega_+| > |\omega_-|$, *then the stability of the zero solution can change a finite number of times, at most, as* τ *is increased, and eventually it becomes unstable.* There are two infinite sets of values of τ, $\tau_{n,1}$ and $\tau_{n,2}$, corresponding to $i\omega_+$ and $i\omega_-$, respectively. They are generated by $\tau_{n,1} = \tau_{0,1} + 2\pi n/\omega_+$ and $\tau_{n,2} = \tau_{0,2} + 2\pi n/\omega_-$ $(n = 0, 1, ...)$. As τ is increased, the multiplicity of roots for which $\operatorname{Re}\lambda > 0$ is increased by two whenever τ passes through a value of $\tau_{n,1}$, and it is decreased by two whenever τ passes through a value of $\tau_{n,2}$. When the zero solution is stable for $\tau = 0$, k switches from stability to instability to stability occur when the parameters are such that

$$\tau_{0,1} < \tau_{0,2} < \tau_{1,1} < \cdots < \tau_{k-1,1} < \tau_{k-1,2} < \tau_{k,1} < \tau_{k+1,1} < \tau_{k,2}, ...,$$

or k switches from instability to stability to instability when

$$\tau_{0,2} < \tau_{0,1} < \tau_{1,2} < \cdots < \tau_{k-1,2} < \tau_{k-1,1} < \tau_{k,1} < \tau_{k,2} < \tau_{k+1,1}, ...,$$

when the zero solution is unstable for $\tau = 0$. The conditions on the parameters in order that the above orderings be valid can be formulated more directly with the help of Eq. (15). Some conditions were explicitly found in the previous sections for the cases with $b = 0$ or $d = 0$.

6. A Population Growth Equation

Cushing [6] has formulated and analyzed some very general population growth models. One of these leads to an equation of the form

$$\frac{dP}{dt} + d(P(t))\,P(t) = \int_{-\infty}^{t} g(t-s)\,m(P(s))\,P(s)\,ds.$$

Here, $d(P)$ is the death rate per unit time, which in this equation is assumed to be a function of the present population size, $P(t)$. The function m is the maternity function, representing the rate of egg-laying. Here it is assumed that the maternity rate at time s depends only on the total population size at time s. The function $g(s)$ represents the proportion of eggs laid at any specific time, that survive and hatch out after time s. That is, g is a gestation-time function, and the integral in the above equation gives the rate of appearance of new individuals at time t due to eggs laid at all previous times.

Assume that the equation has a positive equilibrium P_0. Assume, with no loss of generality, that

$$\int_0^{\infty} g(s)\,ds = 1.$$

Then $d(P_0) = m(P_0)$. Cushing analyzed the local stability of this equilibrium in certain special cases, and found in one of these that there is a change from stability to instability and back to stability as the mean time delay is increased. We will investigate this here, using the same kinds of methods as we have used above. First, we linearize around P_0 by setting $P = P_0 + y$ and dropping non-linear terms. This yields

$$\frac{dy}{dt} + [d(P_0) + P_0\,d'(P_0)]\,y$$

$$= [m(P_0) + P_0\,m'(P_0)] \int_{-\infty}^{t} [g(t-s)\,y(s)\,ds].$$

Cushing [5, p. 9] indicates that under appropriate conditions the integral over $-\infty < s < 0$ may be discarded in discussing local asymptotic stability of P_0. If we do this, we obtain the equation

$$\frac{dx}{dt} + [d(P_0) + P_0\,d'(P_0)]\,x$$

$$= [d(P_0) + P_0\,m'(P_0)] \int_0^{t} g(t-s)\,x(s)\,ds.$$

Further, Cushing specializes to the case in which $d(P)$ is a constant, c, and $m(P) = b(1-P)$ for $0 \leqslant P \leqslant 1$. Then the equation $d(P_0) = m(P_0)$ yields $P_0 = (b-c)/b$ and we assume $0 < c < b$ in order that $0 < P_0 < 1$. Since $m'(P_0) = -b$, the equation becomes

$$\frac{dx}{dt} + cx = (2c-b) \int_0^{t} g(t-s)\,x(s)\,ds. \tag{16}$$

We shall study Eq. (16) for the class of kernels

$$g(t) = \frac{1}{n!}\,T^{-n-1} t^n e^{-t/T},$$

where n is a non-negative integer and $T > 0$ (compare Eq. (24) below). Cushing has given stability results for $n = 0$ and 1. These kernels have been proposed (see also MacDonald [16]) as convenient examples of "distributed delay" because each one (for $n \geqslant 1$) has a single maximum, occurring at $s = nT$, and the width of the peak at $s = nT$ depends on n. By adjusting n and T, it is possible to obtain what looks like a reasonable version of delayed effect with any prescribed mean delay $\tau = nT$. Furthermore, the characteristic equation for (16) is easily obtained for these kernels, corresponding to the fact that (16) is equivalent to a system of ordinary differential equations. This version of "delay" corresponds to the engineer's idea of approximating a pure delay by a cascade or series of first order differential equations.

Here we shall investigate how the stability of Eq. (16) varies with the mean delay τ and with n. In particular, we examine this question as $n \to \infty$, since as $n \to \infty$ the delay kernel more nearly approximates a pure delay. Using the fact that the Laplace transform of $g(t)$ is $(Tz + 1)^{-n-1}$, we see that the condition for stability is that the equation

$$(\lambda + c)(T\lambda + 1)^{n+1} = d, \qquad d = 2c - b, \tag{17}$$

have all its roots in the half-plane $\operatorname{Re} \lambda < 0$.

For $n = 0$, Eq. (17) becomes $(\lambda + c)(T\lambda + 1) = 2c - b$ or

$$T\lambda^2 + (cT + 1)\lambda + b - c = 0.$$

Thus, $b > c$ is necessary and sufficient for stability, and consequently whenever there is an equilibrum P_0 in $0 < P_0 < 1$, it is stable. If $n = 1$, Eq. (17) becomes

$$T^2\lambda^3 + (2T + cT^2)\lambda^2 + (1 + 2cT)\lambda + b - c = 0.$$

As Cushing has pointed out, it follows from the Routh–Hurwitz criterion that all roots lie in the left half-plane if and only if (given that $0 < c < b$)

$$\begin{vmatrix} 2T + cT^2 & T^2 \\ b - c & 1 + 2cT \end{vmatrix} > 0$$

or

$$b < 2/T + 6c + 2c^2T.$$

As T (or $\tau = T$) varies from 0 to ∞, the function of T on the right side of this inequality decreases from ∞ to a minimum value of $10c$ at $T = c^{-1}$ and then increases again. Therefore, if $b < 10c$, the inequality holds for all T, whereas if $b > 10c$, it holds for small T and large T but there is an interval of values of T for which it fails. In the latter case, there is a change from stability to instability and back to stability as T varies.

For larger values of n, it is difficult to use the Routh–Hurwitz criterion. If $\lambda = i\omega$, Eq. (17) yields

$$(\omega^2 + c^2)^{1/2} (T^2\omega^2 + 1)^{(n+1)/2} e^{i[\theta + (n+1)\varphi]} = d,$$

where

$$\theta = \mathrm{tg}^{-1}(\omega/c), \qquad \varphi = \mathrm{tg}^{-1}(\omega T).$$

Hence,

$$\sin[\theta + (n+1)\varphi] = 0,$$

and

$$(\omega^2 + c^2)^{1/2} (T^2\omega^2 + 1)^{(n+1)/2} = |d|.$$

Requiring stability for $T = 0$ (i.e., $\lambda = d - c < 0$), $d < c$ or equivalently $c < b$, and the above conditions can be satisfied only if $c < -d$ or equivalently $c < b/3$. This is just a necessary condition. The result, for all n, is: If $b/3 < c < b$, the steady state is stable for all T. If $c < b/3$ destabilization may be possible as T is increased from zero to infinity. (In fact, c/b must be smaller for small values of n than for large n.) The condition $c < |d|$ has a simple interpretation, as seen from Eq. (16). Only if the coefficient of the delay term is larger in magnitude than that of the instantaneous feedback, does the possibility of switch of stability exist. This is analogous to the situation in the fixed delay case (Section 2).

Some useful information can be obtained from the method we employed before. For fixed n, we obtain, from Eq. (17),

$$\frac{d\lambda}{dT} = \frac{-(n+1)\lambda(\lambda + c)}{(n+2)T\lambda + (n+1)cT + 1}. \qquad (18)$$

At the root $\lambda = i\omega$, if any, we have

$$\frac{d\lambda}{dT} = \frac{-i(n+1)\omega(c + i\omega)}{i(n+2)T\omega + (n+1)cT + 1}$$
$$= \frac{(n+1)\omega^2 - i(n+1)c\omega}{i(n+2)T\omega + (n+1)cT + 1},$$
$$\mathrm{Re}\,\frac{d\lambda}{dT} = \frac{(n+1)\omega^2(1 - cT)}{[(n+1)cT + 1]^2 + (n+2)^2 T^2\omega^2}.$$

Thus, if a root crosses the imaginary axis at a value of $T < c^{-1}$, it must cross from left to right, but if $cT > 1$, from right to left. We have not determined how many crossings may occur for $T < c^{-1}$ and how many for $T > c^{-1}$. However, the formula and the example $n = 1$ suggest that the switching from stability to instability may occur as T increases from zero to c^{-1}. Furthermore, we can show that there is stability for all sufficiently large T, for any fixed positive integer n.

Writing $\lambda = \rho + i\omega$ and using (18) we obtain

$$\frac{d\rho}{dT} = \frac{\left[\begin{array}{c} -(n+1)\{\rho(\rho+c)[(n+2)\rho T + (n+1)cT + 1] \\ + \omega^2[(n+2)\rho T + cT - 1]\} \end{array}\right]}{[(n+2)\rho T + (n+1)cT + 1]^2 + (n+2)^2\omega^2 T^2} \tag{19}$$

Thus, if $T > c^{-1}$, $d\rho/dT < 0$ for any $\rho \geqslant 0$. We can then write (19) as

$$\frac{d\rho}{dT} = -f(T, \rho(T), \omega(T)),$$

or

$$\rho(T) = \rho(c^{-1}) - \int_{c^{-1}}^{T} f(T_1, \rho, \omega)\, dT_1 \qquad (T > c^{-1}).$$

As $T \to \infty$, we see from Eq. (19) that $f \sim (1/T)\tilde{f}(\rho, \omega)$. Since $f > 0$ for $\rho \geqslant 0$ and $T > c^{-1}$, if there is a root $\lambda(T) = \rho(T) + i\omega(T)$ such that $\rho(c^{-1}) \geqslant 0$, then $\rho(T)$ must be a decreasing function of T for $T > c^{-1}$. The only way $\rho(T)$ could remain nonnegative for all $T > c^{-1}$ is if the integral on the right-hand side of the equation for $\rho(T)$ converges, so that $\tilde{f}(\rho, \omega)$ must approach zero sufficiently fast as $T \to \infty$. The form of $\tilde{f}(\rho, \omega)$ would imply that both ρ and ω approach zero in this limit. But then (17) yields $c(k+1)^{n+1} = d$, where $k = \lim T\lambda \geqslant 0$. This is not possible, since $d < c$ if (16) is stable for $T = 0$ and $\lambda = 0$ is not a root. Thus, any root with a positive real part, if such exists for some T, must cross the imaginary axis and undergo an irreversible change of sign of the real part as T is increased. Note that for the average delay $\tau = nT$, this change of sign occurs only for $\tau > nc^{-1}$. The larger n is, the larger must τ be made before restabilization occurs.

The conclusion is that *for the system represented by Eq.* (16), *a switch from stability to instability may occur for a certain range of the parameters and must then be followed by a switch back to stability*. MacDonald found this type of behavior numerically [17]. This is in contrast to the result for the analogous system with fixed delay, Eq. (1), treated in Section 2, where stability can switch to instability but not vice versa. We conclude that *at least with respect to the property of switching of stability the two models manifest different qualitative behavior.*

7. Linear Subsystems Presented as "Delays"

Consider the system given by

$$dx/dt = f(x, \bar{y}) \tag{20}$$

and

$$d\bar{y}/dt = A\bar{y} + \bar{b}x. \tag{21}$$

$\bar{y}$ and $\bar{b}$ are n-vectors, and A is $n \times n$ constant matrix. f is a (generally nonlinear) function of x and $\bar{y}$. The Laplace transform of Eq. (21) with zero initial conditions is

$$s\tilde{\bar{y}}(s) = A\tilde{\bar{y}}(s) + b\tilde{x}(s),$$

from which

$$\tilde{\bar{y}}(s) = (SI - A)^{-1}\,\bar{b}\tilde{x}(s) \equiv \tilde{\bar{G}}(s)\,\tilde{x}(s). \tag{22}$$

The poles of (22) are the values of s for which the determinant is zero, namely,

$$|SI - A| = 0. \tag{23}$$

Using control theory terminology, $\tilde{\bar{G}}(s) = (SI - A)^{-1}\bar{b}$ is a vector of "transfer functions," relating the outputs $\tilde{y}_j(s)$, $j = 1,\ldots, n$, to the input $\tilde{x}(s)$ for the linear subsystem, Eq. (21).

Suppose that Eq. (23) has q different roots λ_i, $i = 1,\ldots, q$, with multiplicities n_i, respectively, so that $\sum_{i=1}^{q} n_i = n$. Then the matrix $(SI - A)^{-1}$ can be expressed by a partial fraction expansion as

$$(SI - A)^{-1} = \sum_{i=1}^{q} \sum_{k=1}^{n_i} \frac{z_{ik}}{(s - \lambda_i)^k}; \qquad z_{ik} \text{ are matrices [23]}.$$

The inverse transform is

$$L^{-1}\{(SI - A)^{-1}\} = \sum_{i=1}^{q} \sum_{k=1}^{n_i} \frac{z_{ik}}{(k-1)!}\, t^{k-1} e^{\lambda_i t}.$$

Define

$$g_a^p(u) = \frac{a^{p+1} u^p}{p!}\, e^{-au}; \tag{24}$$

Eq. (22) yields

$$\bar{y}(t) = \int_0^t \bar{G}(\tau)\, x(t-\tau)\, d\tau = \sum_{i=1}^{q} \sum_{k=1}^{n_i} \frac{z_{ik}}{(-\lambda_i)^k} \bar{b} \int_0^t g^{k-1}_{-\lambda_i}(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_0^t \bar{G}(t-\tau)\, x(\tau)\, d\tau. \tag{25}$$

Equation (25) satisfies the initial conditions $\bar{y}(0) = 0$.

If $\bar{y}(0) \neq 0$, there will be additional contributions to $\tilde{\bar{y}}(s)$, and instead of Eq. (22) we have

$$\tilde{\bar{y}}(s) = \tilde{\bar{G}}(s)\, \tilde{x}(s) + \tilde{\bar{H}}(s), \qquad \bar{H}(s) = (SI - A)^{-1}\, \bar{y}(0). \tag{26}$$

Equation (25) is then replaced by

$$\bar{y}(t) = \sum_{i=1}^{q} \sum_{k=1}^{n_i} \frac{z_{ik}}{(-\lambda_i)^k} \times \left\{ \bar{b} \int_0^t g^{k-1}_{-\lambda_i}(\tau)\, x(t-\tau)\, d\tau + \bar{y}(0)\, g^{k-1}_{-\lambda_i}(t) \right\}. \tag{27}$$

Next, we attempt to rewrite Eq. (27) by extending the range of the integrals from $(0, t]$ to $(-\infty, t]$ and omitting the terms proportional to $\bar{y}(0)$ on the right-hand side. This implies an attempt to *define* the function $x(t)$ for $-\infty < t < 0$ (only $x(0)$ is given initially), by requiring that

$$\int_{-\infty}^0 \bar{G}(t-\tau)\, x(\tau)\, d\tau = \sum_{i=1}^{q} \sum_{k=1}^{n_i} \frac{z_{ik}}{(-\lambda_i)^k} \bar{y}(0)\, g^{k-1}_{-\lambda_i}(t). \tag{28}$$

Note the change of integration variable in the left-hand side of (28) as compared to (25). Equation (28) must be satisfied *for all* $t \geqslant 0$. From definition (24), we can find that the functions $g^{k-1}_{-\lambda_i}(t-\tau)$ may be factored as follows:

$$g^{k-1}_{-\lambda_i}(t-\tau) = \sum_{j=0}^{k-1} f^{ik}_j(\tau)\, g^j_{-\lambda_i(t)};$$

$$f^{ik}_j(\tau) = \frac{(-\lambda_i)^{k-1-j}}{(k-1-j)!} (-\tau)^{k-1-j}\, e^{-\lambda_i \tau}. \tag{29}$$

Thus, the left-hand side of (28) can be written as

$$\int_{-\infty}^0 \bar{G}(t-\tau)\, x(\tau)\, d\tau = \sum_{i=1}^{q} \sum_{k=1}^{n_i} \left\{ \frac{z_{ik}\bar{b}}{(-\lambda_i)^k} \sum_{j=0}^{k-1} g^j_{-\lambda_i}(t) \frac{(-\lambda_i)^{k-1-j}}{(k-1-j)!} \times \int_{-\infty}^0 (-\tau)^{k-1-j}\, e^{-\lambda_i \tau} x(\tau)\, d\tau \right\}.$$

Inserting in (28) and comparing the coefficients of each of the n functions $g^l_{-\lambda_i}(t)$ ($l = 1,..., n_i$, $i = 1,..., q$), we obtain n equations for the n integrals, $\int_{-\infty}^0 (-\tau)^l e^{-\lambda_i \tau} x(\tau)\, d\tau$. In other words, we obtain n constraints on the history function $x(\tau)$ ($\tau < 0$). If all the roots λ_i have negative real parts and $x(\tau)$ is bounded (i.e., $|x(\tau)| < \delta$ for $\tau < 0$), then the integrals exist and the constraints can in general be fulfilled. Note that there is still an infinite number of ways to define $x(\tau)$. This reflects the finite dimensionality of our system; the solution does not actually depend on the entire history function.

In summary: *If the zero solution of the linear subsystem, $d\bar{y}/dt = A\bar{y}$, is asymptotically stable, then the system* (20)–(21) *can be reduced to Eq.* (20) *alone in which $\bar{y}$ is replaced by values of x involving "distributed delay,"*

$$\bar{y}(t) = \int_{-\infty}^{t} \bar{G}(t-\tau)\, x(\tau)\, d\tau, \tag{30}$$

where the vector "memory function" $\bar{G}(t)$ is generally a sum of different moments of the exponential distribution, of the form

$$\bar{G}(t) = \sum_{i=1}^{q} \sum_{k=1}^{n_i} \frac{z_{ik}\bar{b}}{(-\lambda_i)^k}\, g^{k-1}_{-\lambda_i}(t). \tag{31}$$

Consider the particular case

$$dx/dt = f(x, y_n) \tag{32}$$

and

$$dy_1/dt = a(x - y_1); \qquad dy_j/dt = a(y_{j-1} - y_j), \qquad j = 2,..., n. \tag{33}$$

Here, $\tilde{y}_n(s) = \tilde{G}(s)\,\tilde{x}(s)$ (if $\bar{y}(0) = 0$), with $\tilde{G}(s) = a^n/(s+a)^n$,

$$y_n(t) = \int_{-\infty}^{t} g_a^{n-1}(t-\tau)\, x(\tau)\, d\tau. \tag{34}$$

This is the case treated extensively by Cushing [5] and MacDonald [16], with a single distributed delay. There is one multiple, real, negative root, $\lambda = -a$, to $|\lambda I - A| = 0$ in this case.

The equivalence of variables of distributed delays of the form (31) to systems of linear equations was manifested also in the other direction, namely, starting from the integral representation ([11, 26], see also [16]). Our objective in going through the above details is twofold: The first objective is to stress the relationship of generalized exponentially distributed delays to linear systems and the basic difference between these "delays" and fixed delays. As described, *any* subsystem of linear equations can be seen as coupled to the other equations through "input" and "output" variables, and be replaced essentially by its "transfer functions" (in the "complex frequency domain") or by their time representation. The integrals involving "delayed variables" are nothing but an integral representation of the linear system. The extension of the integration region to the entire "history" of the system does not imply a real dependence on the history. Unlike the equations with fixed delays or with other distribution functions, which possess an infinite number of degrees of freedom ("modes"), those with distributions as in Eq. (24) are characterized by a finite number of modes. As far as modelling natural systems by differential equations is concerned, delays may be introduced in order to represent implicitly some linear, intermediate process which the main variables of interest undergo. This naturally suggests distributions such as in (31). *The more complex such a process is and the more information we have about it, the more terms may appear in* $G(t)$. The special distribution of Eq. (34) with large n, or a fixed delay, approximates high dimensionality but lack of "structure."

Our second objective is to introduce here the terms "transfer functions" and "frequency response" in order to give yet a different interpretation to generalized exponential memory distributions. This is done in Section 8.

8. Frequency Dependent Delays

Consider again Eqs. (20)–(22) of the previous section. $x(t)$ is considered as input to (21) and any of the $y(t)$ variables, $y_j(t)$, as output. The corresponding transfer function $\tilde{G}_j(s)$ is a component of $\tilde{\tilde{G}}(s)$ in (22).

If the input is assumed to be sinusoidal, $x = A_0 \sin \omega t$, and if the y-system is *stable*, the "steady state response" is given by

$$y_j = A_0 |\tilde{G}_j(i\omega)| \sin(\omega t + \phi_j(\omega)). \tag{35}$$

$|\tilde{G}_j(i\omega)|$ is the magnitude of $\tilde{G}_j(i\omega)$, and

$$\phi_j(\omega) = \mathrm{tg}^{-1} \frac{\mathrm{Im}\, \tilde{G}_j(i\omega)}{\mathrm{Re}\, \tilde{G}_j(i\omega)}.$$

G_j and ϕ_j, for all ω, constitute the "frequency response" of the linear subsystem [9]. The general solution, $\bar{y}(t)$, will also include transient response functions which decay to zero as $t \to \infty$. A frequency dependent delay, $\tau(\omega)$, can be defined in terms of the phase-shift ϕ_j,

$$\tau_j(\omega) \equiv -\frac{1}{\omega} \phi_j(\omega), \tag{36}$$

so that

$$y_j(t) = A_0 \, |\tilde{G}_j(i\omega)| \sin[\omega(t - \tau_j(\omega))] = |G_j(i\omega)| \, x(t - \tau_j(\omega)). \qquad (37)$$

Thus, apart from transient effects, the effect of the linear system, or of the equivalent integral operator, Eq. (30), on x is to introduce a fixed delay into each frequency component of x and to change its size by a certain factor. This factor and the delay depend on the frequency ω. The combined effect will be referred to as "frequency dependent delay". It is contrasted to the case of a fixed delay, in which τ does not depend on ω, but the expression on the right of Eq. (37) suggests a close analogy.

The analogy may be useful in cases where x is expanded in terms of frequency components. This will be illustrated in Section 10. But first, we discuss the possibility of stability switches in Goodwin's model.

9. Goodwin's Model

The following is a normalized version of the feedback model for the regulation of enzyme synthesis introduced by Goodwin [12].

$$\begin{aligned} dx_1/dt &= f(x_n) - b_1 x_1, \\ dx_j/dt &= x_{j-1} - b_j x_j, \qquad j = 2, \ldots, n. \end{aligned} \qquad (38)$$

$b_j > 0$ for all j. $f(x_n)$ is a positive, monotone decreasing function of $x_n \geqslant 0$. The steady state value of x_n, denoted by x_0, is the (unique) root of

$$f(x_0) = \alpha x_0; \qquad \alpha = \prod_{j=1}^{n} b_j \qquad (\alpha > 0). \qquad (39)$$

The characteristic equation is

$$\prod_{j=1}^{n} (\lambda + b_j) = f'(x_0) \qquad (f'(x_0) < 0). \qquad (40)$$

If $\lambda = i\omega$, Eq. (40) becomes

$$\prod_{j=1}^{n} (b_j^2 + \omega^2)^{1/2} \exp\left\{ i \sum_{j=1}^{n} \varphi_j \right\} = f'(x_0), \qquad \varphi_j = \operatorname{tg}^{-1}(\omega/b_j), \qquad (41)$$

yielding (since $f'(x_0) < 0$)

$$\begin{aligned} \sum_{j=1}^{n} \varphi_j &= (2k+1)\pi; \qquad k = 0, 1, \ldots, \\ \prod_{j=1}^{n} (b_j^2 + \omega^2)^{1/2} &= -f'(x_0). \end{aligned} \qquad (42)$$

Obviously, a necessary condition for the existence of real nonzero ω is

$$\alpha < -f'(x_0). \tag{43}$$

If $f(x) = 1/(1 + x^\rho)$, this becomes, using (39),

$$\frac{\rho x_0^{\rho-1}}{(1 + x_0^\rho)^2} > \frac{1}{x_0(1 + x_0^\rho)},$$

or $x_0^\rho(\rho - 1) > 1$. Hence $\rho > 1$ is a necessary condition. The result that there can be no crossing of the imaginary axis (and hence no change in the stability of the steady state and no bifurcation of periodic solutions) if $\rho \leqslant 1$ in Eq. (38), for this form of $f(x)$, is known and applies also for the fixed delay analogue of (20) (see [14, 18]). Delays in the decay (diagonal) terms of (21) can lead to bifurcations for $\rho = 1$ [14]. However, the biological meaning of such delays is not clear.

The distributed delay model equivalent, in the sense of Section 7, to Eqs. (38) is

$$\frac{dx_1}{dt} = f\left(\int_{-\infty}^{t} G(t-\tau)\, x_1(\tau)\, d\tau\right) - b_1 x_1. \tag{44}$$

$G(t)$ is the function whose Laplace transform is

$$\tilde{G}(s) = \frac{1}{\prod_{j=2}^{n} (s + b_j)}.$$

Note that $\alpha^{-1} = b_1^{-1}\tilde{G}(0)$. Condition (43) can be written as $-\tilde{G}(0) f'(x_0) > b_1$. $f'(x_0)$ is the coefficient of the delay term in the equation arising from Eq. (44) by linearization and b_1 is the coefficient of the instantaneous term. The factor $\tilde{G}(0)$ arises from the fact that the kernel $G(t)$ in (43) has a different normalization as compared to the one in Eq. (16) (there, $\tilde{g}(0) = 1$). Thus, the interpretation of the condition in Eq. (43) is the same as that of $c < |d|$ in Section 6. The analogous equation with one fixed delay is of the form

$$dx/dt = f(x(t-\tau)) - bx \qquad (f(x_0) = bx_0). \tag{45}$$

The characteristic equation

$$\lambda + b - f'(x_0)\, e^{-\lambda\tau} = 0$$

leads to $f'(x_0) \cos \omega\tau = b$ and $f'(x_0) \sin \omega\tau = -\omega$ for $\lambda = i\omega$, from which

$\omega^2 = (f'(x_0))^2 - b^2 > 0$ yields the necessary condition $b < |f'(x_0)|$ for a real ω to exist. If $f(x) = 1/(1 + x^\rho)$, this condition becomes

$$b = \frac{f(x_0)}{x_0} = \frac{1}{x_0(1 + x_0^\rho)} < \frac{\rho x_0^{\rho-1}}{(1 + x_0^\rho)^2},$$

or $x_0^\rho(\rho - 1) > 1$, which is the same condition as that in the non-delay case. $\rho > 1$ is a necessary condition for destabilization of the steady state.

One way to enable destabilization both in the non-delay, "chain-reaction" model (Eq. (38)), and in the delay model (Eq. (45)), is to choose $f(x) = 1/(1 + x^\rho)$ with $\rho \geqslant 2$. (An alternative way is to consider parallel feedback loops [18].) For small values of $\rho > 1$, even when periodic solutions bifurcate from the steady state their frequency is quite restricted. This point was stressed by Tyson [25]. It is easy to observe the restriction on the critical frequency at bifurcation, ω, from Eq. (42). The left-hand side of the equation for ω assumes the value α for $\omega = 0$ and is monotonically increasing with ω^2. For a given ratio $-f'(x_0)/\alpha = r$, ω^2 is restricted to values smaller than some $\bar{\omega}^2$, for which

$$\prod_{j=1}^{n} \left(1 + \frac{\bar{\omega}^2}{b_j^2}\right) = r.$$

For $\rho = 1$, $r < 1$, so that real ω does not exist. For $\rho = 2$, $r = 2x_0^2/(1 + x_0^2) < 2$. Obviously, if b_k is the smallest decay constant for some k, $1 + \bar{\omega}^2/b_k^2 < r$ implies $|\bar{\omega}| < b_k$. If there are several constants of the same order of magnitude as b_k, $\bar{\omega}$ is much smaller. In other words, the root ω must necessarily be smaller than the smallest decay constant, or the period larger than the enzyme's longest life-time.

A way to relax the above restrictions is to consider models with more general feedback functions. One possibility is that x_n actively enhances the removal of x_1. For example,

$$\begin{aligned} dx_1/dt &= f(x_n) - c_1 x_n x_1 - b_1 x_1, \\ dx_j/dt &= x_{j-1} - b_j x_j, \qquad j = 2,\dots, n. \end{aligned} \tag{46}$$

The characteristic equation is

$$(\lambda + b_1 + c_1 x_0) \prod_{j=2}^{n} (\lambda + b_j) = f'(x_0) - (c_1/b_1)\, \alpha x_0; \tag{47}$$

$\alpha = \sum_{j=1}^{n} b_j$, and x_0 is the root (unique if $f(0) > \alpha$ and $f'(x) \leqslant 0$ for $x \geqslant 0$) of

$$f(x_0) = \alpha x_0(1 + (c_1/b_1)\, x_0). \tag{48}$$

A necessary condition for the existence of $\lambda = i\omega$ with real ω can be shown to be, as before, $\alpha < -f'(x_0)$. However, for $f(x) = 1/(1 + x^{\rho})$ we now have

$$r = -f'(x_0)/\alpha = \frac{\rho x_0^{\rho}}{(1 + x_0^{\rho})} [1 + (c_1/b_1) x_0].$$

r can be larger than 1 for $\rho = 1$ and destabilization is possible. r can be made much larger compared to its values for the model with $c_1 = 0$ for the same ρ, so that roots $\lambda = i\omega$ of Eq. (47) with larger ω are possible for the same set $\{b_j\}$.

In Eq. (46), x_n can be replaced by x_1 with distributed delay. The analogous fixed delay equation is

$$dx/dt = f(x(t - \tau)) - cx(t)\, x(t - \tau) - bx(t). \tag{49}$$

The equation for x_0 is $f(x_0) = x_0(cx_0 + b)$, and the characteristic equation is

$$\lambda + [cx_0 - f'(x_0)]\, e^{-\lambda\tau} + cx_0 + b = 0. \tag{50}$$

If $\lambda = i\omega$, $\operatorname{tg} \omega\tau = -\omega/(cx_0 + b)$, and

$$\omega^2 = [cx_0 - f'(x_0)]^2 - (cx_0 + b)^2. \tag{51}$$

If $f'(x_0) < cx_0$, a necessary condition for existence of real ω is $-f'(x_0) > b$, which can be satisfied even if $f(x) = 1/(1 + x)$ $(\rho = 1)$.

Finally, we consider the question of switch from stability to instability and back to stability for system (38). First we choose the parameter to be varied. If we assume that x_0 remains fixed, so does α (Eq. (39)), so that not all the b_j can be varied independently. As an example, consider the case

$$b_1 = \alpha T^{n-1}, \qquad b_j = 1/T \qquad j = 2, \ldots, n.$$

Equation (40) becomes

$$(\lambda + \alpha T^{n-1})(\lambda + 1/T)^{n-1} = f'(x_0). \tag{52}$$

This equation is similar to Eq. (17), Section 6, and we are interested in the variation of $\operatorname{Re} \lambda$ as T is increased from zero to infinity. From (52) we obtain

$$\frac{d\lambda}{dT} = \frac{(n - 1)\, \lambda(1 - \alpha T^n)}{T[n\lambda T + (n - 1)\, \alpha T^n + 1]}. \tag{53}$$

Thus, if $\lambda = \rho + i\omega$,

$$\operatorname{sign}\left\{\frac{d\operatorname{Re}\lambda}{dT}\right\} = \operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{dT}\right)^{-1}\right\}$$
$$= \operatorname{sign}\left\{\frac{nT^2}{(n-1)(1-aT^n)} + \frac{[(n-1)aT^n+1]T\rho}{(n-1)(1-aT^n)(\rho^2+\omega^2)}\right\}$$
$$= \operatorname{sign}\left\{\frac{1}{1-aT^n}\right\}.$$

Thus, for any $\rho \geqslant 0$, $d\operatorname{Re}\lambda/dT > 0$ if $T < a^{-1/n}$ and $d\operatorname{Re}\lambda/dT < 0$ if $T > a^{-1/n}$. Following the same arguments as those in Section 6, we conclude that whenever a switch from stability to instability occurs as T is increased from zero (the critical value of T is then smaller than $a^{-1/n}$), the steady state becomes stable again, irreversibly, for sufficiently large T. Such a switch does not occur for the fixed delay model, as was shown in Section 2. In principle, this qualitative difference might provide some idea on the nature of real systems.

10. Approximate Periodic Solutions

A Poincaré-type expansion method was earlier applied to calculate approximate periodic solutions of a first order differential equation with fixed delay [15] and of second order equations with fixed delay [13, 14]. Morris has also applied this method both for fixed delay and for distributed delay with the distribution given in Eqs. (24) [21]. The calculation in the distributed case is not significantly different from that in the fixed delay case, but appears to be more tedious. Our objective in this section is to make the analogy with the fixed delay case more transparent and to simplify the calculations. In this method, the calculation of integrals is reduced to a single evaluation of a transfer function, which is then used at each step in the iteration process.

Goodwin's model, Eq. (38), will serve as a specific example. For the sake of simplicity, we study in some detail the case where $b_j = b$ for all j and consider b as the "bifurcation parameter." The function f is chosen as $f(x) = 1/(1+x^2)$. If $\lambda = i\omega$, Eq. (42) becomes

$$n\varphi = \pi \quad (k=0), \qquad \varphi = \operatorname{tg}^{-1}(\omega/b), \qquad (b^2+\omega^2)^{n/2} = -f'(x_0), \quad (54)$$

where x_0 is the root of $f(x_0) = b^n x_0$. Thus,

$$\omega^2 = b^2\operatorname{tg}^2(\pi/n) = [-f'(x_0)]^{2/n} - b^2,$$

from which we obtain

$$\cos^n(\pi/n) = b^n/(-f'(x_0)) = f(x_0)/x_0(-f'(x_0))$$
$$= (1 + x_0^2)/2x_0^2 > \tfrac{1}{2}. \tag{55}$$

The smallest positive integer n which fulfills the condition $\cos^n(\pi/n) > \frac{1}{2}$ and allows for a purely imaginary root of the characteristic equation is 8. Thus in the following we choose $n = 8$. Then, from (55),

$$x_0 = x_{0c} = 1/(2\cos^8(\pi/8) - 1)^{1/2} = 4.0298,$$
$$b = b_c = 1/[x_{0c}(1 + x_{0c}^2)]^{1/8} = 0.5885,$$

and $\omega = \omega_c = b_c \operatorname{tg}(\pi/8) = 0.2438$. (It is easily seen that b_c is the largest critical value of b, equivalent to the smallest average delay in Eq. (43). X_{0c} is the corresponding steady state value of x_8.)

We will look for periodic solutions of the nonlinear system, Eq. (38). We are not interested in the transient behaviour, so that the contribution of the region from $-\infty$ to 0 to the integral in (44) can be omitted. Let us introduce the following operator notation:

$$\int_0^t G(\tau)\, x_1(t-\tau)\, d\tau = \hat{G}x_1(t). \tag{56}$$

This defines the linear operator $\hat{G}$ in terms of the kernel $G(\tau)$. Let us assume that $x(t)$ is a periodic function with basic frequency ω. Then

$$x_1(t) = \sum_{m=0}^{\infty} (A_m \cos m\omega t + B_m \sin m\omega t).$$

The operation of $\hat{G}$ on each component is specified by the transfer function $\tilde{G}(s)$, which yields the frequency response as in Eq. (37). In a Poincaré-type expansion method [14, 21], one usually only has to apply $\hat{G}$ to the first few terms of the above series. Since the method is described in detail in [21], we shall only outline the calculations here.

With the help of (56), the equation under consideration is

$$x_1 = \frac{1}{1 + (\hat{G}x_1)^2} - bx_1.$$

$$z \equiv x_8 \equiv \hat{G}x_1, \qquad z_0 = \hat{G}x_0 = b^{-7}x_0, \qquad 1/(1 + z_0^2) = b^8 z_0. \tag{57}$$

Expanding $1/(1+z)^2$ about z_0, $z = z_0 + y$, Eq. (57) can be written as

$$\hat{L}y = N\hat{L}y; \qquad \hat{L}y = \dot{y} + by + \frac{2z_0}{(1+z_0^2)^2}\hat{G}y,$$

$$N\hat{L}y = \sum_{k=1}^{\infty} a_k(z_0)(\hat{G}y)^{k+1}. \tag{58}$$

The first two a_k are

$$a_1(z_0) = (3z_0^2 - 1)/(z_0^2 + 1)^3, \qquad a_2(z_0) = 4z_0(1 - z_0^2)/(z_0^2 + 1)^4. \tag{59}$$

Here (see Eqs. (56), (35)–(37))

$$\tilde{G}(s) = 1/(s+b)^7, \qquad |\tilde{G}(in\omega)| = 1/(b^2 + n^2\omega^2)^{7/2},$$

$$\tau(n\omega) = \frac{1}{n\omega}\, 7 \operatorname{tg}^{-1}(n\omega/b). \tag{60}$$

We make the following magnitude assignments and time renormalization

$$b = b_c + \varepsilon^2\delta, \qquad \omega = \omega_c \sum_{i=0}^{\infty} h_i \varepsilon^i \quad (h_0 = 1), \qquad \bar{t} = \omega t;$$

$$y = \sum_{i=0}^{\infty} \bar{y}_i(\bar{t})\, \varepsilon^{i+1}. \tag{61}$$

$\varepsilon^2\delta$ is the excess of b over the critical value b_c, where ε is an auxiliary parameter. (Note that periodic solutions are expected to exist for $\delta < 0$, when the steady state is locally unstable.) $\bar{y}_i(\bar{t})$ are undetermined periodic functions of frequency 1 on the new time scale. Inserting (61) in (58) and identifying coefficients of ε^j, $j = 0, 1,\ldots$, yields a linear recursive system. Details of the calculations are given in the Appendix.

The approximation method is applicable to more complex systems. The only requisite is the knowledge of the transfer function(s) $\tilde{G}(s)$, corresponding to the linear system, whose number is equal to the number of output variables by which it is coupled to the nonlinear equation(s). (Equivalently, one has to know the Laplace transform(s) of the distribution function(s) in a distributed delay problem.) Distributions $G(t)$ with discontinuities or gaps (corresponding to $\tilde{G}(s)$ of mixed polynomial-transcendental form) can also be treated in the same way. The operator G generalizes the fixed time delay operation (i.e., $\hat{T}y(t) = y(t-\tau)$) in a straightforward way. The calculation does not involve additional integrations (compare with Morris [21, pp. 21–22]).

APPENDIX

We make the following expansions in powers of ε

$$|\tilde{G}(in\omega)| = \sum_{i=0}^{\infty} G_i^{(n)}\varepsilon^i, \qquad \phi(n\omega) \equiv n\omega\tau(n\omega) = \sum_{i=0}^{\infty} \phi_i^{(n)}\varepsilon^i;$$

$$G_0^{(n)} = 1/(b_c^2 + n^2\omega_c^2)^{7/2},$$

$$\phi_0^{(n)} = 7\,\mathrm{tg}^{-1}(n\omega_c/b_c), \qquad n = 0, 1, \ldots, \tag{62}$$

and

$$p(z_0) \equiv 2z_0/(1 + z_0^2)^2 = p(x_{0c}) + p'(x_{0c})\, z_0'\,\delta\varepsilon^2 + O(\varepsilon^4),$$

$$z_0' = \left.\frac{dz_0}{db}\right|_{b=b_c} = -8x_{0c}/b_c(2x_{0c}^3 b_c^8 + 1),$$

$$p'(x_{0c}) = 2(1 - 3x_{0c}^2)/(1 + x_{0c}^2)^3. \tag{63}$$

Thus,

$$\hat{L}\{e^{int}\} = \hat{L}\{e^{in\bar{t}}\}$$
$$= \hat{L}_0\{e^{in\bar{t}}\} + ((\text{terms of order 1 in } \varepsilon \text{ and higher})$$

with

$$\hat{L}_0\{e^{in\bar{t}}\} = \left(\omega_c \frac{d}{d\bar{t}} + b_c\right)\{e^{in\bar{t}}\} + p(x_{0c})\, G_0^{(n)} e^{i(n\bar{t} - \phi_0^{(n)})}. \tag{64}$$

If $n = 1$, the right-hand side of (64) becomes zero since b_c and ω_c satisfy the corresponding characteristic equation.

The first order terms yield

$$\hat{L}_0\,\bar{y}_0(\bar{t}) = 0. \tag{65}$$

Equation (65) is the "generating equation" and $\bar{y}_0$ is the "generating solution,"

$$\bar{y}_0(\bar{t}) = A_0 \cos \bar{t}, \tag{66}$$

where A_0 is unknown. The freedom to choose the phase of the entire periodic solution at will is used in (66). Thus we have appled the phase condition

$$\left.\frac{dy}{d\bar{t}}\right|_{\bar{t}=0} = 0. \tag{67}$$

The second order terms in Eq. (58) yield

$$\hat{L}_0 \bar{y}_1(\bar{t}) = a_1(x_{0c})[G_0^{(1)}\bar{y}_0(\bar{t}-\phi_0^{(1)})]^2$$
$$+ \text{(terms proportional to } h_1 \text{ and linear in } \cos\bar{t} \text{ or } \sin\bar{t}). \quad (68)$$

Since $\cos\bar{t}$ and $\sin\bar{t}$ are solutions of the homogeneous equation $\hat{L}_0\bar{y}_1 = 0$, terms proportional to h_1 would generate "secular terms" (i.e., terms whose magnitudes grow indefinitely with time). Thus for $\bar{y}_1$ to be periodic we must require $h_1 = 0$. Then

$$\bar{y}_1(\bar{t}) = A_1\cos\bar{t} + B_1\sin\bar{t} + A_2\cos 2\bar{t} + B_2\sin 2\bar{t} + K. \quad (69)$$

From Eq. (67), $B_1 = -2B_2$ and A_0 in (66) can be renormalized to include the A_1-term so that we can set $A_1 = 0$. A_2, B_2 and K are determined by inserting (69) into (68) and comparing coefficients

$$K \equiv \bar{K}A_0^2 = a_1(x_{0c})\,G_0^{(1)2}A_0^2/2(b_c + p(x_{0c})\,G_0^{(0)}),$$
$$A_2 \equiv \bar{A}_2A_0^2 = a_1(x_{0c})\,G_0^{(1)2}(\beta\cos 2\phi_0^{(1)} - \alpha\sin 2\phi_0^{(1)})\,A_0^2/2(\alpha^2+\beta^2),$$
$$B_2 \equiv \bar{B}_2A_0^2 = a_1(x_{0c})\,G_0^{(1)2}(\alpha\cos 2\phi_0^{(1)} + \beta\sin 2\phi_0^{(1)})\,A_0^2/2(\alpha^2+\beta^2);$$
$$\alpha = 2\omega_c - p(x_{0c})\,G_0^{(2)}\sin\phi_0^{(2)},$$
$$\beta = b_c + p(x_{0c})\,G_0^{(2)}\cos\phi_0^{(2)}. \quad (70)$$

Comparing terms of the order $O(\varepsilon^3)$ yields

$$\hat{L}_0\bar{y}_2(\bar{t}) = -\omega_c h_2\dot{\bar{y}}_0(\bar{t}) - \delta\bar{y}_0(\bar{t}) - \delta p'(x_{0c})\,z_0'G_0^{(1)}y_0(t-\phi_0^{(1)})$$
$$- p(x_{0c})\,G_2^{(1)}\bar{y}_0(\bar{t}-\phi_0^{(1)}) + p(x_{0c})\,\phi_2^{(1)}\dot{\bar{y}}_0(\bar{t}-\phi_0^{(1)})$$
$$+ a_2(x_{0c})\,A_0^3G_0^{(1)3}\cos^3(\bar{t}-\phi_0^{(1)}) + 2a_1(x_{0c})\,A_0^3G_0^{(1)}\cos(\bar{t}-\phi_0^{(1)})$$
$$\times [G_0^{(0)}\bar{K} + G_0^{(2)})\,\bar{A}_2\cos(2\bar{t}-\phi_0^{(2)}) + G_0^{(2)}\bar{B}_2\sin(2\bar{t}-\phi_0^{(2)})]$$
$$+ \text{(terms which do not generate secular terms)}.$$

Here $G_2^{(1)}$ and $\phi_2^{(1)}$, defined by Eq. (62), are explicitly

$$G_2^{(1)} = -7G_0^{(1)}(b_c\delta + \omega_c^2h_2)/(b_c^2+\omega_c^2),$$
$$\phi_2^{(1)} = 7G_0^{(1)}(-\omega_c\delta + b_c\omega_ch_2)/(b_c^2+\omega_c^2). \quad (72)$$

All the other quantities have been presented earlier. A_0 and h_2, to the leading approximation, are obtained from the requirement of no secular terms, namely, from the conditions that the coefficients of $\cos\bar{t}$ and of $\sin\bar{t}$ on the right-hand side of Eq. (71) should vanish. Let us simplify the notation by denoting

$$G_0^{(1)} = G_1 = (b_c^2 + \omega_c^2)^{-7/2},$$

$$G_0^{(0)} = G_0, \qquad G_0^{(2)} = G_2, \qquad \phi_0^{(1)} = \phi_1,$$

$$\phi_0^{(2)} = \phi_2, \qquad \phi = \phi_2 - 2\phi_1, \qquad p_0(x_{0c}) = p, \qquad p_0'(x_{0c}) = p',$$

$$a_1(x_{0c}) = a_1, \qquad a_2(x_{0c}) = a_2.$$

a_1 and a_2 are given in Eq. (59), p, p' and z_0' in (63), $G_0^{(n)}$ and $\phi_0^{(n)}$ ($n = 0, 1,...$) in Eq. (62). Then, with $\bar{K}$, $\bar{A}_2$ and $\bar{B}_2$ given in Eq. (70), $b_c = 0.5885$, $\omega_c = 0.2438$, $x_{0c} = 4.0298$, and $x_{10} = b^7 z_0$, with z_0 the root of $b^8 z_0(1 + z_0^2) = 1$, we finally have

$$x_1 \cong x_0 + (\varepsilon A_0) \cos \omega\tau,$$

$$(\varepsilon A_0)^2 \cong \frac{g}{G_1 d}(b - b_c), \qquad \omega = \omega_c + \frac{\Omega}{d}(b - b_c),$$

$$g = \mu_4\mu_5 + \mu_3^2, \qquad \Omega = -\mu_1\mu_3 - \mu_2\mu_5, \qquad \mu_1\mu_4 - \mu_2\mu_3,$$

$$\mu_1 = \tfrac{3}{4} a_2 G_1^2 + 2a_1 G_0 \bar{K} + a_1 G_2(\bar{A}_2 \cos\phi - \bar{B}_2 \sin\phi),$$

$$\mu_2 = a_1 G_2(\bar{B}_2 \cos\phi + \bar{A}_2 \sin\phi), \qquad \mu_3 = +\sin\phi_1 + \eta\omega_c,$$

$$\mu_4 = \cos\phi_1 - \eta b_c, \qquad \mu_5 = \cos\phi_1 + G_1 p' z_0' - \eta b_c,$$

$$\eta = 7G_1 p/(b_c^2 + \omega_c^2). \tag{73}$$

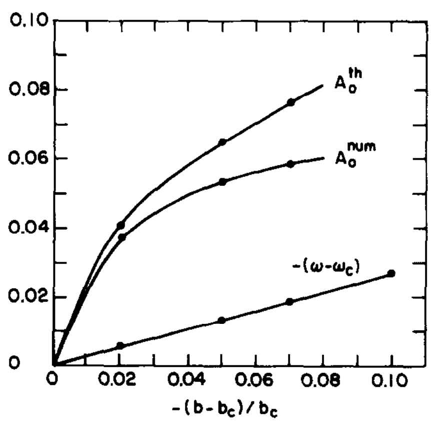

FIG. 1. The theoretical and numerical values of the amplitude (A_0^{th} and A_0^{num}, respectively) and of the deviation from the critical frequency, $(\omega - \omega_c)$, as functions of the excess of the parameter b (see Eqs. (38) and (73)). The numerical and theoretical results for the frequency agree to within 2%.

A_0^2 and $(\omega - \omega_c)$ are computed from (73) as functions of $(b - b_c)$. The results are compared with those of the numerical integration of Eq. (38) with $b_j = b$ for $j = 1, \ldots, 8$ and $f(x) = (1 + x^2)^{-1}$ (Fig. 1).

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