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A Note on the Hecke Hypothesis and the Determination of Imaginary Quadratic Fields with Class-Number 1

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The authors propose a hypothesis whose proof would provide, in particular, a solution of the "class-number 1" problem, recently solved by Stark and Baker.

Let χ be a non-principal character (mod k), $k > 1$. The unproved Hecke hypothesis

$$L(1, \chi) = \sum_1^\infty \frac{\chi(n)}{n} > \frac{c}{\log k},$$

where c is an absolute constant, is considered in this note. In this paper we formulate a hypothesis H which implies the Hecke hypothesis for k a square-free number $\equiv 3 \pmod 4$, and indeed with a computable constant c. Thus, in particular, H would imply a solution of the following problem. Find a computable constant c, such that

$$h(-p) > 1 \qquad \text{for} \quad p > c_1 \, (p : \text{prime}).$$

Here $h(-k)$ denotes the class-number of the imaginary quadratic field $Q(\sqrt{-k})$. This problem was solved recently by Stark and Baker (1967). Now let k be a square-free number $\equiv 3 \pmod 4$. Let $\chi(n) = (n/k)$, the Jacobi symbol. Let k^* be the product of a subset T of the set of prime numbers p with $\chi(p) = -1$ and $p < k$. Let $\chi^*(n) = \chi_T^*(n)$ be the real non-principal character with $\chi^*(n) = 1$ if $(n, k^*) = 1$ and $\chi^*(n) = 0$

if $(n, k^*) > 1$. Then $X(n) = X_T(n) = \chi(n)\,\chi^*(n)$ is a non-primitive character (mod $k\,k^*$).

The hypothesis is as follows:

There exists a subset T such that

$$S(w) = \sum_{1}^{w} X_T(n) \geqslant 0 \qquad \text{for all} \quad w \geqslant 1.$$

To give an example of this hypothesis we take $k^* = 2$ when $k = 19$ i.e., T consists of the single prime 2, which is a quadratic nonresidue (mod k). We leave the details of the verification to the reader. We can also verify the hypothesis when $k = 19$ by taking $k^* = 6$ (i.e., T consists of the primes 2 and 3, both quadratic non-residues (mod k)). A possible choice of T in the general case when k is a prime $\equiv 3 \pmod 4$ is the set of primes $p_1, \ldots, p_{s-2}$ where p_m is the m-th prime and

$$\left(\frac{p_t}{k}\right) = -1 \qquad \text{for} \quad 1 \leqslant t \leqslant s-1, \qquad \text{but} \quad \left(\frac{p_s}{k}\right) = 1.$$

As is well known we have

$$h(-k) = \frac{\sqrt{k}}{\pi}\sum_{1}^{\infty}\frac{\chi(n)}{n} = \frac{\sqrt{k}}{\pi}L(1, \chi) \text{ where}$$

$\chi(n) = (n/k)$ and k is a prime $\equiv 3(4)$. Now

$$\begin{aligned}\sum_{1}^{\infty}\frac{X(n)}{n} &= \frac{S(1)}{1} + \frac{S(2)-S(1)}{2} + \frac{S(3)-S(2)}{3} + \cdots \\ &= S(1)\left(1-\frac{1}{2}\right) + S(2)\left(\frac{1}{2}-\frac{1}{3}\right) + S(3)\left(\frac{1}{3}-\frac{1}{4}\right) + \cdots \\ &> \frac{1}{2}.\end{aligned}$$

Since $S(1) = 1$ and $S(w) \geqslant 0$ for $w \geqslant 1$ by hypothesis H. In what follows we specialize T to be the set of prime numbers p between 0 and k with $\chi(p) = -1$. Then

$$L(1, \chi) \prod_{\substack{p<k \\ \chi(p)=-1}} \left(1+\frac{1}{p}\right) = \sum_{1}^{\infty}\frac{X(n)}{n} > \frac{1}{2}.$$

Hence

$$L(1, \chi) > \frac{1}{2}\prod_{p<k}\left(1+\frac{1}{p}\right)^{-1}.$$

The latter product is, as is well known, greater than $c_2/\log k (c_1, c_2, c_3, \ldots$ etc. denote computable constants). Thus we have

$$h(-k) = \frac{\sqrt{k}}{\pi} L(1, \chi) > \frac{c_3 \sqrt{k}}{\log k}.$$

Since c_3 is a computable constant it follows that, for instance, there are only finitely many square-free $k \equiv 3(4)$ for which $h(-k) = 1$, and these square-free are computable.

Postscript. Our hypothesis H also implies that for primes $k \equiv 3(4)$ we have

$$* \qquad L(s) = \sum_{1}^{\infty} \frac{x(n)}{n^s} > 0 \qquad \text{for} \quad 0 < s < 1.$$

It is well known that *, certainly true under the extended Riemann hypothesis, is still unproved.