Gorenstein Rings as Specializations of Unique Factorization Domains

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Communicated by D. A. Buchsbaum

Received November 10, 1982

INTRODUCTION

It is known that a unique factorization domain which is a Cohen–Macaulay factor ring of a regular local ring is Gorenstein ([22], see, however, [23]). In this paper we show that conversely any Gorenstein ring R which is a factor ring of a regular local ring and a complete intersection locally in codimension one can be realized as a specialization of a unique factorization domain S which is Cohen–Macaulay and an epimorphic image of a regular local ring (Proposition 1). S is obtained as a local blowing-up ring of a generic link of R (for the definitions we refer to the first section of this paper).

As a first application we prove that for certain singularities the highest exterior power and any symmetric power of the module of differentials are never Cohen-Macaulay (Corollary 1). In a second corollary we construct examples of “bad” unique factorization domains S such that each S is a local Cohen–Macaulay ring and the singular locus of S has “big” codimension but nevertheless the completion of S is no longer factorial. This contrasts with a result of H. Flenner [7, 1.5] saying essentially that for quasihomogeneous singularities which fulfill Serre’s conditions \((S_1)\) and \((R_2)\) factoriality is preserved under completion.

Now let \(R\) be the reduced local ring of a quasihomogeneous Gorenstein singularity over a perfect field such that \(R\) is a complete intersection locally in codimension two and the degrees of the homogeneous free resolution of \(R\) are “big enough.” By \(\hat{S}\) we denote the singularity constructed in Proposition 1. In this situation we can use Flenner’s result to conclude that also the completion \(\hat{S}\) of \(S\) is factorial and hence \(\hat{R}\) admits a formal deformation to the complete unique factorization domain \(\hat{S}\). This implies that rigid Gorenstein singularities satisfying the above assumptions are factorial (Proposition 2).
In [20] A. R. Kustin and M. Miller suggest questions similar to those we consider in this paper by asking if unobstructed deformation always improves the divisor class group and if rigid Gorenstein rings in the linkage class of a complete intersection are factorial.

1. Generalities

Let $M$ be a finitely generated module over a ring $R$ and let $Q(R)$ be the total ring of quotients of $R$. If $M \otimes_R Q(R)$ is a free $Q(R)$ module of rank $s$ we say that the $R$ module $M$ has a rank and we set $\text{rank}_R(M) = s$. We denote by $\nu(M)$ the minimal number of generators and by $l(M)$ the length of $M$. For the $n$th symmetric power of $M$ we will write $\text{Sym}_n(M)$. The Gorenstein type of a local Cohen-Macaulay ring $R$ is denoted by $r(R)$, and if the canonical module of $R$ exists we call it $K_R$. We say an ideal $I$ in a local Cohen-Macaulay ring $R$ is a complete intersection ideal if $I$ is generated by an $R$ regular sequence (i.e., $\nu(I) = \text{grade}(I)$) and $I$ is an almost complete intersection ideal if $\nu(I) < \text{grade}(I) + 1$.

We now list some properties of linkage, which is a basic tool in this paper. For proofs we refer to the paper of C. Peskine and L. Szpiro [24].

**Definition.** Let $I$ and $J$ be two ideals in a local Cohen-Macaulay ring $P$. $I$ and $J$ ($P/I$ and $P/J$, respectively) are said to be (algebraically) linked if there exists a $P$ regular sequence $a_1, ..., a_g$ in $I \cap J$ such that $J = \langle a_1, ..., a_g \rangle : I$ and $I = \langle a_1, ..., a_g \rangle : J$.

**Proposition 1.** Let $I$ be an unmixed ideal of grade $g$ in a local Cohen-Macaulay ring $P$. Consider a $P$ regular sequence $a_1, ..., a_g$ contained in $I$ and the ideal $J = \langle a_1, ..., a_g \rangle : I$.

(a) If $I_q = \langle a_1, ..., a_g \rangle_q$ for all $q \in \text{Ass}(P/I)$, or if $P$ is Gorenstein and $P/I$ is Cohen-Macaulay, then $I = \langle a_1, ..., a_g \rangle : J$ (i.e., $I$ and $J$ are linked).

(b) If $P$ is Gorenstein, then $P/J$ is Cohen-Macaulay if and only if $P/I$ is Cohen-Macaulay.

(c) If $P$ is Gorenstein and $P/I$ is Cohen-Macaulay, then $K_{P/J} \cong I/\langle a_1, ..., a_g \rangle$ and $K_{P/I} = J/\langle a_1, ..., a_g \rangle$.

Part (c) of Proposition 1 implies that $\nu(P/I) = \nu(J/\langle a_1, ..., a_g \rangle)$ and hence the linked ideal $J$ has to be an almost complete intersection ideal if $P/I$ is Gorenstein.

Suppose that in addition to the assumptions of Proposition 1(c) the projective dimension of $I$ over $P$ is finite. Then there is a more explicit description of the linked ideal $J$ due to D. Ferrand which can be found in
[24]: Let $F_\cdot$ be a minimal free $P$ resolution of $I$ and let $K_\cdot$ be the Koszul complex of $a_1, \ldots, a_g$ with values in $P$. We consider a homomorphism of complexes $u = (u_1, \ldots, u_g): K_\cdot \to F_\cdot$ lifting the embedding $\langle a_1, \ldots, a_g \rangle \subseteq I$, and its $P$ dual $u^*: F^*_\cdot \to K^*_\cdot$. Then the mapping cone $C(u^*)$ is a free $P$ resolution of $P/J$.

It is known that $F_\cdot \cong P'$, where $r = r(P/I)$. Let $(h_1, \ldots, h_r)$ be the matrix of the homomorphism $u_\cdot: K_\cdot \cong P \to P' \cong F_\cdot$. Then the above description implies that $J$ is generated by $\{a_1, \ldots, a_g, h_1, \ldots, h_r\}$.

Let $P$ be a local Cohen–Macaulay ring and let $I$ be an unmixed ideal of grade $g$ in $P$ such that $I_q$ is a complete intersection ideal in $P_q$ for all $q \in \text{Ass}(P/I)$. We fix a system of generators $\{f_1, \ldots, f_g\}$ of $I$, and for a set of indeterminates $Y = \{Y_{ij} \mid 1 \leq i \leq g, 1 \leq j \leq s\}$ we consider the ring $Q = P[Y]_{(m_p, Y)}$. Then $\{a_i \mid a_i = \sum_{j=1}^s Y_{ij} f_j, 1 \leq i \leq g\}$ is a regular sequence in $Q$ [14, Proposition 21] and because of Proposition I(a) the ideal $J = \langle a_1, \ldots, a_g, f_1, \ldots, f_g \rangle: QI$ is linked to $QI$. In an unpublished manuscript C. Huneke calls $J$ a generic link of $I$ and shows the following:

**Proposition II.** If in addition to the above assumptions $P$ is a domain then a generic link $J$ of $I$ is a prime ideal.

**Proposition III.** If in addition to the above assumptions $P$ is Gorenstein, $s$ is a nonnegative integer with $s \leq 3$, and $I_q$ is a complete intersection ideal in $P_q$ for all $q \in \text{Ass}(P/I)$, then it holds that $J_q$ is a complete intersection ideal in $Q_q$ for all $q \in \{p \mid p \in \text{Spec } Q, J \subseteq p, \dim(Q/J)_p \leq s\}$.

We now recall some basic definitions and known facts concerning deformation theory (see, e.g., [10, 21, 26, 29]).

**Definition.** Let $R$ and $S$ be Noetherian local rings. $R$ is called a specialization of $S$ (with respect to $a$) if there exists an $S$ regular sequence $a$ such that $R \cong S/\langle a \rangle$.

**Definition.** (a) Let $k$ be a field and let $A, R,$ and $S$ be local $k$ algebras such that $A$ has residue class field $k$. A deformation of $R$ over $k$ to $A$ is a flat homomorphism of $k$-algebras $\eta: A \to S$ such that there exists an isomorphism of $k$-algebras $\varphi: S \otimes_A k \cong R$. A deformation is called infinitesimal if $A$ is Artinian and it is called formal if $A$ is a formal power series ring over $k$ and $R$ and $S$ are epimorphic images of power series rings over $k$.

(b) An infinitesimal (respectively formal) deformation is called trivial
if there exists an isomorphism of $A$ algebras $\Phi: S \simeq R \otimes_k A$ (respectively $\Phi: S \simeq R \otimes_k A$) such that $\Phi \otimes_A \text{id}_k = \varphi$.

Because of the local criteria of flatness, to say that $\eta: A = k[[Z]] \to S$ with $\eta(Z) = a$ is a formal deformation of $R$ over $k$ is the same as saying that $R$ is a specialization of $S$ with respect to $a$.

**Definition.** A local $k$ algebra $R$ is called rigid over $k$ if each infinitesimal deformation of $R$ over $k$ is trivial.

By $k[[X_1, \ldots, X_r]]$ we denote the ring of convergent power series over a field $k$ with a valuation. Let $P$ be one of the rings $k[X_1, \ldots, X_r], k[[X_1, \ldots, X_r]]$, or $k[X_1, \ldots, X_r]$, and let $R$ be an epimorphic image of $P$. For the universal, respectively, universally finite, module of differentials of $R$ over $k$ we write $\Omega_k(R)$.

**Proposition IV.** (a) If under the above assumptions $R$ is rigid over $k$ then $\tilde{R}$ is rigid over $k$ and each formal deformation of $\tilde{R}$ over $k$ is trivial.

(b) If $k$ is perfect and $R$ is reduced, the following statements are equivalent:

(i) $R$ is rigid over $k$,

(ii) $\text{Ext}^1_k(\Omega_k(R), R) = 0$.

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### 2. The Main Result

**Proposition 1.** Let $P$ be a regular local ring, $I$ an ideal in $P$, and $R = P/I$. We assume that $R$ is Gorenstein and that $I_q$ is a complete intersection ideal in $P_q$ for all prime ideals $q$ of $P$ containing $I$ with $\dim R_q < 1$.

Then there exists a unique factorization domain $S$ which is a Cohen–Macaulay factor ring of a regular local ring such that $R$ is a specialization of $S$.

**Proof.** We assume that $I$ is an ideal of grade $g$ with a system of generators $\{f_1, \ldots, f_g\}$ and define the ring $Q = P[Y_{(m_p, m_q)}]$, the $Q$ regular sequence $a_1, \ldots, a_g$, and the generic link $J = \langle a_1, \ldots, a_g \rangle: QI$ as in Propositions II and III. Then $J$ is prime (Proposition II), $J_q$ is a complete intersection ideal for all prime ideals $q$ of $Q$ containing $J$ such that $\dim(Q/J) < 1$ (Proposition III), and $J$ admits a generating set of the form $\{a_1, \ldots, a_g, h\}$ (Proposition I(c)).

Under these conditions the associated graded ring $\text{gr}_J(Q)$ is a domain [16, Theorem 3.1], the Rees algebra $\mathcal{R}(J) = Q[It]$ is Cohen–Macaulay ([5, 4.12]; see also [16, Theorem 2.3], and [12, Theorem 2.6]), and the natural
epimorphism from the symmetric algebra $\text{Sym}(J)$ to the Rees algebra $\mathcal{R}(J)$ is an isomorphism [17, Theorem 3.1; 32, Proposition 3.5].

In the following we borrow arguments from the proof of Theorem 2.3 in [18] by considering the blowing-up ring $V = Q[\alpha_1/h, \ldots, \alpha_g/h]$. $V$ is a dehomogenization of $\mathcal{R}(J)$ and because of $\mathcal{R}(J) \cong \text{Sym}(J)$ it follows that

$$V \cong Q[Z_1, \ldots, Z_g] \left/ \left\langle r + \sum_{i=1}^{g} r_i Z_i \mid r \in Q, r_i \in Q, rh + \sum_{i=1}^{g} r_i a_i = 0 \right\rangle \right..$$

So $m = \langle m_0, z_1, \ldots, z_g \rangle$ is a maximal ideal of $V$ and we set $S = V_m$.

The above description shows that $S/\langle z_1, \ldots, z_g \rangle \cong Q/\langle r \mid rh \in \langle \alpha_1, \ldots, \alpha_g \rangle \rangle = Q/\langle \alpha_1, \ldots, \alpha_g \rangle; J = Q/\langle a \rangle$ (see Proposition 1(a)). So for $\langle a \rangle = \langle z_1, \ldots, z_g, Y \rangle$ we get $S/\langle a \rangle \cong R$.

$S$ is Cohen–Macaulay, because $\mathcal{R}(J)$ is Cohen–Macaulay and $S$ is the localization of a dehomogenization of $\mathcal{R}(J)$. Moreover $\text{codim} \langle a \rangle + \dim R = \dim S = \dim Q = g + gc + \dim R$ and therefore $a$ is an $S$-regular sequence.

It remains to show that $S$ is factorial. As in [18, proof of Theorem 2.3], we use [15, Theorem 1] to conclude that $\mathcal{R}(J)[t^{-1}]$ is factorial because $\text{gr}_y(Q)$ is a domain. Then also $U = \mathcal{R}(J)[t^{-1}][(ht)^{-1}] = Q[\alpha_t, (ht)^{-1}]$ is factorial. $U$ has a $Z$ grading such that $U_0 = V$. Because $U$ contains units of degree one, $U_0$ is also factorial. Hence $S$ is factorial.

**Remark.** With the same arguments as in [18, proof of Theorem 2.3], one could show that $S$ is linked to $Q/J[Z_1, \ldots, Z_g]_{m_0, z_1, \ldots, z_g}$, i.e., that $S$ is doubly linked to $R[Y, Z_1, \ldots, Z_g]_{m_0, r, z_1, \ldots, z_g}$.

3. **Applications**

The following Corollary applies Proposition 1 to the highest exterior power, respectively, the symmetric powers, of the module of differentials. Under the given assumptions the statement can be regarded as a generalization of a conjecture of R. Berger claiming that the module of differentials of a reduced curve singularity over a perfect field has nontrivial torsion [3]. Part (a) of Corollary 1 could also be proved by using the fundamental class, a natural homomorphism mapping the highest exterior power of the module of differentials into the canonical module. At least for Gorenstein rings part (b) of Corollary 1 contains as a special case the known fact that Berger's conjecture holds for smoothable curve singularities [2, 2.4; 19; 31, Corollary 1; 13, Corollary 3.6].
COROLLARY 1. Let \( k \) be a perfect field and let \( P \) be one of the rings \( k[X_1, \ldots, X_t]_{(x_1, \ldots, x_t)}, k\langle X_1, \ldots, X_t \rangle, \) or \( k[[X_1, \ldots, X_t]] \). Let \( I \) be an ideal in \( P \) such that \( R = P/I \) is a reduced Gorenstein algebra of dimension \( d \). We assume that \( \bar{R} \) is the specialization of a complete local Noetherian \( k \) algebra \( T \) such that \( T_q \) is a complete intersection for all prime ideals \( q \) of \( T \) with \( \dim T_q \leq 1 \).

(a) With the above assumptions the following statements are equivalent:

(i) \( \bigwedge^d \Omega_k(R) \) is Cohen–Macaulay,

(ii) \( R \) is regular.

(b) If in addition to the above assumptions \( d = 1 \) the following statements are equivalent:

(i) there exists a positive number \( n \) such that \( \text{Sym}_n(\Omega_k(R)) \) is torsion-free,

(ii) \( R \) is regular.

Proof. We show simultaneously that the conditions under (a)(i) and (b)(i) imply regularity of \( R \). We may assume that \( R \) is complete.

According to Proposition 1 applied to \( T \) there exists a unique factorization domain \( S \) which is a local Cohen–Macaulay \( k \) algebra and a regular sequence \( b \) in \( S \) such that \( R \cong S/(b) \).

We consider the subring \( W = k[b] \) of \( S \) and set \( M_1(S) = \bigwedge^d \Omega_w(S), M_2(S) = \text{Sym}_n(\Omega_w(S)), M_1(R) = \bigwedge^d \Omega_k(R), \) and \( M_2(R) = \text{Sym}_n(\Omega_k(R)) \). Let \( i \in \{1, 2\} \).

Obviously \( M_i(S) \otimes_S R \cong M_i(R) \). Under the assumptions of the Corollary \( \text{rank}_S \Omega_k(R) = d \) [28, 6.4] and therefore \( \text{rank}_R M_i(R) = 1 \). Let \( F_i \) denote the first Fitting ideal. Then \( F_i(M_i(R)) \) contains a nonzero divisor and, because \( F_i(M_i(R)) \) is the image of \( F_i(M_i(S)) \) under the projection \( \Pi: S \to R, F_i(M_i(S)) \) cannot be the zero ideal, which means \( \text{rank}_S(M_i(S)) \leq 1 \). On the other hand if \( b = b_1, \ldots, b_s \) then [28, 5.1] implies that

\[
s + d = \dim S \leq \text{rank}_S \Omega_k(S) \leq s + \text{rank} \Omega_w(S),
\]

and therefore \( \text{rank}_S(M_i(S)) \) is greater or equal to one. Hence \( \text{rank}_S(M_i(S)) = 1 \).

Let \( c \) be a system of parameters for \( R \). J. Herzog showed in [9, Proposition 1] that a finitely generated module \( M \) with a nonzero rank over a reduced local Cohen–Macaulay ring is Cohen–Macaulay if and only if \( l_R(M \otimes_R R/\langle c \rangle) = \text{rank}_R(M) \cdot l_R(R/\langle c \rangle) \). So assuming that \( M_i(R) \) is Cohen–Macaulay we conclude that
which implies that \( M_1(S) \) is Cohen–Macaulay.

Therefore \( M_1(S) \) is isomorphic to an ideal in \( S \) which is Cohen–Macaulay as a module over the normal Cohen–Macaulay domain \( S \) and hence divisorial by [11, 4.131]. But \( S \) is factorial and therefore \( M_1(S) \cong S \), hence \( M_1(R) \cong R \).

The equations

\[
1 = v(M_1(R)) = \binom{n}{d} \\
1 = v(M_2(R)) = \binom{n}{d} + n - 1
\]

now imply \( v(\Omega_k(R)) = d \), which means \( \Omega_k(R) \cong R^d \).

H. Flenner shows in [7, 1.51] that for a nonnegatively graded excellent domain \( R = \bigoplus_{t=0}^{\infty} R_t \) which is a finitely generated \( R_0 \) algebra fulfilling Serre's conditions \((R_1)\) and \((S_1)\), the divisor class groups of \( R \) and \( \hat{R} \) are isomorphic (here \( \hat{R} \) denotes the completion of \( R \) with respect to \( R_+ = \bigoplus_{t=1}^{\infty} R_t \)). In particular \( \hat{R} \) is factorial if \( R \) is so. As a second application of Proposition 1 we construct for any integer \( s \geq 2 \) a local Cohen–Macaulay domain \( S \) which fulfills \((R_s)\) such that \( S \) is factorial but \( \hat{S} \) is not factorial. So in contrast to Flenner's result for rings which are not positively graded factoriality is not preserved under completion even if the codimension of the singular locus is arbitrarily high.

**Corollary 2.** Let \( n \geq 3 \) be an integer, let \( k \) be an algebraically closed field of characteristic \( p \geq 0 \), and \( G \neq \{id\} \) a finite cyclic subgroup of \( SL(n, k) \) with order \( N \) acting linearly on \( k^n \) and freely outside the origin. We suppose that \( p \) does not divide \( N \). Then the ring of invariants \( R = (k[x_1, \ldots, x_n]/(x_1, \ldots, x_n))^G \) fulfills the assumptions of Proposition 1. We denote by \( S \) the local blowing-up ring of a generic link of \( R \) which was constructed there. \( S \) has the following properties:

- \( S \) is the localization of an affine \( \mathbb{Z} \)-graded \( k \) algebra, \( S \) is Cohen–Macaulay, \((R_{n-1})\), and factorial, but \( \hat{S} \) is not factorial.

**Proof:** The conditions \( p \nmid N \) and \( G \subseteq SL(n, k) \) imply that \( R \) is Gorenstein [33, Theorem 4]. Because \( G \) acts freely on \( k^n\setminus\{0\} \), \( R \) fulfills \((R_{n-1})\) [4, Section 2, Proposition 4]. So \( R \) satisfies the assumptions of Proposition 1.
and therefore $S$ is Cohen–Macaulay and factorial. Because $\tilde{R}$ fulfills (R$_2$) it is rigid over $k$ [25, 2.6] and hence $\tilde{S} \cong \tilde{R}[W_1, \ldots, W_r]$ for indeterminates $W_1, \ldots, W_r$ (Proposition IV(a)). Therefore $S$ is also (R$_{s-1}$) and, as there exists a faithfully flat homomorphism from $R$ to $\tilde{S}$, factoriality of $\tilde{S}$ would imply that $R$ is factorial [8, 6.11].

But on the other hand for the divisor class group of the invariant ring $R$, it holds by [30, Theorem 2] that $\text{Cl}(R) \cong \text{Hom}(G/H, k^*)$, where $H$ is the subgroup of $G$ generated by all its pseudo-reflections (for the definition we refer to [30]). Because $G$ is a finite subgroup of $SL(n, k)$ and $p \nmid N$ we conclude that $H = \{\text{id}\}$ and because $G$ is cyclic it follows that $|\text{Cl}(R)| = N$. So $R$ cannot be factorial.

It remains to show that $S$ is the localization of a $\mathbb{Z}$ graded affine $k$ algebra. $\tilde{R} = (k[X_1, \ldots, X_n])^G$ is quasihomogeneous, i.e., it has a grading of the form $\tilde{R} = \bigoplus_{i=0}^{\infty} \tilde{R}_i$ with $\tilde{R}_0 = k$. Let us consider a quasihomogeneous representation $\tilde{R} = k[V_1, \ldots, V_t]/J$, where $J$ is an ideal of grade $g$ generated by quasihomogeneous polynomials $f_1, \ldots, f_r$. We set $\mathcal{Q} = k[V_1, \ldots, V_t, \{Y_{ij} \mid 1 \leq i \leq g, 1 \leq j \leq \varepsilon\}]$, $\alpha_i = \sum_{j=1}^{\varepsilon} Y_{ij} f_j$, and $J = \langle \alpha_1, \ldots, \alpha_g \rangle: \mathcal{Q}$. Assigning suitable positive degrees to $Y_{ij}$, the polynomials $\alpha_i$ will be quasihomogeneous and therefore $\mathcal{J}$ will also be a homogeneous ideal. Because $(\mathcal{J}/\langle \alpha_1, \ldots, \alpha_g \rangle)/(V_1, \ldots, V_t, (Y_{ij}))$ is cyclic (Proposition I(c)) we may assume that $\mathcal{J}$ has a system of quasihomogeneous generators $\{\alpha_1, \ldots, \alpha_g, h\}$. Therefore $\mathcal{Q}[\alpha_1/h, \ldots, \alpha_g/h]$ is a $\mathbb{Z}$ graded affine $k$ algebra and $S$ is a localization thereof.

**EXAMPLE.** Let $n \geq 3$ be an integer, let $E$ be the $n$ by $n$ identity matrix over $\mathbb{C}$, $\zeta = e^{2\pi i/n}$ and $G$ the multiplicative group generated by $\zeta E$. Then the obvious action of $G$ on $\mathbb{C}^n$ fulfills the conditions of Corollary 2 and its invariant ring is the Veronese singularity $\mathbb{C}[M(n)]_{(M(n))}$, where $M(n)$ denotes the set of all monomials of degree $n$ in $n$ indeterminates.

### 4. Rigid Singularities

The rings of invariants $R$ we considered in Corollary 2 turned out to be isolated rigid Gorenstein singularities but not factorial. Proposition 2 contains a positive result in a similar direction saying that, under certain conditions concerning grading, rigid Gorenstein singularities are indeed factorial. The additional assumptions are necessary to apply Flenner's result [7, 1.5] and thus to ensure that the completion $\tilde{S}$ of the ring $S$ constructed in Proposition 1 is still factorial. Then one can use rigidity of $R$ to obtain a faithfully flat homomorphism from $R$ to $\tilde{S}$ and to conclude that $R$ is factorial.

The following Lemma is presumably known.
LEMMA. Let $k$ be a perfect field and let $P$ be one of the rings $k[X_1, \ldots, X_t]$, $k\langle X_1, \ldots, X_t \rangle$, or $k[X_1, \ldots, X_t]_I$, $I$ an ideal in $P$, $q$ a prime ideal of $P$ containing $I$, and $R = (P/I)_q$. We suppose that $R$ is reduced. Then the following statements are equivalent:

(i) $R$ is regular,

(ii) $R$ is a complete intersection and rigid over $k$.

Proof. We suppose that $R$ is a complete intersection but not regular. Then the projective dimension of $\Omega_k(R)$ equals one and hence $\Omega_k(R)$ has a minimal free resolution of the form

$$0 \to F_1 \xrightarrow{\varphi} F_0 \to \Omega_k(R) \to 0.$$

$R$ is rigid; therefore $\text{Ext}^1_k(\Omega_k(R), R) = 0$ (Proposition IV(b)) and dualizing the above sequence with respect to $R$ yields an exact sequence

$$0 \to \Omega_k(R)^* \xrightarrow{\varphi^*} F_0^* \xrightarrow{\varphi^*} F_1^* \to 0.$$

All entries of the matrix of $\varphi$ and hence of $\varphi^*$ lie in the maximal ideal $m_R$ and therefore $F_1^* = m_R F_1^*$, which is impossible. 

Let $k$ be a field, let $P$ be one of the rings $k[X_1, \ldots, X_t]_{(k_1, \ldots, k_t)}$, $k\langle X_1, \ldots, X_t \rangle$, or $k[X_1, \ldots, X_t]_I$, $I$ an ideal in $P$, and $R = P/I$. Under these conditions one says that $R$ is quasihomogeneous if there exists a polynomial ring $\tilde{P}$ over $k$ and an ideal $\tilde{I}$ in $\tilde{P}$ such that $\tilde{R} = \tilde{P}/\tilde{I}$ is nonnegatively graded $(\tilde{R} = \oplus_{i=0}^{\infty} \tilde{R}_i)$ with $(\tilde{R})_0 = k$ and such that $\tilde{R} \cong \tilde{R}$. Then $\tilde{R}$ as a graded $\tilde{P}$ module has a homogeneous minimal free resolution $F_\cdot$. Here the $F_i$ have base elements of positive degrees $n_{ij}$ (i.e., $F_i = \oplus_{j} \tilde{P}(-n_{ij})$) such that the homomorphisms in the resolution are homogeneous of degree zero. If $R$ is Gorenstein and $I$ has grade $g$ then $F_g \cong \tilde{P}(-n_g)$ because the Betti numbers are not changed by localization with respect to the irrelevant maximal ideal.

PROPOSITION 2. Let $k$ be a perfect field, $P$ as above, $I$ an ideal in $P$ of grade $g$ and $R = P/I$. We suppose that

(i) $R$ is reduced,

(ii) $R$ is Gorenstein,

(iii) $R_q$ is a complete intersection for all prime ideals $q$ such that $\dim R_q \leq 2$,

(iv) $R$ is quasihomogeneous, where $I$ is generated by the quasihomogeneous polynomials $\{f_1, \ldots, f_e\}$ in $\tilde{P}$.
(v) $n_x > (g - 1) \operatorname{Max}\{\deg f_j \mid 1 \leq j \leq \varepsilon\}$,

(vi) $R$ is rigid over $k$.

Under the above assumptions $\tilde{R}$ is a unique factorization domain.

Proof: It is enough to consider the case that $R$ is a localization of $\tilde{P}/\tilde{I}$ with respect to the irrelevant maximal ideal.

We may assume that $\deg f_v \leq \deg f_u$ if $v < u$. Then let $s$ be the smallest number between 1 and $\varepsilon$ such that $\deg f_s = \deg f_r$ for all $v > s$ and set $\delta = \deg f_s$. For a collection of indeterminates $Y = \{ Y_{ij} \mid 1 \leq i \leq g, 1 \leq j \leq s - 1 \}$ we consider $K = k(\{ Y_{ij} \mid 1 \leq i \leq g, s \leq j \leq \varepsilon \})$, $\tilde{Q} = \tilde{P}[Y] \otimes_k K$, and the $\tilde{Q}$ regular sequence $\{\alpha_i \mid \alpha_i = \sum_{j=1}^{s-1} Y_{ij} f_j, 1 \leq i \leq g\}$. For $1 \leq j \leq s - 1$ we set $\deg Y_{ij} = \delta - \deg f_j > 0$, and thus the elements $\alpha_i$ are homogeneous of degree $\delta$.

The Koszul complex $K = K(\alpha_1, \ldots, \alpha_g ; \tilde{Q}) = \wedge(\tilde{Q} e_1 \oplus \ldots \oplus \tilde{Q} e_g)$ becomes a homogeneous complex by setting $\deg e_i = \delta$. We consider the homomorphism of homogeneous complexes $u = (u_1, \ldots, u_n) : K \rightarrow F, \otimes \tilde{Q}$. As $u_1$ is a homogeneous degree zero homomorphism from $K_1 \cong \tilde{Q}(-g\delta)$ to $(F, \otimes \tilde{Q})_1 \cong \tilde{Q}(-n_x)$ its matrix consists of a homogeneous element $h$ of degree $g\delta - n_x$. Because of $\deg h = g\delta - n_x < \delta = \deg \alpha_i$ for $1 \leq i \leq g$ the ring $\tilde{S} = \tilde{Q}[\alpha_i/h, \ldots, \alpha_g/h]$ is quasihomogeneous and we call its irrelevant maximal ideal $m$. As the linked ideal $\tilde{J} = \langle \alpha_1, \ldots, \alpha_g \rangle ; \tilde{Q}\tilde{I}$ is generated by $\{\alpha_1, \ldots, \alpha_g, h\}$, the algebra $\tilde{S}_m$ is a localization of the ring $S$ constructed in Proposition 1.

Hence $\tilde{S}_m$ is Cohen–Macaulay and specializes to $R' = (R \otimes_k K)_{(m_h)}$. $R'$ is rigid over $K$ and therefore $\hat{S}_m \cong R'[\{ W_1, \ldots, W_r \}]$ for indeterminates $W_1, \ldots, W_r$.

Because of the Lemma, $R$ fulfills $(R_3)$ and so do $R', \hat{S}_m$, and $S_m$. The Jacobian ideal of $\tilde{S}$ over $k$ which describes the singular locus of $\tilde{S}$ is homogeneous and therefore $\tilde{S}$ also satisfies $(R_3)$.

According to Proposition 1, $S$ and hence $\hat{S}_m$ are Cohen–Macaulay and factorial and the same holds for $\tilde{S}$ [27, Proposition 6].

In this situation one can apply Flenner's result [7, 1.5] to conclude that $\hat{S}_m$ is also factorial. The faithfully flat homomorphism $\hat{K} \rightarrow R'[\{ W_1, \ldots, W_r \}] \cong \hat{S}_m$ now yields that $\hat{K}$ is factorial. □

Example. Let $k$ be a field, $n$ a positive integer. Let $P$ denote the power series ring $k[[X_{ij} \mid 1 \leq i < j \leq 2n + 1]]$, $A = (a_{ij})$ the alternating matrix with $a_{ij} = X_{ij}$ for $1 \leq i < j \leq 2n + 1$, $(-1)^{j+i}$ the Pfaffian of $A$ obtained by deleting the $j$th row and column of $A$, and set $(f) = (f_1, \ldots, f_{2n+1})$. Then $\langle f \rangle$ is an ideal of grade 3 in $P$ generated by forms of degree $n$, and $R = P/\langle f \rangle$ is Gorenstein.
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satisfies $(R_{\delta})$, and has a homogeneous resolution of the following form [6, Proposition 6.1]:

$$0 \to P(-2n - 1) \xrightarrow{\delta} (P(-n - 1))^{2n+1} \xrightarrow{\phi} (P(-n))^{2n+1} \xrightarrow{\theta} P \to 0.$$ 

The singularities described above are exactly the generic Gorenstein $k$ algebras of codimension 3 [10, 3.3; 6, Theorem 2.1]. In particular they are rigid over $k$ and hence satisfy the assumptions of Proposition 2. Thus we get a new proof of the known fact that these algebras are factorial (see [1]).

ACKNOWLEDGMENT

The author would like to thank Craig Huneke for many helpful discussions during the preparation of this work.

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