Let $G$ be a group and $\langle x \rangle$ be the free group with generator $x$. Let $w(x)$ be an element of $G \ast \langle x \rangle$ (the free product of $G$ and $\langle x \rangle$) such that the total degree of $w$ with respect to $x$ is positive. An extension of $G$ is constructed in which for some $n$ the equation $w^n(x) = 1$ has a solution.

Let $G$ be a group and $w(x) = x^{i_1}g_1 x^{i_2}g_2 \cdots x^{i_k}g_k$ be an element of $G \ast \langle x \rangle$ (the free product of $G$ and the cyclic group with one generator $x$). Let us assume that $\sum_i i_j = d \neq 0$. Then, changing $x$ to $x^{-1}$ if necessary, we can assume that $d$ is bigger than zero. In what follows $d$ denotes this positive cumulative degree of $w$ with respect to $x$. Under this assumption we will construct an extension of the group $G$ in which the equation $w^n(x) = 1$ has a solution for some $n$.

Let $A$ be the set of all two-sided infinite sequences of elements of $G$. If $a \in A$ we denote by $a_j$ the $j$th term in the sequence $a_j(a_j \in G)$. $G$ can be represented as a group of permutations of $A$ in many different ways. We are going to use the following representation: $g(a) = b$ if $b_j = a_j$ for $j \neq 0$, $b_0 = ga_0$ and identify $G$ with the corresponding group of permutations.

**Lemma 1.** There exists a permutation $p$ of $A$ such that the action of the permutation $w(p) = p^{i_1}g_1 \cdots p^{i_k}g_k$ is $w(p)(a) = b$, where $b_j = a_{j - d}$. (We call such a permutation a shift.) Here $d$ is the total degree as above.

**Proof.** Let $p$ be a permutation such that $c = p(a)$ is given by $c_{j + 1} = p_j a_j$, where the $p_j$ are elements of $G$ which we will find. Then $w(p)(a) = b$ is given by

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where the $g_j$ are elements of the group $G$ which do not depend on $p_j$. So the condition that $w(p)$ is a shift corresponds to the equality of all coefficients of the $a_i$ in (1) to 1. If we fix arbitrarily the elements $p_0, p_1, \ldots, p_{d-2}$ then from this system of equations we can uniquely determine all other $p$. (We will use (1) to denote this system of equations also.)

We are ready now to construct an extension of $G$ in the case $d = 1$. In this case all $p_j$ are determined by the system (1) and it follows that all $p_j$ with sufficiently large $|j|$ (e.g., $|j| > m = \sum_s |t_s|$) are equal to 1. Indeed, all $g_j$ with $|j| > m$ are equal to 1.

We consider now the set $A_{4m}$ of all periodic sequences with period $4m$. Each of these sequences corresponds to the finite sequence of length $4m$ obtained by identifying $g_j$ with $g_j(\mod 4m)$. We define a permutation $p'$ of $A_{4m}$ by $c = p'(a)$ if $c_{j+1} = p_ja_j$ with $p_j$ determined from system (1), $0 \leq j < 4m$, where $c_{4m} = c_0$. Then $w(p')$ is a shift and $w^{4m}(p')$ is the identity permutation.

In the case $d > 1$ groups of permutations of certain sets of finite sequences also give us the extensions we are looking for. But here, generally, speaking, the terms $g_j$ of these sequences should be taken from some extension of $G$: we can not choose $p_0, p_1, \ldots, p_{d-2}$ from $G$ in such a way that the restriction of $p$ on the corresponding set of finite sequences turns $w$ into a shift.

**Lemma 2.** *The solution of system (1) can be given by:*

\[ p_{sd+t} = p_t p_{t-1} \cdots p_0 h_{s,t} p_0^{-1} p_1^{-1} \cdots p_{t-1}^{-1} \]

*Proof.* Formula (2) is obviously correct if $s = 0$ and $h_{0,t} = 1$. To prove this relation for other values of $s$ one can apply induction separately for $s > 0$, $s = -1$ and $s < -1$ and use the equality $p_{d-1} p_{d-2} \cdots p_0 = g_0^{-1}$ (see (1)).

Now for $j$ with large absolute value (1) has the form

\[ p_{j+d-1} p_{j+d-2} \cdots p_j = 1 \]

which shows that $p_{j+d} = p_j$. 

Lemma 3. There exists an extension \( L \) of \( G \) such that it is possible to find \( p_0, p_1, \ldots, p_{d-2} \) from \( L \) which make \( p_{sd+t} = p_{rd+t-1} \) when \( s > 0, r < 0, \) and \( sd + t \) and \( rd + t - 1 \) are sufficiently large.

Proof. It follows from Lemma 2 that we have to solve the following system of equations:

\[
\begin{align*}
p_0 h_{s,0} &= p_{d-1} \cdots p_0 h_{r,d-1} p_0^{-1} \cdots p_{d-2} \\
p_1 p_0 h_{s,1} p_0^{-1} &= p_0 h_{r,0} \\
\vdots \\
p_i \cdots p_0 h_{r,i} p_0^{-1} \cdots p_{i-1}^{-1} &= p_{i-1} \cdots p_0 h_{r,i-1} p_0^{-1} \cdots p_{i-2}^{-1} \\
\vdots \\
p_{d-2} \cdots p_0 h_{s,d-2} p_0^{-1} \cdots p_{d-3}^{-1} &= p_{d-3} \cdots p_0 h_{r,d-3} p_0^{-1} \cdots p_{d-4}^{-1}.
\end{align*}
\]

If \( d = 2 \) we have only one equation \( p_0 h_{s,0} = p_1 p_0 h_{r,1} p_0^{-1} \) which gives \( p_0 h_{s,0} p_0 = p_1 p_0 h_{r,1} = g_0^{-1} h_{r,1}. \) If \( d > 2 \) then it is not difficult to show (starting with the second equation) that \( p_i \cdots p_0 = (p_{i-1} \cdots p_0) f_{i,1} p_0 f_{i,2} \) for \( i < d - 1. \) So \( p_{d-2} \cdots p_0 = p_0 f_0 p_0 f_1 \cdots p_0 f_{d-2} \) and then the first equation gives

\[
p_0 h_{s,0} p_0 f_0 p_0 f_1 \cdots p_0 f_{d-2} = g_0^{-1} h_{r,d-1}. \tag{4}
\]

A construction of an extension of \( G \) where Eq. (4) can be solved is given by F. Levin in [1]. (This construction also is based on permutations of the set of finite sequences using the "diagonal" representation of \( G \).) As soon as we choose \( p_0 \) from this extension all \( p_i \) with \( i < d - 1 \) are uniquely determined by (3) and for all other \( i \) by (2).

Theorem. The equation \( w^n(x) = 1 \) is solvable over group \( G \) for some \( n \leq 4 \sum_i |i_s| + \sum_s i_s. \)

Proof. As a first step we extend group \( G \) to the group \( L \) where Eq. (4) has a solution. Let \( m = \sum_i |i_s| \) and \( u \equiv -4m + 1 \pmod{d}, \) \( 0 \leq u < d. \) Now let us consider the set of sequences \( (a_{-2m-u+1}, a_{-2m-u+2}, \ldots, a_0, \ldots, a_{2m}) \) and represent the group \( G \) as a group of permutations on this set as was done above: \( g \) multiplies the 0th coordinate of \( a \) by \( g \) and fixes all others.

We consider now the permutation \( p' \) such that \( c = p'(a) \) is given by \( c_{j+1} = p_j a_j \) for \( -2m-u+1 \leq j < 2m \) and \( c_{-2m-u+1} = p_{2m} a_{2m}, \) where \( p_j \in L, p_0 \) satisfies Eq. (4), and all other \( p_j \) are determined by the systems (3) and (2). Then \( w(p') \) is a shift and \( w^{4m+u}(p') \) is the identity permutation. So equation \( w^n(x) = 1 \) has a solution over group \( G \) for \( n = 4m + u. \)

Remark. We were not concerned with searching for the best possible estimation of \( n \) in this setting. We were looking rather for further confirmation of the following conjecture:
The equation \( w(x) = 1 \) is solvable over any group \( G \) provided \( \sum_i i \neq 0 \). 

Results of [1] and [2] make us believe that this conjecture is true even though it may not be provable by purely algebraic methods.

REFERENCES