Abstract

In this paper we examine the orders of vertex-transitive self-complementary uniform hypergraphs. In particular, we prove that if there exists a vertex-transitive self-complementary \(k\)-uniform hypergraph of order \(n\), where \(k = 2^\ell\) or \(k = 2^\ell + 1\) and \(n \equiv 1 \pmod{2^\ell + 1}\), then the highest power of any prime dividing \(n\) must be congruent to 1 modulo \(2^\ell + 1\). We show that this necessary condition is also sufficient in many cases – for example, for \(n\) a prime power, and for \(k = 3\) and \(n\) odd – thus generalizing the result on vertex-transitive self-complementary graphs of Rao and Muzychuk. We also give sufficient conditions for the existence of vertex-transitive self-complementary uniform hypergraphs in several other cases. Since vertex-transitive self-complementary uniform hypergraphs are equivalent to a certain kind of large sets of \(t\)-designs, the results of the paper imply the corresponding results in design theory.

1. Introduction

A \(k\)-uniform hypergraph (for short, a \(k\)-hypergraph) is a pair \((V, E)\) of a vertex set \(V\) and edge set \(E \subseteq V^{(k)}\), where \(V^{(k)}\) denotes the set of all \(k\)-subsets of \(V\). The parameters \(k\) and \(|V|\) are called the rank and order, respectively, of the \(k\)-hypergraph \((V, E)\). Observe that a 2-hypergraph is simply a graph. The vertex set and edge set of a hypergraph \(X\) will be denoted by \(V_X\) and \(E_X\), respectively.

E-mail address: msajna@uottawa.ca (M. Šajna).
An isomorphism between $k$-hypergraphs $X$ and $X'$ is a bijection $\varphi: V_X \rightarrow V_{X'}$ inducing a bijection between $E_X$ and $E_{X'}$. An automorphism of $X$ is an isomorphism from $X$ to $X$, and the set of all automorphisms of $X$ is denoted by $\text{Aut}(X)$. A $k$-hypergraph $X$ is vertex-transitive if $\text{Aut}(X)$ acts transitively on $V_X$. In this paper, whenever we refer to the automorphism group of a hypergraph $X$ as being transitive, we mean transitive in its action on the vertex set $V_X$.

The complement $X^c$ of a $k$-hypergraph $X$ is the $k$-hypergraph with vertex set $V_X$ and edge set $V_X^{(k)} \setminus E_X$. If there exists an isomorphism $\tau: X \rightarrow X^c$, then $X$ is said to be self-complementary, and $\tau$ is called an antimorphism of $X$. If $X$ is a vertex-transitive self-complementary uniform hypergraph with an antimorphism $\tau$, then $\tau$ normalizes $\text{Aut}(X)$ and $\tau^2 \in \text{Aut}(X)$, so that $\text{Aut}(X)$ is a normal subgroup of index 2 in $\langle \text{Aut}(X), \tau \rangle$. The set of all antimorphisms of $X$, denoted by $\text{Ant}(X)$, is thus precisely the coset $\tau \text{Aut}(X)$.

A $k$-hypergraph $(V, E)$ is called $t$-subset regular (for $1 \leq t \leq k$) if every $t$-subset of $V$ lies in the same number (called $t$-valency) of edges of $E$. Notice that a vertex-transitive uniform hypergraph is necessarily 1-subset regular.

Clearly, a 2-hypergraph is 1-subset regular if and only if it is regular as a graph. Therefore the notion of $t$-subset regularity for hypergraphs generalizes that of regularity for graphs. On the other hand, it provides a bridge between the theory of hypergraphs and design theory. Namely, a $t$-subset regular $k$-hypergraph of order $n$ is simply a $t$-$(n, k, \lambda)$-design with $\lambda$ equal to the $t$-valency.

If such a hypergraph $X$ is self-complementary, then its complement $X^c$ is also a design with the same set of parameters, which implies that the pair $\{X, X^c\}$ is a large set of $t$-designs $\text{LS}[2](t, k, n)$ (see [1, Section 4.4]), with the additional property that the two $t$-designs that constitute the large set are isomorphic. In this sense self-complementary uniform hypergraphs correspond bijectively to those large sets $\text{LS}[2](t, k, n)$ whose $t$-designs are isomorphic.

Finally, vertex-transitivity of a $t$-regular hypergraph corresponds to point-transitivity of the underlying $t$-design. Hence vertex-transitive self-complementary $k$-hypergraphs of order $n$ correspond bijectively to large sets $\text{LS}[2](t, k, n)$ with $t \geq 1$ whose $t$-designs are point-transitive and isomorphic.

In this paper, however, we shall study vertex-transitive uniform hypergraphs from the perspective of (hyper)graph theory, rather than design theory, and therefore use the former terminology.

2. History and statement of main results

In 1985, Rao [6] determined sufficient conditions on the order $n$ of a vertex-transitive self-complementary graph, while in 1999, following many partial results from numerous researchers, Muzychuk [4] showed in an elegant short paper that these conditions are also necessary. For a positive integer $n$ and a prime $p$, let $n_{(p)}$ denote the largest integer $i$ such that $p^i$ divides $n$. With this notation, the result of Rao and Muzychuk can be formulated as follows.

**Theorem 1** ([6,4]). There exists a vertex-transitive self-complementary graph of order $n$ if and only if $p^{n_{(p)}} \equiv 1 \pmod{4}$ for every prime $p$.

Integers $n$ satisfying the condition in Theorem 1 will be called Muzychuk integers. Note that every Muzychuk integer is odd.

We propose an analogous problem for vertex-transitive self-complementary uniform hypergraphs:
Proposition 3. For a given integer \( k \geq 2 \), determine the set \( \mathcal{H}_k \) of all integers \( n \) for which there exists a vertex-transitive self-complementary \( k \)-hypergraph of order \( n \).

Since a vertex-transitive \( k \)-hypergraph is necessarily 1-subset regular, the necessary condition on its order follows from the following corollary of a result by Khosrovshahi and Tayfeh-Rezaie [2], originally stated in the language of large sets. Note that, for a prime \( p \) and positive integers \( r \) and \( m \), the symbol \( r[m] \) denotes the unique integer in \( \{0, 1, \ldots, m - 1\} \) such that \( r \equiv r[m] \pmod{m} \).

Proposition 3 ([2, Theorem 9]). If there exists a self-complementary \( t \)-subset regular \( k \)-hypergraph of order \( n \), then there exists an integer \( q \), \( k(2) < q \leq \min\{i : 2^i > k\} \), such that \( n_{[2^q]} \in \{t, t + 1, \ldots, k_{[2^q]} - 1\} \).

We remark that the bound on \( q \) in Proposition 3 was not established in [2]; however, it is not difficult to obtain.

The corollary below gives more transparent necessary conditions for \( t = 1 \) and particular values of \( k \).

Corollary 4. Suppose there exists a 1-subset regular self-complementary \( k \)-hypergraph with \( n \) vertices, where \( k = 2^\ell \) or \( k = 2^\ell + 1 \) for some positive integer \( \ell \). Then \( n_{[2^{\ell+1}]} \in \{1, \ldots, k - 1\} \).

In particular,

- if \( k = 2 \), then \( n \equiv 1 \pmod{4} \);
- if \( k = 3 \), then \( n \equiv 1 \) or \( 2 \pmod{4} \);
- if \( k = 4 \), then \( n \equiv 1, 2, \) or \( 3 \pmod{8} \);
- if \( k = 5 \), then \( n \equiv 1, 2, 3, \) or \( 4 \pmod{8} \).

Proof. Let a positive integer \( q \) be as in the statement of Proposition 3, and note that \( \min\{i : 2^i > k\} = \ell + 1 \). If \( k = 2^\ell \), then \( k(2) = \ell \) and so we must have \( q = \ell + 1 \). If \( k = 2^\ell + 1 \), then \( k(2) = 0 \), whence \( q \in \{1, \ldots, \ell + 1\} \). However, \( k_{[2^q]} = 1 \) for all \( i \leq \ell \), which would give \( n_{[2^q]} \in \emptyset \). Hence we must also have \( q = \ell + 1 \). Since in both cases \( k_{[2^{\ell+1}]} = k \), we conclude from Proposition 3 that \( n_{[2^{\ell+1}]} \in \{1, 2, \ldots, k - 1\} \).

In this paper, we consider Problem 2 for some particular values of \( k \) and \( n \). We extend Muzychuk’s result to uniform hypergraphs whose rank is of the form \( 2^\ell \) or \( 2^\ell + 1 \), and present several sufficient conditions for an integer to belong to \( \mathcal{H}_k \). In Section 3, we prove that, when restricted to the case \( n \equiv 1 \pmod{2^{\ell+1}} \), the statement of Corollary 4 can be strengthened as follows.

Theorem 5. Let \( \ell \) be a positive integer, let \( k = 2^\ell \) or \( k = 2^\ell + 1 \), and let \( n \equiv 1 \pmod{2^{\ell+1}} \). If there exists a vertex-transitive self-complementary \( k \)-hypergraph of order \( n \), then \( p^{r(p)} \equiv 1 \pmod{2^{\ell+1}} \) for every prime \( p \).

Observe that Corollary 4 and Theorem 5, used for \( k = 2 \), immediately imply Muzychuk’s result [4].

Let us now turn our attention to sufficient conditions for an integer to belong to \( \mathcal{H}_k \). A crucial step in Rao’s proof of the sufficiency of conditions in Theorem 1 was the observation that every Paley graph of prime power order congruent to 1 modulo 4 is vertex-transitive and self-complementary. In this paper, we generalize the Paley graph construction to the case of \( k \)-hypergraphs, and thus prove the following partial converse to Theorem 5.
Theorem 6. There exists a vertex-transitive self-complementary $k$-hypergraph of order $n$ for every prime power $n$ congruent to 1 modulo $2^{\ell+1}$, where $\ell = \max\{k(2), (k-1)(2)\}$.

The second step towards Rao’s result was the observation that the lexicographic product of two vertex-transitive self-complementary graphs is again vertex-transitive and self-complementary. Unfortunately, no such product construction is available for $k$-hypergraphs with $k \geq 4$ and so Theorem 6 cannot be naturally extended to orders that are not prime powers (except when $k = 3$). However, our rank-increasing method in Section 4.2 yields the following result.

Theorem 7. Let $k$ be a positive integer and $n$ a Muzychuk integer. If $k$ is congruent to 2 or 3 modulo 4, then there exists a vertex-transitive self-complementary $k$-hypergraph of order $n$. If $k$ is congruent to 3 modulo 4, then there exists a vertex-transitive self-complementary $k$-hypergraph of order $2n$.

The final part of the paper considers 3-hypergraphs. For these hypergraphs Problem 2 is solved completely for odd orders, and partially for even orders.

Theorem 8. If $n$ is odd, then there exists a vertex-transitive self-complementary 3-hypergraph of order $n$ if and only if $n$ is a Muzychuk integer.

If $n$ is even, then there exists a vertex-transitive self-complementary 3-hypergraph of order $n$ whenever $n$ is of the form $2m$ or $(1 + q)m$ for a Muzychuk integer $m$ and an odd prime power $q$ congruent to 1 modulo 4.

We prove Theorem 5 in Section 3, and provide constructions that prove Theorems 6–8 in Section 4. In Section 5, we give a list of open problems and possible future research projects.

3. Proof of Theorem 5

The following notation will be used in the proof below. If $\Omega$ is a finite set, $v$ a point in $\Omega$, $\tau$ a permutation on $\Omega$, $G$ a permutation group on $\Omega$, and $p$ a prime, then $v^\tau$, $G_v$, $G^\tau$, and Syl$_p(G)$ will denote the image of $v$ by $\tau$, the stabilizer of the point $v$ in the group $G$, the conjugate of $G$ by $\tau$, and the set of all Sylow $p$-subgroups of $G$, respectively. If the order of $G$ is coprime to $p$, then we define Syl$_p(G)$ to be the singleton containing the trivial group.

Proof of Theorem 5. Let $\ell$, $k$, and $n$ be as in the statement of the theorem; that is, let $\ell$ be a positive integer, let $k \in \{2^\ell, 2^\ell + 1\}$, and suppose that $n \equiv 1 \pmod{2^{\ell+1}}$. Furthermore, let $X$ be a vertex-transitive self-complementary $k$-hypergraph of order $n$, let $p$ be a prime divisor of $n$, and $p^r$ the highest power of $p$ dividing $n$. We shall prove the theorem by finding a vertex-transitive self-complementary $k$-uniform subhypergraph $X'$ of order $p^r$. The result will then follow from Corollary 4, as will be explained later.

Let $M = \text{Aut}(X)$ and let $d$ be the largest positive integer such that $p^d$ divides $|M|$. The subhypergraph $X'$ that we are looking for will be induced by an appropriate orbit of a Sylow $p$-subgroup of $M$. But first we need to examine the orbits of certain antimorphisms of $X$.

Let $G$ be the permutation group on $V_X$ generated by $M$ and an antimorphism of $X$. Then $M$ has index 2 in $G$, and every element of $G \setminus M$ (that is, any antimorphism of $X$) has even order. In particular, for every $\tau \in G \setminus M$ there exists an integer $s$ such that $\tau^s$ is an antimorphism of $X$ whose order is a power of 2. Let $O_2$ be the set of elements of $G \setminus M$ whose orders are powers of 2, and consider any antimorphism $\varphi \in O_2$. What are the lengths of its orbits? Since $n$ is odd, $\varphi$ has an odd number of fixed points, and since its order is a power of 2, all of its orbits have
length a power of 2. Suppose that $\varphi$ has an invariant set $B$ of size $2^s$, where $1 \leq s \leq \ell$. Since $n \equiv 1 \pmod{2^{\ell+1}}$, it follows that $\varphi$ must have another invariant set (disjoint from $B$) of size $2^s$, and thus also one of size $2^{s+1}$. By repeating this argument, we conclude that $\varphi$ has an invariant set of size $2^\ell$, and therefore (by adjoining the leftover fixed point) also an invariant set of size $2^{\ell} + 1$. Hence $\varphi$ has an invariant set of size $k$. Since every set of $k$ vertices is either an edge of $X$ or an edge of the complement $X^C$, no such set can be invariant under an antimorphism of $X$; a contradiction. We have thus shown that every element of $O_2$ has exactly one fixed point and all other orbits of length divisible by $2^{\ell+1}$.

Next, we use Muzychuk’s idea [4] to find an appropriate Sylow $p$-subgroup $P$ of $M$ and an orbit of $P$ that induces a vertex-transitive self-complementary $k$-uniform subhypergraph $X'$ of order $p^\tau$. First, we define the set $P$ of all $p$-subgroups $P$ of $M$ for which there exist $v \in V_X$ and $\tau \in G \setminus M$ such that $v^\tau = v$, $P^\tau = P$ and $P_v \in \text{Syl}_p(M_v)$. We’ll show that a maximal element of $P$ is in fact a Sylow $p$-subgroup of $M$ with precisely the properties we seek.

Next, we prove that $P$ is non-empty. Choose $v \in V_X$, $P \in \text{Syl}_p(M_v)$, and $\sigma \in G \setminus M$. (Note that if $p$ does not divide $|M_v|$, then $P$ is trivial.) Since $M$ is transitive on $V_X$, there exists $h \in M$ mapping $v$ to $\sigma v$. Then $\sigma = h\sigma^{-1}$ is an element of $G \setminus M$ fixing $v$. This implies that $M_v\sigma = M_v$, and thus $P^\sigma \in \text{Syl}_p(M_v)$. Therefore, there exists $g \in M_v$ such that $P^g = P^\sigma$. Let $\tau = g\tau^{-1}$ and observe that $v^\tau = v$ and $P^\tau = P$. Moreover, $P_v = P \in \text{Syl}_p(M_v)$. Hence, $P \in P$ and $P$ is non-empty.

Now let $P$ be a maximal element of $P$ (with respect to inclusion), let $N$ be the normalizer of $P$ in $M$, and let $Q$ be the Sylow $p$-subgroup of $N$ containing $P$. Our next goal is to prove that $P$ is a Sylow $p$-subgroup of $M$. This is achieved by showing that $Q$ lies in $P$, and hence $P = Q$ since $P$ is maximal and $P \leq Q$. It will then follow that $P$ is a Sylow $p$-subgroup of its own normalizer in $M$, and therefore that $P$ is a Sylow $p$-subgroup of $M$.

Let us now show that $Q \in P$. By the definition of $P$, there exist $v \in V_X$ and $\tau \in G \setminus M$ such that $v^\tau = v$, $P^\tau = P$, and $P_v \in \text{Syl}_p(M_v)$. Since $\tau$ normalizes $M$ and $P$, it also normalizes $N$, and hence $Q^\tau \in \text{Syl}_p(N)$. Let $g$ be an element of $N$ such that $Q^\tau = Q^g$, let $\sigma$ be a power of $g\tau^{-1}$ contained in $O_2$, and let $u$ be the unique fixed point of $\sigma$. We thus have a vertex $u$ and an antimorphism $\sigma$ such that $u^\sigma = u$ and $Q^\sigma = Q$, the first two requirements for $Q$ to be in $P$.

It remains to show that $Q_u \in \text{Syl}_p(M_u)$. Let $U$ be the orbit of $N$ containing $v$; that is, $U = v^N$. Observe that $U^\tau = v^{\tau N} = v^{\tau N} = v^N = U$. Since $g \in N$, also $U^g = U$, and so $U^\sigma = U$. Thus the $k$-hypergraph with vertex set $U$ and edge set $E_X \cap U^k$ admits $N$ as a transitive group of automorphisms and $\sigma$ (restricted to $U$) as an antimorphism. By Corollary 4, it follows that its order $|U|$ is congruent to one of $1, 2, \ldots, k - 1$ modulo $2^{\ell+1}$. Moreover, $U$ is a union of orbits of $\sigma$, whose lengths (with the exception of the fixed point) are all divisible by $2^{\ell+1}$. Hence, $|U| \equiv 1 \pmod{2^{\ell+1}}$ and the fixed point $u$ of $\sigma$ is in $U$. Now, since $u$ and $v$ lie in the same orbit of $N$, $P_u$ and $P_v$ are conjugate in $N$, and so $|P_v| = |P_u|$. It follows that $P_u \in \text{Syl}_p(M_u)$. On the other hand, $Q_u$ is a $p$-subgroup of $M_u$ and $P_u \leq Q_u$, whence $Q_u = P_u$. Thus $Q_u \in \text{Syl}_p(M_u)$, and we conclude that $Q \in P$.

It now follows that $P = Q$ and $P$ is a Sylow $p$-subgroup of $M$. It remains to show that the orbit of $P$ containing $v$ induces a $k$-hypergraph with the required properties. First, since $|P| = p^{d}$ and $P_v \in \text{Syl}_p(M_v)$, we have $|P_v| = p^{d-r}$ and thus $|v^P| = p^r$. Second, since $(v^P)^1 = (v^+) = v^P$, the $k$-hypergraph with vertex set $v^P$ and edge set $E_X \cap (v^P)^k$ is vertex-transitive and self-complementary. By Corollary 4, $|v^P|$ is congruent to one of $1, 2, \ldots, k - 1$ modulo $2^{\ell+1}$. But since, for some odd integer $s$, $v^P$ is also a union of orbits of $\tau^s \in O_2$, we have that $p^r = |v^P| \equiv 1 \pmod{2^{\ell+1}}$, as claimed. This concludes the proof.  \[\square\]
The following immediate corollary gives a more transparent result for small values of the rank $k$.

**Corollary 9.** Let $X$ be a vertex-transitive self-complementary $k$-hypergraph of order $n$, and let $p^r$ be the highest power of a prime $p$ that divides $n$. Then:

- if $k = 2$, then $p^r \equiv 1 \pmod{4}$;
- if $k = 3$ and $n$ is odd, then $p^r \equiv 1 \pmod{4}$; and
- if $k = 4$ or $k = 5$, and $n \equiv 1 \pmod{8}$, then $p^r \equiv 1 \pmod{8}$.

4. Constructions

In this section we present several constructions of vertex-transitive self-complementary uniform hypergraphs, and prove Theorems 6–8. We begin with a generalization of Paley graphs.

4.1. Payley hypergraphs

If $\mathbb{F}$ is a field and $a_0, \ldots, a_{k-1} \in \mathbb{F}$, recall that the Van der Monde determinant of $a_0, \ldots, a_{k-1}$ is defined as $\text{VM}(a_0, \ldots, a_{k-1}) = \det(A)$, where $A$ is the $k \times k$ matrix with entries $[A]_{ij} = a_i^j$ for $i, j \in \{0, 1, \ldots, k-1\}$. It is well known that $\text{VM}(a_0, \ldots, a_{k-1}) = \prod_{i \neq j} (a_i - a_j)$.

**Construction 10.** Let $k$ be an integer, $k \geq 2$, and let $n$ be a prime power congruent to 1 modulo $2^{\ell+1}$ where $\ell = \max\{k(2), (k-1)(2)\}$. Let $\mathbb{F}_n$ be the field of cardinality $n$, let $\omega$ be a generator of the multiplicative group $\mathbb{F}_n^\ast$, and let $c = \gcd\left(n - 1, \binom{k}{2}\right)$. For $i = 0, 1, \ldots, 2c - 1$, denote the coset $\omega^i \langle \omega^{k(k-1)} \rangle$ in $\mathbb{F}_n^\ast$ by $F_i$. Finally, let $P_{n,k}$ be the $k$-hypergraph defined by

$$V_{P_{n,k}} = \mathbb{F}_n$$

and

$$E_{P_{n,k}} = \{\{a_1, \ldots, a_k\} \in \mathbb{F}_n^{(k)} : \text{VM}(a_1, \ldots, a_k) \in F_0 \cup \cdots \cup F_{c-1}\}.$$

Construction 10 is a generalization of the Paley graph construction to hypergraphs. If $k = 2$ and $n$ is a prime power congruent to 1 modulo 4, then $c = \gcd(n - 1, 1) = 1$, and $E_{P_{n,2}} = \{\{a_1, a_2\} : a_2 - a_1 \in (\omega^2)^\ast\}$. Hence $P_{n,2}$ is the graph with vertex set $\mathbb{F}_n$ where two elements of $\mathbb{F}_n$ are adjacent whenever they differ by a square. In other words, $P_{n,2}$ is the Paley graph of order $n$. We shall therefore call hypergraphs obtained from Construction 10 Paley hypergraphs. Note that for the particular value $k = 3$ this construction has been previously introduced by Kocay [3, Theorem 3.15].

**Lemma 11.** The Paley hypergraph $P_{n,k}$ defined in Construction 10 is a vertex-transitive self-complementary $k$-hypergraph.

**Proof.** Let $n'$ and $a$ be integers such that $n - 1 = 2^a n'$ and $n'$ is odd. Let $d$ denote the order of $\omega^{\binom{k}{2}}$ in $\mathbb{F}_n^\ast$. First consider the case with $k$ even, that is, $k = 2^\ell k'$ with $\ell > 0$ and $k'$ odd. Since $2^a = 2^{(n-1)(2)} \geq 2^{\ell+1} > 2^\ell$, we know that $a > \ell$. Then

$$d = \frac{\text{lcm}(n - 1, \binom{k}{2})}{\binom{k}{2}} = \frac{n - 1}{\gcd(n - 1, \binom{k}{2})} = \frac{2^a n'}{\gcd(2^a n', 2^{\ell - 1} k' (k - 1))} = 2^{a - \ell + 1} \frac{n'}{\gcd(n', k' (k - 1))}.$$
Lemma 11

For any automorphism \(\sigma\) of hypergraph \(X\) and edge set of the hypergraph \(P_{n,k}\) is well defined. Second, it means that the sets \(A = F_0 \cup \cdots \cup F_{c-1}\) and \(B = F_c \cup \cdots \cup F_{2c-1} = \omega(\frac{q}{2})\) A partition \(\mathbb{P}^*\).

We conclude the subsection with the proof of Theorem 6.

\[\text{Proof of Theorem 6.} \]

The result now follows immediately from Lemma 11.

4.2. Rank-increasing constructions

In this subsection we present two constructions which take an appropriate self-complementary hypergraph as input and return one of a larger rank. We conclude the subsection with the proof of Theorem 7.

Construction 12. Let \(X\) be a graph (that is, a 2-hypergraph) and let \(V^{**} = V_X \times \mathbb{Z}_2\). Define a mapping \(\sigma : (V^{**})^{(2)} \rightarrow \mathbb{Z}_2\) by

\[
\sigma(\{(u,i),(v,j)\}) = \begin{cases} 
1 & \text{if } i = j = 0 \text{ and } \{u,v\} \in E_X, \text{ or } i = j = 1 \text{ and } \{u,v\} \notin E_X \\
0 & \text{otherwise}.
\end{cases}
\]

Furthermore, let

\[E^{**} = \{(x,y,z) \in (V^{**})^{(3)} : \sigma(\{(x,y)\}) + \sigma(\{(y,z)\}) + \sigma(\{(x,z)\}) = 0\}
\]

and let \(X^{**}\) be the 3-hypergraph with vertex set \(V^{**}\) and edge set \(E^{**}\).

Lemma 13. If \(X\) is a vertex-transitive self-complementary graph of order \(n\), then the 3-hypergraph \(X^{**}\) of order \(2n\) defined in Construction 12 is also vertex-transitive and self-complementary.

\[\text{Proof.} \]

For any automorphism \(\alpha\) and antimorphism \(\varphi\) of the graph \(X\), define permutations \(\tilde{\alpha}, \tilde{\varphi}, \) and \(\bar{\varphi}\) on \(V^{**} = V_X \times \mathbb{Z}_2\) by

\[
\tilde{\alpha}(u, i) = (\alpha(u), i)
\]

\[
\tilde{\varphi}(u, i) = (\alpha(u), i + 1)
\]

\[
\bar{\varphi}(u, i) = (\varphi(u), i + 1)
\]

for all \((u, i) \in V^{**}\). It is not difficult to check that both \(\tilde{\alpha}\) and \(\bar{\varphi}\) are automorphisms of the 3-hypergraph \(X^{**}\). Since \(X\) is vertex-transitive, it follows that the group generated by
\[ \{ \bar{\alpha} : \alpha \in \text{Aut}(X) \} \text{ and } \bar{\psi} \text{ for some antimorphism } \varphi \text{ of } X \text{ is a transitive subgroup of } \text{Aut}(X^{\ast\ast}). \]

Furthermore, it is not difficult to check that \( \bar{\alpha} \) is an antimorphism of the 3-hypergraph \( X^{\ast\ast} \). Hence \( X^{\ast\ast} \) is self-complementary. ■

**Theorem 1** and **Lemma 13** immediately yield the following.

**Corollary 14.** If \( n \) is a Muzychuk integer, then there exists a vertex-transitive self-complementary 3-hypergraph of order \( 2n \).

Note that the converse of **Corollary 14** is false; namely, there exist vertex-transitive self-complementary 3-hypergraphs of order \( 2n \) where \( n \) is not a Muzychuk integer. Examples of such 3-hypergraphs are given in [5]. These 3-hypergraphs (more precisely, two-graphs) are of order \( 1 + q \), where \( q \) is a prime power congruent to 1 modulo 4. Note, however, that by **Corollary 4**, if \( 2n \) is the order of a vertex-transitive self-complementary 3-hypergraph, then \( n \) has to be odd. More odd non-Muzychuk integers \( n \) such that there exists a vertex-transitive self-complementary 3-hypergraph of order \( 2n \) will be given in **Corollary 21**.

**Construction 15.** Let \( X \) be a \( k \)-hypergraph and \( k^{\ast} \) an integer, \( k^{\ast} \geq k \). Define \( X^{\ast} \) to be the \( k^{\ast} \)-hypergraph with vertex set \( V^{\ast} = V_{X} \) and edge set

\[ E^{\ast} = \{ f \in (V^{\ast})^{(k^{\ast})} : f \text{ contains an even number of elements of } E_{X} \text{ as subsets} \}. \]

**Lemma 16.** Let \( X \) be a self-complementary \( k \)-hypergraph and \( k^{\ast} \) an integer, \( k^{\ast} \geq k \). If \( \binom{k^{\ast}}{k} \) is odd, then the \( k^{\ast} \)-hypergraph \( X^{\ast} \) defined in **Construction 15** is self-complementary. Moreover, if \( X \) is vertex-transitive, then so is \( X^{\ast} \).

**Proof.** Take any \( e \in V^{(k^{\ast})} \). By definition, \( e \in E^{\ast} \) if and only if \( e \) contains an even number of elements of \( E_{X} \) as subsets. Since \( \binom{k^{\ast}}{k} \) is odd, the latter is equivalent to saying that \( e \) contains an odd number of elements of \( V^{(k)} \setminus E_{X} \). Hence \( e \notin E^{\ast} \) if and only if \( e \) contains an even number of elements of \( V^{(k)} \setminus E_{X} \). This implies that every antimorphism of \( X \) is also an antimorphism of \( X^{\ast} \) and every automorphism of \( X \) is also an automorphism of \( X^{\ast} \). In particular, \( X^{\ast} \) is self-complementary, and if \( X \) is vertex-transitive, then so is \( X^{\ast} \). ■

**Proof of Theorem 7.** Let \( n \) be a Muzychuk integer and \( k \) a positive integer. Suppose first that \( k \equiv 2 \text{ or } 3 \mod 4 \). By [6], there exists a self-complementary vertex-transitive graph \( X \) of order \( n \). Since \( \binom{k}{2} \) is odd in this case, it follows by **Lemma 16** that there exists a vertex-transitive self-complementary \( k \)-hypergraph of order \( n \).

Suppose now that \( k \equiv 3 \mod 4 \). By **Corollary 14**, there exists a vertex-transitive self-complementary 3-hypergraph \( X \) of order \( 2n \). Since \( \binom{k}{3} \) is odd in this case, it follows by **Lemma 16** that there exists a vertex-transitive self-complementary \( k \)-hypergraph of order \( 2n \). ■

### 4.3. Vertex-transitive self-complementary 3-hypergraphs

Let us now turn our attention to self-complementary 3-hypergraphs. In this case, we are able to solve **Problem 2** for odd orders \( n \), and give a partial answer for even orders. In fact, the result for odd orders follows directly from what we have already proved.
Corollary 17. Let $n$ be an odd integer. There exists a vertex-transitive self-complementary 3-hypergraph of order $n$ if and only if $n$ is a Muzychuk integer.

Proof. Suppose first that there is a vertex-transitive self-complementary 3-hypergraph of order $n$. Since $n$ is odd, Corollary 4 implies that $n \equiv 1 \pmod{4}$. By Theorem 5, it follows that $n$ is a Muzychuk integer. The converse follows directly from Theorem 7.

The following construction tries to mimic the lexicographic product construction for graphs. As mentioned in Section 1, this generalization of lexicographic construction to hypergraphs is not straightforward, and only works in a very special setting.

Construction 18. Let $X_2 = (V, E_2)$ and $X_3 = (V, E_3)$ be a graph and a 3-hypergraph, respectively, with the same vertex set, and let $Y = (U, E)$ be a 3-hypergraph. Define a 3-hypergraph $(X_2, X_3) \triangleright Y$ as follows:

$$V_{(X_2, X_3) \triangleright Y} = V \times U,$$

$$E_{(X_2, X_3) \triangleright Y} = \{(x, u), (x, v), (x, w) : x \in V, \{u, v, w\} \in E\},$$

$$\cup \{(x, u), (x, v), (y, w) : \{x, y\} \in E_2, u, v, w \in U, u \neq v\}$$

$$\cup \{(x, u), (y, v), (z, w) : \{x, y, z\} \in E_3, u, v, w \in U\}.$$

Lemma 19. Let $Y = (U, E)$ be a self-complementary 3-hypergraph, and let $X_2 = (V, E_2)$ and $X_3 = (V, E_3)$ be a graph and a 3-hypergraph, respectively, with the same vertex set and a common antimorphism. Then the 3-hypergraph $(X_2, X_3) \triangleright Y$ defined in Construction 18 is self-complementary. If $\text{Aut}(Y)$ and $\text{Aut}(X_2) \cap \text{Aut}(X_3)$ are transitive, then $(X_2, X_3) \triangleright Y$ is vertex-transitive.

Proof. For any permutation $\alpha$ of $V$ and $\beta$ of $U$, define a permutation $\alpha \times \beta$ on $V \times U$ by

$$(\alpha \times \beta)(x, u) = (\alpha(x), \beta(u)).$$

If $\varphi \in \text{Ant}(X_2) \cap \text{Ant}(X_3)$ and $\psi \in \text{Ant}(Y)$, then it is not difficult to see that $\varphi \times \psi$ is an antimorphism of $(X_2, X_3) \triangleright Y$. Hence $(X_2, X_3) \triangleright Y$ is self-complementary. Similarly, if $\alpha \in \text{Aut}(X_2) \cap \text{Aut}(X_3)$ and $\beta \in \text{Aut}(Y)$, then the permutation $\alpha \times \beta$ is an automorphism of $(X_2, X_3) \triangleright Y$. Hence, if $\text{Aut}(Y)$ and $\text{Aut}(X_2) \cap \text{Aut}(X_3)$ are transitive, then $(X_2, X_3) \triangleright Y$ is vertex-transitive.

Corollary 20. Let $q$ be a prime power congruent to 1 modulo 4, and $P_{q, 2}$ and $P_{q, 3}$ the Paley graph and Paley 3-hypergraph, respectively, as defined in Construction 10. For any vertex-transitive self-complementary 3-hypergraph $Y$, the 3-hypergraph $(P_{q, 2}, P_{q, 3}) \triangleright Y$ is vertex-transitive self-complementary.

Proof. Let $\mathbb{F}_q$ be a finite field of order $q$, and $\omega$ a generator of $\mathbb{F}_q^\times$. As we have seen in Lemma 11, $\varphi : \mathbb{F}_q \to \mathbb{F}_q$ defined by $\varphi(x) = \omega x$ is an antimorphism of both $P_{q, 2}$ and $P_{q, 3}$. Moreover, $\{x \mapsto x + a : a \in \mathbb{F}_q\}$ is a transitive subgroup of $\text{Aut}(P_{q, 2}) \cap \text{Aut}(P_{q, 3})$. Hence $(P_{q, 2}, P_{q, 3}) \triangleright Y$ is vertex-transitive and self-complementary by Lemma 19.

In other words, the above corollary shows that if there exists a vertex-transitive self-complementary 3-hypergraph of order $n$, then there exists a vertex-transitive self-complementary 3-hypergraph of order $qn$ for any prime power $q$ congruent to 1 modulo 4. The following corollary focuses on the orders not already covered by Corollary 17.
Corollary 21. Let \( n \) be a positive integer of the form \( n = (1 + q_0)q_1q_2 \ldots q_s \), where \( q_i \) is a prime power congruent to 1 modulo 4 for each \( i = 0, 1, \ldots, s \). Then there exists a self-complementary vertex-transitive 3-hypergraph of order \( n \).

Proof. It was shown in [5] that there exists a vertex-transitive self-complementary 3-hypergraph of order \( 1 + q \) for every prime power \( q \) congruent to 1 modulo 4. (These hypergraphs are in fact two-graphs, introduced by Taylor in [7]. A two-graph is a 3-hypergraph with the additional property that every set of four vertices contains an even number of edges.) We use Corollary 20 inductively, taking \( Y_0 \) to be the self-complementary 2-transitive two-graph of order \( 1 + q_0 \), and \( Y_i = (P_{q_i, 2}, P_{q_i, 3}) \triangle Y_{i-1} \) for \( i = 1, \ldots, s \). Then, by Corollary 20, \( Y_s \) is a self-complementary vertex-transitive 3-hypergraph of order \( n = (1 + q_0)q_1q_2 \ldots q_s \). ■

Proof of Theorem 8. The result now follows directly from Corollaries 17 and 21, and Theorem 7. ■

5. Conclusion

Although several partial results have been proved in this paper, Problem 2 remains mostly unanswered, and it seems to us that no simple solution is possible. Let us therefore suggest a few subproblems that we believe could be solved in the near future.

The first subproblem concerns the hypergraphs of rank 3. Theorem 8 offers a solution for all odd orders \( n \), while for the even orders it shows that if \( 2^m \in H_3 \), then \( m \) is odd. In addition, several values of \( m \) such that \( 2^m \in H_3 \) are given. For example, all Muzychuk integers have this property. It is therefore natural to pose the following problem.

Problem 22. Find all odd non-Muzychuk integers \( m \) such that there exists a vertex-transitive self-complementary 3-hypergraph of order \( 2^m \).

Excluding hypergraphs obtained using the “wreath product” from Construction 18, all known vertex-transitive self-complementary 3-hypergraphs satisfying the condition of Problem 22 (which were given in [5]) have the automorphism group acting doubly transitively on the vertex set. On the other hand, no other self-complementary uniform hypergraph with this property is known to us. Since doubly transitive permutation groups are classified, their careful examination should give a solution to the following problem.

Problem 23. Determine all self-complementary uniform hypergraphs \( X \) such that \( Aut(X) \) acts doubly transitively on the vertex set \( V_X \).

For cases with \( k \) or \( k - 1 \) equal to a power of 2, Theorem 5 gives a necessary condition for \( n \) to belong to \( H_k \). The smallest value of \( k \) for which no such condition has been proved is therefore 6. On the other hand, besides Theorem 7 which ensures existence of vertex-transitive self-complementary uniform 6-hypergraphs of order \( n \) for every Muzychuk integer \( n \), no other sufficient conditions for \( n \) to belong to \( H_6 \) are known to us. Hence the following question.

Question 24. Does there exist a vertex-transitive self-complementary 6-hypergraph whose order is not a Muzychuk integer?
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