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Integration of covariance kernels and stationarity

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The necessary and sufficient matrix condition of Mitchell, Morris and Ylvisaker (1990) for a stationary Gaussian process to have a specified process as kth derivative is investigated. The mean-square smoothing approach of stationary processes requires integration of covariance functions preserving stationarity. By providing a recursive representation of the involved reproducing kernel Hilbert spaces it is possible to analyse another criterion for k-fold integration of a process. This criterion only contains inequalities for the variances of the integrated processes. If the Hilbert space associated with the covariance function has a special form, which often occurs, then it can be shown that such processes can be integrated arbitrarily often. This is especially the case for the Ornstein–Uhlenbeck process. The results are applied to the linear and the exponential kernel and yield explicit norms in the corresponding reproducing kernel Hilbert spaces for each integration.

mean-square integration * stationary process * reproducing kernel Hilbert space * Ornstein–Uhlenbeck process

1. Introduction

In the context of Bayesian prediction of computer experiments Currin, Mitchell, Morris and Ylvisaker (1991) raise the problem of smoothing stationary processes on an interval in the sense of smoothing the covariance function.

Mitchell, Morris and Ylvisaker (1990) — for later references abbreviated MMY — found conditions under which there exists a stationary process whose kth derivative is a given stationary process, for example an Ornstein–Uhlenbeck process. The conditions given there deal with the reproducing kernel Hilbert space (RKHS) associated with the covariance function of the prespecified process, and require nonnegative definiteness of a certain matrix.

In the present paper we propose an alternative approach in which the essential arguments of MMY can be used successfully. We repeatedly apply the procedure of once smoothing a given process. This leads to the problem of handling the RKHSs that are associated with the integrated processes emerging in each step. From a

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characterization of reproducing kernels in terms of the inner product in Sobolev spaces $W_2^{(k)}$ in de Boor and Lynch (1966) only integration of nonstationary kernels can be deduced. The work of Hájek (1962) cannot be used either, because the processes here do not satisfy the necessary conditions for stationary processes with rational spectral density.

Theorem 1 in Section 2 states the theorem of MMY. Based on it Theorem 2 gives another necessary and sufficient condition for the existence of the smoothed process, that involves only inequalities, but different Hilbert spaces. The equivalence of these conditions is made clear in Theorem 3, where a formula for the determinant and the rank of the matrix involved in Theorem 1 is given.

The difficulty with handling the various RKHSs is solved in Section 3 by the main result of the paper, Theorem 4. It derives a recursive relation between RKHSs that arise by integration of the kernel in such a way that stationarity is maintained. Also a nonrecursive result is stated as a remark to the theorem. As an application the problem of existence of a stationary process whose kth derivative is a specified process is solved.

In Section 4 the results are applied to the linear and the exponential covariance kernels. The norms in the RKHS belonging to the covariance function of the one-fold integrated process are evaluated. In particular, we show that processes with exponential covariance may be integrated to arbitrary order, a question which remained open in MMY.

2. Conditions for the existence

All our processes are defined on the interval $[0, T]$ for $T>0$ and have zero mean. All integrals with processes as integrands are defined to be the mean-square limit of their approximating Riemann sums. A derivative of a process is also understood in the mean-square sense. Further we deal with Gaussian processes. (It would suffice to have second-order stationary processes, with some obvious modifications.)

In this section we first state the result of MMY, which gives a condition for the existence of a stationary process $\{X_t, t\in[0, T]\}$, with a given process $\{Y_t, t\in[0, T]\}$ as kth derivative, and use it to derive another condition for the existence.

Let $\{Y_t, t\in[0, T]\}$ be a stationary Gaussian process on a probability space (Ω, Σ, P) with continuous covariance function

$$\mathrm{Cov}(Y_t, Y_s) = E(Y_t\cdot Y_s) = \rho(|t-s|) \quad \forall s, t\in[0, T].$$

The goal is to define recursive processes $\{X_t^{(k-i)}, t\in[0, T]\}$ by

$$X_t^{(k-i)} := X_0^{(k-i)} + \int_0^t X_s^{(k-i+1)}\,\mathrm{d}s \quad \text{for } i=1,\ldots,k, \tag{1}$$

with $X^{(k)} := Y$ and Gaussian random variables $X_0^{(k-i)}$, $i=1,\ldots,k$ on (Ω, Σ, P), in such a way that the processes $X^{(k-i)}$ for $i=1,\ldots,k$ are stationary.

We define $\rho_0 = \rho$ and with k positive constants $\sigma_1^2, \ldots, \sigma_k^2$,

$$\rho_i(t) := \sigma_i^2 - \int_0^t \int_0^u \rho_{i-1}(v)\, \mathrm{d}v\, \mathrm{d}u, \quad i = 1, \ldots, k. \tag{2}$$

The process $X^{(k-i)}$ - if it exists - will have covariance function ρ_i and variance σ_i^2. Hence the process $X := X^{(0)}$, constructed by k-fold integration of Y, has covariance function ρ_k and variance σ_k^2. The function ρ_k and its $2k-2$ derivatives are used to define the $k \times k$ matrix M, according to

$$M_{ij} = (-1)^{i-1} \rho_k^{(i+j-2)}(0), \quad 1 \leq i, j \leq k.$$

The matrix M is symmetric and has, up to the sign ± 1, only $k+1$ different entries: $\sigma_1^2, \ldots, \sigma_k^2$ and 0, because of

$$|\rho_k^{(i+j-2)}(0)| = \begin{cases} |\rho_{k-(i+j-2)/2}(0)| = \sigma_{k-(i+j-2)/2}^2 & \text{for } i+j \text{ even,} \\ |\rho'_{k-(i+j-1)/2}(0)| = 0 & \text{for } i+j \text{ odd.} \end{cases}$$

Under the assumption that the functions $\rho_k^{(k+i-1)}$, $i = 1, \ldots, k$ — depending on the constants $\sigma_1^2, \ldots, \sigma_k^2$—lie in HK_ρ, the RKHS associated with the covariance kernel $K(t, s) = \rho(|t-s|)$, the $k \times k$ matrix Q is defined by inner products in HK_ρ:

$$Q_{ij} = \langle (-1)^{i-1} \rho_k^{(k+i-1)}, (-1)^{j-1} \rho_k^{(k+j-1)} \rangle_\rho, \quad 1 \leq i, j \leq k.$$

The theorem of MMY gives a necessary and sufficient condition for the existence of the process X, depending on the covariance function ρ and the variances σ_i^2 of the processes $X^{(k-i)}$ through the matrices M and Q.

Theorem 1 (MMY). *There exists a stationary Gaussian process $\{X_t, t \in [0, T]\}$ with covariance function ρ_k and kth derivative $\{Y_t, t \in [0, T]\}$ if and only if*

(i) $\rho_k^{(k+i-1)} \in HK_\rho, \quad i = 1, \ldots, k,$

(ii) *$M - Q$ is nonnegative definite.* □

We remark that the theorem of MMY holds without assuming nonsingularity of the finite-dimensional distributions of all processes, as do MMY.

To see this consider the matrices M and Q which are covariance matrices of the random vectors

$$(X_0, X_0^{(1)}, \ldots, X_0^{(k-1)}) \quad \text{and} \quad (\Psi(\rho_k^{(k)}), \Psi((-1)\rho_k^{(k+1)}), \ldots, \Psi((-1)^{k-1}\rho_k^{(2k-1)})),$$

respectively, where Ψ is the isometry between HK_ρ and $L_2(Y_t, t \in [0, T])$. We have (see Parzen, 1961, Theorem 4E)

$$\Psi((-1)^{i-1} \rho_k^{(k+i-1)}) = E[X_0^{(i-1)} \mid Y_t, t \in [0, T]], \quad i = 1, \ldots, k.$$

The nonnegative definiteness of $M - Q$ then follows from

$$\operatorname{Var}(X_0^{(i)}) \geq \operatorname{Var}(E[X_0^{(i-1)} \mid Y_t, t \in [0, T]]), \quad i = 1, \ldots, k,$$

by considering linear combinations of the random variables $X_0^{(i)}$.

We now use Theorem 1 repeatedly for a single smoothing, that is for $k = 1$, in order to obtain equivalent conditions which avoid the matrix condition of Theorem 1.

Theorem 2. *There exists a stationary Gaussian process* $\{X_t, t\in[0,T]\}$ *with covariance function* ρ_k *and kth derivative* $\{Y_t, t\in[0,T]\}$ *if and only if*

$$\begin{aligned}&(\mathrm{i})' \quad \rho_i' \in HK_{\rho_{i-1}},\\ &(\mathrm{ii})' \quad \sigma_i^2 \geq \|\rho_i'\|^2_{\rho_{i-1}},\end{aligned} \qquad i=1,\ldots,k,$$

where $\|\cdot\|_{\rho_{i-1}}$ *is the norm in the RKHS* $HK_{\rho_{i-1}}$ *for the kernel* $K_{i-1}(t,s)=\rho_{i-1}(|t-s|)$.

Proof. Owing to Theorem 1 with $k=1$ the necessary and sufficient conditions for once smoothing a given process Y with covariance function ρ are

$$\rho_1' \in HK_\rho \quad \text{and} \quad \sigma_1^2 - \|\rho_1'\|^2_\rho \geq 0,$$

where ρ_1 is the covariance function and σ_1^2 the variance of X with $X'=Y$. With $X^{(k)} := Y$ we apply this to $X^{(k-i)}$ instead of X and $(X^{(k-i)})' = X^{(k+i+1)}$ instead of Y, for all $i=1,\ldots,k$. This yields conditions (i)′ and (ii)′. □

Remark. The crucial condition for the existence is (i)′, which is in general stronger than (i) of Theorem 1. But we will see in the next section that by strengthening condition (ii)′ we can weaken (i)′ to (i).

The difference between the equivalent conditions (i) and (ii) of Theorem 1, and (i)′ and (ii)′ of Theorem 2 lies in the Hilbert spaces used. In (i) and (ii) we have only the RKHS HK_ρ with respect to the covariance function of the given process. However there is a matrix criterion for the constants $\sigma_1^2,\ldots,\sigma_k^2$ to verify. This is what makes statements about k-fold integration difficult, or even impossible (see MMY, p. 115). In contrast conditions (i)′ and (ii)′ involve k Hilbert spaces $HK_\rho,\ldots,HK_{\rho_{k-1}}$, and their norms. But the matrix has vanished and we are left with only k inequalities for the constants $\sigma_1^2,\ldots,\sigma_k^2$. Before we turn to the problem with the different Hilbert spaces in the next section we state a result which gives more insight in the equivalence of the conditions of Theorem 1 and of Theorem 2.

For proving this result we have to describe the construction of the process $\{X_t, t\in[0,T]\}$ contained in the proofs of sufficiency in Theorem 1 and Theorem 2. The important point for the construction of X consists in finding appropriate Gaussian random variables $X_0^{(k-i)}$ for $i=1,\ldots,k$ in (1).

We first consider the construction of the vector $(X_0, X_0^{(1)},\ldots,X_0^{(k-1)})$ as described in MMY, that is under conditions (i) and (ii). Because of (i) there exist elements

$$\xi_i := \Psi((-1)^{k-i}\rho_k^{(2k-i)}) \in L_2(Y_t, t\in[0,T]), \quad i=1,\ldots,k.$$

The vector $\xi=(\xi_k,\ldots,\xi_1)$ has covariance matrix Q. $M-Q$ is nonnegative definite by assumption (ii). Hence there is a vector $\eta=(\eta_k,\ldots,\eta_1)$ of random variables $\eta_i\in L^2(\Omega)$, orthogonal to $L_2(Y_t, t\in[0,T])$, with covariance matrix $M-Q$. We define

$$X_0^{(k-i)} := \eta_i + \xi_i, \quad i=1,,\ldots,k.$$

Now we turn to the construction of $X_0^{(k-i)}$ for $i=1,\dots,k$ in the proof of Theorem 2. With (i)′ there exists for every $i=1,\dots,k$ a random variable

$$\bar{\xi}_i := \Psi_{i-1}(\rho_i') \in L_2(X_t^{(k-i+1)}, t \in [0,T]),$$

where Ψ_{i-1} is the isometry between $HK_{\rho_{i-1}}$ and $L_2(X_t^{(k-i+1)}, t\in[0,T])$. We note that $\mathrm{Var}(\bar{\xi}_i) = \|\rho_i'\|^2_{\rho_{i-1}}$ and $\bar{\xi}_1 = \xi_1$. Using (ii)′ there is an element $\bar{\eta}_i$ in $L^2(\Omega)$ orthogonal to $L_2(X_t^{(k-i+1)}, t\in[0,T])$ with $\mathrm{Var}(\bar{\eta}_i) = \sigma_i^2 - \|\rho_i'\|^2_{\rho_{i-1}}$ for all $i=1,\dots,k$. The random vector $\bar{\eta} = (\bar{\eta}_k,\dots,\bar{\eta}_1)$ has covariance matrix

$$\mathrm{diag}(\sigma_k^2 - \|\rho_k'\|^2_{\rho_{k-1}}, \dots, \sigma_1^2 - \|\rho_1'\|^2_{\rho}).$$

This follows from the orthogonality of $\bar{\eta}_i$ and $L_2(X_t^{(k-i+1)}, t\in[0,T)]$, which contains all random variables $\bar{\eta}_j$, $j<i$ by construction (1) in combination with the definition

$$X_0^{(k-i)} := \bar{\xi}_i + \bar{\eta}_i, \quad i=1,\dots,k. \tag{3}$$

Theorem 3. *Assume any one of the equivalent conditions from Theorem* 1 *or from Theorem* 2. *Then we have for all dimensions k,*

$$\det(M-Q) = \prod_{i=1}^{k} (\sigma_i^2 - \|\rho_i'\|^2_{\rho_{i-1}}),$$

$$\mathrm{rank}(M-Q) = k - \#\{i=1,\dots,k : \sigma_i^2 = \|\rho_i'\|^2_{\rho_{i-1}}\}.$$

Proof. The important observation is that the set $\{\bar{\eta}_1,\dots,\bar{\eta}_k\}$ of random variables in $L^2(\Omega)$ is the set of orthogonalized random variables $\{\eta_1,\dots,\eta_k\}$. This can be seen as follows.

The elements $\xi_i = \Psi((-1)^{k-i}\rho_k^{(2k-i)})$ of $L_2(Y_t, t\in[0,T])$ for $i=1,\dots,k$ are the conditional expectations $E[X_0^{(k-i)} \mid Y_t, t\in[0,T]]$. In the same way we get

$$\bar{\xi}_i = \Psi_{i-1}(\rho_i') = E[X_0^{(k-i)} \mid X_t^{(k-i+1)}, t\in[0,T]], \quad i=1,\dots,k.$$

Using

$$\begin{aligned} L_2(Y_t, t\in[0,T], \eta_1,\dots,\eta_{i-1}) &= L_2(Y_t, t\in[0,T], X_0^{(k-1)},\dots,X_0^{(k-i+1)}) \\ &= L_2(X_t^{(k-i+1)}, t\in[0,T]) \end{aligned}$$

and the orthogonality of η_i and $L_2(Y_t, t\in[0,T])$ for all $i=1,\dots,k$, we get the following equivalence in $L^2(\Omega)$ for $i=2,\dots,k$:

$$\begin{aligned} \bar{\eta}_i &= X_0^{(k-i)} - \bar{\xi}_i \\ &= X_0^{(k-i)} - E[X_0^{(k-i)} \mid Y_t, t\in[0,T], X_0^{(k-1)},\dots,X_0^{(k-i+1)}] \\ &= \eta_i + E[X_0^{(k-i)} \mid Y_t, t\in[0,T]] - E[X_0^{(k-i)} \mid Y_t, t\in[0,T], X_0^{(k-1)},\dots,X_0^{(k-i+1)}] \\ &= \eta_i - E[X_0^{(k-i)} - E[X_0^{(k-i)} \mid Y_t, t\in[0,T]] \mid Y_t, t\in[0,T], X_0^{(k-1)},\dots,X_0^{(k-i+1)}] \\ &= \eta_i - E[\eta_i \mid Y_t, t\in[0,T], \eta_1,\dots,\eta_{i-1}] \\ &= \eta_i - E[\eta_i \mid \eta_1,\dots,\eta_{i-1}]. \end{aligned}$$

For $i=1$ we have $\bar{\eta}_1 = \eta_1$.

From this representation we conclude $\bar{\eta} = S^{t}\eta$. Here S is a lower-triangular matrix with full rank. The thereoem now follows from

$$\operatorname{diag}(\sigma_k^2 - \|\rho_k'\|_{\rho_{k-1}}^2, \ldots, \sigma_1^2 - \|\rho_1'\|_{\rho}^2) = \operatorname{Cov}(\bar{\eta}) = \operatorname{Cov}(S^{t}\eta) = S^{t}\operatorname{Cov}(\eta)S$$
$$= S^{t}(M-Q)S. \qquad \square$$

Remark. Because of the recursive structure of $M-Q$ the consecutive products

$$\prod_{i=1}^{j} (\sigma_i^2 - \|\rho_i'\|_{\rho_{i-1}}^2)$$

for $j \leqslant k$ are the leading principal minors of $M-Q$ in reversed order.

Theorem 3 studies the relation between the matrix $M-Q$ and the elements $(\sigma_i^2 - \|\rho_i'\|_{\rho_{i-1}}^2)$. But for the full understanding of Theorem 2 the problem with the Hilbert spaces $HK_{\rho}, HK_{\rho_1}, \ldots, HK_{\rho_{k-1}}$ rests to solve. This is done in the next section.

3. A recursive relation between the Hilbert spaces

In this section we will establish the connection between the RKHSs $HK_{\rho_{i-1}}$ and HK_{ρ_i}. The result then can be used to find weaker sufficient conditions for the existence problem of the process X raised in Section 2.

Let $HK_{\rho_{i-1}}$ be given with inner product $\langle\cdot,\cdot\rangle_{\rho_{i-1}}$ associated to the covariance function ρ_{i-1} of $X^{(k-i+1)}$, the $(i-1)$-fold integrated process. In (2) a new covariance function ρ_i is defined under the assumptions $\rho_i' \in HK_{\rho_{i-1}}$ and $\sigma_i^2 \geqslant \|\rho_i'\|_{\rho_{i-1}}^2$.

Theorem 4. *For $i \in \mathbb{N}$ the function space HK_{ρ_i}* and the corresponding inner product are recursively given by (with $\rho_0 = \rho$):

(a) *For $\sigma_i^2 > \|\rho_i'\|_{\rho_{i-1}}^2$*,

$$HK_{\rho_i} = \{f \in C^1[0,T]: f' \in HK_{\rho_{i-1}}\},$$
$$\langle f, g\rangle_{\rho_i} = \langle f', g'\rangle_{\rho_{i-1}} + (\sigma_i^2 - \|\rho_i'\|_{\rho_{i-1}}^2)^{-1}(f(0) - \langle f', \rho_i'\rangle_{\rho_{i-1}})(g(0) - \langle g', \rho_i'\rangle_{\rho_{i-1}}).$$

(b) *For $\sigma_i^2 = \|\rho_i'\|_{\rho_{i-1}}^2$*,

$$HK_{\rho_i} = \{f \in C^1[0,T]: f' \in HK_{\rho_{i-1}}, f(0) = \langle f', \rho_i'\rangle_{\rho_{i-1}}\},$$
$$\langle f, g\rangle_{\rho_i} = \langle f', g'\rangle_{\rho_{i-1}}.$$

Proof. Let f be a given function in $HK_{\rho_{i-1}}$. Hence there is a random variable $\Psi_{i-1}(f) \in L_2(X_t^{(k-i+1)}, t \in [0,T])$ with $E(\Psi_{i-1}(f)\cdot X_t^{(k-i+1)}) = f(t)$ for all $t \in [0,T]$. In the first part we verify:

$$g(t) = c + \int_0^t f(s)\,\mathrm{d}s \in HK_{\rho_i} \quad \text{for } \begin{cases} \text{any } c \in \mathbb{R}, & \text{in case (a)}, \\ c = \langle f, \rho_i'\rangle_{\rho_{i-1}}, & \text{in case (b)}. \end{cases}$$

This will be shown by finding the corresponding elements in $L_2(X_t^{(k-i)}, t \in [0,T])$.

By the construction of $X^{(k-i)}$ as in (1) we have

$$L_2(X_t^{(k-i)}, t\in[0,T]) = L_2(X_t^{(k-i+1)}, t\in[0,T], X_0^{(k-i)})$$
$$= L_2(X_t^{(k-i+1)}, t\in[0,T], \bar{\eta}_i).$$

Now for each $a\in\mathbb{R}$ we define a random variable $Z_a \in L_2(X_t^{(k-i)}, t\in[0,T])$ by the orthogonal sum $Z_a := \Psi_{i-1}(f) + a\cdot\bar{\eta}_i$, and every element in $L_2(X_t^{(k-i)}, t\in[0,T])$ has this form. Using (3) we get

$$E(Z_a\cdot X_t^{(k-i)}) = E(Z_a\cdot X_0^{(k-i)}) + E\left(Z_a\cdot\int_0^t X_s^{(k-i+1)}\,\mathrm{d}s\right)$$
$$= E(\Psi_{i-1}(f)\cdot\Psi_{i-1}(\rho_i')) + a\cdot\mathrm{Var}(\bar{\eta}_i)$$
$$+ E\left(\Psi_{i-1}(f)\cdot\int_0^t X_s^{(k-i+1)}\,\mathrm{d}s\right)$$
$$= \langle f, \rho_i'\rangle_{\rho_{i-1}} + a(\sigma_i^2 - \|\rho_i'\|^2_{\rho_{i-1}}) + \int_0^t f(s)\,\mathrm{d}s.$$

Hence in case (a) the function $g(t) = c + \int_0^t f(s)\,\mathrm{d}s$ is an element of HK_{ρ_i} for any $c\in\mathbb{R}$, because it corresponds to

$$Z_a \in L_2(X_t^{(k-i)}, t\in[0,T]),\quad a = \frac{c - \langle f,\rho_i'\rangle_{\rho_{i-1}}}{\sigma_i^2 - \|\rho_i'\|^2_{\rho_{i-1}}}.$$

In case (b) only the function $g(t) = \langle f,\rho_i'\rangle_{\rho_{i-1}} + \int_0^t f(s)\,\mathrm{d}s$ lies in HK_{ρ_i}, because we have now $\bar{\eta}_i = 0$ and any $Z\in L_2(X_t^{(k-i)}, t\in[0,T]) = L_2(X_t^{(k-i+1)}, t\in[0,T])$ has the form $\Psi_{i-1}(f)$ for some $f\in HK_{\rho_{i-1}}$.

The second part is to investigate the inner product in HK_{ρ_i}. This will be done by evaluating the covariance of the corresponding random variables in $L_2(X_t^{(k-i)}, t\in[0,T])$. In case (a) for $f, g\in HK_{\rho_i}$ the elements $\Psi_i(f)$, $\Psi_i(g)$ are given by the orthogonal sum

$$\Psi_i(f) = \Psi_{i-1}(f') + \frac{f(0) - \langle f',\rho_i'\rangle_{\rho_{i-1}}}{\sigma_i^2 - \|\rho_i'\|^2_{\rho_{i-1}}}\cdot\bar{\eta}_i, \tag{4}$$

and analogue for g. The inner product of f and g now is given by

$$\langle f, g\rangle_{\rho_i} = E(\Psi_i(f)\cdot\Psi_i(g))$$
$$= E(\Psi_{i-1}(f')\cdot\Psi_{i-1}(g'))$$
$$+ (\sigma_i^2 - \|\rho_i'\|^2_{\rho_{i-1}})^{-2}(f(0) - \langle f',\rho_i'\rangle_{\rho_{i-1}})(g(0) - \langle g',\rho_i'\rangle_{\rho_{i-1}})\cdot\mathrm{Var}(\bar{\eta}_i)$$
$$= \langle f', g'\rangle_{\rho_{i-1}} + (\sigma_i^2 - \|\rho_i'\|^2_{\rho_{i-1}})^{-1}(f(0) - \langle f',\rho_i'\rangle_{\rho_{i-1}})(g(0) - \langle g',\rho_i'\rangle_{\rho_{i-1}}).$$

In case (b) the second parts in (4) vanish and therefore the inner product is simply given by the first part of the sum of case (a). □

Remarks. (1) If $M-Q$ has full rank, which is equivalent to $\sigma_i^2>\|\rho_i'\|^2_{\rho_{i-1}}$ for all $i=1,\dots,k$ as we saw in Theorem 3, we can also prove the following nonrecursive form of the inner product in HK_{ρ_k},

$$\langle f, g\rangle_{\rho_k}=\langle f^{(k)}, g^{(k)}\rangle_\rho+\tilde{f}^{\mathrm{t}}(M-Q)^{-1}\tilde{g}.$$

Here the k-vector $\tilde{h}$ of a function $h\in HK_{\rho_k}$ consists of components

$$\tilde{h}_i:=h^{-1}(0)-\langle(-1)^{-1}\rho_k^{-1}, h^{(k)}\rangle_\rho,\quad i=1,\dots,k.$$

In case of singularity, the second part of the sum must be changed to the bilinear form in the preimages under $(M-Q)S$ of $\tilde{f}$ and $\tilde{g}$ induced by the diagonal matrix $S^{\mathrm{t}}(M-Q)S$. This can be proved as in Hájek (1962), where other RKHSs are given.

(2) By use of the orthogonal decomposition (4) of $\Psi_i(f)$ and $\Psi_i(g)$, the inner product relation in case (a) is seen to be the well-known formula:

$$\operatorname{Cov}(\Psi_i(f), \Psi_i(g))=\operatorname{Cov}(\Psi_i(f), \Psi_i(g)\,|\,\bar{\eta}_i)$$
$$+\operatorname{Var}(\bar{\eta}_i)^{-1}\operatorname{Cov}(\Psi_i(f), \bar{\eta}_i)\operatorname{Cov}(\Psi_i(g), \bar{\eta}_i).$$

Thus conditioning on $\bar{\eta}_i$, the random variable that represents the gap between $L_2(X_t^{(k-i+1)}, t\in[0, T])$ and $L_2(X_t^{(k-i)}, t\in[0, T])$, leads to an expression for the inner product in HK_{ρ_i} with the inner product in $HK_{\rho_{i-1}}$.

From Theorem 4 we deduce that condition (i)′ of Theorem 2 is in general stronger than (i) of Theorem 1. This is seen by noting that $f\in HK_{\rho_i}$ implies $f^{(i)}\in HK_\rho$ and $\rho_i^{(i)}\equiv(-1)^{k-i}\rho_k^{(2k-i)}$ for $i=1,\dots,k$.

By defining $m=k-\#\{i=1,\dots,k\colon \sigma_i^2=\|\rho_i'\|^2_{\rho_{i-1}}\}$ we can formulate condition (ii)′ more exactly. The next corollary states that, if we have $m=k$, the strongest form of (ii)′, then (i) implies (i)′.

Corollary 1. *We have equivalence of*

$$\{\text{(i)}'\ \rho_i'\in HK_{\rho_{i-1}},\ \text{(ii)}'\ \textit{with}\ m=k\colon \sigma_i^2>\|\rho_i'\|^2_{\rho_{i-1}},\ i=1,\dots,k\}$$

and

$$\{\text{(i)}\ \rho_k^{(k+i-1)}\in HK_\rho,\ \text{(ii)}'\ \textit{with}\ m=k\colon \sigma_i^2>\|\rho_i'\|^2_{\rho_{i-1}},\ i=1,\dots,k\}.$$

Proof. We only have to prove the converse direction. So assume $\rho_k^{(k+i-1)}\in HK_\rho$ for $i=1,\dots,k$. This implies $\rho_1'\in HK_\rho$. The assumption $m=k$ entails $\sigma_1^2>\|\rho_1'\|^2_\rho$, and Theorem 4 yields $HK_{\rho_1}=\{f\colon f'\in HK_\rho\}$. So $\rho_2'\in HK_{\rho_1}$ is equivalent with $\rho_1\in HK_\rho$, which is the assumption (i) with $i=k-1$. Now we have $\sigma_2^2>\|\rho_2'\|^2_{\rho_1}$ (again from $m=k$) and we continue with Theorem 4 as above. The condition: (i)′ $\rho_j'\in HK_{\rho_{j-1}}$, takes the form $\rho_j^{(j)}\equiv(-1)^{k-j}\rho_k^{(2k-j)}\in HK_\rho$ for all $j=1,\dots,k$. This is condition (i) with $i=k-j+1$. $\square$

Remark. We also see in the proof of Corollary 1 that, if already one σ_i^2 equals its lower bound $\|\rho_i'\|_{\rho_{i-1}}^2$ then condition (i)′ is stronger than (i). We can say that for $m<k$ condition (i)′ contains a part of (ii), the nonnegative definiteness of $M-Q$. So the relations between (i)′, (ii)′ and (i), (ii) are rather delicate. In the extreme case ($m=0$) (i)′ includes knowledge about all nondiagonal elements of $M-Q$ (they are zero), whereas only the diagonal elements are covered by (ii)′ (they are also zero).

This corollary is important for the application of Theorem 2. We only have to fulfil condition (i) and (ii)′ in its strongest form (i.e., $m=k$), that is a combination of the weaker conditions in Theorem 1 and 2, to guarantee existence of the smoothed stationary process.

Corollary 2. *There exists a stationary Gaussian process X with covariance function ρ_k and kth derivative equal to Y, if we have*

$$\text{(i) } \rho_k^{(k+i-1)} \in HK_\rho \text{ and (ii)}' \text{ with } m=k\text{: } \sigma_i^2 > \|\rho_i'\|_{\rho_{i-1}}^2,\ i=1,\ldots,k. \qquad \square$$

Remark. With the Sobolev space $W_2^{(1)}$ as RKHS HK_ρ, as it is often the case, condition (i) is fulfilled, and as long as the constants are chosen big enough integration of the process is possible (see Corollary 3 in Section 4).

4. Examples

We consider the linear covariance on [0, 1],

$$\rho(t)=1-\lambda t, \quad 0<\lambda<2,$$

and the exponential covariance

$$\rho(t)=\frac{1}{2\alpha}\exp(-\alpha t), \quad \alpha>0.$$

Because the RKHS for both covariance functions is (see Hájek, 1962)

$$W_2^{(1)}=\{f\text{: } f \text{ absolutely continuous, } f'\in L^2[0,T]\},$$

all integrals of ρ of arbitrary order are elements of $W_2^{(1)}$. This leads to:

Corollary 3. *If in each integration step $\sigma_i^2 > \|\rho_i'\|_{\rho_{i-1}}^2$ is chosen, then the linear and the exponential covariance can be integrated arbitrarily often.* □

Finally we use Theorem 4 to evaluate the norms arising from one integration first of the linear and then of the exponential covariance. For the ρ-norms stated below see Hájek (1962).

(I) The squared norm for the linear covariance is

$$\|f\|_\rho^2=\frac{1}{2\lambda}\int_0^1 f'(s)^2\,\mathrm{d}s+\frac{1}{2(2-\lambda)}(f(0)+f(1))^2.$$

Choose

$$\sigma_1^2 > \|\rho_1'\|_\rho^2 = \tfrac{1}{24}\lambda - \tfrac{1}{4} + \frac{1}{2\lambda} \quad \text{in } \rho_1(t) = \sigma_1^2 - \tfrac{1}{2}t^2 + \tfrac{1}{6}\lambda t^3.$$

The squared norm in $HK_{\rho_1} = W_2^{(2)}$ is then

$$\|f\|_{\rho_1}^2 = \frac{1}{2\lambda}\int_0^1 f''(s)^2\,\mathrm{d}s + \frac{1}{2(2-\lambda)}(f'(0)+f'(1))^2$$
$$+(\sigma_1^2 - \|\rho_1'\|_\rho^2)^{-1}\left\{\frac{2-\lambda}{4\lambda}(f'(1)-f'(0)) + \tfrac{1}{2}(f(1)+f(0))\right\}^2.$$

(II) The squared norm for the exponential covariance is

$$\|f\|_\rho^2 = \int_0^1 (f'(s)+\alpha f(s))^2\,\mathrm{d}s + 2\alpha f(0)^2.$$

With

$$\sigma_1^2 > \|\rho_1'\|_\rho^2 = \frac{1}{4\alpha^2} \quad \text{for } \rho_1(t) = \sigma_1^2 - \frac{1}{2\alpha^3}\exp(-\alpha t) - \frac{1}{2\alpha^2}t + \frac{1}{2\alpha^3}$$

we get the squared norm in $W_2^{(2)}$,

$$\|f\|_{\rho_1}^2 = \int_0^1 (f''(s)+\alpha f'(s))^2\,\mathrm{d}s + 2\alpha f'(0)^2$$
$$+(\sigma_1^2 - \|\rho_1'\|_\rho^2)^{-1}\left\{\frac{1}{2\alpha}(f'(1)-f'(0)) + \tfrac{1}{2}(f(1)+f(0))\right\}^2.$$

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References

C. de Boor and R.E. Lynch, On splines and their minimum properties, J. Math. Mech. 15(6) (1966) 953-969.

C. Currin, T. Mitchell, M. Morris and D. Ylvisaker, Bayesian prediction of deterministic functions, with applications to the design and analysis of computer experiments, J. Amer. Statist. Assoc. 86 (1991) 953-963.

J. Hájek, On linear statistical problems in stochastic processes, Czechoslovak Math. J. 12(87) (1962) 404-444.

T. Mitchell, M. Morris and D. Ylvisaker, Existence of smoothed stationary processes on an interval, Stochastic Process. Appl. 35 (1990) 109-119.

E. Parzen, An approach to time series analysis, Ann. Math. Statist. 32 (1961) 951-989.