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A Generalization of the Hyers–Ulam–Rassias Stability of Approximately Additive Mappings

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In this paper we prove a generalization of the stability of approximately additive mappings in the spirit of Hyers, Ulam, and Rassias.

INTRODUCTION

Questions concerning the stability of functional equations seem to have been first studied by Ulam [6]. In 1941 Hyers [1] showed that if $\delta > 0$ and $f: E_1 \to E_2$, with E_1 and E_2 Banach spaces, such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \delta, \qquad \text{for all} \quad x, y \in E_1,$$

then there exist a unique $T: E_1 \to E_2$ such that

$$T(x+y) = T(x) + T(y)$$

and

$$\|f(x) - T(x)\| \leqslant \delta,$$

for all $x, y \in E_1$, and if $f(tx)$ is continuous in t for each fixed x, then T is a linear mapping.

In 1978 a generalized solution to the Ulam problem for approximately linear mappings was given by Rassias [5]:

Consider E_1, E_2 to be two Banach spaces and $f: E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta \geqslant 0$ and $p \in [0, 1)$ such that

$$\frac{\|f(x+y) - f(x) - f(y)\|}{\|x\|^p + \|y\|^p} \leqslant \theta, \qquad \text{for any} \quad x, y \in E_1.$$

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Then there exists a unique linear mapping $T: E_1 \to E_2$ such that

$$\frac{\|f(x)-T(x)\|}{\|x\|^p} \leqslant \frac{2\theta}{2-2^p}, \qquad \text{for any} \quad x \in E_1.$$

Isac and Rassias [3, 4] obtain further generalizations of the Hyers–Rassias theorem (see [2] for a report on the development of the subject during the last 50 years).

We denote by $(G, +)$ an abelian group, by $(X, \|\cdot\|)$ a Banach space, and by $\varphi: G \times G \to [0, \infty)$ a mapping such that

$$\tilde{\varphi}(x, y) := \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y) < \infty \tag{1}$$

for all $x, y \in G$.

THEOREM. *Let $f: G \to X$ be such that*

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varphi(x, y), \qquad \textit{for all} \quad x, y \in G. \tag{2}$$

Then there exists a unique mapping $T: G \to X$ such that

$$T(x+y) = T(x) + T(y), \qquad \textit{for all} \quad x, y \in G, \tag{3}$$

and

$$\|f(x)-T(x)\| \leqslant \tfrac{1}{2}\tilde{\varphi}(x, x), \qquad \textit{for all} \quad x \in G. \tag{4}$$

Proof. For $x = y$ inequality (2) implies

$$\|f(2x) - 2f(x)\| \leqslant \varphi(x, x).$$

Thus

$$\|2^{-1}f(2x) - f(x)\| \leqslant \tfrac{1}{2}\varphi(x, x) \qquad \text{for all} \quad x \in G. \tag{5}$$

Replacing x by $2x$, inequality (5) gives

$$\|2^{-1}f(2^2x) - f(2x)\| \leqslant \tfrac{1}{2}\varphi(2x, 2x) \qquad \text{for all} \quad x \in G. \tag{6}$$

From (5) and (6) it follows that

$$\begin{aligned}\|2^{-2}f(2^2x) - f(x)\| &\leqslant \|2^{-2}f(2^2x) - 2^{-1}f(2x)\| \\ &\quad + \|2^{-1}f(2x) - f(x)\| \\ &\leqslant 2^{-1}\tfrac{1}{2}\varphi(2x, 2x) + \tfrac{1}{2}\varphi(x, x).\end{aligned}$$

Hence

$$\|2^{-2}f(2^2x)-f(x)\| \leqslant \tfrac{1}{2}[\varphi(x, x)+\tfrac{1}{2}\varphi(2x, 2x)] \tag{7}$$

for all $x \in G$.

Replacing x by $2x$, inequality (7) becomes

$$\|2^{-2}f(2^3x)-f(2x)\| \leqslant \tfrac{1}{2}[\varphi(2x, 2x)+\tfrac{1}{2}\varphi(2^2x, 2^2x)],$$

and therefore

$$\begin{aligned}\|2^{-3}f(2^3x)-f(x)\| &\leqslant \|2^{-3}f(2^3x)-2^{-1}f(2x)\| + \|2^{-1}f(2x)-f(x)\| \\ &\leqslant 2^{-1}\tfrac{1}{2}[\varphi(2x, 2x)+\tfrac{1}{2}\varphi(2^2x, 2^2x)]+\tfrac{1}{2}\varphi(x, x).\end{aligned}$$

Thus

$$\begin{aligned}&\|2^{-3}f(2^3x)-f(x)\| \\ &\quad\leqslant \frac{1}{2}\left[\varphi(x, x)+\frac{1}{2}\varphi(2x, 2x)+\frac{1}{2^2}\varphi(2^2x, 2^2x)\right]\end{aligned} \tag{8}$$

for all $x \in G$.

Applying an induction argument to n we obtain

$$\|2^{-n}f(2^nx)-f(x)\| \leqslant \tfrac{1}{2}\sum_{k=0}^{n-1} 2^{-k}\varphi(2^kx, 2^kx) \tag{9}$$

for all $x \in G$.

Indeed,

$$\begin{aligned}\|2^{-(n+1)}f(2^{n+1}x)-f(x)\| &\leqslant \|2^{-(n+1)}f(2^{n+1}x)-2^{-1}f(2x)\| \\ &\quad + \|2^{-1}f(2x)-f(x)\|,\end{aligned}$$

and with (9) and (5) we obtain

$$\begin{aligned}\|2^{-(n+1)}f(2^{n+1}x)-f(x)\| &\leqslant 2^{-1}\tfrac{1}{2}\sum_{k=0}^{n-1} 2^{-k}\varphi(2^{k+1}x, 2^{k+1}x)+\tfrac{1}{2}\varphi(x, x) \\ &= \tfrac{1}{2}\sum_{k=0}^{n} 2^{-k}\varphi(2^kx, 2^kx).\end{aligned}$$

We claim that the sequence $\{2^{-n}f(2^nx)\}$ is a Cauchy sequence. Indeed, for $n > m$ we have

$$\|2^{-n}f(2^nx)-2^{-m}f(2^mx)\| = 2^{-m}\|2^{-(n-m)}f(2^{n-m}2^mx)-f(2^mx)\|$$
$$\leqslant 2^{-m}\tfrac{1}{2}\sum_{k=0}^{n-m-1} 2^{-k}\varphi(2^{k+m}x, 2^{k+m}x)$$
$$= \tfrac{1}{2}\sum_{p=m}^{n-1} 2^{-p}\varphi(2^px, 2^px).$$

Taking the limit as $m \to \infty$ we obtain

$$\lim_{m\to\infty} \|2^{-n}f(2^nx)-2^{-m}f(2^mx)\| = 0.$$

Because of the fact that X is a Banach space it follows that the sequence $\{2^{-n}f(2^nx)\}$ converges.

Denote

$$T(x) = \lim_{n\to\infty} \frac{f(2^nx)}{2^n}.$$

We claim that T satisfies (3).

From (2) we have

$$\|f(2^nx+2^ny)-f(2^nx)-f(2^ny)\| \leqslant \varphi(2^nx, 2^ny)$$

for all $x, y \in G$. Therefore

$$\|2^{-n}f(2^nx+2^ny)-2^{-n}f(2^nx)-2^{-n}f(2^ny)\| \leqslant 2^{-n}\varphi(2^nx, 2^ny). \tag{10}$$

From (1) it follows that

$$\lim_{n\to\infty} 2^{-n}\varphi(2^nx, 2^ny) = 0.$$

Then (10) implies

$$\|T(x+y)-T(x)-T(y)\| = 0.$$

To prove (4), taking the limit in (9) as $n \to \infty$, we obtain

$$\|T(x)-f(x)\| \leqslant \tfrac{1}{2}\tilde{\varphi}(x, x), \qquad \text{for all} \quad x \in G.$$

It remains to show that T is uniquely defined. Let $F: G \to X$ be another such mapping with

$$F(x+y) = F(x) + F(y)$$

and (4) satisfied.

Then

$$\begin{aligned}\|T(x)-F(x)\| &= \|2^{-n}T(2^n x)-2^{-n}F(2^n x)\| \\ &\leqslant \|2^{-n}T(2^n x)-2^{-n}f(2^n x)\| + \|2^{-n}f(2^n x)-2^{-n}F(2^n x)\| \\ &\leqslant 2^{-n}\tfrac{1}{2}\tilde{\varphi}(2^n x, 2^n x)+2^{-n}\tfrac{1}{2}\tilde{\varphi}(2^n x, 2^n x) \\ &= 2^{-n}\tilde{\varphi}(2^n x, 2^n x) \\ &= 2^{-n}\sum_{k=0}^{\infty} 2^{-k}\varphi(2^{k+n}x, 2^{k+n}x) \\ &= \sum_{p=n}^{\infty} 2^{-p}\varphi(2^p x, 2^p x).\end{aligned}$$

Thus

$$\|T(x)-F(x)\| \leqslant \sum_{p=n}^{\infty} 2^{-p}\varphi(2^p x, 2^p x) \qquad \text{for all} \quad x\in G. \tag{11}$$

Taking the limit in (11) as $n\to\infty$ we obtain

$$T(x)=F(x) \qquad \text{for all} \quad x\in G. \qquad \text{Q.E.D.}$$

Application

Let G be a normed linear space and define $H\colon \mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ and $\varphi_0\colon \mathbb{R}_+\to\mathbb{R}_+$ such that

$$\begin{aligned}&\varphi_0(\lambda)>0, && \text{for all} \quad \lambda>0,\\ &\varphi_0(2)<2 && \\ &\varphi_0(2\lambda)\leqslant\varphi_0(2)\,\varphi_0(\lambda), && \text{for all} \quad \lambda>0\\ &H(\lambda t, \lambda s)\leqslant\varphi_0(\lambda)\,H(t,s), && \text{for all} \quad t, s\in\mathbb{R}_+,\ \lambda>0.\end{aligned}$$

We take in our theorem

$$\varphi(x,y)=H(\|x\|, \|y\|).$$

Then

$$\begin{aligned}\varphi(2^k x, 2^k y) &= H(2^k\|x\|, 2^k\|y\|)\\ &\leqslant \varphi_0(2^k)\,H(\|x\|, \|y\|)\\ &\leqslant (\varphi_0(2))^k\,H(\|x\|, \|y\|),\end{aligned}$$

and because $\varphi_0(2) < 2$ we have

$$\tilde{\varphi}(x, y) \leqslant \sum_{k=0}^{\infty} 2^{-k}(\varphi_0(2))^k H(\|x\|, \|y\|)$$

$$= \frac{1}{1 - (\varphi_0(2)/2)} H(\|x\|, \|y\|),$$

and the relation (4) becomes

$$\|f(x) - T(x)\| \leqslant \frac{1}{2} \tilde{\varphi}(x, x) \leqslant \frac{1}{2 - \varphi_0(2)} H(\|x\|, \|x\|)$$

or

$$\|f(x) - T(x)\| \leqslant \frac{1}{2 - \varphi_0(2)} \varphi_0(\|x\|) H(1, 1)$$

Remark. The above result generalizes results of Isac and Rassias [3, 4] and Rassias [5] because if $f(tx)$ is continuous in t for each fixed x and

$$T(x) = \lim_{n \to \infty} 2^{-n} f(2^n x),$$

then T is a linear mapping (see [5]).

References

1. D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. U.S.A.* **27** (1941), 222–224.
2. D. H. Hyers and Th. M. Rassias, Approximate homomorphisms, *Aequationes Math.* **44** (1992), 125–153.
3. G. Isac and Th. M. Rassias, On the Hyers–Ulam stability of ψ-additive mappings, *J. Approx. Theory* **72** (1993), 131–137.
4. G. Isac and Th. M. Rassias, New results on the stability of mappings, to appear.
5. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297–300.
6. S. M. Ulam, "Problems in Modern Mathematics," Chap. VI, Science eds., Wiley, New York, 1960.