On HNBUE class after specific age

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Abstract In this article we introduce new classes of life distributions namely harmonic new better (worse) than used in expectation after specific age $t_0$, HNBUE$_{t_0}$ (HNWUE$_{t_0}$). The closure properties under various reliability operations such as convolution, mixture, mixing and the homogeneous Poisson shock model of these classes are studied. Furthermore, nonparametric test is proposed to test exponentiality vs. the HNBUE$_{t_0}$ class. The critical values and the powers of this test are calculated to assess the performance of the test. It is shown that the proposed test have high efficiencies for LFR and Weibull distributions. Sets of real data are used as examples to elucidate the use of the proposed test for practical problems.

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1. Introduction

There are many situations in real life where the phenomena needs to be studied after time $t_0$. For example, before the appearance of a disease there may exist an incubation period $t_0$, during which the disease is unobservable. Concepts of aging play an important role in theory of reliability, life testing, survival analysis and other related fields. For example Barlow and Proschan [1], Deshpand et al. [2] and Cao and Wang [3], gave definitions of several classes of distributions, NBU, NBU(2) and NBUE. Rolski [4] introduced the HNBUE (harmonic new better than used in expectation) class, subsequently the HNBUE has been studied from many points of view by Klefsjo [5] and Basu and Kirmani [6]. Testing exponentiality vs. the classes of life distributions has been a good deal of attention. For testing against IFR class we refer to Barlow and Proschan [1] and Ahmed [7]. For testing IFRA we refer to Ahmed [8] and Deshpande [9], while testing vs. NBU are discussed by Hollander and Proschan [10], Koul [11], Kumazawa [12] and Ahmed [8]. Mahmoud et al. [13–15] for NRBU, RNBU and HNRBUE classes. Testing vs. NBUL are discussed by Diab et al. [16] and Diab [17]. Hollander and Proschan [18], Koul and Susarla [19], Klefsjo [20] and Borges et al. [21] for (NBUE) class. Finally testing vs. HNBUE can be found in the work of Klefsjo [20], Basu and Ebrahim [22], Ahmed [23] and Hendi et al. [24].
Statisticians and reliability analysts studied some aging classes of life distributions at specific age from various points of view. For more details we refer to Hollander et al. [25], Ebrahimi and Habibullah [26], Ahmed [27] and Pandit and Anuradha [28] for NBU – t0, Zehui and Xiaohu [29] for IFRA’ t0 and NBU’ t0 and Elbatal [30] for NBUC – t0 and NBU(2) – t0.

In this article we introduce a new class of life distribution namely HNBUE t0, and its dual class HNWUE t0, which is the generalization of HNBUE (HNWUE) class of life distribution.

Let X be a random variable having distribution function F and μ = \int_0^\infty f(x)dx < \infty, where F denotes the survival function 1 – F. This is known as the aging measure.

**Definition 1.1.** X is harmonic new better than used in expectation (denoted by \(X \in \text{HNBU}E\)) if

\[
\int_0^\infty F(x+t)dt \leq \mu e^{-\mu t} \text{ for all } t \geq 0.
\]  

(1.1)

**Definition 1.2.** X is harmonic new better than used in expectation after specific age \(t_0\) (denoted by \(X \in \text{HNBU}E_{t_0}\)) if

\[
\int_0^{t_0} F(x+t)dt \leq \mu e^{-\mu t} \text{ for all } t \geq t_0 \geq 0.
\]  

(1.2)

In the current investigation, Preservation under convolution, mixture, mixing and the homogeneous Poisson shock model of the HNBUE t0 (HNWUE t0) classes are discussed in Section 2. In Section 3 based on Goodness of fit technique we present a procedure to test X is exponential vs. it is HNBUE t0 and not exponential. Finally numerical examples is presented in Section 4.

2. Some properties of the HNBUE t0 class

In this section some properties of HNBUE t0 and HNWUE t0 classes are introduced under convolution, mixture, mixing and the shock model in homogeneous case.

**Theorem 2.1.** The HNBUE t0 class is preserved under convolution.

**Proof 1.** Suppose that \(F_1\) and \(F_2\) are two independent HNBUE t0 lifetime distributions then their convolution is given by:

\[
\mathcal{F} = \int_0^{\infty} F_1(z-y)df_2(y).
\]

And therefore:

\[
\int_0^{\infty} F_1(x+t)dt = \int_0^{\infty} \int_0^{\infty} F_1(x+u)df_2(u)dx
\]

\[
= \int_0^{\infty} \int_0^{\infty} F_1(x+u)du df_2(u).
\]

Since \(F_1\) is HNBUE t0 then

\[
\int_0^{\infty} F_1(x+t)dt \leq \int_0^{\infty} \int_0^{\infty} e^{-\mu t} F_1(x+u)du df_2(u)
\]

\[
= e^{-\mu t} \int_0^{\infty} \int_0^{\infty} F_1(x+u)du df_2(u)
\]

\[
= e^{-\mu t} \int_0^{\infty} F_1(x)dx.
\]

which complete the proof. □

The following theorem is presented to show that HNWUE t0 class is preserved under convolution.

**Theorem 2.2.** The HNWUE t0 class is preserved under convolution.

**Proof 2.** The proof is obtained by reversing the inequality in the last proof. □

The following theorem is stated and proved to show that the HNBUE t0 class is preserved under mixture.

**Theorem 2.3.** The HNBUE t0 class is preserved under mixture.

**Proof 3.** Suppose \(F_x\) is HNBUE t0 then their mixture is:

\[
\mathcal{F}_3(x) = \int_0^{\infty} \mathcal{F}_2(x)dG(z).
\]

Therefore:

\[
\int_0^{\infty} \mathcal{F}_3(x+t)dt = \int_0^{\infty} \int_0^{\infty} \mathcal{F}_2(x+u)G(z)du dx
\]

\[
= \int_0^{\infty} \int_0^{\infty} \mathcal{F}_2(x+u)du G(z) dx.
\]

(2.1)

Since \(F_x\) is HNBUE t0 then

\[
\int_0^{\infty} \int_0^{\infty} \mathcal{F}_2(x+t)G(z)du dx \leq \int_0^{\infty} \int_0^{\infty} e^{-\mu t} \mathcal{F}_2(x)du G(z) dx
\]

\[
= e^{-\mu t} \int_0^{\infty} \mathcal{F}_2(x)du G(z) dx
\]

\[
= e^{-\mu t} \int_0^{\infty} \mathcal{F}_2(x)dx.
\]

(2.2)

The following theorem is presented to show that HNWUE t0 class is preserved under mixture.

**Theorem 2.4.** The HNWUE t0 class is preserved under mixture.

**Proof 4.** The proof is obtained by reversing the inequality in the last proof. □

The following example illustrates that the HNBUE t0 class is not preserved under mixing.

**Example 2.1.** Let \(\mathcal{F}_3 = e^{-\beta x}\) and \(\mathcal{F}_2 = e^{-\gamma x}\). Let \(\mathcal{F} = \frac{\gamma}{\beta} \mathcal{F}_3 + \frac{\beta}{\gamma} \mathcal{F}_2\). It follows that both \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are HNBUE t0 but \(\mathcal{F}\) is not HNBUE t0.

2.1. Homogeneous Poisson shock model

Suppose that a device is subjected to sequence shocks occurring randomly in the time according to a Poisson process with
constant intensity $\lambda$. Suppose further that the device has probability $p_k$ of surviving the first $k$ shocks, where $1 = p_0 \geq p_1 \geq \cdots$. Then the survival function of the device is given by

$$\mathcal{P}(t) = \sum_{k=0}^{\infty} p_k \frac{(\lambda t)^k}{k!} e^{-\lambda t}. \quad (2.3)$$

This shock model has been studied by Esary [31] for IFR, IFRA, DMRL, NBUE and NBU classes. Klefsjö [32] for HNBU and Mahmoud et al. [33] for NBURFR - $t_0$.

**Definition 2.1.** A discrete distribution $p_k$, $k = 0, 1, \ldots$, is said to have discrete harmonic new better (worse) than expected after specific age $t_0$ (HNBU$^{+}t_0$)(HNWUE$^{-}t_0$) if

$$\sum_{j=0}^{\infty} p_{nj} \leq \left(1 - \frac{1}{m}\right) \sum_{j=0}^{\infty} p_j, \quad j = 0, 1, \ldots. \quad (2.4)$$

Now, let us introduce the following theorem.

**Theorem 2.5.** If $p_k$ is discrete HNBU$^{+}t_0$, then $\mathcal{P}(t)$ given by (2.3) is HNBU$^{+}t_0$.

**Proof 5.** Upon using (2.3), we have

$$\mu_H = \int_0^\infty \mathcal{P}(t) dt = \frac{1}{\lambda} \sum_{k=0}^{\infty} p_k \int_0^\infty \frac{(\lambda t)^k}{k!} e^{-\lambda t} dt = \frac{1}{\lambda} \sum_{k=0}^{\infty} p_k = \frac{m}{\lambda},$$

where $m$ is the mean of the discrete distribution $p_k$.

It must be shown that

$$\int_0^\infty \mathcal{P}(x+t) dx \leq e^{-t/\mu} \int_0^\infty \mathcal{P}(x) dx.$$

Upon using (2.3), we get

$$\int_0^\infty \mathcal{P}(x+t) dx = \int_0^\infty \sum_{k=0}^{\infty} p_k \frac{[\lambda(t+X)]^k}{k!} e^{-\lambda(t+X)} dx = e^{-\lambda t} \sum_{k=0}^{\infty} p_k \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} (X)^j e^{-\lambda X} dx.$$

Integrating by parts yields

$$= \frac{e^{-\lambda t}}{\lambda} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} p_k \frac{\lambda^j}{j!} (X)^j.$$
Thus, the variance is
\[ \sigma^2 = \text{Var}[\phi(X)]. \]

Under \( H_0 \), we get
\[ \sigma_0^2 = e^{-2a} - \frac{1}{2} e^{-a} + \frac{1}{12}. \]  \( (3.3) \)

According to Lemma 3.1, Theorem 3.1 is immediate.

**Theorem 3.1.**

(i) As \( n \to \infty \), \( \sqrt{n} (\hat{\theta}_n - \theta) \sim N(0, \sigma^2) \), where
\[ \sigma^2 = \text{Var} \left[ \frac{1}{2} e^{-\theta} - X e^{-\theta} - e^{-X} \right]. \]

(ii) Under \( H_0 \) the variance is reduced to \( \sigma_0^2 \) in (3.3).

3.1. The Pitman Asymptotic Efficiency (PAE) of \( \hat{\delta} \)

To assess how good this procedure is relative to others in the literature we evaluate its Pitman asymptotic efficiency (PAE) for two alternatives in the class \( HNBUE_t \), these are:

1. Linear failure rate family (LFR): \( \mathcal{F}_\theta(x) = \exp \left( -x - \frac{x^2}{2} \right), \ x > 0, \ \theta \geq 0. \)

2. Weibull family: \( \mathcal{F}_\theta(x) = x^{-\theta}, \ x > 0, \ \theta > 0. \)

The PAE is defined by
\[ \text{PAE}(\theta) = \frac{1}{\sigma_0} \left| \frac{d\theta}{d\theta} \right|_{\theta = \theta_0}. \]

Consider
\[ \delta_0 = t_0 e^{-2a} dt - \frac{1}{\theta} \int_{t_0}^{\infty} e^{-\mathcal{F}_\theta(x + t)} dx dt, \]
then
\[ \frac{\partial}{\partial \theta} \delta_0 = \frac{1}{2} e^{-2a} - \frac{1}{\theta} \int_{t_0}^{\infty} e^{-\mathcal{F}_\theta(x + t)} dx dt. \]

It is easy to prove that

\[ \text{PAE}(\theta, F) = \frac{1}{\sigma_0} \left| \frac{d\theta}{d\theta} \right|_{\theta = \theta_0} + \int_{t_0}^{\infty} e^{-\mathcal{F}_\theta(x)} dx - \int_{t_0}^{\infty} \mathcal{F}_\theta(x) dx. \]

\[ \text{PAE}(\theta, \text{LFR}) = \frac{1}{\sigma_0} \left[ e^{-a} \left( \frac{t_0^2}{2} + t_0 + 1 \right) - e^{-2a} \left( \frac{t_0^2}{2} + t_0 - \frac{3}{2} \right) \right]. \]

**Figure 4.1** The relation between efficiencies and \( t_0 \leq 5. \)

<table>
<thead>
<tr>
<th>Test</th>
<th>LFR</th>
<th>Weibull</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hollander-Proshan</td>
<td>0.8660</td>
<td>1.2007</td>
</tr>
<tr>
<td>Ahmad et al.</td>
<td>0.7490</td>
<td>1.2280</td>
</tr>
</tbody>
</table>

### Table 3.1 The PAE's for LFR and Weibull families.

<table>
<thead>
<tr>
<th>Test</th>
<th>LFR</th>
<th>Weibull</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>0.01</td>
<td>0.05</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>0.90</td>
</tr>
<tr>
<td></td>
<td>0.95</td>
<td>0.99</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n</th>
<th>0.01</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.28</td>
<td>0.30</td>
</tr>
<tr>
<td>0.03</td>
<td>0.29</td>
<td>0.32</td>
</tr>
<tr>
<td>0.04</td>
<td>0.30</td>
<td>0.33</td>
</tr>
<tr>
<td>0.05</td>
<td>0.31</td>
<td>0.34</td>
</tr>
<tr>
<td>0.06</td>
<td>0.32</td>
<td>0.35</td>
</tr>
<tr>
<td>0.07</td>
<td>0.33</td>
<td>0.36</td>
</tr>
<tr>
<td>0.08</td>
<td>0.34</td>
<td>0.37</td>
</tr>
<tr>
<td>0.09</td>
<td>0.35</td>
<td>0.38</td>
</tr>
<tr>
<td>0.10</td>
<td>0.36</td>
<td>0.39</td>
</tr>
</tbody>
</table>

**Table 3.2 Critical values of statistic \( \hat{\delta}_n \) at \( \alpha = 0.05 \).**

<table>
<thead>
<tr>
<th>n</th>
<th>0.01</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.28</td>
<td>0.30</td>
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<td>0.31</td>
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<td>0.35</td>
<td>0.38</td>
</tr>
<tr>
<td>0.10</td>
<td>0.36</td>
<td>0.39</td>
</tr>
</tbody>
</table>
and
\[
PAE(\hat{\delta}, \text{Weibull}) = \frac{1}{\sigma_0} \left( \frac{1}{2} t_0 e^{-\frac{\ln(t)}{\theta}} + \frac{1}{4} e^{-\frac{2\ln(t)}{\theta}} + \left( 1 + \frac{\ln^2(t)}{4} \right) e^{-\frac{\ln(t)}{\theta}} \right) 
\]
\[
+ \frac{3}{4} \int_0^\infty e^{-x} \ln(x) dx .
\]

Fig. 4.1 shows the relation between efficiencies and \(t_0 \leq 5\).

In view of Fig. 4.1 it is noticed that the maximum efficiencies are at \(t_0 = 1.3\) and the PAE for the LFR alternative is greater than the PAE for Weibull alternative.

We compare the above procedure at \(t_0 = 1.3\) to that of Hollander and Proschan [18] and Ahmad et al. [35], and the results are shown in Table 3.1.

Table 3.1 shows that our test outperforms the others tests for the two alternatives.

3.2. Monte Carlo null distribution critical points

Many practitioners, such as applied statisticians, and reliability analysts are interested in simulated percentiles. Table 3.2 gives these percentile points of the statistic \(\hat{\delta}\) given in (3.2) at \(t_0 = 1.3\) and the calculations are based on 1000 simulated samples of sizes \(n = 2(1)50\).

3.3. The power estimates

Table 3.3 shows the power estimates of the test statistic \(\hat{\delta}\) given in (3.2) at the significant level 0.05 using LFR and Weibull distributions. The estimates are based on 1000 simulated samples for sizes \(n = 10, 20\) and 30.

From Table 3.3 we can see that our test has very good power.

4. Applications

Here, we will present real examples to elucidate the applications of our test at 95% confidence level.

Example 4.1. The following data represent 39 liver cancers patients taken from Elminia cancer center Ministry of Health–Egypt, which entered in (1999). The ordered life times (in years) are:

<table>
<thead>
<tr>
<th>Time (years)</th>
<th>0.027</th>
<th>0.038</th>
<th>0.038</th>
<th>0.038</th>
<th>0.038</th>
<th>0.041</th>
<th>0.047</th>
<th>0.049</th>
<th>0.055</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (years)</td>
<td>0.055</td>
<td>0.055</td>
<td>0.055</td>
<td>0.055</td>
<td>0.063</td>
<td>0.063</td>
<td>0.066</td>
<td>0.071</td>
<td>0.082</td>
</tr>
<tr>
<td>Time (years)</td>
<td>0.085</td>
<td>0.110</td>
<td>0.314</td>
<td>0.140</td>
<td>0.143</td>
<td>0.164</td>
<td>0.167</td>
<td>0.184</td>
<td>0.195</td>
</tr>
<tr>
<td>Time (years)</td>
<td>0.203</td>
<td>0.238</td>
<td>0.263</td>
<td>0.288</td>
<td>0.293</td>
<td>0.293</td>
<td>0.318</td>
<td>0.411</td>
<td></td>
</tr>
</tbody>
</table>

It was found that \(\hat{\delta}_0 = -0.141\) and this value less than the tabulated critical value in Table 3.2. Then we accept \(H_0\) which states that the data set has exponential property.

Example 4.2. Consider the following data, which represent failure times in hours, for a specific type of electrical insulation in an experiment in which the insulation was subjected to a continuously increasing voltage stress [36, p. 138]:

<table>
<thead>
<tr>
<th>Voltage (V)</th>
<th>0.205</th>
<th>0.363</th>
<th>0.407</th>
<th>0.477</th>
<th>0.72</th>
<th>0.782</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time (h)</td>
<td>1.178</td>
<td>1.255</td>
<td>1.592</td>
<td>1.635</td>
<td>1.635</td>
<td>2.31</td>
</tr>
</tbody>
</table>

It was found that \(\hat{\delta}_0 = 0.0697\) which is greater than the tabulated critical value in Table 3.2. Then we conclude that this data set has \(HNBUE^*\) property.

References

On HNBUE class after specific age


[26] N. Ebrahimi, M. Habibullah, Testing whether the survival distribution is new better than used of specified age, Biometrika 77 (1990) 212–215.


