Physics Letters B 718 (2012) 638–645

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

A higher order GUP with minimal length uncertainty and maximal momentum II: Applications

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A R T I C L E I N F O

Article history:
Received 20 July 2012
Received in revised form 11 September 2012
Accepted 19 October 2012
Available online 25 October 2012
Editor: T. Yanagida

Keywords:
Quantum gravity
Generalized uncertainty principle
Minimal length uncertainty
Maximal momentum

A B S T R A C T

In a recent paper, we presented a nonperturbative higher order Generalized Uncertainty Principle (GUP) that is consistent with various proposals of quantum gravity such as string theory, loop quantum gravity, doubly special relativity, and predicts both a minimal length uncertainty and a maximal observable momentum. In this Letter, we find exact maximally localized states and present a formally self-adjoint and naturally perturbative representation of this modified algebra. Then we extend this GUP to D dimensions that will be shown it is noncommutative and find invariant density of states. We show that the presence of the maximal momentum results in upper bounds on the energy spectrum of the free particle and the particle in box. Moreover, this form of GUP modifies blackbody radiation spectrum at high frequencies and predicts a finite cosmological constant. Although it does not solve the cosmological constant problem, it gives a better estimation with respect to the presence of just the minimal length.

1. Introduction

The modification of the Heisenberg uncertainty principle in the context of the Generalized Uncertainty Principle (GUP) and the Modified Dispersion Relation (MDR) has attracted much attention in recent years [1]. This interest arises from various theories of quantum gravity such as string theory [2–5], loop quantum gravity [6], noncommutative spacetime [7–9], and doubly special relativity (DSR) [10–12]. All GUP proposals imply the existence of a minimal length scale of the order of the Planck length $\ell_{\mathrm{Pl}}=\sqrt{\frac{G\hbar}{c^3}}\approx 10^{-35}$ m where G is Newton's gravitational constant (see for instance [13–32]). Moreover, a perturbative GUP proposal that is consistent with DSR theories is studied in Refs. [33–38].

Recently, we have proposed a nonperturbative higher order generalized uncertainty principle which implies both a minimal length uncertainty and a maximal observable momentum [39]

$$[X,P]=\frac{i\hbar}{1-\beta P^2}. \tag{1}$$

This commutation relation agrees with Kempf, Mangano and Mann (KMM) [8] and Noucier's [30] proposals to the leading order of the GUP parameter β. In momentum space, the position and momentum operators can be written as [39]

$$P\phi(p)=p\phi(p), \tag{2}$$

$$X\phi(p)=\frac{i\hbar}{1-\beta p^2}\partial_p\phi(p). \tag{3}$$

So the completeness relation and the scalar product take the following form:

$$\langle\psi|\phi\rangle=\int_{-1/\sqrt{\beta}}^{+1/\sqrt{\beta}}\mathrm{d}p\,\left(1-\beta p^2\right)\psi^*(p)\phi(p), \tag{4}$$

$$\langle p|p'\rangle=\frac{\delta(p-p')}{1-\beta p^2}. \tag{5}$$

Also the momentum of the particle is bounded from above

$$P_{\max}=\frac{1}{\sqrt{\beta}}, \tag{6}$$

and the absolutely smallest uncertainty in position is

$$(\Delta X)_{\min}=\frac{3\sqrt{3}}{4}\hbar\sqrt{\beta}. \tag{7}$$

Approximate maximally localization states (using KMM approach) and quantum mechanical and semiclassical solutions of the harmonic oscillator have been also obtained in this framework [39].

Here, we first find maximally localized states using Detournay, Gabriel and Spindel approach. Then we present a formally self-adjoint representation and study the problems of the free particle and the particle in a box and show that their energy spectrum are

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0370-2693
http://dx.doi.org/10.1016/j.physletb.2012.10.059

bounded from above. We also address the generalization to D dimensions, validity of semiclassical approximation, invariant density of states, cosmological constant, and blackbody radiation in this GUP framework.

2. Maximally localized states

In KMM approach the maximally localized states are the solutions of the following equation [8]

$$\left(X-\langle X\rangle+\frac{\langle[X,P]\rangle}{2(\Delta P)^2}\big(P-\langle P\rangle\big)\right)|\psi\rangle=0, \tag{8}$$

where $[X,P]=if(P)$. However, unlike the ordinary quantum mechanics where $f(P)=\mathbb{1}\hbar$ and therefore $\langle f(P)\rangle=\hbar$ for all states, in general, the expectation value of $[X,P]$ depends on the state considered [40,39]. So, except $f(P)\sim 1+\beta P^2$, it is impossible, for an arbitrary function $f(P)$, to write any exact solution for the above equation (see [39] for an approximate solution). On the other hand, Detournay and collaborators proposed an alternative general scheme for finding such states based on a constrained variational principle [40]. In this framework, the maximally localized states are the solutions of the following Euler–Lagrange equation in momentum space

$$\begin{aligned}&\big[-\big(f(p)\partial_p\big)^2-\xi^2+2a\big(if(p)\partial_p-\xi\big)\\&\quad+2b\big(v(p)-\gamma\big)-\mu^2\big]\psi(p)=0,\end{aligned} \tag{9}$$

where a and b are Lagrange multipliers and

$$\begin{aligned}&(\Delta X)^2_{\min}=\min\frac{\langle\psi|X^2-\xi^2|\psi\rangle}{\langle\psi|\psi\rangle}\equiv\mu^2,\qquad \xi=\frac{\langle\psi|X|\psi\rangle}{\langle\psi|\psi\rangle},\\&\gamma=\frac{\langle\psi|v(p)|\psi\rangle}{\langle\psi|\psi\rangle}.\end{aligned} \tag{10}$$

Here $v(p)$ is an arbitrary function whose expectation value is finite (see [40] for details). Now if we define

$$z(p)=\int_0^p f^{-1}(q)\,\mathrm{d}q, \tag{11}$$

and

$$z(+P_{\max})=\alpha_+>0,\qquad z(-P_{\max})=\alpha_-<0, \tag{12}$$

the normalized solution for $b=0$ is [40]

$$\psi^{\mathrm{ML}}_\xi(p)=C\exp\big[-i\xi z(p)\big]\sin\big\{\mu\big[z(p)-\alpha_-\big]\big\}, \tag{13}$$

where

$$|C|=\sqrt{\frac{2/\hbar}{\alpha_+-\alpha_-}},\qquad \mu=\frac{n\pi}{\alpha_+-\alpha_-},\quad n\in\mathbb{N}, \tag{14}$$

and the corresponding spread in position is given by

$$(\Delta X)_{\min}\big|_{b=0}=\frac{\pi}{\alpha_+-\alpha_-}. \tag{15}$$

For our case, i.e., $f(P)=\hbar/(1-\beta P^2)$, we obtain

$$z(p)=\hbar^{-1}\left(p-\frac{\beta}{3}p^3\right), \tag{16}$$

and

$$\alpha_+=+\frac{2}{3\hbar\sqrt{\beta}},\qquad \alpha_-=-\frac{2}{3\hbar\sqrt{\beta}}. \tag{17}$$

So the solution is

$$\begin{aligned}\psi^{\mathrm{ML}}_\xi(p)&=\sqrt{\frac{3\sqrt{\beta}}{2}}\exp\left[\frac{-i\xi}{\hbar}\left(p-\frac{\beta}{3}p^3\right)\right]\\&\quad\times\sin\left[\frac{\mu}{\hbar}\left(p-\frac{\beta}{3}p^3+\frac{2}{3\sqrt{\beta}}\right)\right]\\&=\sqrt{\frac{3\sqrt{\beta}}{2}}\exp\left[\frac{-i\xi}{\hbar}\left(p-\frac{\beta}{3}p^3\right)\right]\\&\quad\times\cos\left[\frac{3\pi}{4}\sqrt{\beta}\left(p-\frac{\beta}{3}p^3\right)\right],\end{aligned} \tag{18}$$

and

$$(\Delta X)_{\min}\big|_{b=0}=\frac{3\pi}{4}\hbar\sqrt{\beta}. \tag{19}$$

Note that $(\Delta X)_{\min}|_{b=0}$ corresponds to a (local) minimum with respect to γ and $\psi^{\mathrm{ML}}_\xi(p)$ is normalized subject to the scalar product presented in Eq. (4). Also the maximally localized states are not mutually orthogonal

$$\begin{aligned}&\big\langle\psi^{\mathrm{ML}}_{\xi'}\big|\psi^{\mathrm{ML}}_\xi\big\rangle\\&\quad=\frac{3\sqrt{\beta}}{2}\int_{-1/\sqrt{\beta}}^{+1/\sqrt{\beta}}\mathrm{d}p\,\big(1-\beta p^2\big)\exp\left[\frac{-i(\xi-\xi')}{\hbar}\left(p-\frac{\beta}{3}p^3\right)\right]\\&\qquad\times\cos^2\left[\frac{3\pi}{4}\sqrt{\beta}\left(p-\frac{\beta}{3}p^3\right)\right]\\&\quad=\frac{3\sqrt{\beta}}{2}\int_{-\frac{2}{3\sqrt{\beta}}}^{+\frac{2}{3\sqrt{\beta}}}\mathrm{d}z\exp\left[\frac{-i(\xi-\xi')z}{\hbar}\right]\cos^2\left[\frac{3\pi}{4}\sqrt{\beta}z\right]\\&\quad=\left[\frac{2(\xi-\xi')}{3\hbar\sqrt{\beta}}-\frac{1}{\pi^2}\left(\frac{2(\xi-\xi')}{3\hbar\sqrt{\beta}}\right)^3\right]^{-1}\sin\left[\frac{2(\xi-\xi')}{3\hbar\sqrt{\beta}}\right],\end{aligned} \tag{20}$$

as well as KMM proposal which is due to the fuzziness of space in both frameworks. Now we can define the quasiposition wave function as

$$\begin{aligned}\psi_{\mathrm{QP}}(\xi)&\equiv\big\langle\psi^{\mathrm{ML}}_\xi\big|\phi\big\rangle\\&=\sqrt{\frac{3\sqrt{\beta}}{2}}\int_{-1/\sqrt{\beta}}^{+1/\sqrt{\beta}}\mathrm{d}p\,\big(1-\beta p^2\big)\exp\left[\frac{i\xi}{\hbar}\left(p-\frac{\beta}{3}p^3\right)\right]\\&\quad\times\cos\left[\frac{3\pi}{4}\sqrt{\beta}\left(p-\frac{\beta}{3}p^3\right)\right]\phi(p).\end{aligned} \tag{21}$$

So the inverse transformation reads

$$\phi(p)=\frac{1}{\sqrt{6\sqrt{\beta}}\pi\hbar}\int_{-\infty}^{+\infty}\mathrm{d}\xi\,\frac{\exp[-\frac{i}{\hbar}\xi(p-\frac{\beta}{3}p^3)]}{\cos[\frac{3\pi}{4}\sqrt{\beta}(p-\frac{\beta}{3}p^3)]}\psi_{\mathrm{QP}}(\xi). \tag{22}$$

Moreover, the scalar product of states in terms of quasiposition wave functions is given by

$$\begin{aligned}\langle\psi|\phi\rangle&=\int_{-1/\sqrt{\beta}}^{+1/\sqrt{\beta}}\mathrm{d}p\,\big(1-\beta p^2\big)\psi^*(p)\phi(p)\\&=\frac{1}{6\sqrt{\beta}\pi^2\hbar^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-1/\sqrt{\beta}}^{+1/\sqrt{\beta}}\mathrm{d}p\,\mathrm{d}\xi\,\mathrm{d}\xi'\end{aligned}$$

$$\times \frac{(1-\beta p^2)}{\cos^2[\frac{3\pi\sqrt{\beta}}{4}(p-\frac{\beta}{3}p^3)]} \times \exp\left[\frac{i}{\hbar}(\xi-\xi')\left(p-\frac{\beta}{3}p^3\right)\right]\psi^*_{QP}(\xi)\psi_{QP}(\xi'). \tag{23}$$

3. Formally self-adjoint representation

Although the set of Eqs. (2) and (3) is an exact representation of the algebra presented in Eq. (1), it does not preserve the ordinary nature of the position operator. Alternatively, we can write $P = f(p)$ and retain the ordinary form of the position operator, i.e., $X = x$ where $[x, p] = i\hbar$. Thus, using Eq. (1) we find $\frac{\mathrm{d}f}{\mathrm{d}p} = \frac{1}{1-\beta f^2}$ which results in

$$f(p) - \frac{1}{3}\beta f^3(p) = p. \tag{24}$$

Consequently, the alternative representation in exact and perturbative forms is

$$X = x, \tag{25}$$

$$P = \frac{1 - i\sqrt{3} + (-2\beta)^{1/3}(3p + \sqrt{9p^2 - 4/\beta})^{2/3}}{(2\beta)^{2/3}(3p + \sqrt{9p^2 - 4/\beta})^{1/3}}, \tag{26}$$

$$= p + \frac{1}{3}\beta p^3 + \frac{1}{3}\beta^2 p^5 + \frac{4}{9}\beta^3 p^7 + \cdots. \tag{27}$$

Note that this representation is formally self-adjoint, i.e., $A = A^\dagger$ for $A \in \{X, P\}$. Also, the presence of the maximal momentum $P_{\max} = 1/\sqrt{\beta}$ is manifest from Eq. (26) which occurs at $p = \frac{2}{3\sqrt{\beta}}$. Now X and P are symmetric operator on the dense domain S_∞ with respect to the following scalar product in the momentum space:

$$\langle\psi|\phi\rangle = \int_{-\frac{2}{3\sqrt{\beta}}}^{+\frac{2}{3\sqrt{\beta}}} \psi^*(p)\phi(p)\,\mathrm{d}p. \tag{28}$$

We have schematically depicted the behavior of P versus p in Fig. 1.

In this representation, to write the Hamiltonian, it is more appropriate to use Eq. (27) and express the Hamiltonian perturbatively as

$$H = \frac{p^2}{2m} + V(x) + \beta\frac{p^4}{3m} + \beta^2\frac{7p^6}{18m} + \mathcal{O}(\beta^3), \tag{29}$$

which agrees with perturbative version of the KMM proposal to $\mathcal{O}(\beta)$ [29]. In the quantum domain, this Hamiltonian results in the following generalized Schrödinger equation in position space representation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + \frac{\beta}{3m}\frac{\partial^4\psi(x)}{\partial x^4} - \frac{7\beta^2}{18m}\frac{\partial^6\psi(x)}{\partial x^6} + \mathcal{O}(\beta^3) + V(x)\psi(x) = E\psi(x), \tag{30}$$

where the extra terms are due to the GUP-corrected terms in Eq. (29). As mentioned before, this representation is naturally perturbative that is apparent from Eq. (30).

Note that for an operator A which is "formally" self-adjoint ($A = A^\dagger$) such as (25) and (27), this does not prove that A is truly self-adjoint because in general the domains $\mathcal{D}(A)$ and $\mathcal{D}(A^\dagger)$ may be different. The operator A with dense domain $\mathcal{D}(A)$ is said to be self-adjoint if $\mathcal{D}(A) = \mathcal{D}(A^\dagger)$ and $A = A^\dagger$. For instance, the position operator (25) is merely symmetric in this representation, but not self-adjoint. To see this point, notice that in this representation and in momentum space, the wave function $\phi(p)$ have to vanish at the boundaries of the p interval ($-2/3\sqrt{\beta} < p < 2/3\sqrt{\beta}$). So X is now the derivative operator $i\hbar\partial/\partial p$ on an interval with Dirichlet boundary conditions. But this means that X cannot be self-adjoint because all candidates for the eigenfunctions of X (the plane waves) are not in the domain of X because they do not obey the Dirichlet boundary conditions. Calculating the domain of the adjoint of X shows that it is larger than that of X, so X is not a true self-adjoint operator, i.e.,

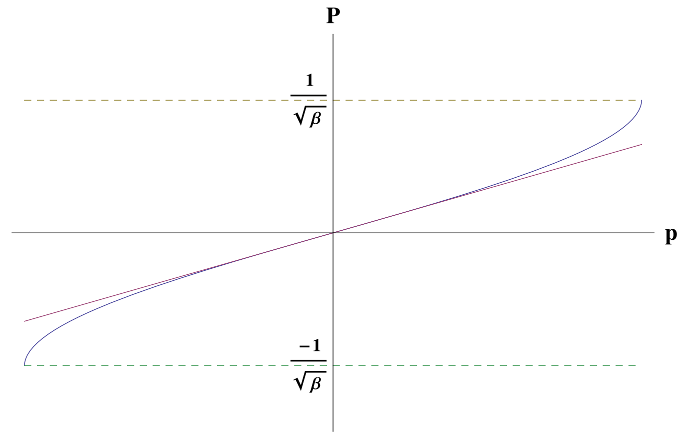

Fig. 1. Schematic behavior of P versus p in the second representation for the ordinary quantum mechanics (red line) and the GUP framework (blue line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this Letter.)

$$\int_{-\frac{2}{3\sqrt{\beta}}}^{+\frac{2}{3\sqrt{\beta}}} \mathrm{d}p\,\psi^*(p)\left(i\hbar\frac{\partial}{\partial p}\right)\phi(p) = \int_{-\frac{2}{3\sqrt{\beta}}}^{+\frac{2}{3\sqrt{\beta}}} \mathrm{d}p\left(i\hbar\frac{\partial\psi(p)}{\partial p}\right)^*\phi(p) + i\hbar\psi^*(p)\phi(p)\Big|_{p=+\frac{2}{3\sqrt{\beta}}} - i\hbar\psi^*(p)\phi(p)\Big|_{p=-\frac{2}{3\sqrt{\beta}}}. \tag{31}$$

Now since $\phi(p)$ vanishes at $p = \pm\frac{2}{3\sqrt{\beta}}$, $\psi^*(p)$ can take any arbitrary value there. Therefore, although its adjoint $X^\dagger = i\hbar\partial/\partial p$ has the same formal expression, it acts on a different space of functions, namely

$$\mathcal{D}(X) = \left\{\phi, \phi' \in \mathcal{L}^2\left(\frac{-2}{3\sqrt{\beta}}, \frac{+2}{3\sqrt{\beta}}\right); \phi\left(\frac{+2}{3\sqrt{\beta}}\right) = \phi\left(\frac{-2}{3\sqrt{\beta}}\right) = 0\right\}, \tag{32}$$

$$\mathcal{D}(X^\dagger) = \left\{\psi, \psi' \in \mathcal{L}^2\left(\frac{-2}{3\sqrt{\beta}}, \frac{+2}{3\sqrt{\beta}}\right); \text{no other restriction on } \psi\right\}. \tag{33}$$

To better clarify this point, we can also use the von Neumann's theorem [41,42]. Thus, we need to find the wave functions that satisfy the eigenvalue equation

$$X^\dagger\phi_\pm(p) = i\hbar\partial_p\phi_\pm(p) = \pm i\lambda\phi_\pm(x). \tag{34}$$

The solutions are

$$\phi_\pm(p) = \mathcal{C}_\pm e^{\mp\lambda p}. \tag{35}$$

Since both $\phi_\pm(p)$ belong to $\mathcal{L}^2(\frac{-2}{3\sqrt{\beta}}, \frac{+2}{3\sqrt{\beta}})$, the deficiency indices are $(1,1)$. Therefore, the position operator is not self-adjoint but has a one-parameter family of self-adjoint extensions which is in agreement with the previous result.

3.1. Free particle

In ordinary quantum mechanics, the free particle wave function $u_p(x)$ is defined as the eigenfunction of the momentum operator, namely $\hat{P}u_p(x) = pu_p(x)$ where p is the eigenvalue. Since the momentum operator in position space is given by $\hat{P} = -i\hbar\frac{\partial}{\partial x}$, we have $-i\hbar\frac{\partial u_p(x)}{\partial x} = pu_p(x)$ which has the following solution

$$u_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ipx}{\hbar}\right), \tag{36}$$

where the constant of integration is chosen to satisfy

$$\int_{-\infty}^{\infty} u_p^*(x)u_p(x')\,\mathrm{d}p = \delta(x-x'). \tag{37}$$

In the GUP scenario, to find the momentum eigenfunction in position space, we write the eigenvalue equation as

$$\frac{1 - i\sqrt{3} + (-2\beta)^{1/3}(-3i\hbar\partial_x + \sqrt{-9\hbar^2\partial_x^2 - 4/\beta})^{2/3}}{(2\beta)^{2/3}(-3i\hbar\partial_x + \sqrt{-9\hbar^2\partial_x^2 - 4/\beta})^{1/3}}u_\wp(x) = \wp u_\wp(x), \tag{38}$$

where $\wp$ is the eigenvalue of P. Now, let us take the solution in the form of Eq. (36)

$$u_\wp(x) = \mathcal{A}\exp\left(\frac{ipx}{\hbar}\right), \tag{39}$$

where $p = f(\wp)$. Inserting this solution in Eq. (38) results in

$$\frac{1 - i\sqrt{3} + (-2\beta)^{1/3}(3p + \sqrt{9p^2 - 4/\beta})^{2/3}}{(2\beta)^{2/3}(3p + \sqrt{9p^2 - 4/\beta})^{1/3}} = \wp, \tag{40}$$

or

$$p = \wp - \frac{\beta}{3}\wp^3, \tag{41}$$

so we have

$$u_\wp(x) = \mathcal{A}\exp\left[\frac{i}{\hbar}\left(\wp - \frac{\beta}{3}\wp^3\right)x\right]. \tag{42}$$

The eigenfunctions are normalizable

$$1 = \mathcal{A}\mathcal{A}^*\int_{-\frac{2}{3\sqrt{\beta}}}^{+\frac{2}{3\sqrt{\beta}}}\mathrm{d}p = \frac{4\mathcal{A}\mathcal{A}^*}{3\sqrt{\beta}}. \tag{43}$$

Therefore

$$u_\wp(x) = \frac{\sqrt{3\sqrt{\beta}}}{2}\exp\left[\frac{i}{\hbar}\left(\wp - \frac{\beta}{3}\wp^3\right)x\right]. \tag{44}$$

The momentum eigenfunctions now satisfy

$$\int_{-\frac{2}{3\sqrt{\beta}}}^{+\frac{2}{3\sqrt{\beta}}} u_\wp^*(x')u_\wp(x)\,\mathrm{d}p = \int_{-1/\sqrt{\beta}}^{+1/\sqrt{\beta}}(1-\beta\wp^2)u_\wp^*(x')u_\wp(x)\,\mathrm{d}\wp, \tag{45}$$

$$= \frac{3\hbar\sqrt{\beta}}{2(x-x')}\sin\left(\frac{2(x-x')}{3\hbar\sqrt{\beta}}\right). \tag{46}$$

Finally, since $\wp_{\max} = 1/\sqrt{\beta}$, the energy of the free particle $E = \frac{\wp^2}{2m}$ is bounded from above

$$E_{\max} = \frac{1}{2m\beta}. \tag{47}$$

To find Eq. (44) we supposed that the coefficient $\mathcal{A}$ does not depend on the momentum. If we relax this assumption, the maximally localized states can be used to find the quasiposition wave function of the momentum eigenstate $\phi_\wp(p) = \delta(p-\wp)$ in a straightforward way. So inserting $\phi_\wp(p)$ in Eq. (21) results in

$$\psi_{\mathrm{QP}}(\xi) = \sqrt{\frac{3\sqrt{\beta}}{2}}(1-\beta\wp^2)\cos\left[\frac{3\pi\sqrt{\beta}}{4}\left(\wp - \frac{\beta}{3}\wp^3\right)\right] \times \exp\left[\frac{i\xi}{\hbar}\left(\wp - \frac{\beta}{3}\wp^3\right)\right], \tag{48}$$

and therefore $\mathcal{A}(\wp) = \sqrt{\frac{3\sqrt{\beta}}{2}}(1-\beta\wp^2)\cos[\frac{3\pi\sqrt{\beta}}{4}(\wp - \frac{\beta}{3}\wp^3)]$. However, for this case the solutions are no longer the eigenfunctions of the position operator which is the consequence of non-self-adjointness property of the position operator. Thus, in comparison, Eq. (48) represents the physically acceptable solutions.

3.2. Particle in a box

As another application, let us consider a particle with mass m confined in an infinite one-dimensional box with length L

$$V(x) = \begin{cases} 0, & 0 < x < L, \\ \infty, & \text{elsewhere.} \end{cases} \tag{49}$$

The corresponding eigenfunctions should satisfy the following generalized Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_n(x)}{\partial x^2} + \frac{\beta\hbar^4}{3m}\frac{\partial^4\psi(x)}{\partial x^4} - \frac{7\beta^2\hbar^6}{18m}\frac{\partial^6\psi(x)}{\partial x^6} + \mathcal{O}(\beta^3) = E_n\psi_n(x), \tag{50}$$

for $0 < x < L$ and they also meet the boundary conditions $\psi_n(0) = \psi_n(L) = 0$. In Refs. [28,18], the above equation is thoroughly solved to $\mathcal{O}(\beta)$ and its exact eigenvalues and eigenfunctions are found. Because of the boundary conditions, if we take the normalized ansatz

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right), \tag{51}$$

Eq. (50) is satisfied and we obtain

$$H\psi_n(x) = \left(\varepsilon_n + \frac{4}{3}\beta m\varepsilon_n^2 + \frac{28}{9}\beta^2m^2\varepsilon_n^3 + \frac{80}{9}\beta^3m^3\varepsilon_n^4 + \cdots\right)\psi_n(x) \tag{52}$$

where $\varepsilon_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$. Now the comparison between Eqs. (50) and (52) shows

$$E_n = \varepsilon_n + \frac{4}{3}\beta m\varepsilon_n^2 + \frac{28}{9}\beta^2 m^2\varepsilon_n^3 + \frac{80}{9}\beta^3 m^3\varepsilon_n^4 + \cdots, \tag{53}$$

$$= \varepsilon_n\left[\frac{1 - i\sqrt{3} + (-2)^{1/3}(3\gamma_n + \sqrt{9\gamma_n^2 - 4})^{2/3}}{4^{1/3}(3\gamma_n + \sqrt{9\gamma_n^2 - 4})^{1/3}}\right]^2, \tag{54}$$

where $\gamma_n = 2\beta m\varepsilon_n$. Therefore, to first order of GUP parameter we have $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} + \beta\frac{n^4\pi^4\hbar^4}{3mL^4}$ which is in agreement with the result of Ref. [28]. These results show that in this GUP scenario there is no change in the particle in a box eigenfunctions but there is a positive shift in the energy spectrum which is proportional to the powers of β.

We now estimate the energy spectrum using the semiclassical scheme. For the particle in a box, the Wilson–Sommerfeld formula

$$\oint p\,\mathrm{d}x = nh, \qquad n = 1, 2, \ldots, \tag{55}$$

results in

$$p_n = \frac{nh}{L}. \tag{56}$$

Since the high energy–momentum P depends on the low energy–momentum through $p_n = P_n - (1/3)\beta P_n^3$ (24), the semiclassical energy spectrum is given by

$$E_n^{(\mathrm{SC})} = \frac{P_n^2}{2m} = \left[\frac{1 - i\sqrt{3} + (-2\beta)^{1/3}(3p_n + \sqrt{9p_n^2 - 4/\beta})^{2/3}}{\sqrt{2m}(2\beta)^{2/3}(3p_n + \sqrt{9p_n^2 - 4/\beta})^{1/3}}\right]^2. \tag{57}$$

It is straightforward to check that the semiclassical result (57) exactly coincide with the quantum mechanical spectrum (53). Therefore, the number of states is finite

$$n_{\max} = \left\lfloor \frac{2L}{3h\sqrt{\beta}} \right\rfloor, \tag{58}$$

where $\lfloor x \rfloor$ denotes the largest integer not greater than x, and the maximal energy of the particle in a box reads

$$E_{\max} = \frac{1}{2m\beta}. \tag{59}$$

So we found that this upper bound is similar to the case of the free particle. However, note that because of the presence of the maximum momentum $P_{\max}$ this result is not surprising. Indeed for both cases we have $E_{\max} = P_{\max}^2/2m$. Moreover, for the case of the harmonic oscillator, the maximal semiclassical energy is $E_{\max}^{(\mathrm{SC})} = 1/m\beta$ [39]. This value can be roughly estimated if we associate the same amount of energy to both kinetic and potential parts of the Hamiltonian, namely $E_{\max}^{(\mathrm{SC})} = E_{\max}^{(K)} + E_{\max}^{(P)} = 2E_{\max}$.

It is now worth mentioning that the existence of the upper bound on the energy spectrum in the GUP scenario is also addressed by Quesne and Tkachuk in the context of Lorentz-covariant deformed algebra with minimal length when it is applied to the $(1+1)$-dimensional Dirac oscillator [43]. For that case the energy spectrum reads

$$|E_n| = \frac{c}{\sqrt{\beta}}\sqrt{1 + \frac{\beta m^2c^2 - 1}{(1 + \beta m\hbar\omega n)^2}}, \quad n = 0, 1, 2, \ldots, \tag{60}$$

where m and ω are the oscillator's mass and frequency, respectively. Therefore both the deformation parameter and the energy spectrum are bounded from above, i.e.,

$$|E|_{\max} = \frac{c}{\sqrt{\beta}}, \quad \beta < \frac{1}{m^2c^2}. \tag{61}$$

In comparison, unlike the particle in a box (58), n is not bounded and ranges from zero to infinity. However, there is no restriction on β in our formulation in contrary to the covariant version of the KMM algebra.

3.3. WKB approximation

To check the validity of the Wilson–Sommerfeld quantization rule for this modified quantum mechanics, we need to show that the zeroth-order wave function, which satisfies the generalized Schrödinger equation (30), can be written as $\psi(x) \simeq \exp[(i/\hbar)\int p\,\mathrm{d}x]$. So let us take

$$\psi(x) = e^{i\varphi(x)}, \tag{62}$$

where $\varphi(x)$ can be expanded as a power series in $\hbar$ in the semiclassical approximation, i.e.,

$$\varphi(x) = \frac{1}{\hbar}\sum_{n=0}^{\infty}\hbar^n\varphi_n(x). \tag{63}$$

So we have

$$\frac{\partial^2\psi(x)}{\partial x^2} = \left(-\varphi'^2 + i\varphi''\right)\psi(x), \tag{64}$$

$$\frac{\partial^4\psi(x)}{\partial x^4} = \left(\varphi'^4 - 6i\varphi'^2\varphi'' - 3\varphi''^2 - 4\varphi'''\varphi' + i\varphi''''\right)\psi(x), \tag{65}$$

$$\vdots$$

where the prime indicates the derivative with respect to x. Now to zeroth-order $\varphi(x) \simeq \varphi_0(x)/\hbar$ and for $\hbar \to 0$ we obtain

$$\varphi_0'^2 + \frac{2}{3}\beta\varphi_0'^4 + \frac{7}{9}\beta^2\varphi_0'^6 + \mathcal{O}(\beta^3) = 2m\big(E - V(x)\big). \tag{66}$$

Thus, the comparison with Eq. (29) shows $\varphi_0' = p$ and consequently

$$\psi(x) \simeq \exp\left[\frac{i}{\hbar}\int p\,\mathrm{d}x\right], \tag{67}$$

which is the usual zeroth-order WKB wave function obeying the Wilson–Sommerfeld quantization rule.

4. Generalization to *D* dimensions

We now extend the developed formalism in previous sections to D spatial dimensions. We then present the generalized Poisson brackets in the classical limit and study the density of states.

4.1. Generalized Heisenberg algebra for *D* dimensions

A natural generalization of the one-dimensional commutation relation (1) that preserves the rotational symmetry is

$$[X_i, P_j] = \frac{i\hbar\delta_{ij}}{1 - \beta P^2}, \tag{68}$$

where $P^2 = \sum_{i=1}^{D} P_iP_i$. This relation implies a nonzero minimal uncertainty and a maximal observable momentum in each position coordinate. If the components of the momentum operator are assumed to be commutative

$$[P_i, P_j] = 0, \tag{69}$$

then the Jacobi identity determines the commutation relations between the components of the position operator as

$$[X_i, X_j] = \frac{2i\hbar\beta}{(1-\beta P^2)^2}(P_i X_j - P_j X_i), \tag{70}$$

which results in a noncommutative geometric generalization of position space. To exactly satisfy these commutation relations, the position and momentum operators in the momentum space representation can be written as

$$P_i \phi(p) = p_i \phi(p), \tag{71}$$

$$X_i \phi(p) = \frac{i\hbar}{1-\beta p^2}\partial_{p_i}\phi(p). \tag{72}$$

X_i and P_j are now symmetric operator on the domain S_∞ with respect to the scalar product:

$$\langle\psi|\phi\rangle = \int_{-1/\sqrt{\beta}}^{+1/\sqrt{\beta}} \mathrm{d}^D p\,\big(1-\beta p^2\big)\psi^*(p)\phi(p), \tag{73}$$

where $p^2 = \sum_{i=1}^{D} p_i p_i$. The identity operator is

$$1 = \int_{-1/\sqrt{\beta}}^{+1/\sqrt{\beta}} \frac{\mathrm{d}^D p}{(1-\beta p^2)}|p\rangle\langle p|, \tag{74}$$

and the scalar product of momentum eigenstates is

$$\langle p|p'\rangle = \frac{\delta^D(p-p')}{1-\beta p^2}. \tag{75}$$

In this representation, the components of the momentum operator are still essentially self-adjoint, however the components of the position operators are merely symmetric and do not have physical eigenstates.

Since the commutation relations (68)–(70) do not break the rotational symmetry, we can express the generators of rotations in terms of the position and momentum operators as

$$L_{ij} \equiv \big(1-\beta P^2\big)(X_i P_j - X_j P_i), \tag{76}$$

as the generalization of the ordinary orbital angular momentum. Now the momentum space representation of the generators of rotations is

$$L_{ij}\psi(p) = -i\hbar(p_i\partial_{p_j} - p_j\partial_{p_i})\psi(p), \tag{77}$$

and

$$[P_i, L_{jk}] = i\hbar(\delta_{ik}P_j - \delta_{ij}P_k), \tag{78}$$

$$[X_i, L_{jk}] = i\hbar(\delta_{ik}X_j - \delta_{ij}X_k), \tag{79}$$

$$[L_{ij}, L_{kl}] = i\hbar(\delta_{ik}L_{jl} - \delta_{il}L_{jk} + \delta_{jl}L_{ik} - \delta_{jk}L_{il}), \tag{80}$$

as well as in ordinary quantum mechanics. However, the geometry is noncommutative, namely

$$[X_i, X_j] = \frac{-2i\hbar\beta}{(1-\beta P^2)^2}L_{ij}. \tag{81}$$

4.2. Density of states

The right hand side of Eq. (1) shows that the "effective" value of $\hbar$ is P dependent. So the size of the unit cell in the phase space that is occupied by each quantum state can be also considered of as being momentum dependent. This fact changes the momentum dependence of the density of states and affects the calculation of cosmological constant, blackbody radiation spectrum, etc. Similar to the KMM algebra [44], we should check that any volume of the phase space evolves such that the number of states inside it does not change with respect to time as the analog of the Liouville theorem.

The Poisson brackets in classical mechanics correspond quantum mechanical commutators via

$$\frac{1}{i\hbar}[A, B] \Longrightarrow \{A, B\}. \tag{82}$$

Thus the classical limits of Eqs. (68)–(70) are given by

$$\{X_i, P_j\} = \frac{\delta_{ij}}{1-\beta P^2}, \tag{83}$$

$$\{P_i, P_j\} = 0, \tag{84}$$

$$\{X_i, X_j\} = \frac{2\beta}{(1-\beta P^2)^2}(P_i X_j - P_j X_i), \tag{85}$$

and the Heisenberg equations for the coordinates and momenta read (i, j run over the spatial dimensions and the summation convention is assumed)

$$\dot{X}_i = \{X_i, H\} = \{X_i, P_j\}\frac{\partial H}{\partial P_j} + \{X_i, X_j\}\frac{\partial H}{\partial X_j}, \tag{86}$$

$$\dot{P}_i = \{P_i, H\} = -\{X_j, P_i\}\frac{\partial H}{\partial X_j}. \tag{87}$$

Note that in one dimension Eq. (86) implies that although the momentum is bounded from above, the velocity

$$\dot{X} = \{X, H\} = \frac{P}{m(1-\beta P^2)}, \tag{88}$$

ranges from $-\infty$ to $+\infty$ as P goes to $\pm\frac{1}{\sqrt{\beta}}$. We now prove that the weighted phase space volume

$$\big(1-\beta P^2\big)^D \mathrm{d}^D X\, \mathrm{d}^D P, \tag{89}$$

is invariant under time evolution as the analog of the Liouville theorem. The evolution of X_i and P_i during an infinitesimal time interval δt is

$$X_i' = X_i + \delta X_i, \tag{90}$$

$$P_i' = P_i + \delta P_i, \tag{91}$$

where

$$\delta X_i = \left[\{X_i, P_j\}\frac{\partial H}{\partial P_j} + \{X_i, X_j\}\frac{\partial H}{\partial X_j}\right]\delta t, \tag{92}$$

$$\delta P_i = -\{X_j, P_i\}\frac{\partial H}{\partial X_j}\delta t. \tag{93}$$

After this infinitesimal evolution, the infinitesimal phase space volume is changed according to

$$\mathrm{d}^D X'\, \mathrm{d}^D P' = \left|\frac{\partial(X_1', \ldots, X_D', P_1', \ldots, P_D')}{\partial(X_1, \ldots, X_D, P_1, \ldots, P_D)}\right| \mathrm{d}^D X\, \mathrm{d}^D P, \tag{94}$$

where

$$\frac{\partial X'_i}{\partial X_j} = \delta_{ij} + \frac{\partial \delta X_i}{\partial X_j}, \qquad \frac{\partial X'_i}{\partial P_j} = \frac{\partial \delta X_i}{\partial P_j},$$
$$\frac{\partial P'_i}{\partial X_j} = \frac{\partial \delta P_i}{\partial X_j}, \qquad \frac{\partial P'_i}{\partial P_j} = \delta_{ij} + \frac{\partial \delta P_i}{\partial P_j}. \tag{95}$$

The Jacobian can be calculated to first order in δt as

$$\left|\frac{\partial(X'_1,\ldots,X'_D,P'_1,\ldots,P'_D)}{\partial(X_1,\ldots,X_D,P_1,\ldots,P_D)}\right| = 1 + \left(\frac{\partial\delta X_i}{\partial X_i} + \frac{\partial \delta P_i}{\partial P_i}\right) + \cdots. \tag{96}$$

So we have

$$\begin{aligned}
&\left(\frac{\partial\delta X_i}{\partial X_i} + \frac{\partial \delta P_i}{\partial P_i}\right)\frac{1}{\delta t} \\
&\quad = \frac{\partial}{\partial X_i}\left[\{X_i,P_j\}\frac{\partial H}{\partial P_j} + \{X_i,X_j\}\frac{\partial H}{\partial X_j}\right] - \frac{\partial}{\partial P_i}\left[\{X_j,P_i\}\frac{\partial H}{\partial X_j}\right] \\
&\quad = \left[\frac{\partial}{\partial X_i}\{X_i,P_j\}\right]\frac{\partial H}{\partial P_j} + \{X_i,P_j\}\frac{\partial^2 H}{\partial X_i \partial P_j} + \left[\frac{\partial}{\partial X_i}\{X_i,X_j\}\right]\frac{\partial H}{\partial X_j} \\
&\qquad + \{X_i,X_j\}\frac{\partial^2 H}{\partial X_i\partial X_j} - \left[\frac{\partial}{\partial P_i}\{X_j,P_i\}\right]\frac{\partial H}{\partial X_j} - \{X_j,P_i\}\frac{\partial^2 H}{\partial P_j \partial X_i} \\
&\quad = \left[\frac{\partial}{\partial X_i}\{X_i,X_j\}\right]\frac{\partial H}{\partial X_j} - \left[\frac{\partial}{\partial P_i}\{X_j,P_i\}\right]\frac{\partial H}{\partial X_j} \\
&\quad = \left[-\frac{2\beta(D-1)}{(1-\beta P^2)^2}P_j\right]\frac{\partial H}{\partial X_j} - \left[\frac{2\beta}{(1-\beta P^2)^2}P_j\right]\frac{\partial H}{\partial X_j} \\
&\quad = \frac{-2\beta D}{(1-\beta P^2)^2}P_j\frac{\partial H}{\partial X_j},
\end{aligned} \tag{97}$$

which to first order in δt results in

$$\mathrm{d}^D X'\,\mathrm{d}^D P' = \mathrm{d}^D X\,\mathrm{d}^D P\left[1 - \frac{2\beta D}{(1-\beta P^2)^2}P_j\frac{\partial H}{\partial X_j}\delta t\right]. \tag{98}$$

Moreover

$$\begin{aligned}
1-\beta P'^2 &= 1-\beta(P_i+\delta P_i)^2 \\
&= 1-\beta\big(P^2 + 2P_i\delta P_i + \cdots\big) \\
&= 1-\beta\left(P^2 - 2P_i\{X_i,P_j\}\frac{\partial H}{\partial X_j}\delta t + \cdots\right) \\
&= 1-\beta\left(P^2 - \frac{2P_i}{1-\beta P^2}\frac{\partial H}{\partial X_i}\delta t + \cdots\right) \\
&= \big(1-\beta P^2\big) + \frac{2\beta P_i}{1-\beta P^2}\frac{\partial H}{\partial X_i}\delta t + \cdots \\
&= \big(1-\beta P^2\big)\left[1 + \frac{2\beta P_i}{(1-\beta P^2)^2}\frac{\partial H}{\partial X_i}\delta t + \cdots\right].
\end{aligned} \tag{99}$$

Therefore, to first order in δt

$$\big(1-\beta P'^2\big)^D = \big(1-\beta P^2\big)^D\left[1 + \frac{2\beta D}{(1-\beta P^2)^2}P_i\frac{\partial H}{\partial X_i}\delta t\right]. \tag{100}$$

Now using Eqs. (98) and (100), it is obvious that the weighted phase space volume (89) is an invariant, i.e.,

$$\big(1-\beta P'^2\big)^D\,\mathrm{d}^D X'\,\mathrm{d}^D P' = \big(1-\beta P^2\big)^D\,\mathrm{d}^D X\,\mathrm{d}^D P. \tag{101}$$

4.3. The cosmological constant

The cosmological constant can be obtained by summing over the zero-point energies of the harmonic oscillator's momentum states. Using the canonical form of the zero-point energy of each oscillator with mass m

$$\frac{1}{2}\hbar\omega = \frac{1}{2}\sqrt{p^2+m^2}, \tag{102}$$

the sum over all momentum states per unit volume is

$$\begin{aligned}
\Lambda(m) &= \int \mathrm{d}^3p\,\big(1-\beta p^2\big)^3\left(\frac{1}{2}\sqrt{p^2+m^2}\right) \\
&= 2\pi\int_0^{1/\sqrt{\beta}}\mathrm{d}p\,\big(1-\beta p^2\big)^3 p^2\sqrt{p^2+m^2} \\
&= \frac{\pi}{20\beta^2}f\big(\beta m^2\big),
\end{aligned} \tag{103}$$

where

$$\begin{aligned}
f(x) = \frac{1}{96}\Big[&\big(96+192x+476x^2+380x^3+105x^4\big)\sqrt{1+x} \\
&- \big(480x^2+720x^3+450x^4+105x^5\big)\cosh^{-1}(\sqrt{x})\Big],
\end{aligned} \tag{104}$$

and $f(0)=1$. In the massless limit we find

$$\Lambda(0) = \frac{\pi}{20\beta^2} = \frac{1}{10}\big[\Lambda(0)\big]^{\mathrm{KMM}}, \tag{105}$$

that is ten times smaller than the massless cosmological constant predicted by the KMM proposal [44]. This finite result is due to the vanishing of the density of states at high momenta where $p=1/\sqrt{\beta}$ plays the role of the UV cutoff. So in this scenario we do not need to put by hand an arbitrary scale as the UV cutoff and the cosmological constant is automatically rendered finite. Note that since $1/\sqrt{\beta}$ is proportional to the Planck mass M_{Pl}, $\Lambda(0)$ is too large in practice and consequently the cosmological constant problem still remains unsolved. However, our formulation gives the better estimation of Λ with respect to that obtained in the KMM framework.

4.4. The blackbody radiation spectrum

Because of the weight factor $(1-\beta P^2)^3$ in 3-dimensions, the average energy of the electromagnetic field per unit volume at temperature T is given by

$$\begin{aligned}
\langle E\rangle &= \frac{8\pi}{c^3}\int_0^\infty \mathrm{d}\nu\left(1-\beta\left(\frac{h\nu}{c}\right)^2\right)^3\left(\frac{h\nu^3}{e^{h\nu/k_BT}-1}\right) \\
&= \int_0^\infty \mathrm{d}\nu\,u_\beta(\nu,T),
\end{aligned} \tag{106}$$

where

$$u_\beta(\nu,T) = \left(1-\left(\frac{\nu}{\nu_\beta}\right)^2\right)^3 u_0(\nu,T). \tag{107}$$

Here

$$u_0(\nu,T) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_BT}-1}, \tag{108}$$

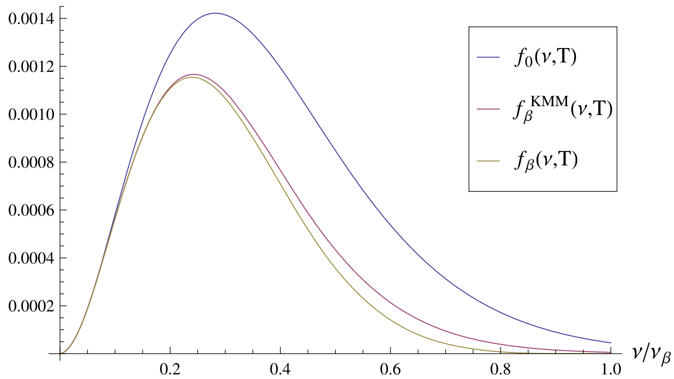

Fig. 2. The blackbody radiation spectrum in the GUP framework at temperature $T = 0.1T_\beta$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this Letter.)

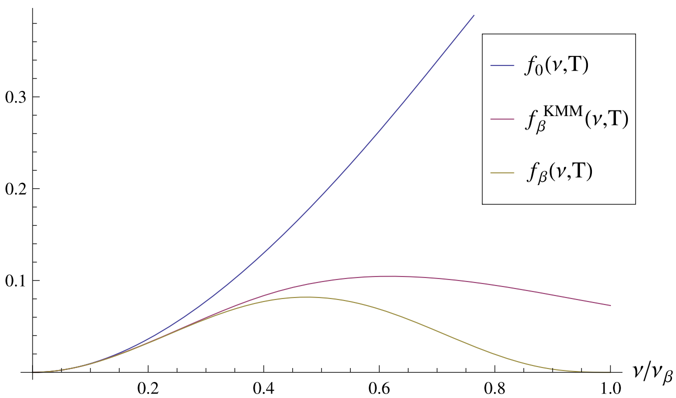

Fig. 3. The blackbody radiation spectrum in the GUP framework at temperature $T = T_\beta$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this Letter.)

is the ordinary spectrum function and $\nu_\beta = c/h\sqrt{\beta}$. To show the effect of the minimal length uncertainty and the maximal momentum on the shape of the spectral function, we have depicted the functions

$$f_0(\nu, T) = \frac{(\nu/\nu_\beta)^3}{e^{(\nu/\nu_\beta)(T_\beta/T)} - 1}, \tag{109}$$

$$f_\beta(\nu, T) = \left(1 - (\nu/\nu_\beta)^2\right)^3 f_0(\nu, T), \tag{110}$$

in Figs. 2 and 3, and compared them with the case of just the minimal length uncertainty [44]

$$f_\beta^{\mathrm{KMM}}(\nu, T) = \frac{1}{(1 + (\nu/\nu_\beta)^2)^3} f_0(\nu, T), \tag{111}$$

where $T_\beta = c/k_B\sqrt{\beta}$. As the figure shows, for small frequencies ($\nu \ll \nu_\beta$), $f_\beta(\nu, T)$ closely coincides with f_β^{KMM}. However, it deviates from f_β^{KMM} as the frequency increases.

5. Conclusions

In this Letter, we studied a higher order generalized uncertainty principle that implies both a minimal length uncertainty and a maximal momentum proportional to $\hbar\sqrt{\beta}$ and $1/\sqrt{\beta}$, respectively. We found maximally localized states and presented a formally self-adjoint representation that preserves the ordinary nature of the position operator and results in the perturbative generalized Schrödinger equation. We exactly solved the problems of the free particle and the particle in a box and showed that the existence of the maximal momentum $P_{\max} = 1/\sqrt{\beta}$ is manifest through this representation. We then generalized this proposal to D dimensions and found the invariant density of states. We showed that the blackbody radiation spectrum are modified at high frequencies and compared the results with the KMM proposal. Although the cosmological constant was rendered finite, the smallness of the GUP parameter resulted in a large cosmological constant that could not solve the cosmological constant problem. However, our calculated cosmological constant is a better estimation with respect to the presence of just the minimal length.

Acknowledgement

I would like to thank Stephane Detournay for introducing his work about maximally localized states.

References

[1] S. Hossenfelder, arXiv:1203.6191.
[2] G. Veneziano, Europhys. Lett. 2 (1986) 199.
[3] E. Witten, Phys. Today 49 (1996) 24.
[4] D. Amati, M. Ciafaloni, G. Veneziano, Phys. Lett. B 216 (1989) 41; D. Amati, M. Ciafaloni, G. Veneziano, Nucl. Phys. B 347 (1990) 550; D. Amati, M. Ciafaloni, G. Veneziano, Nucl. Phys. B 403 (1993) 707.
[5] K. Konishi, G. Paffuti, P. Provero, Phys. Lett. B 234 (1990) 276.
[6] L.J. Garay, Int. J. Mod. Phys. A 10 (1995) 145.
[7] M. Maggiore, Phys. Lett. B 319 (1993) 83.
[8] A. Kempf, G. Mangano, R.B. Mann, Phys. Rev. D 52 (1995) 1108.
[9] A. Kempf, G. Mangano, Phys. Rev. D 55 (1997) 7909.
[10] J. Magueijo, L. Smolin, Phys. Rev. Lett. 88 (2002) 190403, arXiv:hep-th/0112090.
[11] J. Magueijo, L. Smolin, Phys. Rev. D 71 (2005) 026010, arXiv:hep-th/0401087.
[12] J.L. Cortes, J. Gamboa, Phys. Rev. D 71 (2005) 065015, arXiv:hep-th/0405285.
[13] M. Maggiore, Phys. Rev. D 49 (1994) 5182, arXiv:hep-th/9305163.
[14] S. Hossenfelder, et al., Phys. Lett. B 575 (2003) 85, arXiv:hep-th/0305262.
[15] C. Bambi, F.R. Urban, Class. Quantum Gravit. 25 (2008) 095006, arXiv:0709.1965.
[16] K. Nozari, B. Fazlpour, Gen. Relativ. Gravit. 38 (2006) 1661.
[17] R. Banerjee, S. Ghosh, Phys. Lett. B 688 (2010) 224, arXiv:1002.2302.
[18] P. Pedram, Int. J. Mod. Phys. D 19 (2010) 2003.
[19] P. Pedram, Int. J. Theor. Phys. 51 (2012) 1901.
[20] P. Pedram, Physica A 391 (2012) 2100, arXiv:1111.6859.
[21] K. Nozari, P. Pedram, M. Molkara, Int. J. Theor. Phys. 51 (2012) 1268, arXiv:1111.2204.
[22] P. Pedram, Phys. Lett. B 702 (2011) 295.
[23] K. Nozari, P. Pedram, Europhys. Lett. 92 (2010) 50013, arXiv:1011.5673.
[24] B. Vakili, Phys. Rev. D 77 (2008) 044023.
[25] M.V. Battisti, G. Montani, Phys. Rev. D 77 (2008) 023518.
[26] B. Vakili, H.R. Sepangi, Phys. Lett. B 651 (2007) 79.
[27] M.V. Battisti, G. Montani, Phys. Lett. B 656 (2007) 96.
[28] K. Nozari, T. Azizi, Gen. Relativ. Gravit. 38 (2006) 735.
[29] S. Das, E.C. Vagenas, Phys. Rev. Lett. 101 (2008) 221301, arXiv:0810.5333.
[30] K. Nouicer, Phys. Lett. B 646 (2007) 63.
[31] P. Pedram, Phys. Rev. D 85 (2012) 024016, arXiv:1112.2327.
[32] P. Pedram, Phys. Lett. B 710 (2012) 478.
[33] P. Pedram, Europhys. Lett. 89 (2010) 50008, arXiv:1003.2769.
[34] P. Pedram, K. Nozari, S.H. Taheri, JHEP 1103 (2011) 093.
[35] A.F. Ali, S. Das, E.C. Vagenas, Phys. Lett. B 678 (2009) 497.
[36] S. Das, E.C. Vagenas, A.F. Ali, Phys. Lett. B 690 (2010) 407.
[37] A.F. Ali, S. Das, E.C. Vagenas, Phys. Rev. D 84 (2011) 044013.
[38] K. Nozari, A. Etemadi, Phys. Rev. D 85 (2012) 104029.
[39] P. Pedram, Phys. Lett. B 714 (2012) 317.
[40] S. Detournay, Cl. Gabriel, Ph. Spindel, Phys. Rev. D 66 (2002) 125004.
[41] N.I. Akhiezer, I.M. Glazman, Theory of Linear Operators in Hilbert Space, Dover, New York, 1993.
[42] G. Bonneau, J. Faraut, G. Valent, Am. J. Phys. 69 (2001) 322.
[43] C. Quesne, V.M. Tkachuk, J. Phys. A 39 (2006) 10909.
[44] L.N. Chang, D. Minic, N. Okamura, T. Takeuchi, Phys. Rev. D 65 (2002) 125028.