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4-DIMENSIONAL TOPOLOGICAL BORDISM

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We show that two oriented, topological, closed 4-manifolds M_1^4, M_2^4 are topologically cobordant iff $\sigma(M_1^4)=\sigma(M_2^4)$ and either both of M_1^4, M_2^4 are stably smoothable or neither of M_1^4, M_2^4 is stably smoothable.

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Topological cobordism groups stably smoothable

1. Introduction

It is well known that if M^4, a closed, oriented, topological 4-manifold bounds an oriented compact 5-manifold then the signature $\sigma(M^4)$ of M^4 is zero [12]. In [8], Melvin gave a short elegant proof of the following theorem due to Rohlin.

Theorem. *Every closed, oriented, smooth 4-manifold M of signature zero is the boundary of a compact, oriented, smooth 5-manifold.*

In this article, we show that a closed, oriented, topological 4-manifold M^4 bounds an oriented topological 5-manifold iff $\sigma(M^4)=0$ and M^4 is stably smoothable. Consequently, we provide examples of topological oriented 4-manifolds with zero signatures and which do not bound.

2. Preliminaries

Let Ξ denote the set of all closed, connected, oriented, topological 4-manifolds. Because of the Smale Conjecture proved by Hatcher [6], the connected sum operation, $\#$, is well-defined on the topological category. Ξ with the connected sum operation is an Abelian monoid. For each $M\in\Xi$, there is a well-defined Kirby–Siebenmann obstruction $\mathrm{ks}(M)\in H^4(M;\mathbb{Z}_2)$. It is known that $\mathrm{ks}(M)=0$ if and only if M is stably smoothable, i.e. $M \# k(S^2\times S^2)$ is smoothable for some $k\geq 0$, or $M\times\mathbb{R}$ is smoothable. In fact, these two conditions are equivalent. There is a map

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$\mathscr{o}: \Xi \to \mathbb{Z}_2$ which assigns the Kirby–Siebenmann invariant to each $M \in \Xi$, $M \mapsto \mathrm{ks}(M) \in H^4(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$. It is easy to show that this map $\mathscr{o}$ is a homomorphism between the Abelian monoid Ξ and $\mathbb{Z}_2$ [9]. Moreover, $\mathscr{o}(M) = 0$ if and only if M is stably smoothable.

We shall let $\sigma(M)$ denote the signature of M^4, Ω_4^{Top} denote the oriented topological 4-dimensional cobordism group, η_4^{Top} denote the unoriented topological 4-dimensional cobordism group and $F(\mathrm{CP}(2))$ denote the fake CP(2) constructed by Freedman [5]. More generally, we call those simply-connected type 1 4-manifolds which are not stably smoothable fake 4-manifolds.

2.1. Lemma. *If M^4 is a simply-connected, oriented, closed, topological 4-manifold then M^4 bounds a topological, orientable, compact 5-manifold if and only if $\sigma(M^4) = 0$ and $\mathscr{o}(M^4) = 0$. In other words, M^4 bounds if and only if that either $M^4 = n \cdot \mathrm{CP}(2) \# n(-\mathrm{CP}(2))$ or $M^4 = n \cdot (S^2 \times S^2)$* [5].

2.2. Theorem. *If M^4 is an oriented, connected, closed, topological 4-manifold then M^4 bounds a compact, topological, orientable, 5-manifold if and only if $\sigma(M^4) = 0$ and $\mathscr{o}(M^4) = 0$.*

2.3. Theorem. *If M_1^4 and M_2^4 are oriented, connected, closed, topological 4-manifolds then M_1^4 is orientably cobordant to M_2^4 if and only if $\sigma(M_1^4) = \sigma(M_2^4)$ and $\mathscr{o}(M_1^4) = \mathscr{o}(M_2^4)$.*

2.4. Corollary. (a) $\Omega_4^{\mathrm{Top}} = \mathbb{Z} \oplus \mathbb{Z}_2$, *the generators are* CP(2) *and* $F(\mathrm{CP}(2)) \# (-\mathrm{CP}(2))$.

(b) *Kirby–Siebenmann obstructions are cobordism invariants (oriented or unoriented).*

(c) $\eta_4^{\mathrm{Top}} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, *the generators are* $\mathbb{R}P^2 \times \mathbb{R}P^2$, $\mathbb{R}P^4$ *and* $F(\mathrm{CP}(2))$.

2.5. Examples. (a) Since $\mathscr{o}$ is a homomorphism, we observe that $F(\mathrm{CP}(2)) \# (-\mathrm{CP}(2))$ is a simply-connected, closed, oriented, topological 4-manifold with zero signature and $\mathscr{o}(F(\mathrm{CP}(2)) \# (-\mathrm{CP}(2))) = 1$. Hence, this manifold does not bound any compact, oriented or unoriented topological 5-manifold. More generally, for any $n \geq 1$, $F(\mathrm{CP}(2)) \# (n-1)\mathrm{CP}(2) \# n(-\mathrm{CP}(2))$ does not bound.

(b) CP(2) is not cobordant to $F(\mathrm{CP}(2))$ and $F(\mathrm{CP}(2)) \# -F(\mathrm{CP}(2)) = \mathrm{CP}(2) \# -(\mathrm{CP}(2))$.

3. Proofs of statements

We shall use the following observation.

Observation. Given a smooth open 5-manifold W with finitely many ends, if we can complete W by adding a boundary component to each end, then each boundary component must be a stably smoothable 4-manifold. Because, if M_1^4 is one boundary component of the completion, then M_1^4 is collared [1] and hence, $M_1^4 \times (0,1) \hookrightarrow W$ inherits a smoothing structure from W, i.e. $M_1^4 \times \mathbb{R}$ is smoothable. In [11], Silver studied the problem for finding stable-boundaries for open 5-dimensional manifolds.

Proof of Lemma 2.1. If M^4 is simply-connected, $\sigma(M^4)=0$ and $\wp(M^4)=0$ then $M^4 \cong n \cdot \mathbb{CP}(2) \# n \cdot (-\mathbb{CP}(2))$ or $M^4 \cong n \cdot (S^2 \times S^2)$, topologically. It is clear that M^4 bounds a compact oriented 5-manifold.

Conversely, if M^4 bounds a compact, topological, orientable 5-manifold W then $\sigma(M^4)=0$. We only need to show that $\wp(M^4)=0$. Because of the solution of the 4-dimensional Annulus Conjecture provided by Quinn [4], every topological orientable manifold is stable and hence every locally flat simple closed curve has a trivial tubular neighborhood [2]. Therefore, we may do topological surgery on W so that W becomes a simply-connected 5-manifold with boundary M^4. Let Int(W) denote the interior of W. $H^4(\text{Int}(W); \mathbb{Z}_2) \cong H^4(W; \mathbb{Z}_2) \cong H_1(W, M; \mathbb{Z}_2) = 0$. Hence, Int($W$) is smoothable and by the observation we made, we conclude that M is stably smoothable and $\wp(M^4)=0$.

Proof of Theorem 2.2. If $\sigma(M^4)=0$ and $\wp(M^4)=0$ then M^4 bounds an orientable 5-manifold: Because $\wp(M^4)=0$, $M^4 \times \mathbb{R}$ is smoothable. By the splitting theorem [3], we have a smooth $M^4 \# k(S^2 \times S^2)$ imbedded in $M^4 \times \mathbb{R}$ which separates $M^4 \times \mathbb{R}$ into two parts. Choose either one and take a completion. There is a cobordism $(W_1; M^4, M^4 \# k(S^2 \times S^2))$ between M^4 and $M^4 \# k(S^2 \times S^2)$. By the theorem proved by Melvin, there is a compact 5-manifold W_2 with boundary $M^4 \# k(S^2 \times S^2)$. Glue W_1 and W_2 along the manifold $M^4 \# k(S^2 \times S^2)$, we are done.

Conversely, if M^4 bounds a compact, orientable topological 5-manifold W, then $\sigma(M^4)=0$. We need to show that $\wp(M^4)=0$. Consider $(M \times I; M \times \{0\}, M \times \{1\})$. Again, we may do topological surgery on M^4 by attaching handles $\mathbb{D}^2 \times \mathbb{D}^3$'s on $M^4 \times \{1\}$ along the generators of $\pi_1(M^4)$ to obtain a cobordism $(U; M^4, N^4)$. Because N is simply-connected and N bounds the compact oriented 5-manifold obtained by gluing W and U along $M \times \{0\}$, we conclude that $N = n \cdot \mathbb{CP}(2) \# n \cdot (-\mathbb{CP}(2))$ or $N = n \cdot (S^2 \times S^2)$. By Lemma 2.1, N bounds a simply-connected 5-manifold V. Let $X = U \cup_N V$. Clearly, X is simply-connected 5-manifold with boundary M^4. Again, $H^4(\text{Int}(X); \mathbb{Z}_2) \cong H^4(X; \mathbb{Z}_2) \cong H_1(X, M; \mathbb{Z}_2) = 0$. Hence, Int($X$) is a smooth open 5-manifold and M^4 is stably smoothable, i.e. $\wp(M^4)=0$.

Proof of Theorem 2.3. If M_1^4 and M_2^4 are orientably cobordant then $\sigma(M_1^4) = \sigma(M_2^4)$ and $\wp(M_1^4) = \wp(M_2^4)$. This can be seen by drawing a line from M_1^4 to M_2^4 and deleting a tubular neighborhood of this line from the cobordism. We obtain the manifold $M_1^4 \# (-M_2^4)$ which bounds an oriented 5-manifold. The conclusion follows easily.

If $\sigma(M_1^4) = \sigma(M_2^4)$ and $\wp(M_1^4) = \wp(M_2^4)$, we want to produce a cobordism between M_1^4 and M_2^4. We may use the cobordism $(U; M_1^4, N_1^4)$ constructed in the proof of

Theorem 2.2. Furthermore, we may assume that N_1^4 is an indefinite type 1 manifold. Otherwise, we choose a 5-manifold V^5 bounded by $\mathbb{CP}(2) \# (-\mathbb{CP}(2))$. Delete a line segment and its open tubular neighborhood from U and V^5 and denote the resultant manifold by U_- and V_- respectively. Glue U_- and V_- along the boundaries of the open tubular neighborhoods of these lines. Similarly, M_2^4 is cobordant to a type 1 simply-connected 4-manifold N_2^4 with indefinite form. Also, we may assume that $\text{rank}(H_2(N_1^4)) = \text{rank}(H_2(N_2^4))$. Now, if $\wp(M_1^4) = 0 = \wp(M_2^4)$ then $\wp(N_1^4) = 0 = \wp(N_2^4)$. We may glue these cobordisms together along $N_1^4 = N = N_2^4$ and produce a cobordism between M_1^4 and M_2^4. If $\wp(M_1^4) = 1 = \wp(M_2^4)$ then notice that $N_1^4 \cong (\text{F}\mathbb{CP}(2)) \# (n-1)(\mathbb{CP}(2)) \# m(-\mathbb{CP}(2)) \cong N_2^4$ where $n - m = \sigma(M_1^4)$. This yields our Theorem.

Comments on Corollary 2.4. (a) is an easy consequence of Theorem 2.2.

(c) It is known [10] that either $\eta_4^{\text{Top}} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\eta_4^{\text{Top}} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Siebenmann also stated that if there is a 4-manifold M so that $\wp(M) = 1$, then $\eta_4^{\text{Top}} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. From (b), we see that indeed $\eta_4^{\text{Top}} = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(b) For the orientable case, the statement follows from Corollary 2.3. For the general case, we refer to [7] and the observation we made at the beginning of this section. For the reader's convenience, we repeat Lashof and Taylor's argument: Assume that there were a cobordism $(V; M_1, M_2)$ between M_1 and M_2 and $\wp(M_1) = 1$ and $\wp(M_2) = 0$. Since $\wp(M_1) = 1$, $\text{Int}(V)$ is not smoothable, i.e. $\tau(V)$, the topological tangent bundle of V, can not be reduced to a vector bundle. The obstruction for reducing $\tau(V)$ to a vector bundle is $k(V) \in H^4(V; \mathbb{Z}_2)$. Let $a \in H_1(V, M_1 \cup M_2; \mathbb{Z}_2)$ be the dual class. Represent a by n pieces of locally flat imbedded arcs going from M_1 to M_2, $n \geq 1$. Now each arc is the core of a 1-handle $I \times \mathbb{D}^4$ going from M_1 to M_2. Let $P^4 \subset \text{Int}(\mathbb{D}^4)$ be the contratible 4-manifold with ∂P the Poincare homology sphere. Remove $I \times \text{Int}(P^4)$ from each of the above 1-handles. Then the tangent bundle of the new V reduces to a vector bundle, hence $\text{Int}(V)$ is smoothable. Notice that the boundary of the new V becomes $(M_1 \# M_2) \# (n-1)(S^1 \times S^3)$. Clearly, $\wp((M_1 \# M_2) \# (n-1)(S^1 \times S^3)) = 1$, this contradicts the fact that $\text{Int}(V)$ is smoothable. This completes our argument.

4. Problems

Is there any closed, oriented 4-manifold M^4 with zero signature and $\wp(M^4) = 0$ but M^4 is not smoothable? Note, if, in addition, $\pi_1(M^4) = 0$, then M^4 is smoothable. More specifically, is there any homology 4-sphere which is not smoothable?

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