Abstract

We give upper and lower bounds to the number $u_{n-1}(Q_n)$ of edges that one can remove from a hypercube without altering its diameter, namely: 

$$\left( n - 2 \right)2^{n-1} - \left( \binom{n}{\lfloor n/2 \rfloor} \right) + 2 \leq u_{n-1}(Q_n) \leq \left( n - 2 \right)2^{n-1} + 1 - \left( \frac{2^n - 1}{2n - 1} \right).$$

1. Introduction

The $n$-dimensional hypercube $Q_n$ is used as a topology for connecting $2^n$ processors, each processor is labeled by a bit-string of length $n$. Two processors are linked if their binary labels differ by only one bit. We will often use the terms nodes for processors and edges for communication links.

In [2], Bermond, Delorme, and Quisquater presented some strategies for constructing large interconnection networks for a given set of parameters as the maximum degree, the diameter, and so on. The problem of removing as many edges as possible from a hypercube without changing its diameter can be considered as a problem of construction of a graph $G$ of small size and given diameter $n$, with the extra constraint that $G$ must be a subgraph of the hypercube $Q_n$. Using the same notation as in [4], let $u_{n-1}(Q_n)$ be the maximum number of edges of the hypercube whose removal maintains the diameter $n$.

* Corresponding author.
1 Fellowship of Algerian Government.
In what follows, we give a lower bound by constructing a spanning subgraph of $Q_n$ with diameter $n$ and size $2^n + \binom{n}{\lfloor n/2 \rfloor} - 2$. In Section 3, we give an upper bound to $\mu_n^{-1}(Q_n)$.

The following lower bound was found independently by Graham and Harary [4] and ourselves [3].

2. A spanning subgraph of $Q_n$ of diameter $n$ and small size

The construction consists in joining two trees whose roots are the two nodes $0^n$ and $1^n$ ($0^n = 00\ldots0$ and $1^n = 11\ldots1$) in the following way:

Consider nodes of weight $i$ (the weight of a node is the number of 1's in its binary label), each of them can be adjacent only to those of weight $i - 1$ and $i + 1$. Therefore, we can say that nodes of weight $i$ constitute the "level" $i$ which will be denoted by $L_i$.

Lemma 2.1. For any $0 < i < \lfloor n/2 \rfloor$, the bipartite graph $G_i$ induced by $L_i \cup L_{i+1}$ contains a matching which saturates all nodes of $L_i$.

Proof. Note first that $|L_i| \leq |L_{i+1}|$ for any $0 < i < \lfloor n/2 \rfloor$. On the other hand, $\min_{x \in L_i} d_{G_i}(x) \geq \max_{x \in L_{i+1}} d_{G_i}(x)$. Indeed, $\forall x \in L_i$, $d_{G_i}(x) = n - i$ and $\forall x \in L_{i+1}$, $d_{G_i}(x) = i + 1$.

Using a corollary of the theorem of König-Hall (see the book of Berge [1, pp.132–133]), the lemma is proved. □

The same property holds if we replace $L_i$ by $L_{n-i}$ and $L_{i+1}$ by $L_{n-i-1}$.

Now, consider two trees of depth $\lfloor n/2 \rfloor$ rooted at $0^n$ and $1^n$ constructed as follows:

By Lemma 2.1, for any $0 \leq i < \lfloor n/2 \rfloor$ there exists a matching in the bipartite graph induced by $L_i \cup L_{i+1}$ which saturates all nodes of $L_i$. So for any $0 \leq i < \lfloor n/2 \rfloor$ consider such a matching. Connect then each node $x$ of $L_{i+1}$ not already attained by that matching to a single node of $L_i$ adjacent to $x$ in $G_i$. This leads to the construction of the tree rooted at $0^n$. To construct the one rooted at $1^n$, the same idea is used where $L_i$ and $L_{i+1}$ are respectively replaced by $L_{n-i}$ and $L_{n-i-1}$.

Finally, if $n$ is even we obtain two trees both of depth $n/2$ and meeting at level $n/2$, otherwise the two trees are of depth $\lceil n/2 \rceil$ and joined by a perfect matching of the bipartite graph induced by $L_{\lfloor n/2 \rfloor} \cup L_{\lceil n/2 \rceil}$. To construct such a perfect matching we can use the algorithm presented in [5, pp. 99–102].

Let $G$ be the resulting spanning subgraph of $Q_n$.

Lemma 2.2. $G$ is of diameter $n$ and of size $2^n + \binom{n}{\lfloor n/2 \rfloor} - 2$.

Proof. Let $u, v$ be any pair of nodes of $G$, and let $w(u)$ and $w(v)$ be the weights of $u$ and $v$ respectively. It suffices to observe that there exist paths in $G$ of length $w(u)$, $n - w(u)$, $w(v)$ and $n - w(v)$ joining respectively $u$ to $0^n$, $u$ to $1^n$, $v$ to $0^n$ and $v$ to $1^n$. 

Therefore, \( u \) and \( v \) are on a cycle of \( G \) (perhaps not elementary) of length \( 2n \) and the distance between \( u \) and \( v \) is at most \( n \).

To compute the size of \( G \) it suffices to note that the total number of edges in \( G \) is the sum of the sizes of two trees if \( n \) is even, namely: \( 2 \sum_{i=1}^{n/2} \binom{n}{i} \); otherwise it is the sum of the sizes of two trees and the size of the perfect matching: \( 2 \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n}{i} + \binom{n}{\lfloor n/2 \rfloor} \) which yields the required result. \( \square \)

**Theorem 2.3.** The maximum number of edges of \( Q_n \) whose removal leaves the diameter unchanged is bounded by:

\[
un^-(Q_n) \geq (n-2)2^{n-1} - \binom{n}{\lfloor n/2 \rfloor} + 2.
\]

**Proof.** We have constructed a spanning subgraph of \( Q_n \) of diameter \( n \) and of size \( 2^n + \binom{n}{\lfloor n/2 \rfloor} - 2 \), and \( Q_n \) has \( n2^{n-1} \) edges. The difference of these sizes gives the wanted bound. \( \square \)

### 3. An upper bound to \( un^-(Q_n) \)

**Lemma 3.1.** Let \( G \) be a spanning subgraph of \( Q_n \) of diameter \( n \). Then every edge \( e \) of \( G \) is in a cycle of length at most \( 2n \).

**Proof.** Let \( l(P) \) be the length of a path \( P \) of \( G \) and let \( e = \{a, b\} \) be an edge of \( G \). Denote by \( a' \) (respectively \( b' \)) the antipodal node of \( a \) (respectively \( b \)) that is the node at distance \( n \) of \( a \) (respectively \( b \)) in \( Q_n \).

\( G \) is bipartite since it is a subgraph of the bipartite graph \( Q_n \). Therefore, in \( G \), the lengths of a path between \( a \) and \( b' \), and a path between \( b \) and \( b' \) are always of different parities.

Let \( P(a, b') \) be a shortest path between \( a \) and \( b' \) in \( G \) and \( P(b, b') \) a shortest path between \( b \) and \( b' \) in \( G \). We have: \( l(P(b, b')) = n \) since \( G \) is of diameter \( n \) and the distance in \( Q_n \) between \( b \) and \( b' \) is \( n \).

On the other hand, as \( l(P(a, b')) \) has a parity different from that of \( l(P(b, b')) \) and as \( G \) is of diameter \( n \) we have \( l(P(a, b')) = n - 1 \) (if \( l(P(a, b')) < n - 1 \) there would exist a path between \( b \) and \( b' \) via \( a \) of length less than \( n \)). Moreover, \( P(a, b') \) does not contain \( e \) for the same reason.

In the same way, if \( P(b, a') \) is a shortest path between \( b \) and \( a' \) in \( G \), \( l(P(b, a')) = n - 1 \) and \( P(b, a') \) does not contain \( e \).

Now, let us organize the vertices of \( G \) according to their distances to \( a \) and \( b \).

Let us call \( M_i(a) \), for \( 0 \leq i \leq n \), the set of vertices at distance \( i \) from \( a \) and then let us define similarly \( M_i(b) \), for \( 0 \leq i \leq n \). We notice that \( M_i(b) \subset M_{i+1}(a) \cup M_{i-1}(a) \), \( 0 < i < n \), by the usual parity argument and triangular inequality. Then let \( L_i(a) \) be \( M_i(a) \cap M_{i+1}(b) \) and \( L_i(b) = M_i(b) \cap M_{i+1}(a) \), for \( 0 \leq i < n \).
We have \( V(G) = \bigcup_{i=0}^{n-1} L_i(a) \cup \bigcup_{i=0}^{n-1} L_i(b) \). In particular, we see \( L_0(a) = \{a\}, L_{n-1}(b) = \{b'\} \) and \( b' \in L_{n-1}(a) \).

The shortest path in \( G \) from \( a' \) to \( b' \) contains an edge between \( \alpha \in L_d(a) \) and \( \beta \in L_d(b) \) for some \( d \leq n - 1 \); the edge \( \{\alpha, \beta\} \), the shortest paths between \( \alpha \) and \( a \) and between \( \beta \) and \( b \), and the edge \( \{a, b\} \) constitute a cycle of length \( 2d + 2 \leq 2n \).

\[ \text{Lemma 3.2.} \quad \text{Let } G \text{ be a connected multigraph. If every edge of } G \text{ is in a cycle of length at most } l \text{ then } G \text{ has at least } |V(G)| - 1 + \left\lfloor \frac{|V(G)| - 1}{l - 1} \right\rfloor \text{ edges.} \]

\[ \text{Proof.} \quad \text{We prove by induction on the order of } G \text{ that if } G \text{ verifies the hypothesis of the lemma then } (l - 1)|E(G)| \geq l(|V(G)| - 1). \]

If \(|V(G)| = 1\) the property is obvious.

Now, assume that the property is satisfied for all the multigraphs of order less than \( n \) and which verify the hypothesis of the lemma, and let \( G \) be a multigraph of order \( n \) which verifies the hypothesis of the lemma. Consider the multigraph \( G_1 \) obtained from \( G \) by contracting a shortest cycle of \( G \). \( G_1 \) is connected and also verifies that every edge is in a cycle of length at most \( l \).

Note that if \(|V(G)| > 1\) we can always obtain such a multigraph \( G_1 \). If we denote by \( l_1 \) the length of a shortest cycle of \( G \), we have \(|V(G)| = |V(G_1)| + l_1 - 1\) and \(|E(G)| = |E(G_1)| + l_1\) which implies that:

\[ (l - 1)|E(G)| - l(|V(G)| - 1) = (l - 1)|E(G_1)| - l(|V(G_1)| - 1) + l - l_1. \]

Now, since \(|V(G_1)| < |V(G)|\), \( (l - 1)|E(G_1)| - l(|V(G_1)| - 1) \geq 0 \) and by noting that \( l_1 \leq l \) it is easy to see that the property holds for \( G \).

Finally, \( (l - 1)|E(G)| \geq l(|V(G)| - 1) \) implies that:

\[ (l - 1)|E(G)| \geq (l - 1)(|V(G)| - 1) + (|V(G)| - 1) \]

and thus

\[ |E(G)| \geq |V(G)| - 1 + \left\lfloor \frac{|V(G)| - 1}{l - 1} \right\rfloor. \quad \square \]

\[ \text{Theorem 3.3.} \quad \text{The maximum number of edges of } Q_n \text{ whose removal does not alter the diameter is bounded by:} \]

\[ u_n^-(Q_n) \leq (n - 2)2^{n-1} + 1 - \left\lceil \frac{2^n - 1}{2n - 1} \right\rceil. \]

\[ \text{Proof.} \quad \text{Any spanning subgraph } G \text{ of } Q_n \text{ of diameter } n \text{ verifies the hypothesis of Lemma 3.2 with } l = 2n \text{ (see Lemma 3.1), therefore } |E(G)| \geq 2^n - 1 + \left\lceil \frac{2^n - 1}{2n - 1} \right\rceil \text{ and then it is easy to see that the theorem holds.} \quad \square \]

We conclude by giving the interval containing \( u_n^-(Q_n) \) for some values of \( n \):
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
n & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
un^-(Q_n) \leq & 0 & 3 & 14 & 45 & 123 & 311 & 752 \\
un^-(Q_n) \geq & 0 & 3 & 12 & 40 & 110 & 287 & 700 \\
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\end{tabular}

References