Weak compactness in Köthe–Bochner spaces and Orlicz–Bochner spaces*

by Marian Nowak

Institute of Mathematics, T. Kotarbiński Pedagogical University, Pl. Słowiański 9, 65-069 Zielona Góra, Poland,
e-mail: nowakmar@omega.im.wsp.zgora.pl

Communicated by Prof. R. Tijdeman at the meeting of December 24, 1997

ABSTRACT
Let $E$ be a Banach function space over a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$, $E'$-the Köthe dual of $E$
and let $X$ be a reflexive Banach space, $X^*$-the topological dual of $X$. We characterize relatively
$\sigma(E(X), E'(X^*))$-compact subsets of a Köthe–Bochner space $E(X)$ in terms of absolute continuity
of certain seminorm defined on $E'(X^*)$. As an application, we obtain that a solid subset of an
Orlicz–Bochner space $L^\psi(X)$ is relatively $\sigma(L^\psi(X), L^{\psi^*}(X^*))$-compact iff it is norm bounded in
some Orlicz–Bochner space $L^\varphi(X)$, where $\psi$ increases more rapidly than $\varphi$.

1. INTRODUCTION AND PRELIMINARIES
The problem of characterizing of relatively weakly compact subsets of the space
$L^1(X)$ was considered by many authors (see [5], [8], [10], [11], [14], [24], [26]).

J. Batt and W. Hiermeyer [3] characterized relatively $\sigma(L^p(X), L^q(X^*))$-compact subsets of a Lebesgue–Bochner space $L^p(X)$ for $1 \leq p < \infty$ and $q$
conjugate to $p$ over a positive finite measure space.

F. Bombal [4] showed that if $(\Omega, \Sigma, \mu)$ is a finite measure space and $X$ has the
RNP then a subset $H$ of the Orlicz–Bochner space $L^\varphi(X)$ is relatively
$\sigma(L^\varphi(X), L^{\psi^*}(X^*))$-compact ($\varphi^* =$ the complementary Young function) iff the
following conditions are satisfied:

*1991 Mathematics Subject Classification: 46E30, 46E40, 46A50.

Key words and phrases: Köthe–Bochner spaces, Orlicz–Bochner spaces, weak compactness, absolutely continuous seminorms.

Supported by KBN grant: 2P03A 031 10.
(i) $H$ is norm-bounded,
(ii) the set $H(A) = \{\int_A f(\omega) d\mu : f \in H\}$ is relatively weakly compact in $X$ for every $A \in \Sigma$, and
(iii) $\lim_{\mu(A) \to 0} \sup \{\int_A (f(\omega), g(\omega)) d\mu : f \in H\} = 0$ for every $g \in L^{\varphi^*}(X^*)$.

J. Diestel, W.M. Ruess and W. Schachermayer [11] found a characterization of weak compactness in a Köthe–Bochner space $E(X)$ whenever $E$ is an order continuous Banach function space with $L^\infty \subset E \subset L^1$ for some probability space. It is shown (see [11, Theorem 3.2, Corollary 3.3]) that for a subset $H$ of $E(X)$ the following statements are equivalent to relative weak compactness:

(1) The set $\{\|f(.)\|_X : f \in H\}$ of $E$ is relatively weakly compact in $E$, and, given any sequence $(f_n)$ in $H$, there exists a sequence $(g_n)$ with $g_n \in \text{conv} \{f_k : k \geq n\}$, and such that $(g_n(\omega))$ is norm convergent for a.e. $\omega \in \Omega$.

(2) The set $\{\|f(.)\|_X : f \in A\}$ of $E$ is relatively weakly compact in $E$, and $H$ is relatively weakly compact in $L^1(X)$.

In this paper we extend to the vector valued setting the well known criterion for relative $\sigma(E, E')$-compactness in a Banach function space $E$ given in terms of absolute continuity of certain seminorm defined on the Köthe dual $E'$ of $E$ (see [18, Theorem 1.3.5], [19, Theorem 5.1]). As an application, in case of a reflexive Banach space $X$, we characterize solid relatively $\sigma(L^\psi(X), L^{\psi'}(X^*))$-compact subsets of an Orlicz-Bochner space $L^\psi(X)$ as norm bounded sets in some Orlicz–Bochner space $L^{\psi}(X)$, where the Young function $\psi$ increases more rapidly than the Young function $\varphi$. This result is the Bochner version of the Ando’s criterion for relatively $\sigma(L^\psi, L^{\psi'})$-compact sets in an Orlicz space $L^\psi$ (see [2, Theorem 2], [21, Theorem 7.5]).

Let $(\Omega, \Sigma, \mu)$ be a complete $\sigma$-finite measure space, and let $L^0$ denote the corresponding space of equivalence classes of all $\Sigma$-measurable real valued functions. Then $L^0$ is a super Dedekind complete Riesz space under the ordering $u_1 \leq u_2$ whenever $u_1(\omega) \leq u_2(\omega)$ $\mu$-a.e. on $\Omega$. Let $\chi_A$ stand for the characteristic function of a set $A$. Let $E$ be an ideal of $L^0$ with $\text{supp} \ E = \Omega$, and let $\|\cdot\|_E$ be a Riesz norm on $E$. The complete space $(E, \|\cdot\|_E)$ is called a Banach function space or a Köthe function space. The Köthe dual $E'$ of $E$ is defined by

$$E' = \left\{ v \in L^0 : \int_\Omega |u(\omega)v(\omega)| d\mu < \infty \text{ for all } u \in E \right\}.$$  

The associated norm $\|\cdot\|_{E'}$ on $E'$ is defined by

$$\|v\|_{E'} = \sup \left\{ \left| \int_\Omega u(\omega)v(\omega)d\mu \right| : u \in E, \ |u|_E \leq 1 \right\}.$$  

It is well known that $\text{supp} \ E' = \Omega$ and the inclusion $E \subset E''$ holds and $\|u\|_{E''} \leq \|u\|_E$ for $u \in E$ (see [16, Chapter VI, §1]). A Banach function space $(E, \|\cdot\|_E)$ is said to be perfect if $E = E''$ and $\|u\|_E = \|u\|_{E''}$ for $u \in E$. It is well
known that $E$ is perfect if and only if the norm $\| \cdot \|_E$ satisfies both the $\sigma$-Fatou property and the $\sigma$-Levy property (see [16, Theorem 6.1.7]).

We will write $A_n \searrow \emptyset$ if $(A_n)$ is a decreasing sequence in $\Sigma$ such that $\mu(A_n \cap A) \to 0$ for every set $A \in \Sigma$ with $\mu(A) < \infty$. We denote by $E_d$ the ideal of elements of absolutely continuous norm in $E$, i.e.,

$$E_d = \{ u \in E : \| \chi_{A_n} u \|_E \to 0 \text{ as } A_n \searrow \emptyset \}.$$ 

Let $(X, \| \cdot \|_X)$ be a real Banach space, and let $S_X$ and $B_X$ denote the unit sphere and the closed unit ball in $X$ resp. Let $X^*$ stand for the topological dual of $X$. By $L^0(X)$ we will denote the linear space of equivalence classes of all strongly $\Sigma$-measurable functions $f : \Omega \to X$. For $f \in L^0(X)$ let us put

$$\hat{f}(\omega) = \| f(\omega) \|_X \quad \text{for } \omega \in \Omega.$$ 

The linear space $E(X) = \{ f \in L^0(X) : \hat{f} \in E \}$ equipped with the norm $\| f \|_{E(X)} = \| \hat{f} \|_E$ is called a K"{o}the–Bochner space (see [6], [15]).

Now we recall some notions concerning the solid structure of $E(X)$ (see [13]).

A subset $H$ of $E(X)$ is said to be solid whenever $\| f_1(\omega) \|_X \leq \| f_2(\omega) \|_X \mu$-a.e. and $f_1 \in E(X), f_2 \in H$ imply $f_1 \in H$.

A seminorm $p$ on $E(X)$ is said to be solid whenever for every $f \in E(X), \| f_1(\omega) \|_X \leq \| f_2(\omega) \|_X \mu$-a.e. implies $p(f_1) \leq p(f_2)$.

A solid seminorm $p$ on $E(X)$ is said to be absolutely continuous whenever for each $f \in E(X), p(\chi_{A_n} f) \to 0$ as $A_n \searrow \emptyset$.

The following description of absolutely continuous seminorms on $E(X)$ will be needed (see [13, Theorems 5.1, 5.3]).

**Theorem 1.1.** For a solid seminorm $p$ on $E(X)$ the following statements are equivalent:

(i) $p$ is absolutely continuous.

(ii) For every $f \in E(X)$ and $\varepsilon > 0$ there exist $\delta > 0$ and $A_0 \in \Sigma$ with $\mu(A_0) < \infty$ such that $p(\chi_A f) \leq \varepsilon$ for $\mu(A) \leq \delta$ and $p(\chi_{X \setminus A_0} f) \leq \varepsilon$.

(iii) For a sequence $(f_n)$ in $E(X), f_n \overset{\| \cdot \|_X}{\to} 0$ in $E$ implies $p(f_n) \to 0$.

For a linear functional $F$ on $E(X)$ let us put for each $f \in E(X)$

$$|F|(f) = \sup \{ |F(h)| : h \in E(X), \| h(\omega) \|_X \leq \| f(\omega) \|_X \mu$-a.e. \}.$$

The set

$$E(X)^\sim = \{ F \in E(X)^\# : |F|(f) < \infty \text{ for all } f \in E(X) \}$$

will be called the order dual of $E(X)$ (see [22]). (Here $E(X)^\#$ denotes the algebraic dual of $E(X)$.) One can show that $E(X)^\sim$ coincides with the topological dual $(E(X), \| \cdot \|_{E(X)})^*$ (see [22, Theorem 3.5]).

A linear functional $F$ on $E(X)$ is said to be order continuous whenever for a net $(f_n)$ in $E(X), f_n \overset{\| \cdot \|_X}{\rightrightarrows} 0$ in $E$ implies $F(f_n) \to 0$. The set $E(X)_o^\sim$ consisting of all order continuous linear functionals on $E(X)$ will be called the order continuous dual of $E(X)$. 75
In view of the super Dedekind completeness of $L^0$ we can restrict ourselves to the usual sequences $(f_n)$ in $E(X)$. Moreover, we obtain that $E(X)_\infty \subset E(X)$ (see [22, Theorem 2.3]).

From now on we will assume that the Banach space $X^*$ has the Radon–Nikodym property (RNP) (see [10, Chapter IV]). It is well known that $X^*$ has the RNP whenever $X$ is reflexive (see [10, Corollary 3.13]).

The following description of order continuous functionals on $E(X)$ will be of importance (see [6, Theorem 4.1, Theorem 1.1 and (3) p. 24], [7, Theorem 3.5]).

Theorem 1.2 Assume that $X^*$ has the RNP. For a linear functional $F$ on $E(X)$ the following statements are equivalent:

(i) $F$ is order continuous.

(ii) There exists a unique $g \in E'(X^*)$ such that

$$F(f) = F_M(f) = \int_{\Omega} (f(\omega), g(\omega)) d\mu \quad \text{for all } f \in E(X).$$

Moreover, for each $g \in E'(X^*)$

\begin{equation}
|F_M(f)| = \int_{\Omega} \|f(\omega)\| \|g(\omega)\|_{X'} d\mu \quad \text{for all } f \in E(X)
\end{equation}

and

\begin{equation}
\|F_M\|_g(f) = \sup \left\{ \left| \int_{\Omega} (f(\omega), g(\omega)) d\mu \right| : f \in E(X), \|f\|_{E(X)} \leq 1 \right\}
= \|g\|_{E'(X^*)} = \|g\|_{E'}.
\end{equation}

Let $M$ be a $\| \cdot \|_{E'}$-closed ideal of $E'$ with supp $M = \Omega$. Then $M$ can be equipped with the associated norm $\|v\|_{E'} = \sup \{|\int_{\Omega} u(\omega)v(\omega) d\mu| : u \in E, \|u\|_{E} \leq 1\}$. Thus $M(X^*)$ is a Köthe–Bochner space with the norm $\|g\|_{M(X^*)} = \|g\|_{E'}$ for $g \in M(X^*)$.

Assume that $X^*$ has the RNP. In view of Theorem 1.2 we have the dual system $(E(X), M(X^*))$ under its natural duality:

$$(f, g) = F_M(f) = \int_{\Omega} (f(\omega), g(\omega)) d\mu \quad \text{for } f \in E(X), g \in M(X^*).$$

Using the Lebesgue dominant convergence theorem one can define a natural embedding

$$j_M : E(X) \rightarrow M(X^*)_{\infty}$$

by

$$j_M(f)(g) = \int_{\Omega} (f(\omega), g(\omega)) d\mu \quad \text{for } g \in M(X^*).$$

We shall need the following lemma.

Lemma 1.3. Let $(E, \| \cdot \|_E)$ be a perfect Banach function space, and let a Banach
space $X$ be reflexive. Assume that $M$ is a $\| \cdot \|_{E'}$-closed ideal of $E'$ with $\text{supp } M = \Omega$. Then

$$j_{M}(E(X)) = M(X^*)^\sim.$$ 

**Proof.** Let $\kappa : X \to X^{**}$ stand for the canonical isometry. To prove that $M(X^*)^\sim \subset j_{M}(E(X))$, let $G_{0} \in M(X^*)^\sim$. Since $X$ is reflexive, $X^{**}$ has the RNP (see [10, Corollary 3.13]), so by Theorem 1.2 there exists a unique $h_{0} \in M'(X^{**})$ such that

$$G_{0}(g) = \int (g(\omega), h_{0}(\omega)) d\mu \quad \text{for all } g \in M(X^*).$$

Since $\kappa(X) = X^{**}$ we can put $f_{0}(\omega) = \kappa^{-1}(h_{0}(\omega))$ for $\omega \in \Omega$. One can easily show that the function $f_{0}$ is strongly $\Sigma$-measurable, and since $\|f_{0}(\omega)\|_{X} = \|h_{0}(\omega)\|_{X^{*}}$ for all $\omega \in \Omega$, we get $f_{0} \in M'$. But by [20, Theorem 0.1] $M' = (E')' = E$, so $f_{0} \in E(X)$. Thus

$$G_{0}(g) = \int (f_{0}(\omega), g(\omega)) d\mu \quad \text{for all } g \in M(X^*).$$

so $G_{0} = j_{M}(f_{0}) \in J_{M}(E(X))$, as desired. □

The following Eberlein–Šmulian theorem for the locally convex space $(E(X), \sigma(E(X), M(X^*)))$ will be of importance (see [22, Corollary 5.3, Theorem 2.6]).

**Theorem 1.4.** Let $(E, \| \cdot \|_{E})$ be a perfect Banach function space and assume that the Banach space $X^{*}$ has the RNP. Let $M$ be an ideal of $E'$ with $\text{supp } M = \Omega$. Then for a subset $H$ of $E(X)$ the following statements are equivalent:

(i) $H$ is relatively sequentially compact for $\sigma(E(X), M(X^*))$.

(ii) $H$ is relatively countably compact for $\sigma(E(X), M(X^*))$.

(iii) $H$ is relatively compact for $\sigma(E(X), M(X^*))$.

2. WEAKLY COMPACT SETS IN KÖTHE–BOCHNER SPACES

W.A. Luxemburg and A.C. Zaanen [19] obtained some criterion for relative $\sigma(E, M)$-sequential compactness in $E$, where $M$ is a closed ideal of $E'$. In this paper, following the idea of [19] and using the Eberlein–Šmulian theorem for the locally convex space $(E(X), \sigma(E(X), M(X^*)))$ (see [22]) we obtain an equivalent criterion for relative $\sigma(E(X), M(X^*))$-compactness in $E(X)$ when $X$ is a reflexive Banach space, and $M$ is a $\| \cdot \|_{E'}$-closed ideal of $E'$ with $\text{supp } M = \Omega$.

**Theorem 2.1.** Let $(E, \| \cdot \|_{E})$ be a Banach function space and assume that the Banach space $X^{*}$ has the RNP. Let $M$ be a $\| \cdot \|_{E'}$-closed ideal of $E'$ with $\text{supp } M = \Omega$. Then for a solid, relatively $\sigma(E(X), M(X^*))$-compact subset $H$ of $E(X)$ the functional $\rho_{H}$ on $M(X^*)$ defined for each $g \in M(X^*)$ by
\[ \rho_H(g) = \sup_{f \in H} \int_{\Omega} |\langle f(\omega), g(\omega) \rangle| \, d\mu \]

is an absolutely continuous seminorm.

**Proof.** Since \( H \) is solid, for each \( g \in M(X^*) \) we get (see [22, Theorem 1.3])

\[ \rho_H(g) = \sup_{f \in H} \int_{\Omega} \|f(\omega)\|_X \cdot \|g(\omega)\|_X \, d\mu = \sup_{f \in H} \int_{\Omega} \langle f(\omega), g(\omega) \rangle \, d\mu. \]

Thus \( \rho_H \) is a seminorm because \( H \) is \( \sigma(E(X), M(X^*)) \)-bounded. It is also seen that \( \rho_H \) is solid. Since \( \text{supp} \, M = \Omega \), there exists a sequence \( (\Omega_n) \) in \( \Sigma \) such that \( \Omega_n \uparrow \Omega \), \( \mu(\Omega_n) < \infty \) and \( \chi_{\Omega_n} \in M \) for \( n = 1, 2, \ldots \) (see [26, Theorem 8.6.2]). Assume that the seminorm \( \rho_H \) is not absolutely continuous. Then in view of Theorem 1.1 there exist \( g_0 \in M(X^*) \), \( \varepsilon_0 > 0 \) and a sequence \( (A_n) \) in \( \Sigma \) with \( \mu(A_n) < 1/n \) such that either

\[ \rho_H(\chi_{A_n} g_0) > \varepsilon_0 \text{ for } n = 1, 2, \ldots \]

or

\[ \rho_H(\chi_{\Omega_n \setminus A_n} g_0) > \varepsilon_0 \text{ for } n = 1, 2, \ldots. \]

Thus there exist either a sequence \( (f_n) \) in \( H \) or a sequence \( (h_n) \) in \( H \) such that

\[ \int_{A_n} \langle f_n(\omega), g_0(\omega) \rangle \, d\mu > \varepsilon_0 \text{ for } n = 1, 2, \ldots \]

or

\[ \int_{\Omega \setminus A_n} \langle h_n(\omega), g_0(\omega) \rangle \, d\mu > \varepsilon_0 \text{ for } n = 1, 2, \ldots. \]

For each \( A \in \Sigma \) let us put for \( n = 1, 2, \ldots \)

\[ \nu_n^1(A) = \int_{A_n} \langle f_n(\omega), g_0(\omega) \rangle \, d\mu \quad \text{and} \quad \nu_n^2(A) = \int_{A_n} \langle h_n(\omega), g_0(\omega) \rangle \, d\mu. \]

Then \( \nu_n^1 \) and \( \nu_n^2 \) are countably additive set functions on \( \Sigma \), absolutely continuous with respect to the measure \( \mu \).

\(^1\) Assume that (1) holds. By Theorem 1.4 \( H \) is relatively sequentially compact for \( \sigma(E(X), M(X^*)) \), so there exist a subsequence \( (f_{k_n}) \) of \( (f_n) \) and \( f_0 \in E(X) \) such that

\[ \nu_{k_n}^1(A) = \int_{A} \langle f_{k_n}(\omega), g_0(\omega) \rangle \, d\mu \to \int_{A} \langle f_0(\omega), g_0(\omega) \rangle \, d\mu = \nu^1(A) \]

for all \( A \in \Sigma \), because \( \chi_{A} g_0 \in M(X^*) \). Hence by the Vitali–Hahn–Saks theorem (see [9], [18, p. 20]) the family \( \{\nu_{k_n} : n = 1, 2, \ldots\} \) is uniformly absolutely continuous, so there exists \( \delta_0 > 0 \) such that for \( n = 1, 2, \ldots \) and \( A \in \Sigma \) with \( \mu(A) < \delta_0 \)

78
Choose $n_0 \in \mathbb{N}$ such that $1/n_0 \leq \delta_0$. Then for $n \geq n_0$, $\mu(A_{k_n}) < 1/k_n < \delta_0$, so

$$\left| \int_{A_{k_n}} \langle f_{k_n}(\omega), g_0(\omega) \rangle d\mu \right| \leq \frac{\varepsilon_0}{2}.$$ 

But this contradicts (1).

2º. Assume that (2) holds. Thus there exist a subsequence $(h_{i_n})$ of $(h_n)$ and $h_0 \in E(X)$ such that

$$\nu_{i_n}^2(A) = \int_{\Omega \setminus \Omega_{i_n}} (h_{i_n}(\omega), g_0(\omega)) d\mu \rightarrow \int_{\Omega} (h_0(\omega), g_0(\omega)) d\mu$$

for all $A \in \Sigma$, because $\chi_A g_0 \in M(X^*)$. Hence by the Vitali–Hahn–Saks theorem the family $\{\nu_{i_n}^2 : n = 1, 2, \ldots\}$ is uniformly absolutely continuous, so there exists $n_0 \in \mathbb{N}$ such that for $n = 1, 2, \ldots$

$$\left| \nu_{i_n}^2(\Omega \setminus \Omega_{i_n}) \right| = \left| \int_{\Omega \setminus \Omega_{i_n}} (h_{i_n}(\omega), g_0(\omega)) d\mu \right| \leq \frac{\varepsilon_0}{2},$$

so

$$\left| \int_{\Omega \setminus \Omega_{i_n}} (h_{i_n}(\omega), g_0(\omega)) d\mu \right| \leq \frac{\varepsilon_0}{2}.$$ 

But this contradicts (2).

This means that $\rho_H$ is absolutely continuous, as desired. \(\square\)

**Theorem 2.2.** Let $(E, \| \cdot \|_E)$ be a perfect Banach function space and assume that a Banach space $X$ is reflexive. Let $M$ be a $\| \cdot \|_E$-closed ideal of $E'$ with $\text{supp } M = \Omega$ and let $H$ be a solid subset of $E(X)$. Assume that the functional $\rho_H$ on $M(X^*)$ defined for each $g \in M(X^*)$ by

$$\rho_H(g) = \sup_{f \in H} \int_{\Omega} |\langle f(\omega), g(\omega) \rangle| d\mu$$

is an absolutely continuous seminorm. Then the set $H$ is relatively compact for $\sigma(E(X), M(X^*))$.

**Proof.** Since $H$ is solid, for each $g \in M(X^*)$ we have (see [22, Theorem 1.3])

$$\rho_H(g) = \sup_{f \in H} \int_{\Omega} \|f(\omega)\|_X \|g(\omega)\|_X d\mu = \sup_{f \in H} \int_{\Omega} \langle f(\omega), g(\omega) \rangle d\mu.$$ 

Hence the set $\hat{H} = \{ \hat{f} : f \in H \}$ is $\sigma(E, M)$-bounded. Since $M$ is norm fundamental (i.e., $\|u\|_E = \sup \{ \| \int_{\Omega} u(\omega) v(\omega) d\mu \| : v \in M, \|v\|_E \leq 1 \}$) for each $u \in E$ (see [22, Lemma 5.1]), the set $\hat{H}$ is bounded for $\| \cdot \|_E$ (see [18, Lemma 1.3.1]).
Thus $H \subset B_{E(X)}(r) = \{ f \in E(X) : \| f \|_{E(X)} \leq r \}$ for some $r > 0$. Hence $H^0 \supset B_{M(X^*)}(1/r) = \{ g \in M(X^*) : \| g \|_{M(X^*)} \leq 1/r \}$ (see Theorem 1.2). Let $(M(X^*))^* = (M(X^*))^\sigma_{\| \cdot \|_{M(X^*)}}$, and let us consider the dual system $(M(X^*), (M(X^*))^*)$. Then by the Banach–Alaoglu theorem $H^{00}$ is $\sigma((M(X^*))^*, M(X^*))$-compact subset of $(M(X^*))^*$ and $H^{00} = \text{abs conv } j_M(H)^\sigma$, where the closure is taken in $(M(X^*))^*$ for the topology $\sigma((M(X^*))^*, M(X^*))$.

We shall show that

$$j_M(H)^\sigma \subset j_M(E(X)).$$

Indeed, let $G_0 \in j_M(H)^\sigma$. Then for each $g \in M(X^*)$ and $\varepsilon > 0$ there exists $f_0 \in H$ such that $|j_M(f_0)(g) - G_0(g)| \leq \varepsilon$. It follows that

$$|G_0(g)| \leq |j_M(f_0)(g)| + \varepsilon \leq \int |\langle f_0(\omega), g(\omega) \rangle| d\mu + \varepsilon.$$

Hence $|G_0(g)| \leq \rho_H(g)$, and since $\rho_H$ is an absolutely continuous seminorm on $M(X^*)$, by Theorem 1.1 $G_0 \in M(X^*) = j_M(E(X))$ (see Lemma 1.3). Since $j_M(H)^\sigma \subset j_M(E(X)) = M(X^*)^\sigma_n$ and $\sigma((M(X^*))^*, M(X^*))|_{M(X^*)} = \sigma(M(X^*)^*, M(X^*))$ we get

$$j_M(H)^\sigma = j_M(H)^\sigma_n,$$

where $j_M(H)^\sigma_n$ denotes the closure of $j_M(H)$ in $M(X^*)^\sigma_n$ for the topology $\sigma(M(X^*)^*, M(X^*))$. Thus $j_M(H)^\sigma_n$ is a $\sigma(M(X^*)^*, M(X^*))$-compact subset of $M(X^*)^\sigma_n$, because $j_M(H)^\sigma$ is a $\sigma((M(X^*))^*, M(X^*))$-compact subset of $(M(X^*))^*$. It is easy to verify that the mapping

$$j_M : (E(X), \sigma(E(X), M(X^*))) \to (M(X^*)^\sigma_n, \sigma(M(X^*)^\sigma_n, M(X^*)))$$

is a homeomorphism. Thus

$$H^\sigma(E(X), M(X^*)) = j^{-1}_M(j_M(H)^\sigma_n)$$

and $H$ is relatively $\sigma(E(X), M(X^*))$-compact. Thus the proof is complete. $\square$

Now we are in position to present our desired result.

**Theorem 2.3.** Let $(E, \| \cdot \|_E)$ be a perfect Banach function space, and assume that a Banach space $X$ is reflexive. Let $M$ be a $\| \cdot \|_E$-closed ideal of $E'$ with $\text{supp } M = \Omega$. For a solid subset $H$ of $E(X)$ the following statements are equivalent:

(i) $H$ is relatively $\sigma(E(X), M(X^*))$-compact.

(ii) The functional $\rho_H$ on $M(X^*)$ defined for each $g \in M(X^*)$ by

$$\rho_H(g) = \sup_{f \in H} \int_{\Omega} |\langle f(\omega), g(\omega) \rangle| d\mu$$

is an absolutely continuous seminorm.
Now we prove two interesting consequences of Theorem 2.3 (see [19, Theorems 5.2, 5.4]).

**Corollary 2.4.** Let \((E, \| \cdot \|_E)\) be a perfect Banach function space with \(\text{supp} (E')_a = \Omega\) and assume that a Banach space \(X\) is reflexive. Then for a subset \(H\) of \(E(X)\) the following statements are equivalent:

(i) \(\sup_{f \in H} \| f \|_{E(X)} < \infty\).

(ii) \(H\) is relatively \(\sigma(E(X), (E')_a(X^*))\)-compact.

**Proof.** (i) \(\Rightarrow\) (ii). Let \(H \subseteq B_{E(X)}(r)\) for some \(r > 0\). Assume \(A_n \not\subseteq \emptyset\) and \(g \in (E')_a(X^*)\). Then by the Hölder inequality

\[
\rho_{E(X)}(\chi_{A_n}, g) \leq \sup_{f \in B_{E(X)}(r)} \| f(\omega) \|_X \| g(\omega) \|_{X^*} d\mu \leq r \| \chi_{A_n} \|_{E^*}.
\]

Thus \(\rho_{E(X)}(\chi_{A_n}, g) \to 0\) because \(\tilde{g} \in (E')_a\). By Theorem 2.2 the ball \(B_{E(X)}(r)\) is relatively \(\sigma(E(X), (E')_a(X^*))\)-compact, and so is \(H\).

(ii) \(\Rightarrow\) (i). The set \(\tilde{H}\) is \(\sigma(E(X), (E')_a(X^*))\)-bounded, so by [22, Corollary 4.6] its solid hull \(S(\tilde{H})\) is also \(\sigma(E(X), (E')_a(X^*))\)-bounded. By [22, Theorem 1.3] for each \(g \in (E')_a(X^*)\)

\[
\sup_{f \in S(\tilde{H})} \left\| \int_{\Omega} \langle f(\omega), g(\omega) \rangle d\mu \right\| = \sup_{f \in S(\tilde{H})} \left\| \int_{\Omega} \| f(\omega) \|_X \| g(\omega) \|_{X^*} d\mu \right\| = \sup_{f \in S(\tilde{H})} \int_{\Omega} \| f(\omega) \|_X \| g(\omega) \|_{X^*} d\mu.
\]

Hence \(\sup_{f \in H} \int_{\Omega} \| f(\omega) \|_X \| g(\omega) \|_{X^*} d\mu < \infty\) for each \(g \in (E')_a(X^*)\). This shows that the set \(\tilde{H} = \{ f : f \in H \}\) is \(\sigma(E, (E')_a)\)-bounded. Since the space \((E')_a\) is norm fundamental (see [22, Lemma 5.1]) the set \(\tilde{H}\) is bounded for \(\| \cdot \|_E\) (see [18, Lemma 1.3.1]), i.e., \(\sup_{f \in H} \| f \|_{E(X)} < \infty\), as desired. \(\Box\)

**Corollary 2.5.** Let \((E, \| \cdot \|_E)\) be a perfect Banach function space and assume that a Banach space \(X\) is reflexive. Then for each \(f_0 \in E(X)\) its solid hull \(S(f_0) = \{ f \in E(X) : \| f(\omega) \|_X \leq \| f_0(\omega) \|_X \ \mu\text{-a.e.} \}\) is a relatively \(\sigma(E(X), E'(X^*))\)-compact subset of \(E(X)\).

**Proof.** For each \(g \in E'(X^*)\) we have

\[
\rho_{S(f_0)}(g) = \sup_{f \in S(f_0)} \left\| \int_{\Omega} \langle f(\omega), g(\omega) \rangle d\mu \right\| = \sup_{f \in S(f_0)} \int_{\Omega} \| f(\omega) \|_X \| g(\omega) \|_{X^*} d\mu = \int_{\Omega} \| f_0(\omega) \|_X \| g(\omega) \|_{X^*} d\mu.
\]

To prove that the solid seminorm \(\rho_{S(f_0)}\) on \(E(X)\) is absolutely continuous, let \(A_n \not\subseteq \emptyset\). Then
\[
\rho_{S(f_0)}(\chi_{A_0}g) = \int_\Omega \chi_{A_0}(\omega) \tilde{f}_0(\omega) \tilde{g}(\omega) d\mu = \|\chi_{A_0}(\tilde{f}_0 \tilde{g})\|_{L^1},
\]
so \(\rho_{S(f_0)}(\chi_{A_0}g) \to 0\), because \(\tilde{f}_0 \tilde{g} \in L^1\). By Theorem 2.2 the set \(S(f_0)\) is relatively \(\sigma(E(X), E'(X^*))\)-compact, as desired. \(\square\)

### 3. WEAKLY COMPACT SETS IN ORLICZ–BOCHNER SPACES

In this section, as an application of Theorem 2.3 we characterize solid, relatively \(\sigma(L^\varphi(X), L^{\varphi^*}(X^*))\)-compact subsets of an Orlicz–Bochner space \(L^\varphi(X)\) as norm bounded sets in some Orlicz–Bochner space \(L^{\psi}(X)\).

We first recall some notation and terminology concerning Orlicz spaces (see [17], [18], [23] for more details).

By a Young function we mean here a map \(\varphi: [0, \infty) \to [0, \infty)\) that is convex, vanishing only at 0 and \(\lim_{t \to 0} \varphi(t)/t = 0\), \(\lim_{t \to \infty} \varphi(t)/t = \infty\).

For a Young function \(\varphi\) we denote by \(\varphi^*\) the function complementary to \(\varphi\) in the sense of Young, i.e., \(\varphi^*(s) = \sup\{ts - \varphi(t) : t \geq 0\}\) for \(s \geq 0\). It is known that \(\varphi^*\) is also a Young function and \(\varphi^{**} = \varphi\).

The Orlicz space generated by \(\varphi\) is the ideal of \(L^0\) defined by

\[
L^\varphi = \left\{ u \in L^0 : \int_\Omega \varphi(\lambda|u(\omega)|)d\mu < \infty \quad \text{for some } \lambda > 0 \right\}
\]

and equipped with two equivalent norms:

\[
\|u\|_\varphi = \sup\left\{ \int_\Omega u(\omega)v(\omega)d\mu : v \in L^{\varphi^*}, \int_\Omega \varphi^*(|v(\omega)|)d\mu \leq 1 \right\},
\]

\[
\|u\|_{\varphi^*} = \inf\left\{ \lambda > 0 : \int_\Omega \varphi(|u(\omega)|/\lambda)d\mu \leq 1 \right\},
\]
called the Orlicz norm and the Luxemburg norm resp. It is well known that both the norms \(\|\cdot\|_\varphi\) and \(\|\cdot\|_{\varphi^*}\) on \(L^\varphi\) satisfy the \(\sigma\)-Fatou property and the \(\sigma\)-Levy property (see [16, Theorem 4.3.7]). Moreover, \((L^\varphi)' = L^{\varphi^*}\) and

\[
(L^\varphi)' = E^{\varphi^*} = \left\{ u \in L^\varphi : \int_\Omega \varphi(\lambda|u(\omega)|)d\mu < \infty \quad \text{for all } \lambda > 0 \right\}.
\]

The Köthe–Bochner space \(L^\varphi(X) = \{ f \in L^0(X) : \tilde{f} \in L^\varphi \}\) is usually called an Orlicz–Bochner space and is equipped with the corresponding norms

\[
\|f\|_{L^\varphi(X)} = \|\tilde{f}\|_\varphi \quad \text{and} \quad \|f\|_{L^{\psi}(X)} = \|\tilde{f}\|_{\psi^*}.
\]

We shall say that a Young function \(\psi\) is completely weaker than another \(\varphi\), in symbols \(\psi \lesssim \varphi\), if for an arbitrary \(c > 1\) there exists \(d > 1\) such that \(\psi(ct) \leq d\varphi(t)\) for \(t \geq 0\). It is seen that \(\varphi\) satisfies the so-called \(\Delta_2\)-condition if \(\psi \lesssim \varphi\). It is known that the relation \(\psi \lesssim \varphi\) implies that \(L^\varphi \subset E^\psi\) holds (see [2], [23, Theorem 5.3.1]).

We shall say that a Young function \(\varphi\) increases more rapidly than another \(\psi\).
in symbols \( \psi \prec \varphi \), if for \( c > 0 \) there exists \( d > 0 \) such that \( c \psi(t) \leq (1/d) \varphi(dt) \) for \( t \geq 0 \). Note that \( \varphi \) satisfies the so-called \( \nabla_2 \)-condition iff \( \psi \prec \varphi \) (see [2]). It is well known that \( \varphi \) satisfies the \( \nabla_2^\infty \)-condition iff the Simonenko index 
\[
a_{\varphi} = \liminf_{t \to \infty} \left( t \varphi'(t) / \varphi(t) \right) > 1
\]
(see [23, Corollary 2.3.4], [25]). One can verify that for Young functions \( \psi \) and \( \varphi \) the relation \( \varphi \prec \psi \) holds if \( \psi^* \prec \varphi^* \) holds (see [23, Proposition 2.2.4]).

The following characterization of absolutely continuous seminorms on \( L^\varphi(X) \) will be of importance (see [13, Corollary 6.7]).

**Theorem 3.1.** Let \( \varphi \) be a Young function. Then for a solid seminorm \( \rho \) on \( L^\varphi(X) \) the following statements are equivalent:

(i) \( \rho \) is absolutely continuous on \( L^\varphi(X) \).

(ii) There exists a Young function \( \psi \) such that \( \psi \prec \varphi \) and \( \rho(f) \leq a \| f \|_{L^\varphi(X)} \) for some number \( a > 0 \) and all \( f \in L^\varphi(X) \).

The next theorem presents conditions for \( \sigma(L^\varphi(X), L^{\varphi^*}(X^*)) \)-compact embeddings of Orlicz–Bochner spaces.

**Theorem 3.2.** Let \( X \) be a reflexive Banach space and assume that the measure space \( (\Omega, \Sigma, \mu) \) is infinite and atomless. Let \( \psi \) and \( \varphi \) be Young functions such that \( L^{\psi^*}(X) \subset L^\varphi(X) \). Then the following statements are equivalent:

(i) \( \varphi \prec \psi \).

(ii) The embedding \( j : L^\varphi(X) \hookrightarrow L^\psi(X) \) is \( \sigma(L^\varphi(X), L^{\psi^*}(X^*)) \)-compact (i.e., the unit ball in \( L^\varphi(X) \) is a relatively \( \sigma(L^\varphi(X), L^{\psi^*}(X^*)) \)-compact subset of \( L^\psi(X) \)).

**Proof.** (i) \( \Rightarrow \) (ii). Since \( \varphi \prec \psi \), we have \( \psi \prec \psi^* \), so \( L^{\psi^*} \subset E^{\psi^*} = (L^{\psi^*})^* \) (see [23, Theorem 5.3.1]). Let \( B_{L^{\psi^*}(1)}(1) = \{ f \in L^{\psi^*}(X) : \| f \|_{L^{\psi^*}(X)} \leq 1 \} \). Assume that \( g \in L^{\psi^*}(X^*) \) and \( A_n \setminus \emptyset \). Then by the Hölder inequality

\[
\rho_{B_{L^{\psi^*}(1)}(1)}(\chi_{A_n} g) = \sup \left\{ \int_{\Omega} |(f(\omega), \chi_{A_n}(\omega) g(\omega))| d\mu : f \in B_{L^{\psi^*}(1)}(1) \right\}
\]

\[
\leq \sup \left\{ \int_{\Omega} \| f(\omega) \|_X \| \chi_{A_n}(\omega) g(\omega) \|_X \cdot d\mu : f \in B_{L^{\psi^*}(1)}(1) \right\}
\]

\[
\leq \| \chi_{A_n} \tilde{g} \|_{\psi^*}.
\]

Thus \( \rho_{B_{L^{\psi^*}(1)}(1)}(\chi_{A_n} g) \to 0 \), because \( \tilde{g} \in (L^{\psi^*})^* \). By Theorem 2.2 the ball \( B_{L^{\psi^*}(1)}(1) \) is relatively \( \sigma(L^{\varphi}(X), L^{\psi^*}(X^*)) \)-compact.

(ii) \( \Rightarrow \) (i). Since \( L^\varphi(X) \subset L^\psi(X) \) we have \( L^{\psi^*} \subset L^{\varphi^*} \), so \( L^{\psi^*} \subset L^{\varphi^*} \). It is enough to show that \( L^{\varphi^*} \subset E^{\psi^*} \) holds, because this inclusion implies \( \psi^* \prec \varphi^* \) (see [23, Theorem 5.3.1]) and hence \( \varphi \prec \psi \). Indeed, let \( \nu \in L^{\psi^*} \) and \( A_n \setminus \emptyset \). Let \( g = \nu x^* \) for some \( x^* \in S_Y \). Since the Orlicz norm \( \| \cdot \|_{\psi^*} \) on \( L^{\psi^*} \) is the associated norm of the Luxemburg norm \( \| \cdot \|_{\psi^*} \), by Theorem 1.2 we get...
\[
\|\chi_{A_n}g\|_{L^{\varphi^*}(X^*)} = \|\chi_{A_n}v\|_{\varphi^*},
\]
\[
= \sup \left\{ \int_{\Omega} |(f(\omega), \chi_{A_n}(\omega)g(\omega))|d\mu : f \in B_{L^\varphi(X)}(1) \right\}.
\]

Since the unit ball \(B_{L^\varphi(X)}(1)\) is relatively \(\sigma(L^\varphi(X), L^\varphi^*(X^*))\)-compact, by Theorem 2.3 \(\|\chi_{A_n}v\|_{\varphi^*} \to 0\), so \(v \in (L^{\varphi^*})_0 = E^\varphi^*\), as desired. \(\square\)

**Corollary 3.3.** Let \(\varphi\) be a Young function, and let \(X\) be a reflexive Banach space. Assume that the measure space \((\Omega, \Sigma, \mu)\) is infinite (resp. finite) and atomless. Then the following statements are equivalent:

(i) \(\varphi\) satisfies the \(\nabla_2\)-condition (resp. \(\nabla_2^\infty\)-condition).

(ii) The unit ball in \(L^\varphi(X)\) is relatively \(\sigma(L^\varphi(X), L^\varphi^*(X^*))\)-compact.

Example. Let \(X\) be a reflexive Banach space and let the measure space \((\Omega, \Sigma, \mu)\) be finite and atomless. Let \(\varphi(t) = e^{-t} - t - 1\) for \(t \geq 0\). Then \(a_\varphi = \infty\), so \(\varphi\) satisfies the \(\nabla_2^\infty\)-condition. Thus every norm bounded subset of \(L^\varphi(X)\) is relatively \(\sigma(L^\varphi(X), L^\varphi^*(X^*))\)-compact.

Now we are in position to prove our desired result.

**Theorem 3.4.** Let \(\varphi\) be a Young function, and assume that \(X\) is a reflexive Banach space. Then for a solid subset \(H\) of \(L^\varphi(X)\) the following statements are equivalent:

(i) \(H\) is relatively \(\sigma(L^\varphi(X), L^\varphi^*(X^*))\)-compact.

(ii) There exists a Young function \(\psi\) with \(\varphi \preceq \psi\) such that \(H \subset L^\psi(X)\) and \(\sup \{ \|f\|_{L^\varphi(X)} : f \in H \} < \infty\).

**Proof.** (i) \(\Rightarrow\) (ii). By Theorem 2.1 the functional \(\rho_H\) defined on \(L^\varphi^*(X^*)\) by
\[
\rho_H(g) = \sup_{f \in H} \int_{\Omega} |(f(\omega), g(\omega))|d\mu = \sup_{f \in H} \int_{\Omega} \|f(\omega)\|_X \|g(\omega)\|_Y d\mu
\]
is an absolutely continuous seminorm. In view of Theorem 3.1 there exist a number \(a > 0\) and a Young function \(\psi_0\) with \(\psi_0 \preceq \varphi^*\) such that
\[
(1) \quad \rho_H(g) \leq a\|g\|_{L^{\varphi^*}(X^*)} \quad \text{for all} \ g \in L^{\varphi^*}(X^*).
\]
Putting \(\psi = \psi_0^*\) we have \(\varphi = \varphi^{**} \preceq \psi^*_0 = \psi\) and it is enough to show that \(H \subset L^\psi(X)\) and \(\sup \{ \|f\|_{L^\psi(X)} : f \in H \} < \infty\). Indeed, let \(f_0 \in H\). Then by (1) for each \(g \in L^\varphi^*(X^*) \subset E^\psi_0(X^*)\)
\[
(2) \quad \int_{\Omega} \|f_0(\omega)\|_X \|g(\omega)\|_Y d\mu \leq a\|g\|_{L^{\varphi^*}(X^*)} = a\|g\|_{E^\psi_0(X^*)}.
\]

Let \((\Omega_n)\) be a sequence in \(\Sigma\) with \(\Omega_n \uparrow \Omega\) and \(\mu(\Omega_n) < \infty\) for \(n = 1, 2, \ldots\). Let \(v \in L^{\psi_0}\) and let us put for \(n = 1, 2, \ldots\)
\[
v^{(n)}(\omega) = \begin{cases} v(\omega) & \text{if } |v(\omega)| \leq n \quad \text{and} \quad \omega \in \Omega_n, \\ 0 & \text{elsewhere.} \end{cases}
\]
Then $v^{(n)} \in L^{\psi^n}$ for $n = 1, 2, \ldots$ and $|v^{(n)}(\omega)| < \infty$ for $\omega \in \Omega$. By applying the Fatou lemma and (2) we get

$$\int_{\Omega} \tilde{f}_0(\omega)|v(\omega)|d\mu \leq \sup_n \int_{\Omega} \tilde{f}_0(\omega)|v^{(n)}(\omega)|d\mu$$

$$\leq a \sup_n \|v^{(n)}\|_{c_0} \leq a \|v\|_{c_0}.$$  

Hence $\tilde{f}_0 \in (L^{\psi^n})' = L^{\psi_0} = L^\varphi$ and since $\psi - \psi_0$ by (3) we get

$$\|f_0\|_{L^\varphi(X)} = \|\tilde{f}_0\|_\varphi = \left\{ \int_{\Omega} \tilde{f}_0(\omega)v(\omega)d\mu : v \in L^{\psi_0}, \|v\|_{c_0} \leq 1 \right\} \leq a.$$  

Thus $\sup \{\|f\|_{L^\varphi(X)} : f \in Z\} < \infty$, as desired.

(ii) $\Rightarrow$ (i). Assume that $H \subset L^\varphi(X)$ and $\sup \{\|f\|_{L^\varphi(X)} : f \in H\} = a < \infty$ for some Young function $\psi$ with $\varphi \prec \psi$. Thus $\psi^* \ll \varphi^*$, so $L^{\psi^*} \subset E^{\varphi^*} = (L^\varphi)^*$. Let $g \in L^{\psi^*}(X^*)$ and $A_\alpha \setminus \emptyset$. Then by the Hölder inequality

$$\rho_H(\chi_{A_\alpha}g) = \sup \left\{ \int_{\Omega} \langle f(\omega), \chi_{A_\alpha}(\omega)g(\omega) \rangle d\mu : f \in H \right\}$$

$$\leq \sup \left\{ \int_{\Omega} \|f(\omega)\|_X \|\chi_{A_\alpha}(\omega)g(\omega)\|_{X^*} d\mu : f \in H \right\}$$

$$\leq \sup \{\|f\|_{L^\varphi(X)} : f \in H\} \cdot \|\chi_{A_\alpha}g\|_{L^{\psi^*}(X^*)} \leq a \|\chi_{A_\alpha}g\|_{c_\alpha}.$$  

Thus $\rho_H(\chi_{A_\alpha}g) \to 0$, because $\tilde{g} \in (L^\varphi)^*$. By Theorem 2.2 the set $H$ is relatively $\sigma(L^{\varphi^*}(X), L^{\psi^*}(X^*))$-compact, as desired.

ACKNOWLEDGEMENT

The author wishes to thank the referee for the valuable remarks.

REFERENCES


Received November 25, 1996, revised November 28, 1997