Optimality Conditions for Lower Semi-continuous Functions*

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Let \( f \) be a lower semi-continuous and bounded below function from a Banach space \( X \) into \((-\infty, +\infty)\) where \( X \) is assumed to admit a Lipschitz smooth “bump-function.” Generalizing results of Chaney, we study optimality conditions for \( x \in X \) to be a local minimum point of \( f \). These conditions are described in terms of generalized Chaney’s subdifferentials and second-order derivatives.

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1. INTRODUCTION

Let $X$ be a Banach space with dual denoted by $X^*$. Given a locally Lipschitz real-valued function $f$, Clarke's directional derivative and subdifferential are defined [5] by

$$f^0(x; u) = \limsup_{t \downarrow 0, y \to x} \frac{f(y + tu) - f(y)}{t}, \quad x, u \in X,$$

$$\partial f(x) = \{ x^* \in X^* : x^*(\cdot) \leq f^0(x; \cdot) \text{ on } X \}.$$

For any unit vector $u$ in $X$, we write $x_k \to u x$, if $\{x_k\}$ is a sequence convergent to $x \in X$ in the direction $u$ in the sense that $(x_k - x)/\|x_k - x\| \to u$. It is equivalent to say that there exist a positive number sequence $t_k \to 0$ and a vector sequence $u_k \to u$ such that $x_k = x + t_k u_k$ for all $k$.

Chaney's subdifferential $\partial_u f(x)$ is defined to consist of all $x^*$ for each of which there exist $\{x_k\}$ in $X$ and $\{x_k^*\}$ in $X^*$ such that $x_k \to u x$ and $\|x_k^* - x^*\| \to 0$ with $x_k^* \in \partial f(x_k)$ for all $k$. In this case, Chaney [2-4] further introduced the second-order directional derivatives $f''(x; x^*, u)$, and used them to provide optimality conditions for minimization problems mainly for $X = \mathbb{R}^n$. His results were recently improved considerably in [9, 10].

In this paper, we shall look at a more general class of functions: for the rest of the paper we consider that $f : X \to (-\infty, +\infty]$ is a bounded below l.s.c. (lower semi-continuous) function on $X$ with dom$(f) \neq \emptyset$, i.e., there exists $\bar{x} \in X$ such that $f(\bar{x})$ is finite. Our approach is then to replace Clarke's subdifferential by $\beta$-subdifferentials in the investigation.

2. OPTIMALITY CONDITIONS AND $\beta$-SUBDIFFERENTIALS

Let $f : X \to (-\infty, +\infty]$ be a bounded below l.s.c. function on $X$ with dom$(f) \neq \emptyset$ and $\bar{x} \in X$ be a local minimum point of $f$ on $X$. A trivial, but the most strong necessary condition for $\bar{x}$ is that

$$\liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x})}{\|x - \bar{x}\|} \geq 0. \quad (1)$$
The problem is what will happen when the equality holds in (1). If \( X = \mathbb{R}^n \) and a sequence \((x_k)\) in \( X \) satisfies

\[
\lim_{k \to \infty} \frac{f(x_k) - f(x)}{\|x_k - x\|} = 0,
\]

then there exists a subsequence of \((x_k)\), denoted also by \((x_k)\), such that \((x_k - x)/\|x_k - x\|\) converges to a unit vector \( u \in X \), i.e., \( x_k \to_u x \). Improving a result of Chaney [3], it is known from [9] for any locally Lipschitz function \( f \) and such a unit vector \( u \), 0 must belong to Chaney subdifferential \( \partial_u f(x) \). For a general Banach space, as we cannot deduce from (2) to the existence of \( u \) and a subsequence \((x_{k_j})\) of \((x_k)\) such that \( x_{k_j} \to_u x \), if we want to conclude a necessary condition about a direction \( u \), our starting point would be (2) with \( x_k \to_u x \). That is, we would replace (1) and (2) by

\[
\liminf_{t \downarrow 0, u' \to u} \frac{f(x + tu') - f(x)}{t} \geq 0, \quad \forall u \in X \text{ with } \|u\| = 1
\]

and

\[
\lim_{x_k \to_u x} \frac{f(x_k) - f(x)}{\|x_k - x\|} = 0.
\]

Recall that Dini’s directional derivative and subdifferential are defined (see, e.g., [1, 11]) by

\[
D_- f(x; u) := \liminf_{t \downarrow 0, u' \to u} \frac{f(x + tu') - f(x)}{t},
\]

\[
\partial^- f(x) = \{x^* \in X^* : x^*(\cdot) \leq D_- f(x; \cdot) \text{ on } X\}
\]

and, similar to Chaney’s subdifferential, we define \( \partial^- u f(x) \) to consist of all \( x^* \) for each of which there exist \((x_k)\) in \( X \) and \((x^*)\) in \( X^* \) such that \( x_k \to_u x \) and \( \|x_k^* - x^*\| \to 0 \) with \( x_k^* \in \partial^- f(x_k) \) for all \( k \) (thus \( \partial^- f(x) \) and \( \partial^- u f(x) \) are subsets of \( \partial f(x), \partial u f(x) \), respectively.) Then (3) and (4) would become

\[
D_- f(x; v) \geq 0, \quad \forall v \in X, \|v\| = 1 \text{ or } 0 \in \partial^- f(x)
\]

and

\[
D_- f(x; u) = 0.
\]
A natural generalization of Chaney's result stated above would be

\[ D_- f(\bar{x}; u) = 0 \Rightarrow 0 \in \partial^- u f(\bar{x}). \] (9)

In this paper we shall show this is indeed the case for any l.s.c. bounded below function \( f \) on a Banach space which admits a smooth bump function (for definition, see the next section).

For shrinking \( \partial f(\bar{x}) \) and its various subsets along a given unit direction, we shall make use of the concept of \( \beta \)-differential. Recall that [6, 7, 12–14] a bornology of \( X \), denoted by \( \beta \), is a family of bounded subsets of \( X \) which forms a covering of \( X \), i.e., \( \bigcup_{\mathcal{S} \in \beta} S = X \). A (extended real-valued) function \( f : X \to [-\infty, +\infty] \) is \( \beta \)-subdifferentiable at \( x \in X \) if \( f \) is finite at \( x \) and there is an \( x^* \in X^* \) such that

\[ \liminf_{t \downarrow 0} \inf_{u \in \mathcal{S}} \left( \frac{f(x + tu) - f(x)}{t} - x^*(u) \right) \geq 0, \quad \forall \mathcal{S} \in \beta. \]

Such an \( x^* \) is called a \( \beta \)-subderivative of \( f \) at \( x \) and the set of all \( \beta \)-subderivatives of \( f \) at \( x \) is denoted by \( \partial_\beta f(x) \) and is called the \( \beta \)-subdifferential of \( f \) at \( x \); i.e.,

\[ \partial_\beta f(x) = \left\{ x^* \in X^* : \liminf_{t \downarrow 0} \inf_{u \in \mathcal{S}} \left( \frac{f(x + tu) - f(x)}{t} - x^*(u) \right) \geq 0, \quad \forall \mathcal{S} \in \beta \right\}. \] (10)

If \(-f\) is \( \beta \)-subdifferentiable at \( x \), then \( f \) is said to be \( \beta \)-superdifferentiable at \( x \), and its \( \beta \)-superdifferential at \( x \) is defined by \( \partial^\beta f(x) = -\partial_\beta (-f)(x) \). If \( f \) is both \( \beta \)-subdifferentiable and \( \beta \)-superdifferentiable at \( x \), then \( f \) is said to be \( \beta \)-differentiable at \( x \). This is the case if and only if \( \partial^\beta f(x) = \partial_\beta f(x) \) consists of a single element which will then be denoted by \( f^\beta(x) \). If \( f \) is \( \beta \)-differentiable everywhere on a open subset \( D \) of \( X \), then \( f \) is called \( \beta \)-smooth on \( D \). Notice that if \( \beta = F \) (resp. \( G \)) is the collection of all bounded (resp. finite) subsets of \( X \), then \( \beta \)-differentiability coincides with Fréchet-(resp. Gâteaux-)differentiability. Obviously, if \( f \) is locally Lipschitz near \( x \), then we have always

\[ \partial_\beta f(x) \subseteq \partial f(x). \]

In addition, it is easy to show

\[ 0 \in \partial f(x) \Leftrightarrow (1). \]
On the other hand, Dini's subdifferential \( \partial^- f(x) \) lies between \( \partial f(x) \) and \( \partial_G f(x) \),

\[
\partial f(x) \subseteq \partial^- f(x) \subseteq \partial_G f(x)
\]
and if the dimension of \( X \) is finite, then \( \partial f(x) = \partial^- f(x) \) \[1\].

Now, similar to Chaney's subdifferential, we define

\[
\partial_{\beta} f(x) = \limsup_{x_k \to x} \partial_{\beta} f(x_k),
\]

i.e.,

\[
\partial_{\beta} f(x) = \{ x^* \in X^* : \exists x_k \to u x, \exists x_k^* \in \partial_{\beta} f(x_k), x_k^* \to x^* \text{ in norm} \}.
\]

For \( x^* \in \partial_{\beta} f(x) \) (resp. \( x^* \in \partial_{\beta}^- f(x) \)), we define Chaney's second-order derivative \( f_{\beta}^*(x; x^*, u) \) (resp. \( f_{\beta}^-(x; x^*, u) \)) to be the infimum of all extended real numbers

\[
\liminf_{k \to \infty} \frac{f(x_k) - f(x) - x^*(x_k - x)}{\|x_k - x\|^2}
\]

taken over the set of all sequences \( (x_k) \) satisfying the properties

(a) \( x_k \to x \);

(b) \( \forall k, \exists x_k^* \in \partial_{\beta} f(x_k) \) (resp. \( x_k^* \in \partial_{\beta}^- f(x_k) \)), such that \( \|x_k^* - x^*\| \to 0 \).

If condition (b) is dropped from the above definition, one obtains yet another type of derivative which we will denote by \( f_{\beta}^- (x; x^*, u) \) (resp. \( f_{\beta}^-- (x; x^*, u) \)). Note that \( f_{\beta}^-- (x; x^*, u) \leq f_{\beta}^*(x; x^*, u) \) (resp. \( f_{\beta}^- (x; x^*, u) \leq f_{\beta}^- (x; x^*, u) \)), and that

\[
f_{\beta}^- (x; x^*, u) \quad \text{(resp. } f_{\beta}^- (x; x^*, u) \text{)}
\]

\[
= \inf \liminf_{\{x_k\} \to_{x_k \to u x}} \frac{f(x_k) - f(x) - x^*(x_k - x)}{\|x_k - x\|^2}.
\]

In addition, if \( X = \mathbb{R}^* \), we have \( f_{\beta}^*(x; x^*, u) = f_{\beta}^- (x; x^*, u) \) and \( f_{\beta}^- (x; x^*, u) = f_{\beta}^-- (x; x^*, u) \).

The main results of this paper are as follows.

**Theorem 2.1** (Necessary Condition Theorem). Let \( X \) be a Banach space with a Lipschitz \( \beta \)-smooth bump function and \( f : X \to (-\infty, +\infty] \) be a bounded below l.s.c. function on \( X \). Let \( \bar{x} \) be a local minimum point of \( f \) on \( X \). Then

\[
D_- f(\bar{x}; u) \geq \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x})}{\|x - \bar{x}\|} \geq 0, \quad \forall u \in X, \|u\| = 1.
\]
If $u$ is a unit vector such that $D_-f(\bar{x}; u) = 0$, then $0 \in \partial_{\beta u} f(\bar{x})$ and

$$0 \leq f'_{\beta} (\bar{x}; 0, u) \leq f''_{\beta} (\bar{x}; 0, u).$$

**Corollary 2.1.** Let $X$ be a Banach space with a Lipschitz Fréchet-smooth bump function (in particular, $X = \mathbb{R}^n$) and $f: X \to (-\infty, +\infty]$ be a bounded below l.s.c. function on $X$. Let $\bar{x}$ be a local minimum point $f$ on $X$. Then

$$D_-f(\bar{x}; u) \geq \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x})}{\|x - \bar{x}\|} \geq 0, \quad \forall u \in X, \|u\| = 1. \quad (15)$$

If $u$ is a unit vector such that $D_-f(\bar{x}; u) = 0$, then $0 \in \partial_{\alpha} f(\bar{x})$ and

$$0 \leq f''_{\alpha} (\bar{x}; 0, u) \leq f''_{\alpha} (\bar{x}; 0, u).$$

For stating a sufficient condition theorem, we propose a new concept about a minimum point. $\bar{x} \in X$ is called a **local strict minimum point in the direction** $v$ with $\|v\| = 1$ if

$$\exists \delta_v > 0, \forall t \in (0, \delta_v), \forall w \in B[v, \delta_v], \quad f(\bar{x} + tw) > f(\bar{x}), \quad (16)$$

where $B[v, \alpha] := \{w \in X : \|w - v\| \leq \alpha\}$, the closed ball with centre $v$ and radius $\alpha \geq 0$. $\bar{x} \in X$ is called a **weak local strict minimum point of** $f$ if for any $v \in X$ with $\|v\| = 1$, $\bar{x} \in X$ is a local strict minimum point in the direction $v$. A local strict minimum point is always a weak one, and vice versa for $X = \mathbb{R}^n$. Similarly, we can also define a “weak local minimum point” of $f$. In fact, Theorem 2.1 and its corollaries hold for weak local minimum points.

A sufficient condition for a weak local strict minimum point is that

$$D_-f(\bar{x}; v) > 0, \quad \forall v \in X \text{ with } \|v\| = 1. \quad (17)$$

A non-trivial sufficient condition would be for the case that there exists a direction $v$ such that $D_-f(\bar{x}; v) = 0$.

**Theorem 2.2 (Sufficient Condition Theorem).** Let $X$ be a Banach space with a Lipschitz $\beta$-smooth bump function and $f$ be a l.s.c. bounded below function on $X$. Let $\bar{x} \in X$ satisfy

(i) $\liminf_{x \to \bar{x}} (f(x) - f(\bar{x}))/\|x - \bar{x}\| \geq 0$;

(ii) $f'_{\beta} (\bar{x}; 0, u) > 0$ whenever $D_-f(\bar{x}; u) = 0$ and $\|u\| = 1$.

Then $\bar{x}$ is a weak local strict minimum point of $f$. 
COROLLARY 2.2. Let $X$ be a Banach space with a Lipschitz Fréchet-smooth bump function and let $f$ be a l.s.c. bounded below function on $X$. Let $\bar{x} \in X$ satisfy

\begin{enumerate}[(i)]
  \item $\liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x})}{\|x - \bar{x}\|} \geq 0$.
  \item $f'(\bar{x}; 0, u) > 0$ whenever $D_-f(\bar{x}; u) = 0$ and $\|u\| = 1$.
\end{enumerate}

Then $\bar{x}$ is a weak local strict minimum point of $f$.

COROLLARY 2.3. Let $f$ be a l.s.c. bounded below function on $X = \mathbb{R}^n$. Let $\bar{x} \in X$ satisfy

\begin{enumerate}[(i)]
  \item $D_-f(\bar{x}; v) \geq 0$ for all $v \in X$ with $\|v\| = 1$;
  \item $f'(\bar{x}; 0, u) > 0$ whenever $D_-f(\bar{x}; u) = 0$ and $\|u\| = 1$.
\end{enumerate}

Then $\bar{x}$ is a local strict minimum point of $f$.

Our results on optimality conditions generalize those in [9] dealing with locally Lipschitz functions which in turn improve results given by Chaney [2–4]. Our arguments are based on a generalization of the Ekeland variational principle proposed by [12].

3. A GENERALIZED EKELAND VARIATIONAL PRINCIPLE USING BUMP FUNCTION

A bump function on a Banach space $X$ means a real valued function $b: X \to \mathbb{R}$ with bounded non-empty support, saying $b(0) > 0$, $b(x) = 0$ if $\|x\| \geq 1$. We will need the existence of a Lipschitz $\beta$-smooth bump function on $X$. Replacing $b$ by $\phi \circ b$ if necessary, where $\phi \in C^\infty(\mathbb{R}, [0,1])$ with $\phi(0) = 0$ and $\phi(b(0)) = 1$, it is equivalent to say that the following hypothesis holds:

\begin{enumerate}[(H)]
  \item There exists a Lipschitz $\beta$-smooth function $b: X \to [0,1]$ such that
  \begin{enumerate}[(i)]
    \item $b(0) = 1$;
    \item $\|x\| > 1 \Rightarrow b(x) = 0$.
  \end{enumerate}
\end{enumerate}

PROPOSITION 3.1. Assume that a Banach space $X$ admits a Lipschitz $\beta$-smooth bump function $b$ with a Lipschitz constant $L$ such that (H) holds. Then

\begin{enumerate}[(P)]
  \item There exists a Lipschitz $\beta$-smooth function $\rho: X \to [0,1]$ such that
    \begin{enumerate}[(i)]
      \item $0 \leq \rho(x) \leq L\|x\|$
      \item $\rho(x) \geq \|x\|^2/4$ if $\|x\| \leq 1$.
    \end{enumerate}
\end{enumerate}
Proof. We take
\[ \rho(x) = \sum_{n=1}^{\infty} \frac{1}{2^{2n}} [1 - b(2^n x)] . \]

Obviously, \( \rho \) is a Lipschitz \( \beta \)-smooth function with the same Lipschitz constant \( L \) as that of \( b \) and then, for all \( x \in X \),
\[ 0 = \rho(0) \leq \rho(x) \leq \sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \frac{1}{3} , \]
\[ \rho(x) = \rho(x) - \rho(0) \leq L \|x\| . \]

On the other hand, if
\[ 1/2^{n-1} \geq \|x\| > 1/2^n , \quad n = 1, 2, \ldots , \]
then
\[ \rho(x) \geq [1 - b(2^n x)] / 2^{2n} = 1/2^{2n} \geq \|x\|^2 / 4 . \]
Therefore, \( \rho \) satisfies (P). \( \blacksquare \)

Remark 3.1. We have also (P) \( \Rightarrow \) (H). See [12, Proposition 2].

Remark 3.2. If we take
\[ \rho(x) = \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} [1 - b(2^n x)] , \]
then for any \( \varepsilon \in (0, 1) \), we can replace \( \rho(x) \geq \|x\|^2 / 4 \) by \( \rho(x) \geq C(\varepsilon) \|x\|^{1-\varepsilon} \), where \( C(\varepsilon) \) is a constant depending on \( \varepsilon \).

Remark 3.3. If we take \( \rho_1(x) = 2/\sum_{n=0}^{\infty} b(nx) \), then \( \rho_1 \) satisfies [8]

\[ (P_1) \begin{cases} 
\text{There exists a Lipschitz function } \rho_1 : X \to [0, 2] \\
\text{such that} \\
\quad (i) \rho_1 \text{ is } \beta \text{-smooth on } X \setminus \{0\}; \\
\quad (ii) \|x\| \leq \rho_1(x) \leq C \|x\| \text{ if } \|x\| \leq 1 \text{ and } \rho_1(x) = 2 \text{ if } \|x\| \geq 1 . 
\end{cases} \]

We shall need the following generalized Ekeland variational principle due to [12].

Theorem 3.1. Let \( X \) be a Banach space, \( F : X \to (-\infty, +\infty] \) be a bounded below l.s.c. function, and \( \lambda > 0 \). Suppose that \( \rho : X \to (-\infty, +\infty] \)
is a lower semi-continuous function such that
\[
\begin{cases}
(i) \ \rho(0) = 0; \\
(ii) \ \forall \{y_k\} \subseteq X, \ \rho(y_k) \to 0 \Rightarrow \|y_k\| \to 0;
\end{cases}
\] (18)

and that \(\delta_n > 0, \ n = 0, 1, 2, \ldots\), is a positive number sequence. Then, for every \(x_0 \in X\) and \(\varepsilon > 0\) with
\[
F(x_0) \leq \inf_X F + \varepsilon,
\] (19)

there exists a sequence \(\{x_n\} \subseteq X\) which converges to some \(x_\varepsilon \in X\) such that
\[
\rho(x_\varepsilon - x_n) \leq \varepsilon / 2^n \delta_0, \quad n = 0, 1, 2, \ldots;
\] (20)
\[
F(x_\varepsilon) + \sum_{n=0}^\infty \delta_n \rho(x_\varepsilon - x_n) \leq F(x_0) \leq \inf_X F + \varepsilon
\] (21)
\[
\forall x \neq x_\varepsilon, \quad F(x) + \sum_{n=0}^\infty \delta_n \rho(x - x_n) > F(x_\varepsilon) + \sum_{n=0}^\infty \delta_n \rho(x_\varepsilon - x_n).
\] (22)

**Theorem 3.2.** Let \(X\) be a Banach space with a Lipschitz \(\beta\)-smooth bump function, \(F: X \to (-\infty, +\infty]\) be a bounded below l.s.c. function, and \(\lambda > 0\). Then, for every \(x_0 \in X\) and \(\varepsilon > 0\) small enough with
\[
F(x_0) \leq \inf_X F + \varepsilon,
\] (23)

there exists an \(x_\varepsilon \in X\) such that
\[
\|x_\varepsilon - x_0\| \leq 2\sqrt{\lambda},
\] (24)
\[
F(x_\varepsilon) \leq F(x_0) \leq \inf_X F + \varepsilon
\] (25)

and
\[
\forall x \neq x_\varepsilon, \quad F(x) + \Phi_\varepsilon(x) > F(x_\varepsilon) + \Phi_\varepsilon(x_\varepsilon),
\] (26)
where \(\Phi_\varepsilon: X \to \mathbb{R}\) is a Lipschitz \(\beta\)-smooth function such that
\[
\forall x \in X, \quad 0 \leq \Phi_\varepsilon(x) \leq 2\varepsilon \lambda^{-1}
\] (27)

and
\[
\left\|\Phi_\varepsilon'(x)\right\| \leq 2\varepsilon L \lambda^{-1},
\] (28)
where \(L\) is a constant.
Proof. We can assume that \((P)\) holds. Take a sequence \(\{\delta_n\}\) of positive real numbers with \(\delta_0 = \varepsilon / \lambda = \sum_{n=1}^\infty \delta_n\). From Theorem 3.1, there exists a sequence \(\{x_n\} \subseteq X\) which converges to some \(x_0 \in X\) such that
\[
\rho(x_n - x_0) \leq \varepsilon / 2^n \delta_0, \quad n = 0, 1, 2, \ldots ;
\]
and
\[
F(x_n) + \sum_{n=0}^{\infty} \delta_n \rho(x_n - x_0) \leq F(x_0) \leq \inf_X F + \varepsilon,
\]
and
\[
\forall x \neq x_0, \quad F(x) + \Phi_\varepsilon(x) > F(x_0) + \Phi_\varepsilon(x_0),
\]
where
\[
\Phi_\varepsilon(x) = \sum_{n=0}^{\infty} \delta_n \rho(x - x_n).
\]
From (30) and (31), we obtain (25) and (26). Then, when \(\varepsilon\) is small enough, (29) and (P) imply
\[
\|x_n - x_0\| \leq \sqrt{\lambda / 2^n} - \varepsilon
\]
which includes (24) when \(n = 0\).

On the other hand, we have
\[
0 \leq \Phi_\varepsilon(x) = \sum_{n=0}^{\infty} \delta_n \rho(x - x_n) \leq \sum_{n=0}^{\infty} \delta_n = 2 \varepsilon \lambda^{-1}
\]
and
\[
\|\Phi_\varepsilon'(x)\|^* \leq \sum_{n=0}^{\infty} \delta_n \|\rho'(x - x_n)\|^* \leq 2 \varepsilon L \lambda^{-1},
\]
where \(L\) is a Lipschitz constant of \(\rho\). Hence, (27) and (28) are proved. 

Remark 3.4. Replacing \(\rho\) by \(\rho_1\) in \((P)\), we can replace (24) by \(\|x_n - x_0\| \leq \lambda\) as in the classic Ekeland variational principle. But in this case, \(\Phi_\varepsilon\) may not be \(\beta\)-differentiable at \(x_0\).

Remark 3.5. A similar result without (24) appeared in [6, 7].

The key points of our results are the following two theorems.

Theorem 3.3. Let \(X\) be a Banach space with a Lipschitz \(\beta\)-smooth bump function, \(f: X \to (-\infty, +\infty]\) be a bounded below l.s.c. function. Let \(D \subseteq X\) be closed, \(y_0 \in D\) with \(f(y_0)\) finite, and \(\varepsilon, t > 0\) such that
\[
f(y_0) \leq \inf_D f + \varepsilon t.
\]
Then there exist a Lipschitz $\beta$-smooth function $g$ and $z_0 \in D$ such that

(i) $\forall x \in X, |g(x)| \leq 3\varepsilon t$ and $\|g'(x)\|_* \leq 3L\sqrt{\varepsilon t}$;
(ii) $f(x) + g(x) \geq f(z_0) + g(z_0), \forall x \in D$;
(iii) $f(z_0) \leq f(y_0)$;
(iv) $\|z_0 - y_0\| \leq \sqrt{\varepsilon t}$.

Proof. We can assume that (H) holds. Define

$$h(x) = \begin{cases} f(x) - 2\varepsilon t b \left( \frac{x - y_0}{\varepsilon t} \right) & \forall x \in D, \\ +\infty & \forall x \in X \setminus D, \end{cases}$$

where $b$ is as in (H) with a Lipschitz constant $L$. Then $h$ is l.s.c. bounded below and

(a) $h(y_0) = f(y_0) - 2\varepsilon t \leq \inf_D f - \varepsilon t$,
(b) $h(x) = f(x) \geq \inf_D f, \forall x \in D \setminus B[y_0, \sqrt{\varepsilon t}]$.

If $y_0$ is a minimum point of $h$ on $D$, then the theorem is seen to hold by taking $g(x) = -2\varepsilon t b(x - y_0)/\sqrt{\varepsilon t}$ and $z_0 = y_0$. Therefore, we may suppose that $h(y_0) > \inf_D h$. Then take $\xi$ with

$$0 < \xi < \min\{\varepsilon t, L\sqrt{\varepsilon t}\}$$

such that

$$h(y_0) > \xi + \inf_D h.$$

By Theorem 3.2, there exist a Lipschitz $\beta$-smooth function $\phi$ and $z_0 \in X$ such that

$$\forall x \in X, \quad 0 \leq \phi(x) \leq \xi/2,$$

$$\|\phi'(x)\|_* \leq \xi,$$

and

$$\forall x \neq z_0, \quad h(x) + \phi(x) > h(z_0) + \phi(z_0).$$

Note that $z_0$ must be in $D$ and

$$h(z_0) \leq h(x) + \phi(x) - \phi(z_0) \leq h(x) + \xi, \quad \forall x \in D.$$ 

This implies by virtue of (a) that

$$h(z_0) \leq \inf_D h + \xi < h(y_0) < \inf_D f.$$
It follows from (b) that \( z_0 \in B[y_0, \sqrt{e_1}] \), showing (iv). By (32), we also have
\[
 f(z_0) - 2e t b \left( \frac{z_0 - y_0}{\sqrt{e_1}} \right) < f(y_0) - 2e t b \left( \frac{y_0 - y_0}{\sqrt{e_1}} \right) = f(y_0) - 2e t ,
\]
implying (iii) as \( 0 \leq b(x) \leq 1 \). Finally by the minimality of \( h + \phi \) at \( z_0 \), we see that \( z_0 \) is a strict minimum point of \( f(x) - 2e t b(h(x_0)/\sqrt{e_1}) + \phi(x) \) on \( D \). Hence (i) and (ii) hold by taking \( g(x) = -2e t b(h(x_0)/\sqrt{e_1}) + \phi(x) \) for all \( x \).

**Remark 3.6.** Instead of applying Theorem 3.2, one can alternatively apply the main result of [6, 7].

**Theorem 3.4.** Let \( X \) be a Banach space with a Lipschitz \( \beta \)-smooth bump function, \( f : X \to (-\infty, +\infty) \) be a bounded below l.s.c. function. Let \( \bar{x} \in X \) with \( f(\bar{x}) \) finite, \( u \) be a unit vector in \( X \), and \( \{x_k\} \) be a sequence convergent to \( \bar{x} \) in the direction \( u \). Let \( 1 > e_k \downarrow 0, \ t_k > 0 \) with \( \|x_k - \bar{x}\|/t_k \to 1 \), and \( \gamma_k > 0 \) be such that \( 3t_k \leq \gamma_k \) and
\[
 f(x_k) \leq \inf_{B_{\bar{x}}} f + e_k t_k , \quad \forall k ,
\]
where \( B_{\bar{x}} := B[\bar{x}, \gamma_k] \), the closed ball with centre \( \bar{x} \) and radius \( \gamma_k \). Then \( 0 \in \partial_{\mu^k} f(\bar{x}) \), and in fact there exist sequences \( \{z_k\} \) and \( \{z_k^k\} \) with each \( z_k^k \in \partial_{\mu^k} f(z_k) \) such that \( z_k \to_{\mu} \bar{x} \), \( \|z_k^k\| \to 0 \), and \( f(z_k) \leq f(x_k) \) for all \( k \).

**Proof.** Without loss of generality we may suppose that \( \|x_k - \bar{x}\| < 2t_k \) for all \( k \). For each \( k \), by Theorem 3.3 (applied to \( x_k, B_{\bar{x}} \) in place of \( y_0, D \)), there exist Lipschitz \( \beta \)-smooth function \( g_k \) and \( z_k \in B_{\bar{x}} \) such that
\[
 (i) \quad \forall x \in X , \ g_k(x) \leq 3e_k t_k \text{ and } \|g'_k(x)\| \leq 3L_{\sqrt{\varepsilon_k}} ;
\]
\[
 (ii) \quad \forall x \in B_{\bar{x}} , \ f(x) + g_k(x) \geq f(z_k) + g_k(z_k) ;
\]
\[
 (iii) \quad f(z_k) \leq f(x_k) ;
\]
\[
 (iv) \quad \|z_k - x_k\| \leq \sqrt{\varepsilon_k} t_k .
\]
By (iv) it follows that
\[
 \|z_k - \bar{x}\| \leq \|z_k - x_k\| + \|x_k - \bar{x}\| \leq \left( \sqrt{\varepsilon_k} + 2 \right) t_k \leq 3t_k ,
\]
showing that \( z_k \in \text{int } B_{\bar{x}} \). Moreover, since
\[
 \left\| \frac{z_k - \bar{x}}{t_k} - \frac{x_k - \bar{x}}{t_k} \right\| = \left\| \frac{z_k - x_k}{t_k} \right\| \leq \sqrt{\varepsilon_k} \to 0 ,
\]
and \( (x_k - \bar{x})/t_k \to u \), we have \( (z_k - \bar{x})/t_k \to u \), showing that \( z_k \to_{\mu} \bar{x} \). Since \( z_k \) is an interior point of \( B_{\bar{x}} \), (ii) implies that
\[
 0 \in \partial_{\mu}(f + g_k)(z_k) = \partial_{\mu} f(z_k) + g'_{\mu^k}(z_k) .
\]
Letting \( z_k^* = -g_k' (z_k) \), we see that \( z_k^* \in \partial f(z_k) \) and \( \|z_k^*\| \to 0 \) by (i). Hence, \( 0 \in \partial \mu f(x) \).

**Corollary 3.1**. Let \( X, f \) be as in Theorem 3.3. Let \( x = x \in X \), \( 1/4 > \varepsilon_k \downarrow 0 \), and \( \delta_k > 0 \) be such that

\[
f(x) \leq f(x) + \varepsilon_k \|x - \bar{x}\|, \quad \forall x \in B[\bar{x}, \delta_k].
\] (33)

Suppose that \( u \in X \) is a unit vector such that \( D f(x; u) = 0 \). Then \( 0 \in \partial \mu f(x) \).

*Proof.* Since \( D f(x; u) = 0 \) for each \( k \), we can find inductively sequences \( t_k \downarrow 0 \) and \( u_k \to u \) such that \( 3t_k < \delta_k \) and

\[
f(x + t_k u_k) - f(x) < \varepsilon_k t_k.
\]

Let \( x_k = x + t_k u_k \) and \( \gamma_k = 3t_k \). If \( x \in B[\bar{x}, \gamma_k] \), then

\[
f(x_k) < f(x) + \varepsilon_k t_k \leq f(x) + \varepsilon_k \|x - \bar{x}\| + \varepsilon_k t_k \leq f(x) + 4 \varepsilon_k t_k.
\]

By Theorem 3.4 (applied to \( 4 \varepsilon_k \) in place of \( \varepsilon_k \)), we have \( 0 \in \partial \mu f(x) \).

Equivalently, Corollary 3.1 can be restated in the following form.

**Corollary 3.1**. Let \( X, f \) be as in Theorem 3.3. Let \( x \in X \) satisfy

\[
\liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x})}{\|x - \bar{x}\|} \geq 0.
\] (34)

Suppose that \( u \in X \) is a unit vector such that \( D f(x; u) = 0 \). Then \( 0 \in \partial \mu f(x) \).

*Proof.* Let \( \varepsilon > 0 \). Then there exists \( \delta > 0 \) such that

\[
f(x) \leq f(x) + \varepsilon \|x - \bar{x}\|, \quad \forall x \in B[\bar{x}, \delta].
\]

For, otherwise, there exists a sequence \( (x_k) \) such that \( \|x_k - \bar{x}\| \leq 1/k \), and

\[
f(x) > f(x_k) + \varepsilon \|x_k - \bar{x}\|, \quad \forall k.
\]

Note that \( x_k \neq \bar{x} \) and

\[
-\varepsilon \geq \frac{f(x_k) - f(\bar{x})}{\|x_k - \bar{x}\|},
\]

contradicting (34). This shows that for any sequence \( \{\varepsilon_k\} \) with \( \varepsilon_k \downarrow 0 \), there exists \( \{\delta_k\} \) with \( \delta_k > 0 \) satisfying (33). Thus the result follows from Corollary 3.1.
Remark 3.7. Conversely it is easy to show that if (33) holds with some \( \{e_k\} \) and \( \{\delta_k\} \), then (34) must hold.

Corollary 3.2. Let \( f, X \) be as in Theorem 3.3, \( \bar{x} \in X \), and suppose (34) holds. Let \( u \) be a unit vector and \( \{x_k\} \) be a sequence convergent to \( \bar{x} \) in the direction \( u \) such that \( f(x_k) \leq f(\bar{x}) \) for all \( k \). Then \( 0 \in \partial_{\bar{x}}f(\bar{x}) \) and \( f_{\bar{x}}^+(\bar{x};0,u) \leq 0 \).

Proof. Take \( 1/4 > e_k \downarrow 0 \). By the proof of Corollary 3.1*, there exists a sequence \( \{\delta_k\} \) with \( \delta_k > 0 \) such that (33) holds. Without loss of generality we may suppose that \( 3\|x_k - \bar{x}\| < \delta_k \) for all \( k \). Let \( t_k = \|x_k - \bar{x}\| \) and \( \gamma_k = 3t_k \). Then, whenever \( x \in B[\bar{x}, \gamma_k] \), one has from (33) that

\[
f(x_k) \leq f(\bar{x}) \leq f(x) + e_k \gamma_k = f(x) + 3e_k t_k, \quad \forall k.
\]

By Theorem 3.4, \( 0 \in \partial_{\bar{x}}f(\bar{x}) \) and there exist sequences \( \{z_k\} \) and \( \{z^+_k\} \) with \( z^+_k \in \partial f(z_k) \), \( f(z_k) \leq f(x_k) \leq f(\bar{x}) \) for all \( k \), such that \( z_k \to u \bar{x} \) and \( \|z^+_k\| \to 0 \). Therefore,

\[
f^+_u(\bar{x};0,u) \leq \liminf_{k \to \infty} \frac{f(z_k) - f(\bar{x})}{\|z_k - \bar{x}\|^2} \leq 0.
\]

4. PROOF OF THE MAIN RESULTS

Proof of Theorem 2.1. The first conclusion follows from Corollary 3.1 with any \( e_k \downarrow 0 \) and the last inequality from the definition. Thus we need only prove the first inequality. Let \( x_k \to u \bar{x} \). Then

\[
\liminf_{k \to \infty} \frac{f(x_k) - f(\bar{x})}{\|x_k - \bar{x}\|^2} \geq 0,
\]

because \( f(x_k) \geq f(\bar{x}) \) by minimality of \( \bar{x} \). This implies that \( f^+_u(\bar{x};0,u) \geq 0 \).

Corollaries 2.1 and 2.2 are the consequences of Theorem 2.1 by using (11) and \( X = \mathbb{R}^n \).

Proof of Theorem 2.2. Suppose not: there exist a unit vector \( u \) and a sequence \( \{x_k\} \) in \( X \) such that \( x_k \to u \bar{x} \), \( x_k \neq \bar{x} \), and \( f(x_k) \leq f(\bar{x}) \). Then \( D_-f(\bar{x};u) \leq 0 \). But by (i), \( D_-f(\bar{x};u) \geq 0 \); hence, \( D_-f(\bar{x};u) = 0 \). Thus, \( f^-u(\bar{x};0,u) > 0 \) by (ii), contradicting Corollary 3.2.
Corollary 2.3 follows from Theorem 2.2 by taking $\beta = F$, the family of all bounded subsets of $X$ and by noting

$$f^u_\beta(x;0,u) \geq f^\ast_\beta(\bar{x};0,u), \quad 0 \in \partial^+_u f(\bar{x}) \subseteq \partial^+_u f(\bar{x})$$

as readily verified from the definitions. Finally Corollary 2.4 follows from Corollary 2.3 as, for $X = \mathbb{R}^n$, the assumptions of these corollaries are equivalent as well as their conclusions.

REFERENCES