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Coxeter group actions on Saalschützian ${}_4F_3(1)$ series and very-well-poised ${}_7F_6(1)$ series

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ABSTRACT

In this paper we consider a function $L(\vec{x}) = L(a, b, c, d; e; f, g)$, which can be written as a linear combination of two Saalschützian ${}_4F_3(1)$ hypergeometric series or as a very-well-poised ${}_7F_6(1)$ hypergeometric series. We explore two-term and three-term relations satisfied by the L function and put them in the framework of group theory. We prove a fundamental two-term relation satisfied by the L function and show that this relation implies that the Coxeter group $W(D_5)$, which has 1920 elements, is an invariance group for $L(\vec{x})$. The invariance relations for $L(\vec{x})$ are given two classifications based on two double coset decompositions of the invariance group. The fundamental two-term relation is shown to generalize classical results about hypergeometric series. We derive Thomae's identity for ${}_3F_2(1)$ series, Bailey's identity for terminating Saalschützian ${}_4F_3(1)$ series, and Barnes' second lemma as consequences. We further explore three-term relations satisfied by $L(a, b, c, d; e; f, g)$. The group that governs the three-term relations is shown to be isomorphic to the Coxeter group $W(D_6)$, which has 23 040 elements. Based on the right cosets of $W(D_5)$ in $W(D_6)$, we demonstrate the existence of 220 three-term relations satisfied by the L function that fall into two families according to the notion of L-coherence. The complexity of the coefficients in front of the L functions in the three-term relations is studied and is shown to also depend on L-coherence.

1. Introduction

Hypergeometric series of type ${}_2F_1$ were studied in detail by Gauss [12], who developed many of their properties. Generalized hypergeometric series of type ${}_AF_B$, where A and B are positive integers, were studied in the late nineteenth and early twentieth century by Thomae [28], Barnes [3,4], Ramanujan (see [15]), Whipple [31–33], Bailey [1,2], and others.

There has been a renewed interest in hypergeometric series over the last twenty-five years. Relations among hypergeometric and basic hypergeometric series were put into a group-theoretic framework in papers by Beyer et al. [5], Srinivasa Rao et al. [22], Formichella et al. [10], Van der Jeugt and Srinivasa Rao [30], Lievens and Van der Jeugt [18,19]. Other papers include Groenevelt [13], van de Bult et al. [6], and Krattenthaler and Rivoal [17].

Hypergeometric series have also appeared in recent papers by Bump [7], Stade [23–26], and Stade and Taggart [27] with applications in the theory of automorphic functions. Other recent works, with applications in physics, were written by Drake [8], Grozin [14], and Raynal [21].

The goal of this paper is to describe two-term and three-term relations among linear combinations of Saalschützian ${}_4F_3(1)$ hypergeometric series and put them in the framework of group theory. We examine a function $L(a, b, c, d; e; f, g)$ (see (2.2) for the definition) which is a linear combination of two Saalschützian ${}_4F_3(1)$ series. This particular linear

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combination of two Saalschützian ${}_4F_3(1)$ series appears in [24] in the evaluation of the Mellin transform of a spherical principal series $GL(4,\mathbb{R})$ Whittaker function.

In Section 3 we derive a fundamental two-term relation (see (3.3)) satisfied by $L(a,b,c,d;e;f,g)$. The fundamental two-term relation (3.3) is derived through a Barnes integral representation of $L(a,b,c,d;e;f,g)$ and generalizes both Thomae's identity (see [2, p. 14]) and Bailey's identity (see [32, Eq. (10.11)] or [2, p. 56]) in the sense that the latter two identities can be obtained as limiting cases of our fundamental two-term relation (see Section 5).

In Section 4 we show that the two-term relation (3.3) combined with the trivial invariances of $L(a,b,c,d;e;f,g)$ under permutations of a, b, c, d and interchanging f, g implies that the function $L(a,b,c,d;e;f,g)$ has an invariance group G_L isomorphic to the Coxeter group $W(D_5)$, which is of order 1920. (See [16] for general information on Coxeter groups.) The invariance group G_L is given as a matrix group of transformations of the affine hyperplane

$$V=\left\{(a,b,c,d,e,f,g)^T\in\mathbb{C}^7:\ e+f+g-a-b-c-d=1\right\}. \tag{1.1}$$

The 1920 invariances of the L function that follow from the invariance group G_L are classified first into six types based on a double coset decomposition of G_L with respect to its subgroup Σ consisting of all the permutation matrices in G_L (see Theorem 4.4). Another classification of the 1920 invariances of the L function is given through a double coset decomposition of G_L with respect to the subgroup generated by Σ and the invariance corresponding to interchanging the two ${}_4F_3(1)$ series in the definition of $L(a,b,c,d;e;f,g)$ (see Theorem 4.5). Such double coset decompositions with respect to subgroups of trivial invariances can also be seen in other works. One example is [33, Eq. (2.2)] concerning a very-well-poised ${}_7F_6(1)$ series analog of the L function (see also Remark 4.6), and another example is [19, Eqs. (24a)–(24e)] describing the invariances coming from Bailey's four-term transformations for non-terminating ${}_{10}\phi_9$ basic hypergeometric series.

Some consequences of the fundamental two-term relation (3.3) are shown in Section 5. In particular, as already mentioned, we show that Thomae's and Bailey's identities follow as limiting cases of (3.3). We also show that Barnes' second lemma (see [4] or [2, p. 42]) follows as a special case of (3.3) when we take $d=g$.

In Section 6 we lay the group-theoretic foundation for the three-term relations satisfied by $L(a,b,c,d;e;f,g)$. The three-term relations are governed by the right coset space $G_L\setminus M_L$ described in that section. We also introduce the notion of L-coherence (see Definition 6.5), which will be used in the classification of the three-term relations in the following section.

Section 7 describes the three-term relations satisfied by $L(a,b,c,d;e;f,g)$. We show that for every $\sigma_1,\sigma_2,\sigma_3\in M_L$ such that σ_1, σ_2, σ_3 are in different right cosets of G_L in M_L, there exists a three-term relation involving the functions $L(\sigma_1\vec{x})$, $L(\sigma_2\vec{x})$, and $L(\sigma_3\vec{x})$. Thus we demonstrate the existence of 220 three-term relations. The 220 three-term relations are shown to fall into two families based on the notion of L-coherence. We explicitly find a three-term relation of each family in (7.1) and (7.7), and then show that every other three-term relation is obtained from one of those two through a change of variable of the form $\vec{x}\mapsto\mu\vec{x}$ applied to all terms and coefficients. In addition, the complexity of the coefficients in front of the L functions in the three-term relations is studied. It is shown that each coefficient can be written as a rational combination of transformations of the function $\gamma_1(\vec{x})$ given in (7.13), and the number of monomials in a coefficient depends on L-coherence related to the other two L functions in the three-term relation.

Versions of the L function (in terms of very-well-poised ${}_7F_6(1)$ series, see (2.3)) were examined in the past by Bailey [1], Whipple [33], and Raynal [21]. The authors examine two-term and three-term relations, but do not put their results in a group-theoretic framework. The two-term and three-term relations for the L function obtained in the present paper are analogous to the two-term and three-term relations for very-well-poised ${}_7F_6(1)$ series obtained by Whipple in [33]. The difference is that in the present paper we use the theory of Coxeter groups to describe the structure behind the two-term and three-term relations for the L function. The group-theoretic approach presents us with structured and convenient ways to derive and classify all 1920 two-term relations and all 220 three-term relations.

The L function appears as a Wilson function (a nonpolynomial extension of the Wilson polynomial) in [13]. Van de Bult et al. [6] examine generalizations to elliptic, hyperbolic, and trigonometric hypergeometric functions.

A basic hypergeometric series analog of the L function (in terms of ${}_8\phi_7$ series) was studied by Van der Jeugt and Srinivasa Rao [30] and by Lievens and Van der Jeugt [18]. There are, however, some significant differences between the present paper and the papers [30] and [18]:

(i) While mentioning that ${}_8\phi_7$ series relations are q-analogs of ${}_7F_6$ series relations, the papers [30] and [18] deal specifically with ${}_8\phi_7$ series relations. The results of the present paper translate directly to ${}_7F_6$ series relations and thus the algebraic structure of those relations is verified, as has been expected, to be in agreement with the algebraic structure of their basic hypergeometric series analogs.

(ii) The authors of [30] establish an invariance group isomorphic to $W(D_5)$ for the ${}_8\phi_7$ series. In the present paper, we establish an invariance group isomorphic to $W(D_5)$ for the L function and, in addition to that, we classify all two-term relations for the L function in two different ways in Theorem 4.4 and in Theorem 4.5.

(iii) The authors of [18] obtain formulas for every three-term relation for the ${}_8\phi_7$ series and list three groups of formulas containing 160, 30, and 30 three-term relations (see [18, Theorem 3]). In the present paper, we examine the orbits of the action of $W(D_6)$ on triples from the right coset space $W(D_5)\setminus W(D_6)$ and show, based on the introduced notion of L-coherence, that the 220 three-term relations for the L function fall into two families of sizes 160 and 60.

(iv) The authors of [18] do not notice the symmetry of the coefficients in the three-term relations. In the present paper, the symmetry of the coefficients in the three-term relations for the L function is apparent, and, in addition, we discuss the complexity of those coefficients and show how the number of monomials in each coefficient depends on the notion of L-coherence (see the last paragraph of Chapter 7). Furthermore, we observe that we need only one function to generate all the coefficients (see (7.13)).

Very recently Formichella et al. [10] explored a function $K(a; b, c, d; e, f, g)$ which is a different linear combination of two Saalschützian ${}_4F_3(1)$ series from the function $L(a, b, c, d; e; f, g)$. The particular linear combination of ${}_4F_3(1)$ series studied by Formichella et al. appears in the theory of archimedean zeta integrals for automorphic L functions (see [26, 27]). The function $K(a; b, c, d; e, f, g)$ behaves very differently from $L(a, b, c, d; e; f, g)$. Formichella et al. obtain in [10] a two-term relation satisfied by $K(a; b, c, d; e, f, g)$ and show that their two-term relation implies that the symmetric group S_6 is an invariance group for $K(a; b, c, d; e, f, g)$. In addition, the existence of 4960 three-term relations satisfied by the K function is demonstrated and the 4960 three-term relations are classified into five families based on the notion of Hamming type. In a future work by the author of the present paper and by Green and Stade, the connection between the K and the L functions will be studied.

2. Hypergeometric series and Barnes integrals

The hypergeometric series of type ${}_{p+1}F_p$ is the power series in the complex variable z defined by

$$
{}_{p+1}F_p\left[\begin{matrix} a_1, a_2, \ldots, a_{p+1}; \\ b_1, b_2, \ldots, b_p; \end{matrix} \quad z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{p+1})_n}{n!(b_1)_n (b_2)_n \cdots (b_p)_n} z^n, \tag{2.1}
$$

where p is a positive integer, the numerator parameters $a_1, a_2, \ldots, a_{p+1}$ and the denominator parameters $b_1, b_2, \ldots, b_p$ are complex numbers, and the rising factorial $(a)_n$ is given by

$$
(a)_n = \begin{cases} a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, & n > 0, \\ 1, & n = 0. \end{cases}
$$

The series in (2.1) converges absolutely if $|z| < 1$. When $|z| = 1$, the series converges absolutely if $\operatorname{Re}(\sum_{i=1}^{p} b_i - \sum_{i=1}^{p+1} a_i) > 0$ (see [2, p. 8]). We assume that no denominator parameter is a negative integer or zero. If a numerator parameter is a negative integer or zero, the series has only finitely many nonzero terms and is said to terminate.

When $z = 1$, the series is said to be of unit argument and of type ${}_{p+1}F_p(1)$. If $\sum_{i=1}^{p} b_i = \sum_{i=1}^{p+1} a_i + 1$, the series is called Saalschützian. If $1 + a_1 = b_1 + a_2 = \cdots = b_p + a_{p+1}$, the series is called well-poised. A well-poised series that satisfies $a_2 = 1 + \frac{1}{2}a_1$ is called very-well-poised.

Our main object of study in this paper will be the function $L(a, b, c, d; e; f, g)$ defined by

$$
\begin{aligned}
L(a, b, c, d; e; f, g) = {} & \frac{{}_4F_3\left[\begin{smallmatrix} a,b,c,d; \\ e,f,g; \end{smallmatrix}\ 1\right]}{\sin\pi e\,\Gamma(e)\Gamma(f)\Gamma(g)\Gamma(1+a-e)\Gamma(1+b-e)\Gamma(1+c-e)\Gamma(1+d-e)} \\
& - \frac{{}_4F_3\left[\begin{smallmatrix} 1+a-e, 1+b-e, 1+c-e, 1+d-e; \\ 2-e, 1+f-e, 1+g-e; \end{smallmatrix}\ 1\right]}{\sin\pi e\,\Gamma(2-e)\Gamma(1+f-e)\Gamma(1+g-e)\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)},
\end{aligned} \tag{2.2}
$$

where $a, b, c, d, e, f, g \in \mathbb{C}$ satisfy $e + f + g - a - b - c - d = 1$.

The function $L(a, b, c, d; e; f, g)$ is a linear combination of two Saalschützian ${}_4F_3(1)$ series. Other notations we will use for $L(a, b, c, d; e; f, g)$ are $L\left[\begin{smallmatrix} a,b,c,d; \\ e; f, g \end{smallmatrix}\right]$ and $L(\vec{x})$, where we will always have $\vec{x} = (a, b, c, d, e, f, g)^T \in V$ (see (1.1)).

It should be noted that by [2, Eq. (7.5.3)], the L function can be expressed as a very-well-poised ${}_7F_6(1)$ series:

$$
\begin{aligned}
L(a, b, c, d; e; f, g) = {} & \frac{\Gamma(1+d+g-e)}{\pi\,\Gamma(g)\Gamma(1+g-e)\Gamma(f-d)\Gamma(1+a+d-e)\Gamma(1+b+d-e)\Gamma(1+c+d-e)} \\
& \times {}_7F_6\left[\begin{matrix} d+g-e, 1+\frac{1}{2}(d+g-e), g-a, g-b, g-c, d, 1+d-e; \\ \frac{1}{2}(d+g-e), 1+a+d-e, 1+b+d-e, 1+c+d-e, 1+g-e, g; \end{matrix} \quad 1\right],
\end{aligned} \tag{2.3}
$$

provided that $\operatorname{Re}(f - d) > 0$. Therefore our results on the L function can also be interpreted in terms of the very-well-poised ${}_7F_6(1)$ series given in (2.3). However, in this paper we choose to primarily view L as a linear combination of two Saalschützian ${}_4F_3(1)$ series because this representation connects L to the K function studied in [10] (see the Introduction) and will be used in a future work to deduce new relations among Saalschützian ${}_4F_3(1)$ series. Also, it can be noted that the very-well-poised ${}_7F_6(1)$ series analog of the L function does not converge for all values of the parameters whereas the Saalschützian condition guarantees the convergence of the two ${}_4F_3(1)$ series in the definition of the L function.

If we use [11, Eq. (6.3.11)], the L function can also be written as:

$$L(a,b,c,d;e;f,g)=\frac{1}{\left[\begin{smallmatrix}4\pi^2\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(1+a-e)\Gamma(1+b-e)\Gamma(1+c-e)\\ \times\Gamma(g-a)\Gamma(g-b)\Gamma(g-c)\Gamma(f-d)\end{smallmatrix}\right]}$$
$$\times\int_{-\infty}^{\infty}\frac{\left[\begin{smallmatrix}\Gamma(A+it)\Gamma(A-it)\Gamma(B+it)\Gamma(B-it)\Gamma(C+it)\Gamma(C-it)\\ \times\Gamma(D+it)\Gamma(D-it)\Gamma(F+it)\Gamma(F-it)\end{smallmatrix}\right]}{\Gamma(2it)\Gamma(-2it)\Gamma(G+it)\Gamma(G-it)}\,dt, \tag{2.4}$$

where

$$A=\frac{g-a-b+c}{2},\qquad B=\frac{g-a+b-c}{2},\qquad C=\frac{g+a-b-c}{2},$$
$$D=\frac{f-d-e+1}{2},\qquad F=\frac{f-d+e-1}{2},\qquad G=\frac{f+d-e+1}{2},$$

and we require that $\mathrm{Re}(A,B,C,D,F)>0$.

Fundamental to the derivation of a nontrivial two-term relation for the L function will be the notion of a Barnes integral, which is a contour integral of the form

$$\int_t\prod_{i=1}^{n}\Gamma^{\epsilon_i}(a_i+t)\prod_{j=1}^{m}\Gamma^{\epsilon_j}(b_j-t)\,dt, \tag{2.5}$$

where $n,m\in\mathbb{Z}^+$; $\epsilon_i,\epsilon_j=\pm1$; and $a_i,b_j,t\in\mathbb{C}$. The path of integration is the imaginary axis, indented if necessary, so that any poles of $\prod_{i=1}^{n}\Gamma^{\epsilon_i}(a_i+t)$ are to the left of the contour and any poles of $\prod_{j=1}^{m}\Gamma^{\epsilon_j}(b_j-t)$ are to the right of the contour. This path of integration always exists, provided that, for $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant m$, we have $a_i+b_j\notin\{0,-1,-2,\ldots\}$ whenever $\epsilon_i=\epsilon_j=1$.

From now on, when we write an integral of the form (2.5), we will always mean a Barnes integral with a path of integration as just described.

A Barnes integral can often be evaluated in terms of hypergeometric series using the Residue Theorem, provided that we can establish the necessary convergence arguments. This is the approach we take in the next section. We will make use of the extension of Stirling's formula to the complex numbers (see [29, Section 4.42] or [34, Section 13.6]):

$$\Gamma(a+z)=\sqrt{2\pi}z^{a+z-1/2}e^{-z}\bigl(1+O\bigl(1/|z|\bigr)\bigr)\quad\text{uniformly as }|z|\to\infty, \tag{2.6}$$

provided that $-\pi+\delta\leqslant\arg(z)\leqslant\pi-\delta$, $\delta\in(0,\pi)$.

When applying the Residue Theorem, we will use the fact that the gamma function has simple poles at $t=-n$, $n=0,1,2,\ldots,$ with

$$\mathrm{Res}_{t=-n}\Gamma(t)=\frac{(-1)^n}{n!}. \tag{2.7}$$

When simplifying expressions involving gamma functions, the reflection formula for the gamma function will often be used:

$$\Gamma(t)\Gamma(1-t)=\frac{\pi}{\sin\pi t}. \tag{2.8}$$

Finally, we will use a result about Barnes integrals known as Barnes' first lemma (see [3] or [2, p. 6]):

Lemma 2.1 *(Barnes' first lemma). If $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, we have*

$$\frac{1}{2\pi i}\int_t\Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(\gamma-t)\Gamma(\delta-t)\,dt=\frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}, \tag{2.9}$$

provided that none of $\alpha+\gamma,\alpha+\delta,\beta+\gamma$, and $\beta+\delta$ belongs to $\{0,-1,-2,\ldots\}$.

3. Fundamental two-term relation

In this section we show that the function $L(a,b,c,d;e;f,g)$ defined in (2.2) can be represented as a Barnes integral. The Barnes integral representation will then be used to derive a fundamental two-term relation satisfied by the L function.

Proposition 3.1.

$$L(a,b,c,d;e;f,g)=\frac{1}{\pi\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(1+a-e)\Gamma(1+b-e)\Gamma(1+c-e)\Gamma(1+d-e)} \times\frac{1}{2\pi i}\int_t\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(c+t)\Gamma(d+t)\Gamma(1-e-t)\Gamma(-t)}{\Gamma(f+t)\Gamma(g+t)}\,dt. \tag{3.1}$$

Proof. Let

$$I\begin{bmatrix} a,b,c,d; \\ e;f,g \end{bmatrix}=\frac{1}{2\pi i}\int_t\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(c+t)\Gamma(d+t)\Gamma(1-e-t)\Gamma(-t)}{\Gamma(f+t)\Gamma(g+t)}\,dt. \tag{3.2}$$

For $N\geqslant 1$, let C_N be the semicircle of radius ρ_N on the right side of the imaginary axis and center at the origin, chosen in such a way that $\rho_N\to\infty$ as $N\to\infty$ and

$$\varepsilon:=\inf_N \operatorname{dist}\bigl(C_N,\mathbb{Z}\cup(\mathbb{Z}-e)\bigr)>0.$$

The formula (2.8) gives

$$G(t):=\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(c+t)\Gamma(d+t)\Gamma(1-e-t)\Gamma(-t)}{\Gamma(f+t)\Gamma(g+t)} =\frac{-\pi^2\Gamma(a+t)\Gamma(b+t)\Gamma(c+t)\Gamma(d+t)}{\Gamma(f+t)\Gamma(g+t)\Gamma(e+t)\Gamma(1+t)\sin\pi t\sin\pi(e+t)}.$$

By Stirling's formula (2.6),

$$\frac{\Gamma(a+t)\Gamma(b+t)\Gamma(c+t)\Gamma(d+t)}{\Gamma(f+t)\Gamma(g+t)\Gamma(e+t)\Gamma(1+t)}\sim t^{a+b+c+d-e-f-g-1}=t^{-2}.$$

It follows that we can find a constant $K>0$ such that

$$\bigl|G(t)\bigr|\leqslant K/|t|^2\quad \text{if } t\in C_N,\ N=1,2,\ldots,$$

which implies

$$\int_{C_N}G(t)\,dt\to 0\quad\text{as } N\to\infty.$$

Therefore the integral given by $I\left[\begin{smallmatrix} a,b,c,d; \\ e;f,g \end{smallmatrix}\right]$ is equal to the sum of the residues of the integrand at the poles of $\Gamma(1-e-t)$ and $\Gamma(-t)$, which yields the result. □

The fundamental two-term relation satisfied by $L(a,b,c,d;e;f,g)$ is given in the next proposition.

Proposition 3.2.

$$L\begin{bmatrix} a,b,c,d; \\ e;f,g \end{bmatrix}=L\begin{bmatrix} a,b,g-c,g-d; \\ 1+a+b-f;1+a+b-e,g \end{bmatrix}. \tag{3.3}$$

Proof. Let $I\left[\begin{smallmatrix} a,b,c,d; \\ e;f,g \end{smallmatrix}\right]$ be as given in (3.2). As a first step, we will prove that

$$\frac{I\left[\begin{smallmatrix} a,b,c,d; \\ e;f,g \end{smallmatrix}\right]}{\Gamma(c)\Gamma(d)\Gamma(1+a-e)\Gamma(1+b-e)}=\frac{I\left[\begin{smallmatrix} a,b,g-c,g-d; \\ 1+a+b-f;1+a+b-e,g \end{smallmatrix}\right]}{\Gamma(f-a)\Gamma(f-b)\Gamma(g-c)\Gamma(g-d)}. \tag{3.4}$$

By Barnes' first lemma,

$$\frac{\Gamma(a+t)\Gamma(b+t)}{\Gamma(f+t)}=\frac{1}{2\pi i\Gamma(f-a)\Gamma(f-b)}\int_u\Gamma(t+u)\Gamma(f-a-b+u)\Gamma(a-u)\Gamma(b-u)\,du$$

and

$$\frac{\Gamma(c+t)\Gamma(d+t)}{\Gamma(g+t)}=\frac{1}{2\pi i\Gamma(g-c)\Gamma(g-d)}\int_v\Gamma(t+v)\Gamma(g-c-d+v)\Gamma(c-v)\Gamma(d-v)\,dv.$$

We re-write the integral for $I\left[\begin{smallmatrix} a,b,c,d; \\ e;f,g \end{smallmatrix}\right]$ by substituting for the above expressions, changing the order of integration, so that we integrate with respect to t first (the change in the order of integration is readily justified using Stirling's formula and Fubini's theorem), and then applying Barnes' first lemma again to the integral with respect to t. We obtain

$$\begin{aligned}
&\frac{I\left[\begin{smallmatrix} a,b,c,d; \\ e;f,g \end{smallmatrix}\right]}{\Gamma(c)\Gamma(d)\Gamma(1+a-e)\Gamma(1+b-e)} \\
&\quad = \frac{-1}{4\pi^2\Gamma(c)\Gamma(d)\Gamma(1+a-e)\Gamma(1+b-e)\Gamma(f-a)\Gamma(f-b)\Gamma(g-c)\Gamma(g-d)} \\
&\qquad \times \int_u \Gamma(f-a-b+u)\Gamma(a-u)\Gamma(b-u)\Gamma(u)\Gamma(1-e+u) \\
&\qquad \times \left(\int_v \frac{\Gamma(g-c-d+v)\Gamma(c-v)\Gamma(d-v)\Gamma(v)\Gamma(1-e+v)}{\Gamma(1-e+u+v)}\,dv \right) du. \qquad (3.5)
\end{aligned}$$

After the substitution $v \mapsto c+d-f+v$ in the inside integral, it is easily checked (using the Saalschützian condition $e+f+g-a-b-c-d=1$) that the right-hand side of (3.5) is invariant under the transformation

$$(a,b,c,d;e;f,g) \mapsto (a,b,g-c,g-d;1+a+b-f;1+a+b-e,g),$$

which proves (3.4). The result in the proposition now follows immediately from (3.4) upon writing the two L functions in (3.3) in terms of their Barnes integral representations (3.1). □

Remark 3.3. If we write the L function as a very-well-poised ${}_7F_6(1)$ series, Proposition 3.2 is apparent from [33, Eq. (2.2)].

4. Invariance group

In the previous section we showed that the function $L(a,b,c,d;e;f,g)$ satisfies the two-term relation (3.3). If we define

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in GL(7,\mathbb{C}), \qquad (4.1)$$

then (3.3) can be expressed as $L(\vec{x}) = L(A\vec{x})$.

If $\sigma \in S_7$, we will identify σ with the matrix in $GL(7,\mathbb{C})$ that permutes the standard basis $\{e_1, e_2, \ldots, e_7\}$ of the complex vector space $\mathbb{C}^7$ according to the permutation σ. For example,

$$(123) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let

$$G_L = \big\langle (12), (23), (34), (67), A \big\rangle \leqslant GL(7,\mathbb{C}). \qquad (4.2)$$

The two-term relation (3.3) along with the trivial invariances of the function $L(a,b,c,d;e;f,g)$ under permutations of a, b, c, d and interchanging f, g implies that G_L is an invariance group for $L(a,b,c,d;e;f,g)$, i.e. $L(\vec{x}) = L(\alpha\vec{x})$ for every $\alpha \in G_L$.

The goal of this section is to find the isomorphism type of the group G_L and further to describe the two-term relations for the L function in terms of two different double coset decompositions of G_L. We begin by defining the subgroup Σ of G_L by:

$$\Sigma = \big\langle (12), (23), (34), (67) \big\rangle. \qquad (4.3)$$

The group Σ is a subgroup of G_L consisting of permutation matrices. It is clear that $\Sigma \cong S_4 \times S_2$ and so $|\Sigma| = 48$. We note that if $\sigma \in \Sigma, \alpha \in G_L$, the multiplication $\sigma\alpha$ permutes the rows of α, and the multiplication $\alpha\sigma$ permutes the columns of α. A double coset of Σ in G_L is a set of the form

$$\Sigma\alpha\Sigma = \{\sigma\alpha\tau\colon \sigma,\tau\in\Sigma\},\quad \text{for some } \alpha\in G_L. \tag{4.4}$$

The distinct double cosets of the form (4.4) partition the group G_L and give us a double coset decomposition of G_L with respect to Σ. (See [9, p. 119] for more on double cosets.)

In Theorem 4.1 below we show that the group G_L is isomorphic to the Coxeter group $W(D_5)$, which is of order 1920. In Theorem 4.4 we show that the subgroup Σ is the largest permutation subgroup of G_L and obtain a double coset decomposition of G_L with respect to Σ. We list a representative for each of the six double cosets obtained and give the six invariance relations induced by those representatives (see (4.6)–(4.11)). The six invariance relations (4.6)–(4.11) listed are all the "different" types of invariance relations in the sense that every other invariance relation can be obtained by permuting the first four entries and permuting the last two entries on the right-hand side of a listed invariance relation (which corresponds to permuting the rows of the accompanying matrix), and by permuting a,b,c,d and permuting f,g on the right-hand side of a listed invariance relation (which corresponds to permuting the columns of the accompanying matrix).

Theorem 4.1. *The group G_L is isomorphic to the Coxeter group $W(D_5)$, which is of order* 1920.

Proof. The Dynkin diagram of the Coxeter group $W(D_n)$ is given by the graph with vertices labeled $1',1,2,\dots,n-1$, where $i,j\in\{1,2,\dots,n-1\}$ are connected by an edge if and only if $|i-j|=1$, and $1'$ is connected to 2 only. The presentation of $W(D_n)$ is given by

$$W(D_n) = \langle s_{1'}, s_1, s_2,\dots,s_{n-1}\colon (s_is_j)^{m_{ij}} = 1\rangle,$$

where $m_{ii}=1$ for all i; and for i and j distinct, $m_{ij}=3$ if i and j are connected by an edge, and $m_{ij}=2$ otherwise. It is well-known that the order of $W(D_n)$ is $2^{n-1}n!$ (see [16, Section 2.11]).

Consider the generators of G_L given by

$$a_1=(34),\qquad a_2=(23),\qquad a_3=(34)A,\qquad a_4=(67),\qquad a_{1'}=(12). \tag{4.5}$$

A direct computation shows that

$$(a_ia_j)^{m_{ij}}=1,\quad \text{for all } i,j\in\{1,2,3,4,1'\}.$$

Therefore if we define $\varphi(s_i)=a_i$ for every $i\in\{1,2,3,4,1'\}$, φ extends (uniquely) to a surjective homomorphism from $W(D_5)$ onto G_L. Since $W(D_5)$ is a finite group, we need to show that G_L and $W(D_5)$ have the same order to complete the proof. To that end, it is enough to just show that $|G_L|>960=\frac{|W(D_5)|}{2}$. An estimate on the order of G_L is obtained by computing the sizes of the distinct double cosets $\Sigma A\Sigma$ and $\Sigma[(123)(67)A]^2\Sigma$ of Σ in G_L, where Σ is as given in (4.3). Indeed, by permuting columns that are different as multisets and then permuting the rows of the resulting matrices in every possible way, we see that the number of matrices that belong to each of those double cosets is $12\cdot 48$, which shows that $|G_L|>960$ and completes the proof. □

Remark 4.2. The result in Theorem 4.1 is apparent from [33, Section 3] with the difference that no groups are mentioned in the paper [33].

Remark 4.3. Let $x_0,x_1,x_2,x_3,x_4,x_5\in\mathbb{C}$. If we reparameterize the L function and define a new function $\tilde{L}(x_0;x_1,x_2,x_3,x_4,x_5)$ by

$$\tilde{L}(x_0;x_1,x_2,x_3,x_4,x_5) = L\begin{bmatrix}\tilde{A},\tilde{B},\tilde{C},\tilde{D};\\ \tilde{E};\tilde{F},\tilde{G}\end{bmatrix},$$

where

$$\tilde{A}=\frac{1}{2}-x_0+x_1+x_2-x_3+x_4+x_5,\qquad \tilde{B}=\frac{1}{2}-x_0+x_1+x_2+x_3-x_4+x_5,$$
$$\tilde{C}=\frac{1}{2}-x_0+x_1+x_2+x_3+x_4-x_5,\qquad \tilde{D}=\frac{1}{2}-x_0+x_1+x_2-x_3-x_4-x_5,$$
$$\tilde{E}=1+2x_1+2x_2,\qquad \tilde{F}=1-2x_0+2x_1,\qquad \tilde{G}=1-2x_0+2x_2,$$

then the action of the invariance group $W(D_5)$ translates to the equivalence of the 1920 $\tilde{L}$ functions of the form $\tilde{L}(x_0;\pm x_{i_1},\pm x_{i_2},\pm x_{i_3},\pm x_{i_4},\pm x_{i_5})$, where (i_1,i_2,i_3,i_4,i_5) is a permutation of $(1,2,3,4,5)$ and the number of negative signs is even. The above reparameterization of the L function is analogous to the reparameterization given in (3.1) in [33].

As stated before Theorem 4.1, we are interested in the complete double coset decomposition of G_L with respect to Σ since this will classify all the invariance relations for the function $L(a,b,c,d;e;f,g)$ in a convenient way. We use the same technique as in the proof of Theorem 4.1 given by permuting columns that are different as multisets and then permuting the rows of the resulting matrices in every possible way. We find that there are six double cosets of Σ in G_L. Representative matrices for the double cosets are $I_7, A, [(123)(67)A]^2, [(123)(67)A]^3, [(123)A]^3, [(123)(67)A]^4$. The corresponding double coset sizes are $1\cdot 48,\ 12\cdot 48,\ 12\cdot 48, 12\cdot 48,\ 2\cdot 48,\ 1\cdot 48$. Furthermore, the representative matrices are all seen to have different entries, so that Σ must indeed be the largest permutation subgroup of G_L. Each representative matrix gives rise to an invariance relation. Theorem 4.4 summarizes the result.

Theorem 4.4. *Let Σ be as defined in* (4.3)*. Then Σ consists of all the permutation matrices in G_L. There are six double cosets in the double coset decomposition of G_L with respect to Σ. Representative matrices for the double cosets are I_7, A, $[(123)(67)A]^2$, $[(123)(67)A]^3$, $[(123)A]^3$, $[(123)(67)A]^4$ and the corresponding double coset sizes are* $1\cdot 48$, $12\cdot 48$, $12\cdot 48$, $12\cdot 48$, $2\cdot 48$, $1\cdot 48$. *The corresponding invariances of the L function are given by*

$$L\begin{bmatrix} a,b,c,d; \\ e;f,g \end{bmatrix} = L\begin{bmatrix} a,b,c,d; \\ e;f,g \end{bmatrix}, \tag{4.6}$$

$$L\begin{bmatrix} a,b,c,d; \\ e;f,g \end{bmatrix} = L\begin{bmatrix} a,b,g-c,g-d; \\ 1+a+b-f;1+a+b-e,g \end{bmatrix}, \tag{4.7}$$

$$L\begin{bmatrix} a,b,c,d; \\ e;f,g \end{bmatrix} = L\begin{bmatrix} 1+a-e,g-c,a,f-c; \\ 1+a-c;1+a+b-e,1+a+d-e \end{bmatrix}, \tag{4.8}$$

$$L\begin{bmatrix} a,b,c,d; \\ e;f,g \end{bmatrix} = L\begin{bmatrix} 1+d-e,1+a-e,g-c,g-b; \\ 1+g-b-c;1+a+d-e,1+g-e \end{bmatrix}, \tag{4.9}$$

$$L\begin{bmatrix} a,b,c,d; \\ e;f,g \end{bmatrix} = L\begin{bmatrix} g-a,g-b,g-c,g-d; \\ 1+g-f;1+g-e,g \end{bmatrix}, \tag{4.10}$$

$$L\begin{bmatrix} a,b,c,d; \\ e;f,g \end{bmatrix} = L\begin{bmatrix} 1+c-e,1+d-e,1+a-e,1+b-e; \\ 2-e;1+g-e,1+f-e \end{bmatrix}. \tag{4.11}$$

The group Σ in Theorem 4.4 consists of the trivial invariances of the L function given by permutations of a, b, c, d and interchanging f, g. Another trivial invariance of the L function may be considered

$$L\begin{bmatrix} a,b,c,d; \\ e;f,g \end{bmatrix} = L\begin{bmatrix} 1+a-e,1+b-e,1+c-e,1+d-e; \\ 2-e;1+f-e,1+g-e \end{bmatrix}, \tag{4.12}$$

which corresponds to the interchanging of the two ${}_4F_3(1)$ series in the definition of the L function (this invariance is from the same double coset as (4.11) above). Let A' be the matrix that corresponds to the invariance (4.12). Define the group Σ' by

$$\Sigma' = \langle (12),(23),(34),(67),A' \rangle, \tag{4.13}$$

i.e. Σ' is generated by the group Σ and the matrix A'. It is straightforward to see that $\Sigma' \cong S_4\times S_2\times S_2$ and so $|\Sigma'| = 96$. The following theorem, which is readily verified if we use the result in Theorem 4.4, describes the double cosets of Σ' in the invariance group G_L:

Theorem 4.5. *Let Σ' be as defined in* (4.13)*. There are four double cosets in the double coset decomposition of G_L with respect to Σ'. Representative matrices for the double cosets are I_7, A, $[(123)(67)A]^2$, $[(123)A]^3$ and the corresponding double coset sizes are* $1\cdot 96$, $12\cdot 96$, $6\cdot 96$, $1\cdot 96$. *The corresponding invariances of the L function are given by the relations* (4.6), (4.7), (4.8), (4.10) *from Theorem* 4.4.

Theorem 4.5 implies that every invariance of the L function can be obtained by starting with $L(a,b,c,d;e;f,g)$ and applying the following sequence of operations to the function (where some or all of steps (i), (ii), (iv), and (v) may be skipped for some of the invariances):

(i) Interchange the two ${}_4F_3(1)$ series appearing in the L function.
(ii) Permute the first four entries and permute the last two entries.
(iii) Apply one of the relations (4.6), (4.7), (4.8), or (4.10).
(iv) Permute the first four entries and permute the last two entries.
(v) Interchange the two ${}_4F_3(1)$ series appearing in the L function.

Remark 4.6. Different representations of the L function lead to different groups of trivial symmetries. If we consider the function $\psi[a;b,c,d,e,f]$ defined as a very-well-poised ${}_7F_6(1)$ series in [33, Eq. (2.1)], then $\psi[a;b,c,d,e,f]$ has trivial symmetries given by permutations of the parameters b, c, d, e, f, thus having a trivial symmetry group isomorphic to S_5. A double coset decomposition of the invariance group $W(D_5)$ with respect to the trivial symmetry subgroup S_5 is implied in [33, Eq. (2.2)], and we can deduce that there are three double cosets of sizes $1 \cdot 120$, $10 \cdot 120$, $5 \cdot 120$. The integral representation of a very-well-poised ${}_7F_6(1)$ series given in [11, Eq. (6.3.11)] also implies a trivial symmetry group isomorphic to S_5 given by permutations of the parameters A, B, C, D, F.

5. Applications of the fundamental two-term relation

In this section we prove some consequences of the fundamental two-term relation given in Proposition 3.2. As a first step, we write the two L functions in (3.3) in terms of their definitions as linear combinations of two ${}_4F_3(1)$ series. We obtain

$$\frac{{}_4F_3\left[\begin{smallmatrix} a,b,c,d; \\ e,f,g; \end{smallmatrix}\ 1\right]}{\sin\pi e\,\Gamma(e)\Gamma(f)\Gamma(g)\Gamma(1+a-e)\Gamma(1+b-e)\Gamma(1+c-e)\Gamma(1+d-e)}$$
$$-\frac{{}_4F_3\left[\begin{smallmatrix} 1+a-e,1+b-e,1+c-e,1+d-e; \\ 1+f-e,1+g-e,2-e; \end{smallmatrix}\ 1\right]}{\sin\pi e\,\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(1+f-e)\Gamma(1+g-e)\Gamma(2-e)}$$
$$=\frac{{}_4F_3\left[\begin{smallmatrix} a,b,g-c,g-d; \\ 1+a+b-f,1+a+b-e,g; \end{smallmatrix}\ 1\right]}{\left[\begin{smallmatrix} \sin\pi(1+a+b-f)\Gamma(1+a+b-f)\Gamma(1+a+b-e)\Gamma(g) \\ \times\Gamma(f-b)\Gamma(f-a)\Gamma(1+d-e)\Gamma(1+c-e) \end{smallmatrix}\right]}$$
$$-\frac{{}_4F_3\left[\begin{smallmatrix} f-b,f-a,1+d-e,1+c-e; \\ 1+f-e,f+g-a-b,1+f-a-b \end{smallmatrix}\ 1\right]}{\left[\begin{smallmatrix} \sin\pi(1+a+b-f)\Gamma(a)\Gamma(b)\Gamma(g-c)\Gamma(g-d) \\ \times\Gamma(1+f-e)\Gamma(f+g-a-b)\Gamma(1+f-a-b) \end{smallmatrix}\right]}. \tag{5.1}$$

We fix $b, c, d, f, g \in \mathbb{C}$ in such a way that

$$\operatorname{Re}(f+g-b-c-d) > 0, \qquad \operatorname{Re}(f-b) > 0. \tag{5.2}$$

Let $a \in \mathbb{C}$ and let $e = 1+a+b+c+d-f-g$ depend on a. In Eq. (5.1) we let $|a| \to \infty$. Using Stirling's formula (2.6) and the conditions (5.2), we obtain

$$\frac{{}_3F_2\left[\begin{smallmatrix} b,c,d; \\ f,g; \end{smallmatrix}\ 1\right]}{\Gamma(f)\Gamma(g)\Gamma(f+g-b-c-d)} = \frac{{}_3F_2\left[\begin{smallmatrix} b,g-c,g-d; \\ f+g-c-d,g; \end{smallmatrix}\ 1\right]}{\Gamma(f+g-c-d)\Gamma(g)\Gamma(f-b)}. \tag{5.3}$$

We note that the conditions (5.2) are needed for the absolute convergence of the two ${}_3F_2(1)$ series in (5.3). Applying (5.3) twice yields Thomae's identity

$$\frac{{}_3F_2\left[\begin{smallmatrix} b,c,d; \\ f,g; \end{smallmatrix}\ 1\right]}{\Gamma(f)\Gamma(g)\Gamma(f+g-b-c-d)} = \frac{{}_3F_2\left[\begin{smallmatrix} f-b,g-b,f+g-b-c-d; \\ f+g-b-d,f+g-b-c; \end{smallmatrix}\ 1\right]}{\Gamma(b)\Gamma(f+g-b-d)\Gamma(f+g-b-c)}. \tag{5.4}$$

In fact, applying (5.4) twice gives (5.3), so that (5.3) and (5.4) are equivalent.

Next in Eq. (5.1) we let $a \to -n$, where n is a nonnegative integer. Using the fact that $\lim_{a\to -n}\frac{1}{\Gamma(a)} = 0$ and then formula (2.8) to simplify the result, we obtain Bailey's identity

$${}_4F_3\left[\begin{matrix} -n,b,c,d; \\ e,f,g; \end{matrix}\quad 1\right] = \frac{(e-b)_n(f-b)_n}{(e)_n(f)_n}\,{}_4F_3\left[\begin{matrix} -n,b,g-c,g-d; \\ 1-n+b-f,1-n+b-e,g; \end{matrix}\quad 1\right], \tag{5.5}$$

which holds provided that $e+f+g-b-c-d+n = 1$.

Thomae's and Bailey's identities have been shown in [10] in a similar way to be limiting cases of a fundamental two-term relation satisfied by the function $K(a; b, c, d; e, f, g)$.

As a final application, in the fundamental two-term relation (3.3) we let $d = g$. We express the left-hand side as a Barnes integral according to Proposition 3.1, and we write the right-hand side in terms of two ${}_4F_3(1)$ series according to the definition of the L function. The condition $d = g$ causes one of the terms on the right-hand side to go to zero and the ${}_4F_3(1)$ series in the other term to be trivially equal to one. If we simplify the result further using (2.8), we obtain

$$\frac{1}{2\pi i}\int_t \frac{\Gamma(a+t)\Gamma(b+t)\Gamma(c+t)\Gamma(1-e-t)\Gamma(-t)}{\Gamma(f+t)}\,dt$$
$$=\frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(1+a-e)\Gamma(1+b-e)\Gamma(1+c-e)}{\Gamma(f-a)\Gamma(f-b)\Gamma(f-c)}, \tag{5.6}$$

which holds provided that $e+f-a-b-c = 1$. Eq. (5.6) is precisely the statement of Barnes' second lemma.

6. Group-theoretic structure of the three-term relations

Let

$$M_L = \langle (12), (23), (34), (56), (67), A \rangle \leqslant GL(7, \mathbb{C}). \tag{6.1}$$

The group M_L is generated by the invariance group G_L and the transposition matrix (56). We will show that the right cosets of G_L in M_L govern the three-term relations for the function $L(a, b, c, d; e; f, g)$. In particular, in the next section we will show that for every $\sigma_1, \sigma_2, \sigma_3 \in M_L$ such that σ_1, σ_2, σ_3 are in different right cosets of G_L in M_L, there exists a three-term relation involving the functions $L(\sigma_1\vec{x})$, $L(\sigma_2\vec{x})$, and $L(\sigma_3\vec{x})$.

In this section we lay the foundation of the group-theoretic structure of the three-term relations satisfied by the L function by studying the right coset space $G_L \setminus M_L$ and then introducing the notion of L-coherence (see Definition 6.5), which will lead us to the classification of the three-term relations through Proposition 6.6.

Theorem 6.1. *The group M_L is isomorphic to the Coxeter group $W(D_6)$, which has* 23 040 *elements.*

Proof. The proof follows the same lines as the proof of Theorem 4.1. We let a_1, a_2, a_3, $a_4, a_{1'}$ be as in (4.5), and we also let $a_5 = (56)$. We obtain a surjective homomorphism from $W(D_6)$ onto M_L, which homomorphism is then shown to be an isomorphism by estimating the order of M_L using the counting technique from the proof of Theorem 4.1. $\square$

There are $23\,040/1920 = 12$ right cosets of G_L in M_L. The group M_L acts by right multiplication on those right cosets through

$$(G_L\mu) \cdot \nu = G_L(\mu\nu), \quad \text{for } \mu, \nu \in M_L.$$

It is well-known that the Coxeter group $W(D_6)$ has a center consisting of two elements (see [16, pp. 82 and 132]). We let w_0 denote the unique nonidentity element in the center of M_L. The element w_0 is called the central involution. The following proposition follows from standard facts about the Coxeter group $W(D_n)$ (see [16, Section 2.10]):

Proposition 6.2. *We can label the* 12 *right cosets of G_L in M_L by*

$$1, 2, \ldots, 6, \bar{1}, \bar{2}, \ldots, \bar{6}$$

in such a way that i and $\bar{i}$ are interchanged under the action of w_0, for every $i \in \{1, 2, \ldots, 6\}$.

Indeed, the central involution w_0 is computed to be

$$w_0 = (12)(34)\big[\big[(1234)(567)\big]^2 A\big]^4,$$

and from here, representatives of the twelve right cosets $6, \ldots, 1, \bar{6}, \ldots, \bar{1}$ are computed to be

$$\begin{aligned}
&\mu_6 = I_7, &&\mu_5 = (56), &&\mu_4 = (57),\\
&\mu_3 = w_2, &&\mu_2 = (56)w_2, &&\mu_1 = (57)w_2,\\
&\mu_{\bar{6}} = w_0, &&\mu_{\bar{5}} = (56)w_0, &&\mu_{\bar{4}} = (57)w_0,\\
&\mu_{\bar{3}} = w_1, &&\mu_{\bar{2}} = (56)w_1, &&\mu_{\bar{1}} = (57)w_1,
\end{aligned}$$

respectively, where

$$\begin{aligned}
&w_1 = (1234)\big[(1234)(567)A\big]^3(1432),\\
&w_2 = w_0 w_1.
\end{aligned}$$

Every matrix $\mu \in M_L$ induces a permutation of the twelve element set $\{1, \ldots, 6, \bar{1}, \ldots, \bar{6}\}$. Let $\Phi : M_L \to S_{12}$ be the induced permutation representation. We note that if $\mu \in M_L$, then the permutation $\Phi(\mu)$ can be uniquely described by specifying its effect on the set $\{1, \ldots, 6\}$, since the elements in $\{\bar{1}, \ldots, \bar{6}\}$ will be permuted based on where $\{1, \ldots, 6\}$ are permuted with the addition or omission of a bar. We give the images under Φ of the generators of M_L in the following proposition:

Proposition 6.3. *The images of the generators a_1, a_2, a_3, a_4, a_5, and $a_{1'}$ of M_L under the permutation representation $\Phi : M_L \to S_{12}$ are given by*

$$\Phi(a_1) = \Phi\big((34)\big) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \end{pmatrix},$$

$$\Phi(a_2) = \Phi\big((23)\big) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 4 & 5 & 6 \end{pmatrix},$$

$$\Phi(a_3) = \Phi\big((34)A\big) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 3 & 5 & 6 \end{pmatrix},$$

$$\Phi(a_4) = \Phi\big((67)\big) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 4 & 6 \end{pmatrix},$$

$$\Phi(a_5) = \Phi\big((56)\big) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 6 & 5 \end{pmatrix},$$

$$\Phi(a_{1'}) = \Phi\big((12)\big) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \bar{2} & \bar{1} & 3 & 4 & 5 & 6 \end{pmatrix}.$$

Given the decomposition of an element $\mu \in M_L$ in terms of the generators a_1, a_2, a_3, a_4, a_5, $a_{1'}$, Proposition 6.3 can be used to find the permutation corresponding to $\Phi(\mu)$.

The next proposition is a restatement of standard facts about the construction of the Coxeter group $W(D_n)$ as a semidirect product (see [16, Section 2.10] for more details).

Proposition 6.4. *The permutation representation $\Phi : M_L \to S_{12}$ is faithful, i.e.* $\ker(\Phi) = \{I_7\}$. *The embedding of M_L into S_{12} consists of all permutations of the form $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ j_1 & j_2 & j_3 & j_4 & j_5 & j_6 \end{smallmatrix}\right)$, where each j_i belongs to the set $\{1, \dots, 6, \bar{1}, \dots, \bar{6}\}$, the j_i's are all distinct if we remove the bars, and an even number, i.e.* 0, 2, 4, *or* 6, *of the j_i's contain a bar.*

Central to the classification of the three-term relations in the next section is the notion of L-coherence defined as follows:

Definition 6.5. If S is a subset of $G_L \setminus M_L$, we say that S is L-coherent if no two elements of S are interchanged by the action of the central involution w_0. If S is not L-coherent, then S is called L-incoherent.

It is clear that a set S is L-coherent if and only if S does not contain *both* elements of the form i and $\bar{i}$, for any $i \in \{1, \dots, 6\}$.

The group M_L acts by right multiplication on triples of cosets of $G_L \setminus M_L$ according to

$$\{i, j, k\} \cdot \mu = \{i \cdot \mu, j \cdot \mu, k \cdot \mu\},$$

for $i, j, k \in G_L \setminus M_L$ and $\mu \in M_L$.

There are $\binom{12}{3} = 220$ triples of right cosets of $G_L \setminus M_L$. The final proposition of this section describes the orbits of the action of M_L on those triples. It will be used in the next section in the classification of the three-term relations satisfied by the L function.

Proposition 6.6. *There are two orbits of the action of M_L on triples of cosets of $G_L \setminus M_L$. One orbit is of length* 160 *and consists of all triples that are L-coherent. The other orbit is of length* 60 *and consists of all triples that are L-incoherent.*

Proof. From Proposition 6.4, it is easily seen that there are two orbits given by the L-coherent and L-incoherent triples. A simple counting argument establishes the lengths of the two orbits. $\square$

7. Three-term relations

For every $i \in G_L \setminus M_L$, we define the function

$$L_i(\vec{x}) = L(\mu\vec{x}),$$

where $\vec{x} = (a, b, c, d, e, f, g)^T \in V$ and μ is any representative of the right coset i. We give below a list of the twelve functions of the form $L_i(\vec{x})$, $i \in G_L \setminus M_L$:

$$L_6(\vec{x}) = L\begin{bmatrix} a, b, c, d; \\ e; f, g \end{bmatrix},$$

$$L_5(\vec{x}) = L\begin{bmatrix} a, b, c, d; \\ f; e, g \end{bmatrix},$$

$$L_4(\vec{x}) = L\begin{bmatrix} a, b, c, d; \\ g; f, e \end{bmatrix},$$

$$L_3(\vec{x}) = L\begin{bmatrix} a, 1+a-e, 1+a-f, 1+a-g; \\ 1+a-b; 1+a-c, 1+a-d \end{bmatrix},$$

$$L_2(\vec{x}) = L\begin{bmatrix} a, 1+a-e, 1+a-f, 1+a-g; \\ 1+a-c; 1+a-b, 1+a-d \end{bmatrix},$$

$$L_1(\vec{x}) = L\begin{bmatrix} a, 1+a-e, 1+a-f, 1+a-g; \\ 1+a-d; 1+a-c, 1+a-b \end{bmatrix},$$

$$L_{\bar{6}}(\vec{x}) = L\begin{bmatrix} 1-a, 1-b, 1-c, 1-d; \\ 2-e; 2-f, 2-g \end{bmatrix},$$

$$L_{\bar{5}}(\vec{x}) = L\begin{bmatrix} 1-a, 1-b, 1-c, 1-d; \\ 2-f; 2-e, 2-g \end{bmatrix},$$

$$L_{\bar{4}}(\vec{x}) = L\begin{bmatrix} 1-a, 1-b, 1-c, 1-d; \\ 2-g; 2-f, 2-e \end{bmatrix},$$

$$L_{\bar{3}}(\vec{x}) = L\begin{bmatrix} 1-a, e-a, f-a, g-a; \\ 1+b-a; 1+c-a, 1+d-a \end{bmatrix},$$

$$L_{\bar{2}}(\vec{x}) = L\begin{bmatrix} 1-a, e-a, f-a, g-a; \\ 1+c-a; 1+b-a, 1+d-a \end{bmatrix},$$

$$L_{\bar{1}}(\vec{x}) = L\begin{bmatrix} 1-a, e-a, f-a, g-a; \\ 1+d-a; 1+c-a, 1+b-a \end{bmatrix}.$$

A three-term relation involving the functions $L_i(\vec{x})$, $L_j(\vec{x})$, and $L_k(\vec{x})$ is said to be of type $\{i, j, k\}$.

Remark 7.1. If we consider the reparameterization of the L function given by the function $\tilde{L}(x_0; x_1, x_2, x_3, x_4, x_5)$ defined in Remark 4.3, then we have that the 23 040 $\tilde{L}$ functions are given by $\tilde{L}(\pm x_{i_0}; \pm x_{i_1}, \pm x_{i_2}, \pm x_{i_3}, \pm x_{i_4}, \pm x_{i_5})$, where $(i_0, i_1, i_2, i_3, i_4, i_5)$ is a permutation of $(0, 1, 2, 3, 4, 5)$ and the number of negative signs is even. Representatives of the twelve cosets of $W(D_5)$ in $W(D_6)$ are given by twelve $\tilde{L}$ functions whose first parameter is $\pm x_i$, $i \in \{0, 1, 2, 3, 4, 5\}$.

Proposition 7.2. *A three-term relation of the L-coherent type $\{6, 5, 4\}$ is given by*

$$\frac{\sin\pi(f-g)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)} L\begin{bmatrix} a, b, c, d; \\ e; f, g \end{bmatrix} + \frac{\sin\pi(g-e)}{\Gamma(f-a)\Gamma(f-b)\Gamma(f-c)\Gamma(f-d)} L\begin{bmatrix} a, b, c, d; \\ f; e, g \end{bmatrix}$$
$$+ \frac{\sin\pi(e-f)}{\Gamma(g-a)\Gamma(g-b)\Gamma(g-c)\Gamma(g-d)} L\begin{bmatrix} a, b, c, d; \\ g; f, e \end{bmatrix} = 0. \tag{7.1}$$

Proof. We have

$$\frac{1}{2\pi i}\int_t \frac{\Gamma(a+t)\Gamma(b+t)\Gamma(c+t)\Gamma(d+t)\Gamma(1-e-t)\Gamma(1-f-t)\Gamma(-t)}{\Gamma(g+t)}$$
$$\times \big[\sin\pi e \sin\pi(1-f-t) - \sin\pi f \sin\pi(1-e-t) - \sin\pi(f-e)\sin\pi(-t)\big]\,dt = 0,$$

since the quantity in square brackets in the above integral is equal to zero, which can be seen by applying elementary trigonometric identities. We break up the integral into three parts and use (2.8) to simplify the result to obtain

$$\frac{1}{2\pi i}\int_t \frac{\sin\pi e\,\Gamma(a+t)\Gamma(b+t)\Gamma(c+t)\Gamma(d+t)\Gamma(1-e-t)\Gamma(-t)}{\Gamma(f+t)\Gamma(g+t)}\,dt$$

$$-\frac{1}{2\pi i}\int_t \frac{\sin\pi f\,\Gamma(a+t)\Gamma(b+t)\Gamma(c+t)\Gamma(d+t)\Gamma(1-f-t)\Gamma(-t)}{\Gamma(e+t)\Gamma(g+t)}\,dt$$

$$-\frac{1}{2\pi i}\int_t \frac{\begin{bmatrix} \sin\pi(f-e)\Gamma(a+t)\Gamma(b+t)\Gamma(c+t) \\ \times\Gamma(d+t)\Gamma(1-e-t)\Gamma(1-f-t) \end{bmatrix}}{\Gamma(1+t)\Gamma(g+t)}\,dt = 0.$$

After the substitution $t \mapsto t+1-e$ in the third integral, we express each of the three integrals as an L function according to (3.1). The end result is a three-term relation of type $\{6, 5, \bar{4}\}$ which can be written as

$$\frac{\sin\pi e}{\Gamma(1+a-f)\Gamma(1+b-f)\Gamma(1+c-f)\Gamma(1+d-f)} L\begin{bmatrix} a, b, c, d; \\ e; f, g \end{bmatrix} + \frac{\sin\pi(-f)}{\Gamma(1+a-e)\Gamma(1+b-e)\Gamma(1+c-e)\Gamma(1+d-e)} L\begin{bmatrix} a, b, c, d; \\ f; e, g \end{bmatrix} + \frac{\sin\pi(e-f)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)} L\begin{bmatrix} 1-a, 1-b, 1-c, 1-d; \\ 2-g; 2-f, 2-e \end{bmatrix} = 0. \tag{7.2}$$

If we let

$$\mu = (14)(23)\big[(123)A\big]^3 \in M_L,$$

we have

$$\{6, 5, \bar{4}\} \cdot \mu = \{6, 5, 4\}.$$

Applying the change of variable $\vec{x} \mapsto \mu\vec{x}$ to all terms and coefficients in the relation (7.2) yields the result. □

Remark 7.3. Analogs of Proposition 7.2, when we write the L function as a very-well-poised ${}_7F_6(1)$ series, are given in [33, Eqs. (3.3) and (3.4)].

If $\vec{x} = (a, b, c, d, e, f, g)^T \in V$, we define $\gamma_1(\vec{x})$, $\gamma_2(\vec{x})$, and $\gamma_3(\vec{x})$ to be the respective coefficients in front of the three L functions in (7.1). This way the three-term relation (7.1) can be written as

$$\gamma_1(\vec{x})L_6(\vec{x}) + \gamma_2(\vec{x})L_5(\vec{x}) + \gamma_3(\vec{x})L_4(\vec{x}) = 0, \tag{7.3}$$

or, equivalently, as

$$\sum_{i=0}^{2} \gamma_1\big((576)^i\vec{x}\big)L\big((576)^i\vec{x}\big) = 0. \tag{7.4}$$

Let $\{i, j, k\}$ be any L-coherent triple. Since, by Proposition 6.6, $\{i, j, k\}$ is in the same orbit as $\{6, 5, 4\}$, there exists $\mu \in M_L$ such that

$$\{6, 5, 4\} \cdot \mu = \{i, j, k\}.$$

Then a three-term relation of type $\{i, j, k\}$ is given by

$$\gamma_1(\mu\vec{x})L_i(\vec{x}) + \gamma_2(\mu\vec{x})L_j(\vec{x}) + \gamma_3(\mu\vec{x})L_k(\vec{x}) = 0, \tag{7.5}$$

or, equivalently, by

$$\sum_{i=0}^{2} \gamma_1\big((576)^i\mu\vec{x}\big)L\big((576)^i\mu\vec{x}\big) = 0. \tag{7.6}$$

Proposition 7.4. *A three-term relation of the L-incoherent type $\{6, 5, \bar{6}\}$ is given by*

$$\frac{\sin\pi(f-g)\Gamma(g-a)\Gamma(g-b)\Gamma(g-c)\Gamma(g-d)}{\sin\pi(e-f)\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)} L\begin{bmatrix} a, b, c, d; \\ e; f, g \end{bmatrix} + \left(\frac{\sin\pi(g-e)\Gamma(g-a)\Gamma(g-b)\Gamma(g-c)\Gamma(g-d)}{\sin\pi(e-f)\Gamma(f-a)\Gamma(f-b)\Gamma(f-c)\Gamma(f-d)} + \frac{\sin\pi f\,\Gamma(1+a-f)\Gamma(1+b-f)\Gamma(1+c-f)\Gamma(1+d-f)}{\sin\pi g\,\Gamma(1+a-g)\Gamma(1+b-g)\Gamma(1+c-g)\Gamma(1+d-g)}\right) L\begin{bmatrix} a, b, c, d; \\ f; e, g \end{bmatrix} + \frac{\sin\pi(f-g)\Gamma(1+a-f)\Gamma(1+b-f)\Gamma(1+c-f)\Gamma(1+d-f)}{\sin\pi g\,\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)} \times L\begin{bmatrix} 1-a, 1-b, 1-c, 1-d; \\ 2-e; 2-f, 2-g \end{bmatrix} = 0. \tag{7.7}$$

Proof. Let

$$\sigma=(14)(23)\big[(123)A\big]^3(576)\in M_L.$$

We have

$$\{6,5,4\}\cdot\sigma=\{5,4,\bar{6}\}.$$

This and (7.3) lead to a three-term relation of type $\{5,4,\bar{6}\}$ given by

$$\gamma_1(\sigma\vec{x})L_5(\vec{x})+\gamma_2(\sigma\vec{x})L_4(\vec{x})+\gamma_3(\sigma\vec{x})L_{\bar{6}}(\vec{x})=0.$$

We combine the above three-term relation of type $\{5,4,\bar{6}\}$ with the three-term relation of type $\{6,5,4\}$ given in (7.3) and cancel the terms involving the function $L_4(\vec{x})$ to obtain (7.7). □

Remark 7.5. An analog of Proposition 7.4, when the L function is written as a very-well-poised ${}_7F_6(1)$ series, is given in [33, Eq. (3.5)].

If $\vec{x}=(a,b,c,d,e,f,g)^T\in V$, we define $\beta_1(\vec{x}),\beta_2(\vec{x})$, and $\beta_3(\vec{x})$ to be the respective coefficients in front of the three L functions in (7.7). This way the three-term relation (7.7) can be written as

$$\beta_1(\vec{x})L_6(\vec{x})+\beta_2(\vec{x})L_5(\vec{x})+\beta_3(\vec{x})L_{\bar{6}}(\vec{x})=0. \tag{7.8}$$

Let $\sigma=(14)(23)[(123)A]^3(576)$ be as given in the proof of Proposition 7.4 and let $\nu=(576)^2\sigma$. The transformation ν has the following effect on a vector $(a,b,c,d,e,f,g)^T\in V$:

$$\begin{pmatrix} a,b,c,d,\\ e,f,g\end{pmatrix}^T\to\begin{pmatrix} e-d,e-c,e-b,e-a,\\ e,1+e-g,1+e-f\end{pmatrix}^T.$$

Eq. (7.8) can now be written as

$$\beta_1(\vec{x})L(\vec{x})+\big[\beta_1\big((56)\vec{x}\big)-\beta_1\big((56)\nu\vec{x}\big)\big]L\big((56)\vec{x}\big)-\beta_1(\nu\vec{x})L(\nu\vec{x})=0. \tag{7.9}$$

We also note that $\beta_1(\vec{x})$ can be written in terms of $\gamma_1(\vec{x})$ (appearing in the L-coherent three-term relations) through

$$\beta_1(\vec{x})=\frac{\gamma_1(\vec{x})}{\gamma_1((576)^2\vec{x})}. \tag{7.10}$$

Let $\{i,j,\bar{i}\}$ be any triple that is L-incoherent. By Proposition 6.6, $\{i,j,\bar{i}\}$ is in the same orbit as $\{6,5,\bar{6}\}$ and so there exists $\mu\in M_L$ such that

$$\{6,5,\bar{6}\}\cdot\mu=\{i,j,\bar{i}\}.$$

Then a three-term relation of type $\{i,j,\bar{i}\}$ is given by

$$\beta_1(\mu\vec{x})L_i(\vec{x})+\beta_2(\mu\vec{x})L_j(\vec{x})+\beta_3(\mu\vec{x})L_{\bar{i}}(\vec{x})=0, \tag{7.11}$$

or, equivalently, by

$$\beta_1(\mu\vec{x})L(\mu\vec{x})+\big[\beta_1\big((56)\mu\vec{x}\big)-\beta_1\big((56)\nu\mu\vec{x}\big)\big]L\big((56)\mu\vec{x}\big)-\beta_1(\nu\mu\vec{x})L(\nu\mu\vec{x})=0. \tag{7.12}$$

To summarize our results, we have shown that for every $\sigma_1,\sigma_2,\sigma_3\in M_L$ such that σ_1, σ_2, σ_3 are in different right cosets of G_L in M_L, there exists a three-term relation involving the functions $L(\sigma_1\vec{x}),L(\sigma_2\vec{x})$, and $L(\sigma_3\vec{x})$. We have a total of 220 three-term relations. The three-term relations fall into two families based on L-coherence. The two fundamental three-term relations for L-coherent and for L-incoherent triples are given in (7.3) and (7.8) respectively. Any other three-term relation can be obtained from one of those two through a change of variable of the form $\vec{x}\mapsto\mu\vec{x}$ applied to all terms and coefficients. The result is a three-term relation of the form (7.5) or (7.11). The appropriate matrix $\mu\in M_L$ can be found through Proposition 6.3, which describes the actions of the generators a_1, a_2, a_3, a_4, a_5, $a_{1'}$ of the group M_L.

By examining the three-term relations (7.6) and (7.12), combined with Eq. (7.10), we notice that in a three-term relation involving the functions $L(\sigma_1\vec{x}),L(\sigma_2\vec{x})$, and $L(\sigma_3\vec{x})$, the coefficient in front of each $L(\sigma_i\vec{x})$, $i\in\{1,2,3\}$, is a rational combination of transformations of the function

$$\gamma_1(\vec{x})=\frac{\sin\pi(f-g)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)\Gamma(e-d)}. \tag{7.13}$$

The number of monomials in the coefficient of each $L(\sigma_i\vec{x})$ depends on the other two $L(\sigma_j\vec{x})$ and $L(\sigma_k\vec{x})$ functions in the three-term relation as follows: the number of monomials is one if the cosets $G_L\sigma_j$ and $G_L\sigma_k$ form an L-coherent set, and the number of monomials is two if the cosets $G_L\sigma_j$ and $G_L\sigma_k$ form an L-incoherent set.

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