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ON NORMAL TORSION-FREE FINITE INDEX SUBGROUPS OF POLYHEDRAL GROUPS

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Suppose that P is a convex polyhedron of finite volume in the hyperbolic 3-space such that each dihedral angle is an integer (>1) submultiple of π. We show that the abelianization of any normal torsion-free finite index subgroup of the polyhedral group associated to P is not isomorphic to $\mathbb{Z}$, the group of integers.

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hyperbolic 3-space   polyhedral group
normal torsion-free finite index   group action

1. Introduction

A polyhedron in the hyperbolic 3-space, $\mathbb{H}^3=\{(x+iy, z): x, y, z\in\mathbb{R}, z>0\}$, is in this paper assumed to be a convex subset of $\mathbb{H}^3$ of finite volume with finitely many faces such that each dihedral angle is π/n for some positive integer n (>1). Polyhedra of this type have been classified in [1] and [12].

The polyhedral group associated to a polyhedron is defined to be the subgroup of the orientation preserving isometries in the reflection group generated by the reflections of $\mathbb{H}^3$ in the planes containing the faces of the polyhedron. A polyhedral group admits a presentation which can be easily written down from a diagram of the polyhedron [5]. Generators are the rotations about the edges of the polyhedron, and, in addition to the standard defining relation that a proper power of each rotation is the trivial element, each vertex induces a relation which says that the product of the rotations about the edges sharing the vertex is the identity of the group.

A polyhedral group is a discrete subgroup of PSL(2, $\mathbb{C}$), the full group of orientation preserving isometries of $\mathbb{H}^3$. A torsion-free finite index (t.f.i.) subgroup of a polyhedral group is discrete in PSL(2, $\mathbb{C}$) and it gives a hyperbolic 3-manifold of finite volume as its orbit space of the action on $\mathbb{H}^3$. The subgroup is isomorphic to the fundamental group of the 3-manifold. The subgroups of polyhedral groups have been studied by many people [2, 4, 7, 10, 13], and the first few examples of hyperbolic 3-manifolds were discovered by studying the subgroups. If the polyhedron has an ideal vertex, then the hyperbolic 3-manifold corresponding to a t.f.i. subgroup of the polyhedral group is a link complement in a closed 3-manifold. The complements of many links in the 3-sphere arise this way, and among knots the figure-eight knot is such an example.

The figure-eight knot group is an index 12 subgroup of a tetrahedral group [10], [12]. This seems to be the only knot group known to be contained in a polyhedral group. Other known t.f.i. subgroups of polyhedral groups have abelian rank greater than one, or if the rank is one, the abelianization contains a torsion part. It might be conjectured that the figure-eight knot group is the only knot group contained in a polyhedral group. The main theorem of this paper shows that a weaker conjecture is true: No knot group is a normal finite index subgroup of a polyhedral group. Thus, the figure-eight knot group is not a normal subgroup of the tetrahedral group.

Theorem. *No polyhedral group contains a normal t.f.i. subgroup whose abelianization is isomorphic to* $\mathbb{Z}$.

Corollary. *No knot group is contained in a polyhedral group as a normal finite index subgroup.*

A knot group is torsion-free and the abelianization is isomorphic to $\mathbb{Z}$. The Corollary follows from the Theorem.

It is shown in [9] that if a polyhedral group contains a normal t.f.i. subgroup of rank one (the rank of the abelianization), then the polyhedron is a special kind described in the following definition.

Definition. A polyhedron P (or the associated polyhedral group G) is *exceptional* if

(i) P has exactly one ideal vertex and it is of type $(2, 2, 2, 2)$ and

(ii) G has an index 2 subgroup, which does not contain any one of the four standard generators (rotations of π about the edges sharing the vertex) of the stabilizer of the ideal vertex.

Remark. The Theorem is true for the 3-dimensional Euclidean polyhedral groups, because the arguments in [9] work for the Euclidean polyhedral groups, and no Euclidean groups are exceptional. But the Theorem is not as interesting as in the hyperbolic case since there are only a few Euclidean polyhedra [6].

There are four types of ideal vertices: $(2, 2, 2, 2)$, $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$ [12], where π divided by an integer in the description is the dihedral angle between two adjacent faces (faces meeting at an edge) sharing the vertex. There are four types of regular vertices: Type 1: $(2, 2, n)$, $n > 1$; type 2: $(2, 3, 3)$; type 3: $(2, 3, 4)$; type 4: $(2, 3, 5)$.

To prove the Theorem, we need to show that an exceptional polyhedral group cannot contain a normal t.f.i. subgroup whose abelianization is isomorphic to $\mathbb{Z}$. We do this by proving two propositions which contain the Theorem.

Proposition 1. *If an exceptional polyhedral group contains a normal t.f.i. subgroup of rank* 1, *then all the regular vertices of the polyhedron are of type* 1.

Remark. If there is no exceptional polyhedron whose regular vertices are of type 1, then Proposition 1 contains the Theorem. But as the example in Fig. 1 shows, there are many exceptional polyhedra with only type 1 regular vertices. In describing a polyhedron or a union of its copies by a diagram, we use the following convention in this paper. Let P be a polyhedron or a union of its copies. In all cases we consider in this paper, the pair $(P, \partial P)$ is homeomorphic to $(\mathbb{H}^3, \mathbb{R}^2)$, where $\mathbb{R}^2 = \{(x+iy, 0): x, y \in \mathbb{R}\} = \partial\mathbb{H}^3$. We will call $\mathbb{R}^2$ the complex plane or the x-y-plane. To describe P, we draw the complex structure (faces, edges, vertices) of ∂P on the x-y-plane after ∂P is identified with the plane by a homeomorphism of the pair $(P, \partial P)$ in a convenient way. The dihedral angle of an edge can be computed by dividing π by the integer assigned to the edge in the diagram. Note that a polyhedron P is uniquely determined by the complex structure of ∂P and the dihedral angles of P [1]. We make statements about P, while viewing P as it sits canonically (up to isometries) in the hyperbolic 3-space or as the whole of $\mathbb{H}^3$ after the pair $(P, \partial P)$ is identified with $(\mathbb{H}^3, \mathbb{R}^2)$. We will not identify the viewpoint in each case and, actually, we will move back and forth between these two viewpoints.

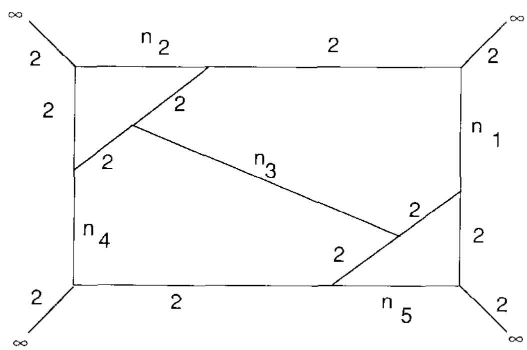

Fig. 1.

The polyhedron in Fig. 1 has 8 faces, 17 edges and 11 vertices. The integers n_1, n_2, n_3, n_4 and n_5 may be any positive integers greater than 1, where n_3 is greater than 2. The existence of the polyhedron in the hyperbolic 3-space can be easily verified by checking the necessary and sufficient conditions given in [1]. Let K be the subgroup of the polyhedral group generated by the rotations of angle $2\pi/n_i$ about the edge whose dihedral angle is π/n_i, $1 \leq i \leq 5$. Then K is an index 2 subgroup of the polyhedral group and it does not contain the rotation of angle π about any one of the four edges containing the ideal vertex ∞. Therefore, the polyhedron is an exceptional polyhedron with regular vertices only of type 1.

Proposition 2. *Suppose that all the regular vertices of an exceptional polyhedron are of type* 1. *If the polyhedral group contains a normal t.f.i. subgroup of rank* 1, *then the abelianization of the subgroup contains torsion elements.*

The above two propositions are proved by using heavily some of the basic constructions and theorems from transformation group theory. To explain the main idea, let N be a normal t.f.i. subgroup of an exceptional polyhedral group G, such that the rank of the abelianization of N is 1. Then there exists a special index 2 subgroup K of G containing N. The quotient group K/N acts naturally on the orbit space $\mathbb{H}^3/N$ and the orbit space of this action is $\mathbb{H}^3/K$. By viewing $\mathbb{H}^3/K$ as the space obtained from a fundamental domain of K by identifying the faces, we prove that $\mathbb{H}^3/K$ is homeomorphic to the open solid torus, $S^1 \times \mathring{D}^2$ (Lemma 4). On the other hand, $\mathbb{H}^3/N$ is an open 3-manifold, and there exists a compact 3-manifold Y containing $\mathbb{H}^3/N$ such that $\partial Y = Y - \mathbb{H}^3/N \cong S^1 \times S^1$. We observe that the action of K/N on $\mathbb{H}^3/N$ extends to an action of Y. The restriction of the extended action of K/N over ∂Y turns out to be free (Lemma 5), and the orbit space of this action is homeomorphic to $S^1 \times S^1$. It follows that K/N is an abelian group isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_q$ for some positive integers p and q (Lemma 6). This implies that given any regular vertex v of P the stabilizer S_v of v in G is either abelian or has an index 2 abelian subgroup. Type 1 regular vertices are the only ones with this property, thus proving Proposition 1.

The proof of Proposition 2 is done by a contradiction. Suppose that G contains a normal t.f.i. subgroup N whose abelianization is isomorphic to $\mathbb{Z}$. Proposition 1 and the related lemmas hold under the new hypothesis. In Lemma 7, we prove that there exists an edge of P whose dihedral angle is π/n, $n > 2$. The lemma allows us to pick an edge E in P with the properties given in Section 3.4. From the properties, a rotation e about E is an element of K, and e can be regarded as an element of K/N. In Lemma 8, it is proven that the fixed point set of any non-trivial element of K/N acting on $\mathbb{H}^3/N$ is homeomorphic to a circle when it is not empty. By studying the fixed point set of e, we prove that K/N is cyclic (Lemma 10). Using this lemma, we show that there exists an element of K/N such that the fixed point set of the element contains two disjointly embedded circles in $\mathbb{H}^3/N$, thus contradicting Lemma 8. The details of the proof of Proposition 1 and 2 are given in the Sections 2 and 3, respectively. Finally, it may be interesting to find more knot groups contained in polyhedral groups as finite index subgroups or to prove that the figure-eight knot group is the only such group.

2. Proof of Proposition 1

Let P be an exceptional polyhedron and G the associated polyhedral group. Suppose that G contains a normal t.f.i. subgroup N of rank 1. It is shown in the proof of the main Theorem, Lemma 1 and Lemma 2 of [9], that N is contained in an index 2 subgroup K of G, where K does not contain any one of the standard generators of the stabilizer of the ideal vertex of P and the rank of $H_1(\mathbb{H}^3/K;\mathbb{Z})$ is 1. We give a sketch of the proof given in [9] for this assertion to introduce the notation.

G/N can be regarded as acting on $\mathbb{H}^3/N$ simplicially and the orbit space of this action is homeomorphic to $\mathbb{H}^3/G$, where $\mathbb{H}^3/G$ is homeomorphic to the open 3-ball. On the other hand, $H_1(\mathbb{H}^3/N;\mathbb{Z})/\mathrm{Tor}\cong\mathbb{Z}$ (/Tor means modulo the torsion subgroup), since the rank of N is 1. The action of G/N on $\mathbb{H}^3/N$ induces a homomorphism $\psi: G/N\to \mathrm{Aut}(H_1(\mathbb{H}^3/N;\mathbb{Z})/\mathrm{Tor})\cong\mathbb{Z}_2$, and ψ is not a trivial homomorphism by a theorem in Section 3 of [3]. Let $\Gamma: G\to G/N \xrightarrow{\psi} \mathbb{Z}_2$ be the composition of the quotient map with ψ and define $K=\mathrm{Ker}(\Gamma)$. From the construction, K is an index 2 subgroup of G containing N and $H_1(\mathbb{H}^3/K;\mathbb{Z})/\mathrm{Tor}\cong\mathbb{Z}$. If K contains one of the four standard generators of the stabilizer of the ideal vertex of P, then $H_1(\mathbb{H}^3/K;\mathbb{Z})\cong 0$ by the proof of Lemma 2 of [9]. Therefore, K does not contain any one of the standard generators, thus proving the above assertion. □

2.1. Let A, B, C and D be the faces of P containing the ideal vertex ∞ as in Fig. 2. The diagram does not show all of ∂P. (Recall the convention for describing a polyhedron as explained in the Introduction.) For example, the face B may contain many more edges than the 3 edges shown in the diagram.

We denote by d the reflection of $\mathbb{H}^3$ in D and by r the rotation of π about the edge contained in C and D. Note that $d\notin G$ but $r\in G$. It is easy to see that $P\cup d(P)$ is a fundamental domain of G and $Q=P\cup d(P)\cup r(P)\cup rd(P)$ is a fundamental domain of K, since $\{1, r\}$ is a set of right coset representatives of K in G [8]. The second diagram in Fig. 2 describes Q, where the face C of P is contained in the

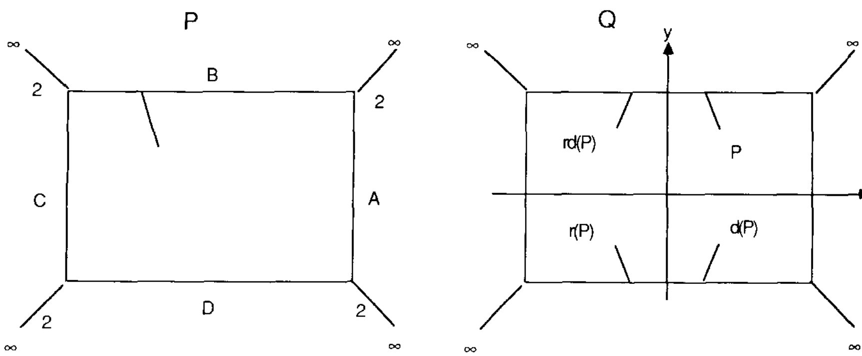

Fig. 2.

y-z-plane and the face D is contained in the x-z-plane. Now $\mathbb{H}^3/K$ is homeomorphic to Q with faces identified by elements of K. We say that a face F of Q is identified to a face F' by K if there exists g in K such that $g(Q) \cap Q$ is a union of faces of Q and $g(F) = F'$. Every face of Q is identified by K to some other face of Q. If U is a union of faces of Q, and if U is identified with another union V of faces by elements of K, then we say that U is identified with V by K.

Given an edge E of P with dihedral angle π/n, let $\sigma(E)$ be the rotation of $2\pi/n$ about E. The definition does not give $\sigma(E)$ uniquely because we could have two distinct elements of G for $\sigma(E)$ depending on the direction of the rotation, with one the inverse of the other. It is necessary to choose the correct direction for some of the arguments of this paper to work. To save space, we will assume that the correct choice has been made whenever it is necessary without mentioning it. We define E to be a negative edge if $\sigma(E) \in K$ and a positive edge otherwise. Define an equivalence relation $\sim$ for the set of positive edges of P; for two positive edges E and E', $E \sim E'$ if there exists a sequence of positive edges $E_1, E_2, \ldots, E_k$ such that $E = E_1$, $E' = E_k$, and E_i and E_{i+1} meet at a regular vertex for $i = 1, 2, \ldots, k-1$.

If E, E' and E'' are edges sharing a regular vertex of P, then $\sigma(E)\sigma(E')\sigma(E'') = 1$ in G [5]. Therefore, either all three edges are negative or exactly two edges are positive since $G/K \cong \mathbb{Z}_2$. Hence an equivalence class of $\sim$ is an open arc or is homeomorphic to a circle, and there are exactly two open arcs in the set of all equivalence classes. An example of the equivalence classes is given in Fig. 3.

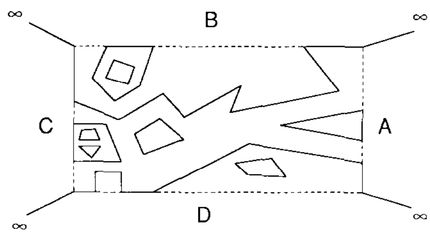

Fig. 3.

We may assume without loss of generality that the two open arcs are horizontal in the x-y-plane as in the figure, where the x-axis is assumed to be horizontal. Let J be the set of equivalence classes of $\sim$, and extend the definition of negative edges, positive edges and $\sim$ to the edges of Q, i.e. an edge E of Q is negative if $\sigma(E) \in K$ etc. Let J' be the set of equivalence classes of $\sim$ on the set of positive edges of Q. The following lemma is the key to the understanding of the identification of faces of Q by K and in turn, $\mathbb{H}^3/K$.

Lemma 1. *Let F and F' be faces of Q meeting at an edge E. Suppose that F is identified with $d(F)(rd(F))$ by an element $g \in K$ such that $g = d(rd)$ when restricted to F. Then*

(a) *if E is a negative edge, then F' is identified with $d(F')(rd(F'))$ by an element $h \in K$ such that $h = d(rd)$ when restricted to F', and*

(b) *if* E *is a positive edge, then* F' *is identified with* $rd(F')(d(F'))$ *by an element* $h' \in K$ *such that* $h' = rd(d)$ *when restricted to* F'.

Proof. Suppose that F is identified with $d(F)$ by $g \in K$ and $g = d$ when restricted to F. Suppose that E is a negative edge. Let $h = \sigma(d(E))g$. Since K is a normal subgroup of G, $\sigma(d(E)) = g\sigma(E)g^{-1} \in K$ and $h \in K$. Now h identifies F' with $d(F')$ and $h = d$ when restricted to F'. This claim is easily verified by studying the sectional view of Q near E and its images as in Fig. 4, where Q is cut by a plane perpendicular to the edge E. If we denote the dihedral angle along E by θ, then the rotation, $\sigma(d(E))$, of 2θ about $d(E)$ identifies $g(F')$ with $d(F')$, thus h identifies F' with $d(F')$. From the construction of h, $Q \cap h(Q) \subset \partial Q$ and $h = d$ when restricted to F'. This proves the first statement (given without the parentheses) of part (a).

To prove the first statement of part (b), suppose that E is a positive edge. Let $h' = \sigma(rd(E))rg$. Then $\sigma(rd(E)) = rd\sigma(E)rd \notin K$ and $r \notin K$. Since K is an index 2 subgroup of G, $h' \in K$, and h' identifies F' with $rd(F')$ such that $h' = rd$ when restricted to F' (refer to Fig. 4). The rest of the cases of the lemma can be shown similarly. $\square$

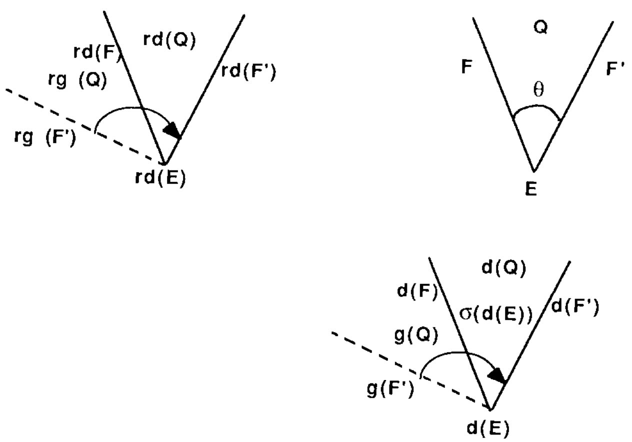

Fig. 4.

It follows from Lemma 1 that J' is symmetric in the x- and y-axes when Q is described by the diagram in Fig. 2. Therefore, J' is completely determined by J; $J' = (\partial Q) \cap (J \cup d(J) \cup rd(J) \cup r(J))$ (see Fig. 5). To verify this claim, let s be the rotation of π about the common edge of the faces A and D of P. Now rs is an element of K and it identifies the face A with $rd(A)$. Furthermore, $rs = rd$ when restricted to A. Suppose that F is a face of P adjacent to A. By Lemma 1, F is identified with $d(F)$ or $rd(F)$ by an element of K depending on whether the common edge of F and A is positive or negative. It is clear by induction that each

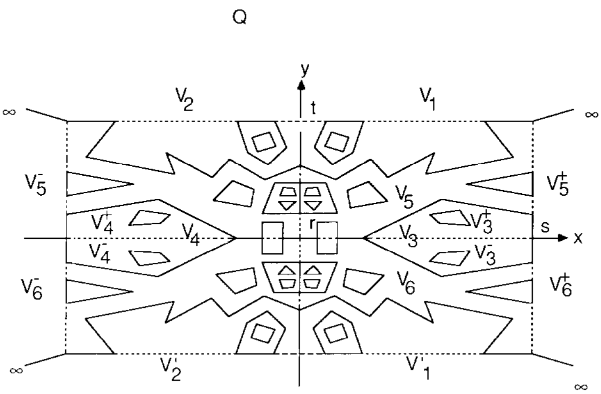

Fig. 5.

face F of P is either identified to $d(F)$ or $rd(F)$ by K. We now make the following observations: Let E be an edge of $(\partial Q)\cap P$. From the above, we can derive that there exists $g\in K$ such that g identifies E with $d(E)$ or $rd(E)$. Suppose that E is identified with $d(E)$. Then $g\sigma(E)g^{-1}=\sigma(d(E))$. Hence $\sigma(d(E))\in K$ if and only if $\sigma(E)\in K$. On the other hand, $rg^{-1}r\sigma(rd(E))rgr=\sigma(r(E))$. Hence $\sigma(rd(E))\in K$ if and only if $\sigma(r(E))\in K$. If E is identified with $rd(E)$, we can show similarly that $\sigma(E)\in K$ if and only if $\sigma(rd(E))\in K$, and $\sigma(d(E))\in K$ if and only if $\sigma(r(E))\in K$. Finally, for any edge E of $(\partial Q)\cap P$, $\sigma(r(E))=r\sigma(E)r$. Hence $\sigma(r(E))\in K$ if and only if $\sigma(E)\in K$. this implies that J' is symmetric in the origin ((0, 0) in the x-y-plane). Now the above observations imply that J' is symmetric in the x- and y-axes.

2.2. We assume that ∂Q is identified with the x-y-plane as in Fig. 5 and let $R=\partial Q-J'$. If U is a component of R, then U is an open subset of the plane. The closure of U, $\bar{U}$, has a boundary consisting of open arcs and circles, where an open arc or a circle is a union of edges of Q.

Definition. Let U be a component of R. If a component of $\partial\bar{U}$ is an open arc, then define the union of all open arcs in $\partial\bar{U}$ to be the *outer boundary* of U and the union of circles in $\partial\bar{U}$ to be the *inner boundary* of U. If every component of $\partial\bar{U}$ is a circle,

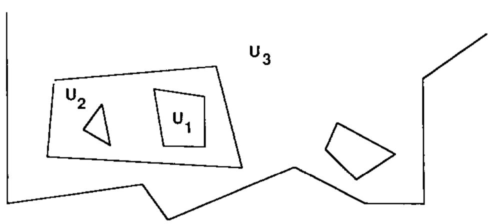

Fig. 6.

then there exists a largest circle in $\partial\bar{U}$ which contains the rest of the circles in its interior. We call the maximal circle the *outer boundary* and the union of the rest of the circles the *inner boundary.*

Figure 6 shows three components of R. U_3 has an open arc as the outer boundary, and a union of two circles as the inner boundary. U_1 has an empty inner boundary.

Definition. Let U_1 and U_2 be two components of R. We call U_2 *a predecessor* of U_1 if the outer boundary of U_1 and the inner boundary of U_2 have a common edge.

In Fig. 6, U_3 is a predecessor of U_2 and U_2 is a predecessor of U_1. We observe that a component has at most one predecessor. Therefore, given a component U, there exists a unique sequence of components, $U_1 = U, U_2, \ldots, U_k$ such that U_{i+1} is a predecessor of U_i, $1 \leq i \leq k-1$, and U_k does not have a predecessor.

Definition. In the above notation, we call U_k *the maximal component* associated to U and k *the depth* of U. A component without a predecessor will be called *a maximal component.*

It is clear from the diagram of ∂Q that the faces B, $d(B)$, $r(B)$ and $rd(B)$ belong to distinct maximal components. On the other hand, the faces A and $rd(A)$ belong to one maximal component, and the faces $d(A)$ and $r(A)$ belong to another maximal component. Hence there are at least six maximal components in R. These are denoted by V_1, V_1', V_2, V_2', V_5 and V_6 in Fig. 5. In addition to these maximal components, if the face D contains negative edges, then there exist more maximal components like V_3 and V_4 in Fig. 5. We call this type of maximal components *inessential components.* An inessential component is characterized as a maximal component containing a negative edge of D or $r(D)$. We partition some of the maximal components as follows.

For any inessential component V,

$$V^+ = V \cap \{(x+iy): y \geq 0\}, \qquad V^- = \text{Closure}(V - V^+),$$

$$V_5^+ = V_5 \cap \{(x+iy): x \geq 0\}, \qquad V_5^- = \text{Closure}(V_5 - V_5^+),$$

$$V_6^+ = V_6 \cap \{(x+iy): x \geq 0\}, \qquad V_6^- = \text{Closure}(V_6 - V_6^+).$$

The following two lemmas describe the face identification on Q by K in terms of the components in R.

Lemma 2. *Suppose that U_1 and U_2 are components of R (the components could be the same) and F is a face of Q contained in $\bar{U}_1$. If F is identified with a face F' in $\bar{U}_2$ by an element $g \in K$ such that $g = d$ or rd when restricted to F, then $\bar{U}_1$ is identified with $\bar{U}_2$ by elements of K such that the identification is equivalent to the one induced by d or rd, respectively.*

Proof. If $\bar{U}_1 = F$, then the lemma holds trivially. Suppose that $\bar{U}_1 \neq F$. Then there exists a face F_1' in $\bar{U}_2$ such that F' and F_1' share a negative edge. If F is identified with F' by g, where $g = d$ or rd when restricted to F, then by Lemma 1, F_1' is identified with $d(F_1')$ or $rd(F_1')$, respectively, such that the identification is equivalent to the one induced by d or rd, respectively. The proof is completed inductively since every face in $\bar{U}_2$ can be joined to F' by a finite sequence of faces in $\bar{U}_2$, where any two consecutive faces share a negative edge. □

Lemma 3. *Suppose that U is a component of R, V is the maximal component associated to U, and k is the depth of U.*

(a) *Suppose that $V = V_1$, V_2 or an inessential component.*

(i) *$\bar{U}$ is identified with $d(\bar{U})$ by elements of K if k is odd and the identification is equivalent to the one induced by d.*

(ii) *$\bar{U}$ is identified with $rd(\bar{U})$ if k is even and the identification is equivalent to the one induced by rd.*

(b) *Suppose that $V = V_5$ or V_6.*

(i) *$\bar{U}$ is identified with $rd(\bar{U})$ by K if k is odd and the identification is equivalent to the one induced by rd.*

(ii) *$\bar{U}$ is identified with $d(\bar{U})$ if k is even and the identification is equivalent to the one induced by d.*

Proof. We use induction on k.

(a) Suppose that $V = V_1$ and $k = 1$. (The argument is similar if $V = V_2$ and $k = 1$.) Let t be the rotation of π about the edge common to B and C. We also use the rotations, r and s, defined in Section 2.1. Since $r \notin K$ and $t \notin K$, $rt \in K$. Now rt identifies the face B with $d(B)$ in $\bar{V}_1'$ and $rt = d$ when restricted to B. By Lemma 2, $\bar{V}_1$ is identified with $d(\bar{V}_1) = \bar{V}_1'$ and the identification is equivalent to the one induced by d.

Suppose that V is an inessential component, say V_3, and $k = 1$. (The argument is similar for any inessential component.) There exists a face F in V_3 such that F and $d(F)$ share a negative edge, say E. We may imagine that F is a face of V_3^+ and $d(F)$ a face of V_3^- such that E is an edge along the x-axis in Fig. 5. Now $\sigma(E)$ identifies F with $d(F)$ and $\sigma(E) = d$ when restricted to F. Lemma 2 implies that K identifies V_3 with itself, more precisely, V_3^+ with V_3^-, and the identification is equivalent to the one induced by d.

Suppose that (a) is true for $k - 1$.

(i) Suppose that U has depth k and k is odd. Let U_1 be the predecessor of U. Then $\bar{U}_1$ is identified with $rd(\bar{U}_1)$ by the induction hypothesis. There exists a face F of $\bar{U}$ containing a positive edge E such that E is contained in the inner boundary of U_1. Let F_1 be the face contained in $\bar{U}_1$ meeting F at the edge E. By Lemma 1, since F_1 is identified with $rd(F_1)$, F is identified with $d(F)$ and the identification is equivalent to the one induced by d. By Lemma 2, $\bar{U}$ is identified with $d(\bar{U})$ as described.

(ii) If k is even, the induction step can be shown the same way as the above.

(b) The proof of case (b) is again similar. We observe that V_5^+ and V_6^+ are identified with V_5^- and V_6^- by K, respectively, since A and $d(A)$ are identified with $rd(A)$ and $r(A)$ by $rs \in K$, respectively. □

2.3. We now have enough information to describe $\mathbb{H}^3/K$.

Lemma 4. *$\mathbb{H}^3/K$ is homeomorphic to the open solid torus, $S^1 \times \mathring{D}^2$.*

Proof. According to Lemma 3, K identifies V_1 with V_1', V_2 with V_2', and V^+ with V^- for any inessential component V, where the identification is equivalent to the one induced by d. K also identifies V_5^+ with V_5^- and V_6^+ with V_6^-, where the identification is equivalent to the one induced by rd. Therefore, if every component of R is maximal, then the above completely describes the face identification of Q and $\mathbb{H}^3/K$ is homeomorphic to the open solid torus. We show below that R has only maximal components.

Suppose that not all the components of R are maximal. Then there exists a component U, such that it is not a predecessor of other components and it has a predecessor. Denote the predecessor of U by U_1. U is homeomorphic to the 2-dimensional open disk. Suppose that $\bar{U}_1$ is identified with $d(\bar{U}_1)$ by elements of K. (If $\bar{U}_1$ is identified with $rd(\bar{U}_1)$, the argument below still holds.) Then $rd(\bar{U}_1)$ is identified with $r(\bar{U}_1)$ by elements of K, $\bar{U}$ with $rd(\bar{U})$, and $d(\bar{U})$ with $r(\bar{U})$ (Fig. 7) as described in Lemma 3.

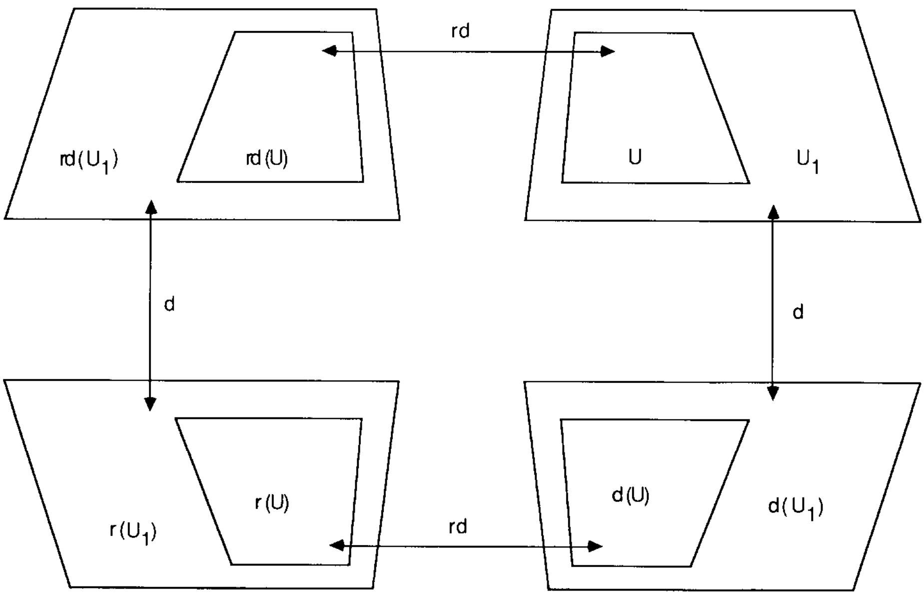

Fig. 7.

We now slightly alter the identification of the faces of Q by K as follows. The new identification is the same as the old one, except that $\bar{U}$ is identified with $d(\bar{U})$ and $rd(\bar{U})$ with $r(\bar{U})$, where the identification is done by d. Let M be the space obtained from Q by the new face identification. M is an orientable 3-manifold (topological) and $\mathbb{H}^3/K$ is obtained from M by a 0-surgery (orientation preserving), where a 0-surgery can be described as follows.

We first give $S^0 \times D^3$ the orientation induced from that of the standard orientation of $D^1 \times D^3$. Let M be an orientable connected 3-manifold with a fixed orientation and α be an orientation preserving embedding of $S^0 \times D^3$ into the interior of M. Then the result of a 0-surgery on M using α is

$$(\text{Closure of } (M - \text{Im}(\alpha)) \cup D^1 \times S^2,$$

where the identification is done by

$$\alpha \,|\, S^0 \times S^2 = \alpha \,|\, (\partial D^1) \times (\partial D^3).$$

In general, if M' is the result of a 0-surgery on M, then the rank of $H_1(M'; \mathbb{Z})$ is one larger than that of $H_1(M; \mathbb{Z})$. To see this, let $M_0 = \text{Closure}(M - \text{Im}(\alpha))$. Using $\mathbb{Z}$-coefficients, $H_1(M) \cong H_1(M_0)$ and we have the following long exact sequence

$$H_2(M', M_0) \to H_1(M_0) \to H_1(M') \to H_1(M', M_0) \to 0.$$

$$H_*(M', M_0) \cong H_*(D^1 \times S^2, S^0 \times S^2) \text{ by excision,}$$

$$H_2(M', M_0) \cong H_2(D^1 \times S^2, S^0 \times S^2) \cong H^1(D^1 \times S^2) \cong 0$$

by Poincaré duality [11], and

$$H_1(M', M_0) \cong H^2(D^1 \times S^2) \cong \mathbb{Z}.$$

The above exact sequence reduces to

$$0 \to H_1(M_0) \to H_1(M') \to \mathbb{Z} \to 0.$$

Hence the rank of $H_1(M')$ is one larger than that of $H_1(M)$.

We see inductively that $\mathbb{H}^3/K$ is obtained from the open solid torus by a sequence of 0-surgeries and the sequence is not empty if not all the components in R are maximal. But if the sequence is not empty, then the rank of $H_1(\mathbb{H}^3/K; \mathbb{Z})$ is greater than 1, which contradicts our assumption that $H_1(\mathbb{H}^3/K; \mathbb{Z})$ has rank 1. This finishes the proof of the lemma. □

Remark. The proof of Lemma 4 shows that the equivalence classes J of positive edges of P defined in Section 2.1 consist exactly of two open arcs.

2.4. The abelianization of $\pi_1(\mathbb{H}^3/N)(\cong N)$ is isomorphic to $H_1(\mathbb{H}^3/N; \mathbb{Z})$. Hence the rank of $H_1(\mathbb{H}^3/N; \mathbb{Z})$ is 1 by the assumption. On the other hand, the rank of $H_1(\mathbb{H}^3/N; \mathbb{Z})$ is greater than or equal to the number of cusps in $\mathbb{H}^3/N$. Since P has an ideal vertex, $\mathbb{H}^3/N$ has at least one cusp. Consequently, $\mathbb{H}^3/N$ has exactly one cusp. Therefore, $\mathbb{H}^3/N$ is an orientable open 3-manifold with one end homeomorphic to $S^1 \times S^1 \times \mathbb{R}$ [12].

Let Y be a compact 3-manifold containing $\mathbb{H}^3/N$ as a submanifold such that $\partial Y = Y - \mathbb{H}^3/N$ is homeomorpic to $S^1 \times S^1$ (Y is a compactification of $\mathbb{H}^3/N$). We observe now that the natural action of K/N on $\mathbb{H}^3/N$ extends to an action over Y. Let $H = K/N$. We may write $H = \{[a_i]: a_i \in K, 1 \leq i \leq n, a_1 = \text{the identity of } K\}$. Then $Q' = \bigcup_{1 \leq i \leq n} a_i(Q)$ is a fundamental domain of N [8], and $\mathbb{H}^3/N$ can be obtained from Q' by identifying the faces by elements of N. Topologically, Q is homeomorphic to $I \times I \times [0, 1)$, a solid cube with no top face, where I is the unit interval. By viewing $I \times I \times \{0\}$ as the base of the cube, we may assume that the four faces on the side of the cube correspond to the four faces of Q containing ∞. It is important to note that a face on the side of $a_i(Q)$ can only be identified with a face on the side of $a_j(Q)$ by an element on N. Thus, ∂Y admits a rectangular decomposition consisting of n rectangles.

Given $x \in Q'$, let $[x]$ be the point of $\mathbb{H}^3/N$ represented by x. Then for any i, j and $x \in Q$, $[a_i]([a_j(x)]) = [a_k(x)]$, where $[a_i] \cdot [a_j] = [a_k]$ in H. Therefore, an element of H leaves the $\mathbb{R}$ factor fixed in the end ($\cong S^1 \times S^1 \times \mathbb{R}$) of $\mathbb{H}^3/N$. It follows that the action of H on $\mathbb{H}^3/N$ extends to an action over Y. If we restrict the extended action to ∂Y, an element of H permutes the rectangles in the rectangular decompositions of ∂Y.

Let X be the compact solid torus. Then the orbit space of the extended action of H on Y is homeomorphic to X such that $X - \partial X$ is homeomorphic to $\mathbb{H}^3/K$. By restricting this action to ∂Y, we obtain an action of H on $S^1 \times S^1$ such that its orbit space is homeomorphic to $S^1 \times S^1$.

Lemma 5. *The action of H on ∂Y is free.*

Proof. Suppose that $[h] \in H$ is not the identity of H and $[h]$ fixes a point in ∂Y. Then the fixed point set of $[h]$ in Y is a regularly embedded 1-dimensional submanifold whose boundary is in ∂Y since $[h]$ can be regarded as an orientation preserving diffeomorphism of Y. Therefore, the fixed point set of $[h]$ in $\mathbb{H}^3/N$ must contain an open arc. Let α be such an arc, and let $y \in f_1^{-1}(\alpha)$, where $f_1 : Q' \to \mathbb{H}^3/N$ is the quotient map induced by the identification of the faces. Then there exist $x \in Q$ and a_i, $i = 1, 2, \ldots, n$, such that $y = a_i(x)$. On the other hand, since $[h](f_1(y)) = f_1(y)$, there exists $g \in N$ such that $gh(y) = y$. Then $a_i^{-1} gha_i(x) = x$. Since a non-trivial element of K can only fix a point in an edge of Q, x is a point of an edge. Hence y is a point of an edge of Q'. This implies that $f_1^{-1}(\alpha)$ must contain an interior point of an edge containing ∞ since α is an open arc and Q' has only finitely many edges. Let y be such a point in the above argument. Then x is an interior point of an edge E of Q containing ∞. From the definition of Q (see Section 2.1), there exist $g_1 \in G$ and an edge E_1 of P containing ∞ such that $E = g_1(E_1)$. Now $g_1^{-1} a_i^{-1} gha_i g_1$ is an element of G, and it fixes $g_1^{-1}(x)$, an interior point of E_1. It is a standard fact that if an element of a polyhedral group fixes an interior point of an edge E_1 of the polyhedron, then the element is equal to a power of the rotation,

$\sigma(E_1)$. In the above discussion, the order of $\sigma(E_1)$ is 2 since the dihedral angle along E_1 is $\pi/2$. Therefore, $g_1^{-1}a_i^{-1}gha_ig_1 = \sigma(E_1) \in K$ since $gh \in K$ and K is an index 2 normal subgroup of G (any index 2 subgroup is normal). But K cannot contain such an element from the definition given in Section 2. □

Remark. The proof of Lemma 5 shows that $f_1^{-1}(\alpha)$ is a union of edges.

Lemma 6. *K/N is an abelian group.*

Proof. By Lemma 5, we may regard H as the group of covering transformations of the covering projection $f: S^1 \times S^1 \xrightarrow{H} S^1 \times S^1$. A theorem in covering space theory [11] implies that H is isomorphic to $\pi_1(S^1 \times S^1)/f_*(\pi_1(S^1 \times S^1))$. Since $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$, H is an abelian group. Furthermore, there exist positive integers p and q such that $K/N \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$. □

2.5. We now finish the proof of Proposition 1. Let v be a regular vertex of P and S_v be the stabilizer of v in G. Suppose that $S_v \not\subset K$. Let $g \in S_v - K$. Then

$$S_v = (S_v \cap K) \cup (S_v \cap Kg) = (S_v \cap K) \cup (S_v \cap Kg)g^{-1}g$$
$$= (S_v \cap K) \cup (S_v \cap K)g.$$

Therefore, $S_v \cap K$ is either S_v or an index 2 subgroup of S_v. On the other hand, $S_v \cap K \cap N$ is a trivial group since $S_v \cap K$ is finite and N is torsion-free. Here, we are using the fact that for any regular vertex v, S_v is a finite group because it is a so-called spherical group acting on a 2-sphere. We will see below what S_v precisely is. Therefore, $S_v \cap K \cong (S_v \cap K)N/N \subset K/N$. By Lemma 6, $S_v \cap K$ is abelian. Hence for any regular vertex v, S_v is abelian or it contains an index 2 abelian subgroup.

The stabilizer S_v of a regular vertex v has the following presentation.

(1) 1st type: $(2, 2, n)$, $n > 1$,

$$S_v = \langle a, b;\ a^2 = b^2 = (ab)^n = 1 \rangle \cong D_n.$$

(2) 2nd type: $(2, 3, 3)$,

$$S_v = \langle a, b;\ a^2 = b^3 = (ba)^3 = 1 \rangle \cong A_4.$$

(3) 3rd type: $(2, 3, 4)$,

$$S_v = \langle a, b;\ a^2 = b^3 = (ba)^4 = 1 \rangle \cong S_4.$$

(4) 4th type: $(2, 3, 5)$,

$$S_v = \langle a, b;\ a^2 = b^3 = (ba)^5 = 1 \rangle \cong A_5.$$

S_n denotes the permutation group on n letters and A_n the alternating subgroup of S_n.

It is obvious from the presentation that none of A_4, S_4 and A_5 is abelian or contains an index 2 abelian subgroup. Therefore, any regular vertex of P must be of the 1st type, and this finishes the proof of Proposition 1. We note that the dihedral group $D_n (n \geqslant 3)$ does contain a unique index 2 abelian subgroup and D_2 is abelian.

Remark. Suppose that v is a regular vertex of Q of type 1 with $n>2$ and let E be the edge containing v with dihedral angle equal to π/n. Then S_v is isomorphic to D_n and D_n is not abelian. Therefore, the above argument shows that $S_v \cap K$ must be an index 2 abelian subgroup of S_v. The only index 2 abelian subgroup of S_v is the subgroup generated by $\sigma(E)$, the rotation of $2\pi/n$ about E. Therefore, E is a negative edge and the other two edges containing v are positive edges.

3. Proof of Proposition 2

We prove Proposition 2 by a contradiction. Suppose that G is an exceptional polyhedral group associated to polyhedron P and all the regular vertices of P are of type 1. Suppose that G contains a normal t.f.i. subgroup N and $H_1(N;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$. Notice that all the results obtained in the last section hold under the new hypothesis. We use the notations of the last section.

3.1. We first show that not all of the regular vertices of P are of type $(2,2,2)$.

Lemma 7. *There exists a negative edge in P whose dihedral angle is π/n, $n>2$.*

Proof. Suppose that all the regular vertices are of type $(2,2,2)$. Given a group H, let $\#H$ denote the smallest number of generators for H. By the proof of Lemma 6, $K/N \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$ for some positive integers p and q. From the short exact sequence $1 \to N \to K \to K/N \to 1$ we obtain an exact sequence of homology groups with $\mathbb{Z}$ coefficients

$$H_1(N) \to H_1(K) \to \mathbb{Z}_p \oplus \mathbb{Z}_q \to 0.$$

$\#H_1(K) < 4$ since $\#H_1(N) = 1$. On the other hand,

$$H_1(K) \to H_1(G) \to \mathbb{Z}_2 \to 0.$$

Therefore, $\#H_1(G) < 5$.

Suppose that P has m vertices. Then P has $k = \frac{1}{2}(3m+1)$ edges and a presentation of G can be given as follows [5].

$$G = \langle g_1, g_2, \ldots, g_k;\ g_i^2 = 1,\ 1 \leqslant i \leqslant k,\ R_j,\ 1 \leqslant j \leqslant m-1 \rangle,$$

where R_j is a relation of the form, $g_\alpha g_\beta g_\gamma = 1$, and for each vertex there is one relation of this type. The relation corresponding to the ideal vertex is redundant [5]. This shows that $H_1(G)$ is isomorphic to the direct sum of at least $(k-m+1)$ copies of $\mathbb{Z}_2$'s. Hence $\#H_1(G) \geqslant k-m+1 = \frac{1}{2}(m+3)$ and $\frac{1}{2}(m+3) < 5$. Therefore, $m = 5$ or 6, since P must have at least 5 vertices.

(i) If $m=5$, the only possible polyhedron is given in Fig. 8.
But P cannot be a hyperbolic polyhedron [1], because two non-adjacent faces containing ∞ meet a third face at angle $\pi/2$.

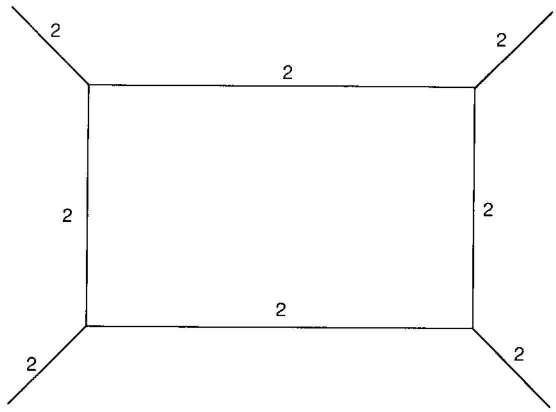

Fig. 8.

(ii) It is easy to check that there is no exceptional polyhedra with 6 vertices. This finishes the proof of the lemma. □

3.2. We consider the natural action of $H = K/N$ on Y, where Y is defined as in Section 2.4.

Lemma 8. *Suppose that* $[h] \in H$ *and* $[h]$ *is not the trivial element. If* $[h]$ *fixes a point in* Y, *then the fixed point set of* $[h]$ *is homeomorphic to a circle.*

Proof. Let S be the fixed point set of $[h]$. Then S is a union of circles (see the proof of Lemma 5). Let w be a prime number dividing the order of $[h]$. We regard $\mathbb{Z}_w$ as a subgroup of the group generated by $[h]$. Then $\mathbb{Z}_w$ acts on Y and the fixed point set of $\mathbb{Z}_w$ contains S.

If we use $\mathbb{Z}_w$ coefficients for the homology groups, then $\mathrm{rank}(H_1(S)) \leq \mathrm{rank}(H_1(Y)) + \mathrm{rank}(H_2(Y))$ by [3]. On the other hand, we have a homology long exact sequence

$$0 \to H_3(Y, \partial Y) \to H_2(\partial Y) \to H_2(Y) \to H_2(Y, \partial Y)$$
$$\to H_1(\partial Y) \to H_1(Y) \to H_1(Y, \partial Y) \to 0.$$

By Poincaré duality and the universal coefficient theorem [11] the above sequence reduces to

$$0 \to \mathbb{Z}_w \to \mathbb{Z}_w \to H_2(Y) \to \mathbb{Z}_w \to \mathbb{Z}_w \oplus \mathbb{Z}_w \to \mathbb{Z}_w \to H_2(Y) \to 0.$$

The sequence implies that $H_2(Y) = 0$. Therefore, $\mathrm{rank}(H_1(S)) \leq \mathrm{rank}(H_1(Y)) = 1$ and S is homeomorphic to a circle. □

3.3. We define the quotient maps f, f_1, f_2 and f_3 as in the following diagram, where for any $x \in Q$ and i, $1 \leq i \leq n$, $f_2(a_i(x)) = x$. The diagram clearly commutes. Recall that $\{a_i\}$ is defined to be a set of right cosets of N in K in Section 2.4.

$$\begin{array}{ccc}
Q' & \xrightarrow{f_2} & Q \\
\downarrow f_1 & & \downarrow f_3 \\
\mathbb{H}^3/N & \xrightarrow[f]{/H} & \mathbb{H}^3/K \\
\cap & & \cap \\
Y & \xrightarrow[f]{/H} & X
\end{array}$$

Suppose that E is a negative edge of Q and let $e = \sigma(E)$. Since e has a finite order, e does not belong to N. Hence we may regard the group generated by e as a subgroup of H and it acts on Y. The fixed point set of e is non-empty and it is homeomorphic to a circle by Lemma 8. Let S be the fixed point set of e.

Lemma 9. *Under the above notation, $f_3^{-1}(f(S))$ is a union of negative edges of Q.*

Proof. From the commutativity of the above diagram, $f_3^{-1}(f(S)) = f_2(f_1^{-1}(S))$ and from the remark of Section 2.4, $f_1^{-1}(S)$ is a union of edges of Q'. Hence $f_3^{-1}(f(S))$ is a union of edges since f_2 maps edges to edges.

Suppose that E' is an edge in $f_3^{-1}(f(S))$. If $\sigma(E')$ has order greater than 2, then the lemma holds trivially by the remark of Section 2.5. Suppose that $\sigma(E')$ has order 2. Then there exist $a_i \in K$ and $g \in N$ such that for any $x \in E'$ $gea_i(x) = a_i(x)$. Therefore, $a_i^{-1}gea_i = \sigma(E')$ or the trivial element of K since the order of $\sigma(E')$ is 2. Recall that any element of K fixing E' pointwise is a power of $\sigma(E')$. If $a_i^{-1}gea_i$ is the trivial element of K, then $g = e^{-1}$, which is not possible. Hence $a_i^{-1}gea_i = \sigma(E') \in K$, thus E' is a negative edge. □

3.4. We have shown in the remark of Section 2.3 that J consists of exactly two open arcs. We assume without loss of generality that they are horizontal as in Fig. 9.

Lemma 10. *K/N is cyclic.*

Proof. We choose a smoothly embedded horizontal open arc Λ in ∂P (or the x-y-plane) extending to ∞ at both ends with the following properties (see Fig. 9).

(i) Λ is disjoint from J and passes through the faces A and C.

(ii) Λ does not contain any regular vertex of P, and if it intersects an edge, then it does so transversely.

(iii) There exists a negative edge E of P such that E cannot be joined to a point of Λ by a path consisting only of points of negative edges.

We see the existence of Λ as follows. If there exists an open arc satisfying (i) and (iii), then we isotop the arc slightly to make it satisfy (ii). If an edge of B or D is negative, then any arc Λ satisfying (i) satisfies (iii). Suppose that B and D do not contain any negative edges, then the dihedral angles of all edges of B and D

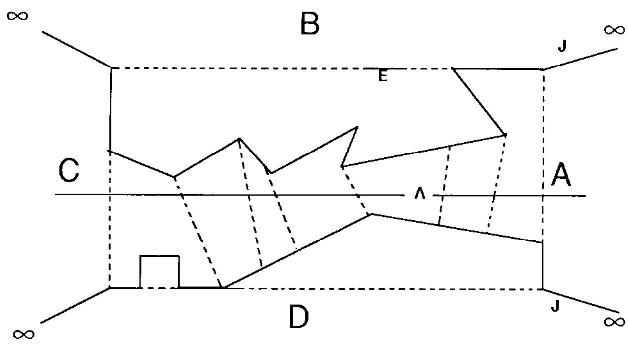

Fig. 9.

are $\pi/2$ by the remark of Section 2.5. On the other hand, there exists by Lemma 7 an edge E in P whose dihedral angle is π/n, $n>2$. The endpoints of E are in J again by the remark of Section 2.5. Let J_1 and J_2 be the components of J. Then both endpoints of E must be either in J_1 or J_2, because if not, the faces B and D intersect a third face at a dihedral angle $\pi/2$, which is impossible for a hyperbolic polyhedron [1]. Now we can choose Λ such that it satisfies (i) and is disjoint from E. There is no path of negative edges joining E to Λ.

We assume that $(Q, \partial Q)$ is identified with $(\mathbb{H}^3, \mathbb{R}^2)$ as in Fig. 5. Let $\Lambda' = \partial Q \cap (\Lambda \cup rd(\Lambda))$, and W be the half plane in Q perpendicular to the x-y-plane with Λ' as its boundary. Then $f_3(W)$ is an open disk embedded in $\mathbb{H}^3/K$. Furthermore, there exists a smoothly embedded closed disk Δ in X such that $\Delta - \partial\Delta$ is $f_3(W)$.

The map $f\colon Y \to X$ is transverse regular to Δ. If not, there exists a $y \in Y$ such that f is not transverse regular at y and y must be a fixed point of some element $[h] \in H$. Then there exists an embedded circle S containing y in Y such that $[h]$ fixes the points of S. Now $f_3^{-1}(f(S))$ is a union of edges (negative) in Q. The property (ii) of Λ implies that Δ intersects $f(S)$ transversely at $f(y)$. Therefore, the tangent space of Δ at $f(y)$ and the tangent space of $f(S)$ at $f(y)$ generate the tangent space of X at $f(y)$, which is a contradiction. By the transverse regularity theorem, $f^{-1}(\Delta)$ is a 2-dimensional orientable compact submanifold of Y.

We now fix two generators, μ and ν, of $H_1(\partial X; \mathbb{Z})$. Let ν be an element represented by the embedded circle $\partial\Delta$ in ∂X. Then ν may be regarded as a meridian of ∂X. To define μ, let W' be the upper half of the y-z-plane in Q in Fig. 5. Then the closure of $f_3(W')$ in X intersects ∂X in a circle. We define μ to be an element represented by this circle. We may regard μ as a longitude of ∂X. Since $f|\partial Y$ is a covering projection by Lemma 4, there exist generators a and b of $H_1(\partial Y; \mathbb{Z})$ represented by embedded circles in ∂Y such that $f_*(a) = p\mu$ and $f_*(b) = q\nu$, where $K/N \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$ as in the proof of Lemma 6.

Now $f^{-1}(f_3(W))$ is $\bigcup_{1\le i\le n} a_i(W)$ with the boundaries identified by elements of N, where $n = pq$ and $\{a_i, 1 \le i \le n\}$ is a set of cosets of N in K. Since each $a_i(W)$ is non-compact, each component of $f^{-1}(f_3(W))$ is non-compact. Hence each component of $f^{-1}(\Delta)$ must have a non-empty boundary contained in $\partial Y \cap f^{-1}(\Delta) = f^{-1}(\partial\Delta)$, where $f^{-1}(\partial\Delta)$ is a union of p circles.

Suppose that $\tilde{\Delta}$ is a component in $f^{-1}(\Delta)$. Then $\partial\tilde{\Delta}$ consists of some of the p circles which make up $f^{-1}(\partial\Delta)$. Since f is transverse regular to Δ, f induces a bundle map from the normal bundle of $f^{-1}(\Delta)$ in Y into the normal bundle of Δ in X. Both bundles are line bundles and we may assume that f respects positive directions of the bundles if we choose the directions properly. On the other hand, the positive direction of the bundles restricted over the boundaries must be as in Fig. 10 if we identify the total spaces of the normal bundles with tubular neighborhoods.

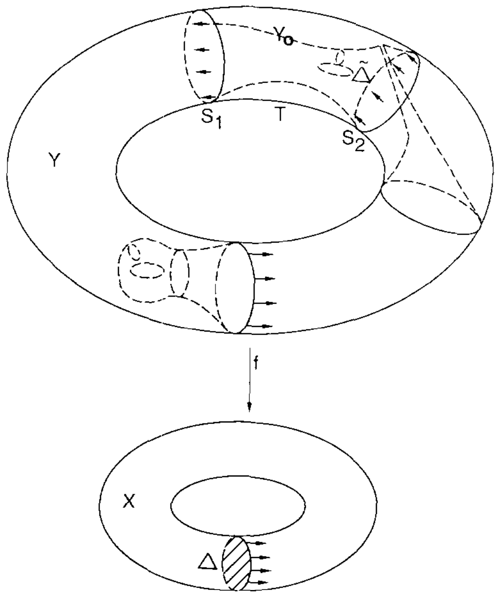

Fig. 10.

Suppose that the boundary of $\tilde{\Delta}$ has more than one component. Let S_1 and S_2 be two components such that they form the boundary of a compact annulus T in ∂Y, where T does not contain any other components of $\partial\tilde{\Delta}$. Let Z be the closure of the component of $Y - \tilde{\Delta}$ containing T. Z is an orientable 3-manifold. Hence there exists a consistent inward normal direction along ∂Z, and we may assume that this direction agrees with the positive direction along a copy of $\tilde{\Delta}$ in ∂Z. But if we translate the positive direction along S_1 across T to S_2 using the inward normal direction of ∂Z in Z, we obtain an inconsistent direction. This implies that Z is not an orientable manifold. Therefore, each component of $f^{-1}(\Delta)$ contains exactly one component of $f^{-1}(\partial\Delta)$, thus $f^{-1}(\Delta)$ has p components.

Let S be the fixed point set of $\sigma(E)$, where E is the edge given in property (iii) in the definition of Λ. Now $f_3^{-1}(f(S))$ is a union of negative edges of Q by Lemma 9, and if $f_3^{-1}(f(S)) \cap \Lambda' \neq \emptyset$, then there must be a path in ∂P from E to a point in

Λ consisted of points of negative edges. But E and Λ are chosen such that this is not possible. Hence $f_3^{-1}(f(S))\cap\Lambda'=\emptyset$. Therefore, $f(S)\cap\Delta=\emptyset$ or $S\cap f^{-1}(\Delta)=\emptyset$.

We choose a thin tubular neighborhood $f^{-1}(\Delta)\times(-1,1)$ of $f^{-1}(\Delta)$ in Y, such that it does not intersect S, and $f(f^{-1}(\Delta)\times(-1,1))$ is a thin tubular neighborhood of Δ in X, which does not intersect $f_3(E)$. Let $Y_0=Y-f^{-1}(\Delta)\times(-1,1)$. S is contained in one of the components of Y_0. By the Lefschetz duality [11], $H_0(Y_0)\cong H^3(Y,\partial Y\cup f^{-1}(\Delta))$, where the homology and cohomology groups are over $\mathbb{Z}$ coefficients. From the triple $(Y,\partial Y\cup f^{-1}(\Delta),\partial Y)$, we get a long exact sequence

$$\to H^2(Y,\partial Y)\to H^2(\partial Y\cup f^{-1}(\Delta),\partial Y)$$
$$\to H^3(Y,\partial Y\cup f^{-1}(\Delta))\to H^3(Y,\partial Y)\to 0.$$
$$H^2(Y,\partial Y)\cong H_1(Y)\cong H_1(N)\cong\mathbb{Z},$$

$H^2(\partial Y\cup f^{-1}(\Delta),\partial Y))\cong p\mathbb{Z}$ (direct sum of p copies of $\mathbb{Z}$) and $H^3(Y,\partial Y)\cong\mathbb{Z}$. Therefore, the rank of $H^3(Y,\partial Y\cup f^{-1}(\Delta))$ is at least p and Y_0 has at least p components.

Let $X_0=X-f(f^{-1}(\Delta)\times(-1,1))=f(Y_0)$. Suppose that Y' is a component of Y_0. Y' is a compact connected manifold and $f(\partial Y')=\partial X_0$. We now observe that f is an open map when restricted to Y'. Suppose that $y\in Y'$. If y is in $\partial Y'$, then there exists an open neighborhood of y, whose image under f is open in X_0 from the fact that f is a covering projection on ∂Y and is transverse regular to Δ. If y is an interior point of Y', we can still find an open neighborhood of y, whose image under f is open in X_0 since f is a quotient map induced by a finite group action. From the observation, $f(Y')$ is an open subset of X_0. On the other hand, $f(Y')$ is closed since it is a continuous image of a compact space. Hence $f(Y')=X_0$. In particular, $f^{-1}(f_3(E))\cap Y'\neq\emptyset$ and Y' contains $f_1(a_i(E))$ for some $i=1,2,\dots,n$.

Suppose that $x\in E$. Then $\sigma(E)(f_1a_i(x))=\sigma(E)[a_i](f_1(x))=[a_i]\sigma(E)(f_1(x))=[a_i]f_1(\sigma(E)(x))=[a_i]f_1(x)=f_1(a_i(x))$. (In the computation, we used the fact that K/N is abelian.) Hence $f_1(a_i(E))\subset S$ and $S\cap Y'\neq\emptyset$. This implies that S is contained in every component of Y_0, thus Y_0 has only one component. Therefore, $p=1$ and $K/N\cong\mathbb{Z}_q$. □

3.5. We now finish the proof of Proposition 2. First, observe that the endpoints of any negative edge of P lie in J (see Section 3.4 for the definition of J). Because if not, there must be three negative edges meeting at a vertex, and the stabilizer of the vertex can be regarded as a subgroup of H, which implies that H is not cyclic.

Choose two negative edges E and E_1 of P (Fig. 11) with the following properties:

The endpoints of E are in distinct components of J,

the order of $\sigma(E_1)$ is greater than 2, and

there is no path joining E to E_1 consisting only of points of negative edges.

Edges E and E_1 exist from the above observation and Lemma 7. Note that $f_3(E)$ is disjoint from $f_3(E_1)$ by Lemma 9.

Let S be the fixed point set of $\sigma(E)\in H$. If $y\in S$ and $h\in H$, $\sigma(E)h(y)=h\sigma(E)(y)=h(y)$. Hence $h(y)\in S$ for any $y\in S$, thus H acts on S. Let H_0 be the subgroup of H

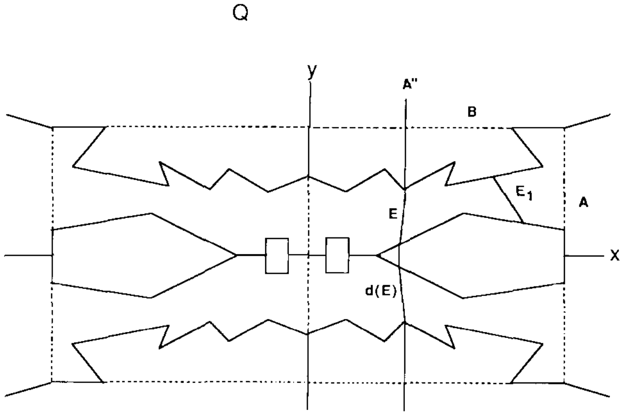

Fig. 11.

of elements fixing S pointwise. We claim that $f(S)=S/H=S/(H/H_0)$ is a circle in X.

Suppose that it is not. Then $f(S)$ is a compact arc and H/H_0 must contain an element which reverses the orientation of S. By identifying S with the standard unit circle in the x-y-plane, we may assume that the orientation reversing element is the involution η reflecting the plane in the x-axis. On the other hand, the subgroup of H/H_0 of orientation preserving elements is generated by a rotation ξ of the plane about the origin in an angle $2\pi/k$ for some integer $k \geqslant 1$. It is easy to check that $H/H_0 \cong \langle\eta\rangle \oplus \langle\xi\rangle$. If we regard the circle as $\{\Theta\colon 0 \leqslant \Theta \leqslant 2\pi\}/(0=2\pi)$, then for any integer i,

$$\eta\xi^i(\Theta)=\eta\left(\Theta+\frac{2i\pi}{k}\right)=-\Theta-\frac{2i\pi}{k}$$

and

$$\xi^i\eta(\Theta)=\xi^i(-\Theta)=-\Theta+\frac{2i\pi}{k}.$$

Since H/H_0 is abelian,

$$-\Theta-\frac{2i\pi}{k}=-\Theta+\frac{2i\pi}{k}+2n\pi$$

for some integer n.

$-4i\pi/k=2n\pi$. Hence $2i/k$ is an integer for every integer i. Hence $k=1$ or 2. If $k=2$, then $H/H_0 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, but this is not possible since H is cyclic. If $k=1$, then $H/H_0 \cong \mathbb{Z}_2$. This implies that there exists $h \in \langle\alpha(E_1)\rangle$ such that $[h] \in H_0$, because the order of $\sigma(E_1)$ is greater than 2. Let S_1 be the fixed point set of $\sigma(E_1)$. S_1 is disjoint from S by the choice of E and E_1. Furthermore, S and S_1 are fixed pointwise by $[h]$, which contradicts Lemma 8. We have shown that $f(S)$ is an embedded circle in X.

Let $E' = E \cup d(E)$. Then $f(S) = f_3(E')$ and $S = f_1(\bigcup_{1 \leq i \leq q} a_i(E'))$. Since H/H_0 is a cyclic group, say of order k, and $f: S \to f(S)$ is a k-fold covering projection of a circle onto a circle. Choose a vertical open arc Λ'' containing E' in ∂Q (Fig. 11) with the following properties:

The arc extends to ∞ at both ends, does not contain vertices except for the ones in E', is transverse regular to the edges it intersects except for E' and is symmetric in the x-axis.

Let W'' be the plane perpendicular to the x-y-plane containing Λ''. Then $f_3(W'')$ is a non-compact annulus in $\mathbb{H}^3/K$ whose boundary is $f(S)$ and there exists an embedded compact annulus Δ' in X such that $\Delta' \cap \partial X$ is a circle and $\Delta' - (\Delta' \cap \partial X) = f_3(W'')$. Clearly, $\Delta' \cap \partial X$ represents the homology class μ of $H_1(\partial X; \mathbb{Z})$ which generates $H_1(X; \mathbb{Z})$. (See Section 3.4 for the notation.)

From the construction, f is transverse regular to $\Delta' - f(S)$. Hence $f^{-1}(\Delta' - f(S))$ is an orientable 2-manifold. Since the closure of each component of this manifold is obtained by identifying the boundaries of some of the $a_i(W'')$, $1 \leq i \leq q$, it has S as a boundary component. Hence $f^{-1}(\Delta')$ is a union of orientable 2-manifolds, where they are identified along the common boundary S. Therefore, $Y - f^{-1}(\Delta')$ is an orientable 3-manifold. As in the proof of Lemma 10, we can show that each component of $f^{-1}(\Delta' - f(S))$ has exactly one boundary component. Let $\tilde{\Delta}'$ be a component of $f^{-1}(\Delta' - f(S))$. Then $\partial\tilde{\Delta}'$ is an embedded circle in ∂Y and it represents the homology class a in $H_1(\partial Y; \mathbb{Z})$, where a is defined in Section 3.4. Since $\tilde{\Delta}' \cup S$ is a cobordism between S and $\partial\tilde{\Delta}'$, $[S] = [\partial\tilde{\Delta}']$ in $H_1(Y; \mathbb{Z})$, where [ ] denotes a homology class. Now $f_*([S]) = \pm k([f(S)]) = \pm k\mu$ since $f: S \to f(S)$ is a k-fold covering projection and $f_*([\partial\tilde{\Delta}']) = f_*(a) = \mu$. Therefore, $k = 1$, thus H/H_0 is a trivial group. This implies that every element of H fixes the points of S, in particular, $\sigma(E_1)$ fixes the points of S which is a contradiction. □

References

[1] E. Andreev, On convex polyhedra of finite volume in Lobacevskii spaces, Mat. Sb. 83 (125) (1970); english translation: Math. USSR-Sb. 12 (1970) 255–259.

[2] L. Best, On torsion-free discrete subgroups of PSL(2, ℂ) with compact orbit space, Canad. J. Math. 23 (1971) 451–460.

[3] G. Bredon, Introduction to Compact Transformation Groups (Academic Press, New York, 1972).

[4] A. Brunner, M. Frame, Y. Lee and N. Wielenberg, Classifying torsion-free subgroups of Picard group, Trans. Amer. Math. Soc. 282 (1984) 205–235.

[5] A. Brunner, Y. Lee and N. Wielenberg, Polyhedral groups and graph amalgamation products, Topology Appl. 20 (1985) 289–304.

[6] H. Coxeter, Discrete groups generated by reflections, Ann. of Math. 35 (1934) 588–621.

[7] A. Hatcher, Hyperbolic structures of arithmatic type on some link complements, J. London Math. Soc. 2 (1983) 345–355.

[8] Y. Lee, Geometric method for presenting subgroups of discrete groups, Topology Appl. 18 (1984) 179–195.

[9] Y. Lee, Abelian rank of normal torsion-free finite index subgroups of polyhedral groups, Trans. Amer. Math. Soc. 290 (2) (1985) 735–745.
[10] R. Riley, A quadratic parabolic group, Math. Proc. Cambridge Philos. Soc. 77 (1975) 281–288.
[11] E. Spanier, Algebraic Topology (McGraw-Hill, New York, 1966).
[12] W. Thurston, The Geometry and Topology of 3-Manifolds (Princeton Univ. Press, Princeton).
[13] N. Wielenberg, The structure of certain subgroups of the Picard group, Math. Proc. Cambridge Philos. Soc. 84 (1978) 427–436.