Nonnegative Solutions of a Nonlinear Recurrence

JOHN S. LEW AND DONALD A. QUARLES, JR.

Mathematical Sciences Department, IBM T. J. Watson Research Center,
Yorktown Heights, New York 10598, U.S.A.

Communicated by Paul G. Nevai

Received June 28, 1982

Orthonormal polynomials with weight $|\tau|^\rho \exp(-\tau^r)$ have leading coefficients with recurrence properties which motivate the more general equations

$$\xi_m(\xi_{m-1} + \xi_{m} + \xi_{m+1}) = \gamma_m, \quad m = 1, 2, \ldots,$$

where $\gamma_0$ is a fixed nonnegative value and $\gamma_1, \gamma_2, \ldots$ are positive constants. For this broader problem, the existence of a nonnegative solution is proved and criteria are found for its uniqueness. Then, for the motivating problem, an asymptotic expansion of its unique nonnegative solution is obtained and a fast computational algorithm, with error estimates, is given.

1. INTRODUCTION

Given $\rho > -1$ and $r = 1, 2, \ldots$, let $w(\tau) = |\tau|^\rho \exp(-|\tau|^r)$, where $-\infty < \tau < +\infty$. Then the weight function $w(\tau)$ defines unique orthonormal polynomials $p_0(\tau), p_1(\tau), p_2(\tau), \ldots$ such that

$$\int_{-\infty}^{\infty} p_m(\tau) p_n(\tau) w(\tau) d\tau = \delta_{mn}$$

and $p_m(\tau) = \pi_m \tau^m + \text{lower terms}$, where the coefficients $\pi_m > 0$. We set $p_{-1}(\tau) = 0, \pi_{-1} = 0$. Freud [3-5] and Nevai [8] have studied the ratios $\pi_{m-1}/\pi_m$ because these determine the polynomials $p_m(\tau)$. When $r = 2$, Freud shows

$$\frac{(\pi_{m-1}/\pi_m)^2}{\pi_m} = \left(\frac{1}{4}\right)[2m + \rho - \rho(-1)^m], \quad m = 0, 1, 2, \ldots.$$  \hspace{1cm} (1.2)

When $r = 4$, he finds that these ratios obey a nonlinear recurrence relation. Indeed, if $\xi_m = (\pi_{m-1}/\pi_m)^2$, then $\xi_0 = 0$ and all higher $\xi_m > 0$, while

$$\xi_m(\xi_{m-1} + \xi_{m} + \xi_{m+1}) = \left(\frac{1}{4}\right)[2m + \rho - \rho(-1)^m], \quad m = 1, 2, \ldots.$$  \hspace{1cm} (1.3)

Freud [5] shows that $\lim_{m}(12/m)^{1/2} \xi_m = 1$. 

357
Here we extend and sharpen these results. Taking any positive sequence \((\gamma_1, \gamma_2, \ldots)\) we fix \(\xi_0 \geq 0\) and study the real sequences \((\xi_1, \xi_2, \ldots)\) such that

\[
(\xi_m(\xi_{m-1} + \xi_m + \xi_{m+1}) = \gamma_m^2, \quad m = 1, 2, \ldots. \quad (1.4)
\]

Since the value \(\xi_0\) is already given, each real \(\xi_1\) inductively determines an infinite sequence \((\xi_1, \xi_2, \ldots)\) unless some component \(\xi_m\) is precisely zero—and this last condition excludes just countably many \(\xi_1\). If \(\gamma_m = \gamma\) and \(\xi_0 \neq 0\) then (1.4) has a period-4 solution, namely,

\[
(\xi_0, \xi_1, \xi_2, \ldots) = (\xi_0, \gamma^2/\xi_0, -\gamma^2/\xi_0, -\xi_0, \xi_0, \ldots), \quad (1.5)
\]

and this sequence has some negative terms; but our prime concern will be nonnegative \(\xi_m\). Unless we state otherwise, hereafter the \(\gamma_m\) are positive and the \(\xi_m\) are nonnegative.

Other sequences give useful illustrations. If \(\gamma_m = \beta m + \gamma\), where \(\beta \geq 0\) and \(\gamma = \xi_0\sqrt{3} > 0\), then the following satisfies (1.4):

\[
\xi_m = (\beta m + \gamma)/\sqrt{3} \quad \text{for } m = 1, 2, \ldots. \quad (1.6)
\]

When \(\beta = 0\), these constants \(\gamma_m\) include the previous example, but the values (1.6) exceed 0; so these \(\xi_m\) furnish a different solution. Yet another solution provides a later counterexample. If \(\gamma_m = \gamma(\sigma^m - \sigma^{-m})\), where \(\sigma > 1\) and \(\gamma > \xi_0 = 0\), then the following satisfies (1.4):

\[
\xi_m = \gamma(\sigma + 1 + \sigma^{-1})^{-1/2} (\sigma^m - \sigma^{-m}) \quad \text{for } m = 1, 2, \ldots. \quad (1.7)
\]

Further notation simplifies this work. The set \(\mathbb{R}^n\) (resp. \(\mathbb{R}^\infty\)) of all real \(n\)-tuples (resp. real infinite sequences) is a vector space under componentwise addition and real scalar multiplication. The scalar 0, with no ambiguity, will denote the zero vector in either space. Typical elements \(x, y\) of these spaces will have respective components \(\xi_m, \eta_m\) for positive indices \(m\); the scalars \(\xi_0, \eta_0\) with subscript zero will not be components of the associated vectors. Given any \(x = (\xi_1, \xi_2, \ldots)\) and \(y = (\eta_1, \eta_2, \ldots)\), write \(x \leq y\) (resp. \(x < y\)) if all \(\xi_m \leq \eta_m\) (resp. \(\xi_m < \eta_m\)). Call \(x\) nonnegative when \(0 \leq x\); call \(x\) positive when \(0 < x\).

The finite dimensional space \(\mathbb{R}^n\) will use the standard \(l^\infty\) norm:

\[
||x|| = \max \{|\xi_m|: m = 1, \ldots, n\}. \quad (1.8)
\]

More generally, let \(0 < a = (a_1, a_2, \ldots) \in \mathbb{R}^\infty\) and let \(0 < b = (\beta_1, \beta_2, \ldots) \in \mathbb{R}^\infty\). Then the set \(\mathbb{R}^{\infty}_a\), with the norm

\[
||x||_a = \sup \{|\xi_m/\alpha_m|: m = 1, 2, \ldots\} \quad (1.9)
\]
is a complete metric space with distance \( d(x, y) = \|x - y\|_a \). (This definition allows infinite distances, but these cause no difficulties.) If \( \| \cdot \|_b \)-convergence implies \( \| \cdot \|_a \)-convergence, then \( \| b \|_a < +\infty \); however, \( \| x \|_a \leq \| x \|_b \| b \|_a \), and this proves the converse. Thus \( a \) and \( b \) define the same topology (so that \( a \) and \( b \) yield equivalent norms) if and only if \( \| a \|_b \), \( \| b \|_a \) < +\( \infty \).

This work fixes positive \( c = (y_1, y_2, \ldots) \in \mathbb{R}^\infty \) and \( \epsilon_0 > 0 \); it seeks nonnegative \( x = (x_1, x_2, \ldots) \in \mathbb{R}^\infty \) where \( x \) satisfies (1.4). Section 2 obtains some basic results for an auxiliary linear recurrence with variable coefficients. Section 3 discusses the truncated problem of finite sequences \( \xi_1, \ldots, \xi_n \) with fixed \( \xi_0, \xi_{n+1} \). Sections 4, 5, 6, for infinite sequences, using an equivalent formulation as a fixed-point problem, prove the existence of nonnegative solutions, and give several criteria for uniqueness. Conversely, an argument in Section 6 generates multiple nonnegative solutions for some vectors \( c \). Also, the stronger hypotheses of Section 5 yield important results for norm-convergence. Section 7, on sequence computation, shows the instability of forward iteration and gives a stable algorithm, with error estimates. Sections 8 and 9, for the original problem (1.3), find an asymptotic expansion of the unique nonnegative solution, and report computational experience with a still more refined algorithm.

2. LINEAR RECURRENCE

Given any complex sequence \((\omega_1, \omega_2, \ldots)\), consider the complex sequence \((\zeta_0, \zeta_1, \zeta_2, \ldots)\) such that

\[
\zeta_{m+1} = \omega_m \cdot \zeta_m + \zeta_{m-1} = 0
\]

when \( m = 1, 2, \ldots \). This section, for certain recurrences (2.1), extends a well-known theorem of Poincaré (Montel [7, Chap. 5]) and collects auxiliary results for later use. If \((\zeta_0, \zeta_1) = (1, 0)\) or \((0, 1)\), respectively, then \( \omega_1, \ldots, \omega_m \) determine the value \( \zeta_{m+1} \), whence \( \zeta_{m+1} \) is some function \( B_m(\omega_1, \ldots, \omega_m) \) or \( C_m(\omega_1, \ldots, \omega_m) \), and \( B_0 = 0, C_0 = 1 \). Then the linearity of (2.1) gives the solution for any \((\zeta_0, \zeta_1)\):

\[
\zeta_{m+1} = \zeta_0 B_m(\omega_1, \ldots, \omega_m) + \zeta_1 C_m(\omega_1, \ldots, \omega_m).
\]

If \( \omega_1 = \cdots = \omega_m = \omega \) then we adopt a simpler notation: \( B_m(\omega, \ldots, \omega) = B_m(\omega) \) and \( C_m(\omega_1, \ldots, \omega_m) = C_m(\omega) \). Also, \( U_m(\tau) \), for each integer \( m \), is the \( m \)'th Chebyshev polynomial of the second kind (Abramowitz and Stegun [1, (22.2.5)]).

**Lemma 2.1.** If \( m = 0, 1, 2, \ldots \) then \( C_m(\omega) = U_m(\omega/2) \). Indeed, \( C_m(2) = m + 1 \) and \( C_m(-2) = (-1)^m (m + 1) \). If \( \omega \neq \pm 2 \) and \( \sigma \) is either complex
number such that \( \sigma + \sigma^{-1} = \omega \), then \( C_m(\omega) = (\sigma^{m+1} - \sigma^{-m-1})/(\sigma - \sigma^{-1}) \).
Also, \( B_m(\omega) = -C_{m-1}(\omega) \) for \( m = 0, 1, 2, \ldots \), where these explicit forms define \( C_m(\omega) \) for \( m < 0 \).

**Proof.** The explicit forms \( C_m(\omega) \) and \(-C_{m-1}(\omega)\) satisfy the recurrence (2.1). The resulting expressions \( B_m(\omega) \) and \( C_m(\omega) \) reproduce the stated \( \zeta_0 \) and \( \zeta_1 \). If \( \omega = 2 \cos \theta \) and \( 0 < \theta < \pi \), then \( \sigma = \exp(\pm i\theta) \) and (Abramowitz and Stegun [1, Eq. (22.3.16)]):

\[
C_m(\omega) = \sin(m + 1)\theta \sin \theta = U_m(\cos \theta) = U_m(\omega/2). \tag{2.3}
\]

Analytic continuation admits all other complex \( \omega \).

Now let \( I_n \) be the \( n \times n \) identity matrix, where \( n = 1, 2, \ldots \), and let \( E_n \) be the \( n \times n \) matrix \((e_{ij})\) such that \( e_{ij} = 1 \) when \( |i - j| = 1 \) but otherwise \( e_{ij} = 0 \). If \( \theta_{mn} = m\pi/(n + 1) \), where \( m = 1, \ldots, n \), then

\[
|E_n \cdot (\sin \theta_{mn}, \ldots, \sin n\theta_{mn})^{\text{transpose}}|_k = 2 \cos \theta_{mn} \cdot \sin k\theta_{mn} \tag{2.4}
\]

for \( k = 1, \ldots, n \). Thus \( E_n \) has the \( n \) distinct eigenvalues \( 2 \cos \theta_{mn} \), and these have the corresponding nonzero eigenvectors \((\sin \theta_{mn}, \ldots, \sin n\theta_{mn})^{\text{transpose}}\).

**Lemma 2.2.** If \( n = 0, 1, 2, \ldots \) then

\[
C_n(\omega_1, \ldots, \omega_n) = \det(E_n + \text{diag}(\omega_1, \ldots, \omega_n)). \tag{2.5}
\]

If all \( \omega_m \geq 2 \), where \( m = 1, \ldots, n \), then \( E_n + \text{diag}(\omega_1, \ldots, \omega_n) \) is a positive definite matrix, and

\[
0 < C_n(\min(\omega_1, \ldots, \omega_n)) \leq C_n(\omega_1, \ldots, \omega_n) \leq C_n(\max(\omega_1, \ldots, \omega_n)). \tag{2.6}
\]

If also \( \omega'_m \geq \omega_m \) and some \( \omega'_k > \omega_k \), where \( 1 \leq k \leq n \), then

\[
C_n(\omega_1, \ldots, \omega_n) < C_n(\omega'_1, \ldots, \omega'_n). \tag{2.7}
\]

**Proof.** A bottom-row expansion of \( \det(E_n + \text{diag}(\omega_1, \ldots, \omega_n)) \) obtains the recurrence (2.1) for these determinants. (Any such expansion by minors assigns the value 1 to a zero-order determinant.) But inspection yields (2.5) when \( n = 0, 1, 2 \); so induction proves (2.5) when \( n = 0, 1, \ldots \). The matrix \( E_n + 2I_n \) is positive definite, because the principal minors have the form \( C_m(2) = m + 1 > 0 \). The matrix \( \text{diag}(\omega_1 - 2, \ldots, \omega_n - 2) \) is nonnegative definite whenever all \( \omega_m \geq 2 \). These matrices have positive definite sum \( E_n + \text{diag}(\omega_1, \ldots, \omega_n) \), which has strictly positive determinant \( C_n(\omega_1, \ldots, \omega_n) \). But expansion via the \( m \)th row gives

\[
\partial C_n(\omega_1, \ldots, \omega_n)/\partial \omega_m = C_{m-1}(\omega_1, \ldots, \omega_{m-1}) \cdot C_{n-m}(\omega_{m+1}, \ldots, \omega_n) > 0; \tag{2.8}
\]

and this fact, for all variables \( \omega_m \), yields (2.7). Finally, (2.7) implies (2.6).
THEOREM 2.3. If $2 \leq \omega_1, \omega_2, \ldots$ and $0 < \varepsilon$ such that $2 + \varepsilon < \omega_- = \lim \inf_m \omega_m < +\infty$ and $\omega_+ = \lim \sup_m \omega_m < +\infty$, then this $\varepsilon$ determines constants $\gamma_-(\varepsilon), \gamma_+(\varepsilon) > 0$ such that
\[ \gamma_-(\varepsilon) C_m(\omega_- - \varepsilon) \leq C_m(\omega_1, \ldots, \omega_m) \leq \gamma_+(\varepsilon) C_m(\omega_+ + \varepsilon), \quad m = 0, 1, 2, \ldots. \tag{2.9} \]

Proof. Any positive $\varepsilon$ fixes an integer $n$ such that $\omega_m \geq \omega_- - \varepsilon$ whenever $m > n$. If $\omega_m' = \omega_m$ for $m \leq n$, while $\omega_m' = \omega_- - \varepsilon$ for $m > n$, then
\[ m + 1 = C_m(2) \leq C_m(\omega_1, \ldots, \omega_m') \leq C_m(\omega_1, \ldots, \omega_m) \] \tag{2.10}
for $m = 0, 1, 2, \ldots$. Choose the root $\sigma > 1$ satisfying the equation $\sigma + \sigma^{-1} = \omega_- - \varepsilon$. If $m > n$ then Lemma 2.1 implies constants $\alpha, \beta$ such that
\[ m + 1 \leq C_m(\omega_1, \ldots, \omega_m') = \alpha \sigma^{m-n} + \beta \sigma^{n-m}. \] \tag{2.11}
(To prove this, replace $m$ by $m - n$.) Moreover, $\alpha > 0$, since $m + 1$ may be larger than $|\beta|$. Thus $C_m(\omega_1, \ldots, \omega_m')/C_m(\omega_- - \varepsilon) > 0$ when $m = 0, 1, 2, \ldots$, and this ratio has a positive limit as $m \to \infty$, whence
\[ C_m(\omega_1, \ldots, \omega_m)/C_m(\omega_- - \varepsilon) \geq \inf_m C_m(\omega_1, \ldots, \omega_m')/C_m(\omega_- - \varepsilon) = \gamma_-(\varepsilon) > 0, \tag{2.12} \]
when $m = 0, 1, 2, \ldots$. Similar arguments find $\gamma_+(\varepsilon)$ unless $\omega_+ = +\infty$, but then $C_m(\omega_+ + \varepsilon) = +\infty$ whenever $m > 0$.

Remark. One easily finds examples where $\gamma_-(\varepsilon) \to 0$ or $\gamma_+(\varepsilon) \to +\infty$ as $\varepsilon \to 0$.

3.FINITE SEQUENCES

Here, keeping the positive constants $\gamma_1, \gamma_2, \ldots$, we specify a positive integer $n$, and, fixing real numbers $\xi_0, \xi_{n+1}$, we seek real values $\xi_1, \ldots, \xi_n$ such that
\[ \xi_m(\xi_{m-1} + \xi_m + \xi_{m+1}) = \gamma_m^2 \quad \text{for } m = 1, \ldots, n. \tag{3.1} \]
First, we restate the problem. Given $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $\xi_1, \ldots, \xi_n \neq 0$, define
\[ F(x) = -\sum_{m=1}^n \gamma_m^2 \log |\xi_m| + \left(\frac{1}{2}\right) \sum_{m=0}^{n+1} \xi_m^2 + \sum_{m=0}^n \xi_m \xi_{m+1}; \] \tag{3.2}
then $F'(x) = (\partial F/\partial \xi_1, \ldots, \partial F/\partial \xi_n)$, where
\[ \partial F/\partial \xi_m = -\gamma_m^2 \xi_m^{-1} + \xi_{m-1} + \xi_m + \xi_{m+1} \quad \text{for } m = 1, \ldots, n. \tag{3.3} \]
Clearly, $x$ satisfies (3.1) if and only if $F'(x) = 0$, because necessarily $\xi_1, \ldots, \xi_n \neq 0$ when $x$ satisfies either of these conditions. Hence the solutions of (3.1) are precisely the stationary points of $F(x)$.

Truncations of sequence (1.5) yield solutions for equal $\gamma_m$ having negative values for some $\xi_m$. (Appendix B gives further such solutions.) Nevertheless, here we suppose nonnegative $\xi_0, \xi_{n+1}$, and again we admit only nonnegative $\xi_1, \ldots, \xi_n$. But if $x \geq 0$, then, by (1.8),

$$F(x) \leq -\left( \sum_{m=1}^{n} \gamma_m^2 \right) \log \|x\| + \left( \frac{1}{2} \right) \|x\|^2; \quad (3.4)$$

while if $k = 1, \ldots, n$, then, for each $k$,

$$F(x) \geq -\gamma_k^2 \log |\xi_k| - \left( \sum_{m \neq k} \gamma_m^2 \right) \log \|x\|. \quad (3.5)$$

Hence $\{x \in \mathbb{R}^n : x \geq 0; \sigma \geq F(x)\}$, for any real $\sigma$, is bounded in the norm (1.8); and this level set, via this fact, is bounded away from the coordinate hyperplanes. Also, the set is closed, whence it is compact. Thus $F(x)$, on the orthant $\{x \in \mathbb{R}^n : x \geq 0\}$, achieves its minimum at some strictly positive $x^*$. This $x^*$ is a nonnegative solution; our next theorem shows that no other $x$ is a nonnegative solution.

**Lemma 3.1.** If $x$ satisfies (3.1), where $\xi_0, \xi_1, \ldots, \xi_{n+1} \geq 0$, then either $0 < \xi_m < \gamma_m$ for $m = 1, \ldots, n$ or, specifically, $n = 1$ and $(\xi_0, \xi_1, \xi_2) = (0, \gamma_1, 0)$.

**Proof:** If nonnegative $\xi_1, \ldots, \xi_n$ satisfy (3.1) then each exceeds zero. If $n = 1$ and $\xi_0 = \xi_2 = 0$, then $\xi_1 = \gamma_1$ since $\xi_1 > 0$. Otherwise $\xi_{m-1} + \xi_{m+1} > 0$ for $m = 1, \ldots, n$; whence $\xi_m^2 < \xi_m (\xi_{m-1} + \xi_{m+1}) = \gamma_m^2$ and $\xi_m < \gamma_m$.

**Theorem 3.2.** If $\xi_0, \xi_{n+1} \geq 0$ then $F(x)$ has a unique stationary point $x^*$ in $\{x \in \mathbb{R}^n : x \geq 0\}$, and this is the unique nonnegative solution of (3.1). Moreover, $F(x)$, on this orthant, achieves its minimum at $x^*$.

**Proof:** Let the set $K = \{x \in \mathbb{R}^n : 0 \leq \xi_m \leq \gamma_m, m = 1, \ldots, n\}$. Then the nonnegative stationary points of $F(x)$ are just the nonnegative solutions of (3.1), and any such solutions, by Lemma 3.1, are necessarily elements of the set $K$. By definition, the Hessian $F''(x)$ is the $n \times n$ matrix $(\partial^2 F/\partial \xi_i \partial \xi_j)$; by (3.3),

$$F''(x) = F_n + \text{diag}(1 + \gamma_1^2 \xi_1^{-2}, \ldots, 1 + \gamma_n^2 \xi_n^{-2}). \quad (3.6)$$

Here Section 2 defines the matrix $E_n$. If $x$ has domain $K$ then, by definition, $1 + \gamma_m^2 \xi_m^{-2} \geq 2$ when $m = 1, \ldots, n$; so, by Lemma 2.2, $F''(x)$ is positive definite and $F(x)$ is strictly convex (Ortega and Rheinboldt [10, p. 87]). Thus $F(x)$,
on the domain $K$, can have no stationary point but a unique minimum. However, $F(x)$ on $\{x \in R^n : x \geq 0\}$ has a global minimum by our previous remarks.

**Corollary 3.3.** For $n(1)$, $n(2)$ any nonnegative integers and $k = 1, 2$, let $\xi_{0}^{(k)}, \ldots, \xi_{n(k)+1}^{(k)} \geq 0$. For $\gamma_1, \gamma_2, \ldots$ any positive reals and $m = 1, \ldots, n(k)$, let

$$
\xi_m^{(k)}(\xi_{m-1}^{(k)} + \xi_{m}^{(k)} + \xi_{m+1}^{(k)}) = \gamma_m^2.
$$

(3.7)

If $\xi_i^{(1)} = \xi_i^{(2)}$ for any distinct subscripts $p$, $q$, then $\xi_i^{(1)} = \xi_i^{(2)}$ for all common indices $i$.

**Proof.** Assuming $p < q$, replace $m$ by $m - p$. Setting $n = q - p - 1$, use Theorem 3.2 if $n > 0$. Then $\xi_i^{(1)} = \xi_i^{(2)}$ when the new index $i = 0, \ldots, n + 1$ or the old index $i = p, \ldots, q$. Forward or backward recurrence determines uniquely all other $\xi_i^{(k)}$.

**Corollary 3.4.** If $\xi_0$ is a constant and $\xi_1, \ldots, \xi_n$ obey (3.1), then $\xi_{n+1}(\xi_1) = q_{n+1}(\xi_1)/r_{n+1}(\xi_1)$, where $q_{n+1}$, $r_{n+1}$ are real polynomials. Indeed,

$$
\frac{q_{m+1}}{r_{m+1}} = \frac{\gamma_m^2 r_m^2 r_{m-1} - q_m^2 r_m r_{m-1} - q_m r_m q_{m-1}}{q_m r_m r_{m-1}}, \quad m = 1, \ldots, n.
$$

(3.8)

where $q_0(\xi)$ is the constant $\xi_0$, $q_1(\xi) = \xi$, $r_0(\xi) = r_1(\xi) = 1$. Moreover, $\xi_{n+1}$ has an open domain $S_{n+1}$, satisfying $[0, +\infty) \subset S_{n+1}$, whereon $\xi_{n+1}$ uniquely determines $\xi_1$ provided $\xi_0, \ldots, \xi_n \geq 0$. The function $\xi_1(\xi_{n+1})$ for each $n$ is monotone and bicontinuous on this domain.

**Proof.** Direct use of (3.1) proves everything through (3.8). By Corollary 3.3, if $S = \{\xi_1 : 0 \leq \xi_0, \ldots, \xi_{n+1}\}$ then $\xi_{n+1}(\xi_1)$, on $S$, has a well-defined inverse function, whence $\xi_{n+1}(\xi_1)$, on $S$, has no local extrema. Specifically, this function has no minimum where $\xi_{n+1}(\xi_1) = 0$. Hence $\xi_{n+1}(\xi_1)$ is continuous and monotonic on some larger open domain $S'$ including $S$, and its inverse is continuous and monotonic on an open domain $S_{n+1}$ including $[0, +\infty)$.

**4. Infinite Sequences**

Here we treat the infinite system (1.4): demanding nonnegative solutions, we prove a general existence theorem and obtain a uniqueness criterion. First, we state an equivalent fixed-point problem, using an auxiliary function $g$. Specifically, we let

$$
\sigma = g(\tau) = -\tau + (1 + \tau^2)^{1/2},
$$

(4.1)
LEW AND QUARLES

where \(-\infty < \tau < +\infty\); equivalently, we have

\[ 1 = \sigma^2 + 2\sigma \tau \quad \text{and} \quad \sigma > 0. \quad (4.2) \]

Thus \(g'(\tau) = da/d\tau = -2/(1 + \sigma^{-2})\), and \(g(0) = -g'(0) = 1\). Hence \(g(\tau)\) and \(-g'(\tau)\) decrease from 1 to 0 as \(\tau\) increases from 0 to \(+\infty\). Fixing real \(\xi_0 \geq 0\) and \(0 < c = (y_1, y_2, \ldots) \in R^\infty\), define the map \(T: R^\infty \rightarrow R^\infty\) via

\[ (Tx)_m = y_m \cdot g((\xi_{m-1} + \xi_{m+1})/2y_m), \quad m = 1, 2, \ldots \quad (4.3) \]

Then (1.4), for each positive \(m\), is a quadratic equation in \(\xi_m\), and the nonnegative root of this is the right side of (4.3). Therefore the nonnegative solutions of (1.4) are precisely the fixed points of \(T\). Our first lemmas concern this map \(T\).

**Lemma 4.1.** If \(0\) is the zero vector then \(T_0 \leq c\); if \(x\) is any real vector then \(0 < Tx\). If \(x \leq y\) (resp. \(x < y\)) then \(Ty \leq Tx\) (resp. \(Ty < Tx\)) and \(T^2x \leq T^2y\) (resp. \(T^2x < T^2y\)). If \(\xi_0 - \eta_0\) and \(x, y \in R^\infty\), while \(\tau_m \leq (1/2\gamma_m) \min(\xi_{m-1} + \xi_{m+1}, \eta_{m-1} + \eta_{m+1})\) \(\quad (4.4)\)

for \(m = 1, 2, \ldots, \) then

\[ |(Ty)_m - (Tx)_m| \leq |g'(\tau_m)/2| \cdot |\eta_{m-1} - \xi_{m-1}| + |\eta_{m+1} - \xi_{m+1}| \]

\[ \quad \text{for} \quad m = 1, 2, \ldots \quad (4.5) \]

**Proof.** Clearly \((T_0)_1 = y_1g(\xi_0/2y_1) \leq y_1\), since \(\xi_0 \geq 0\); and \((T_0)_m - y_m\) when \(m \geq 2\). Also, any \((Tx)_m > 0\) because \(g\) is strictly positive, and \(T\) reverses inequalities because \(g\) is strictly decreasing. Finally, the mean value theorem asserts

\[ (Ty)_m - (Tx)_m = (\frac{1}{2})g'(\tau_m) \left[ (\eta_{m-1} + \eta_{m+1}) - (\xi_{m-1} + \xi_{m+1}) \right], \quad (4.6) \]

where the interval containing \(\tau_m\) has bounds \((\xi_{m-1} + \xi_{m+1})/2y_m\) and \((\eta_{m-1} + \eta_{m+1})/2y_m\). But we increase \(|g'(\tau_m)|\) when we decrease \(\tau_m\).

**Lemma 4.2.** If \(k = 0, 1, 2, \ldots, \) then

\[ 0 = T^00 \leq T^{2k}0 < T^{2k+2}0 < T^{2k+3}0 < T^{2k+1}0 \leq T0. \quad (4.7) \]

Thus, componentwise, \(T^{2k}0\) has monotone increasing limit \(x^-\) and, similarly, \(T^{2k+1}0\) has monotone decreasing limit \(x^+\) as \(k \to \infty\). Also, \(Tx^- = x^+\) and \(Tx^+ = x^-\), while

\[ Tc < x^- \leq x^+ < c. \quad (4.8) \]

If \(T\) has fixed point \(x^*\), so that \(Tx^* = x^*\), then \(x^- \leq x^* \leq x^+.\)
Proof. Lemma 4.1 shows, first, that $0 < T^k 0$ if $k = 1, 2, \ldots$, whence $T^{k+1} 0 < T 0$. Specializing these facts gives $0 < T^2 0 < T 0$; then using $T$ gives $T^3 0 < T^2 0 < T 0$. Together, these are (4.7) when $k = 0$. Repeatedly using $T^2$ inductively completes (4.7). If $j$ is any positive integer, then the increasing sequence $T^0 0, T^1 0, T^2 0, \ldots$ has upper bound $T^{j+1} 0$, and the decreasing sequence $T 0, T^2 0, T^3 0, \ldots$ has lower bound $T^j 0$; so these sequences have componentwise limits $x^-, x^+$. Thus $T 0 \leq T^2 0 < x^- \leq T^{j+1} 0$, while $T^{j+1} 0 \leq x^+ < T 0 \leq c$, and these yield (4.8) when $j \to \infty$. But $g$ is continuous, whence

$$
(Tx^-)_m = \gamma_m \cdot g(\lim_k ((T^{2k} 0)_{m-1} + (T^{2k} 0)_{m+1})/2\gamma_m)
$$

$$= \gamma_m \cdot \lim_k g(((T^{2k} 0)_{m-1} + (T^{2k} 0)_{m+1})/2\gamma_m)
$$

$$= \lim_k (T^{2k+1} 0)_m = (x^+)_m. \quad (4.9)
$$

Therefore $Tx^- = x^+$; likewise $Tx^+ = x^-$. If $Tx^* = x^*$ then $0 < x^*$; so $T^{2k} 0 < T^{2k} x^* = x^* = T^{2k+1} x^* < T^{2k+1} 0$, where $k = 0, 1, 2, \ldots$. Let $k \to \infty$ to find $x^- \leq x^* \leq x^+$. 

Remark. The sequences $x^-$ and $x^+$ underlie many later results.

**Theorem 4.3.** If $\xi_0 \geq 0$ and $c > 0$ then $T$ has a fixed point, whence (1.4) has a nonnegative solution. Either $x^- = x^+$ and this solution is unique, or $x^- < x^+$ and this solution is not unique. Specifically, it is unique when $\inf_m \gamma_m/m = 0$.

Proof. Given $x^- = (\xi^-_1, \xi^-_2, \ldots)$ and $x^+ = (\xi^+_1, \xi^+_2, \ldots)$, define

$$x^z = (\xi^z_1, \xi^z_2, \xi^z_3, \xi^z_4, \ldots) \quad \text{and} \quad x^z = (\xi^z_1, \xi^z_2, \xi^z_3, \xi^z_4, \ldots). \quad (4.10)
$$

Clearly $Tx^z = x^z$ and $Tx^z = x^z$, because $Tx^- = x^+$ and $Tx^+ = x^-$. If solution $x^*$ is unique then $x^z = x^z$, whence $x^- = x^+$; conversely, solution $x^*$ is unique whenever $x^- = x^+$, since $x^- < x^+$, and vice versa. But $x^z = x^z$, by Corollary 3.3, if $\xi^- = \xi^+$ for any positive $m$. If $\xi^z = \xi^z - \xi^z$ for $m = 0, 1, 2, \ldots$, then $\xi^- = 0$, since $\xi^- = \xi^- = \xi_0$. Also,

$$\xi^z_{m+1} = (\xi^z_{m-1} + \xi^z_{m+1}) - (\xi^z_{m-1} + \xi^z_{m+1})
$$

$$= (\gamma^z_m/\xi^z_{m-1} - \xi^z_{m-1}) - (\gamma^z_m/\xi^z_{m-1} - \xi^z_{m+1}) = \omega^z_m \cdot \xi^z_m \quad (4.11)
$$

by (4.2) and Lemma 4.2, where $\omega^z_m = 1 + \gamma^z_m/\xi^z_m < 2$ by (4.8). However, $\xi^z_1, \xi^z_2, \ldots > 0$ for multiple solutions, whence

$$\gamma_m > \xi_m = \xi_1 \cdot C_{m-1}(\omega_1, \ldots, \omega_{m-1}) > \xi_1 \cdot C_{m-1}(2) = m^2 \xi^z_1 \quad (4.12)
$$

by (2.2), (2.7), and (4.8), where $m = 1, 2, \ldots$; thus $\inf_m \gamma_m/m \geq \xi_1 > 0$. 

5. Norm-Convergence

Here we add a hypothesis on the given vector \( c \), and we treat convergence in the sequence norms (1.9). Choosing any \( \gamma_0 \geq 0 \), define \( \phi_c(\gamma_0) = \inf S(c) \) and \( \psi_c(\gamma_0) = \sup S(c) \), where \( S(c) = \{(\gamma_{m-1} + \gamma_{m+1})/2 \gamma_m : m = 1, 2, \ldots \} \).

Previously, we required only \( 0 < c \in \mathbb{R}^\infty \), and thus \( 0 \leq \phi_c \leq \psi_c \leq +\infty \). Hereafter, we assume also

\[
0 < \inf \{ \gamma_{m+1}/\gamma_m : m = 1, 2, \ldots \} \leq \sup \{ \gamma_{m+1}/\gamma_m : m = 1, 2, \ldots \} < +\infty, \tag{5.1}
\]

whence now \( 0 < \phi_c \leq \psi_c < +\infty \). Conversely, the finiteness of \( \psi_c \) implies (5.1), independently of \( \gamma_0 \). Also, \( \phi_c(\gamma_0) \) and \( \psi_c(\gamma_0) \) are continuous, and \( \phi_c(\gamma_0) \) is bounded, while \( \psi_c(\gamma_0) \to +\infty \) as \( \gamma_0 \to +\infty \). If \( 0 \leq \beta_0 \) and \( 0 < b \in \mathbb{R}^\infty \), then \( \phi_b(\beta_0) \) and \( \psi_b(\beta_0) \) have similar definitions and properties. Moreover, \( b \) is a concave sequence when \( \psi_b \leq 1 \).

If \( 0 < k < \lambda \leq 1 \) then \( 0 \leq \kappa \phi_c \leq \lambda \psi_c < +\infty \), whence \( 0 < g(\lambda \psi_c) < g(k\phi_c) \leq 1 \). Taking \( \kappa_0 = 0 \), define

\[
\lambda_j = g(k_j \phi_c), \quad \kappa_{j+1} = g(\lambda_j \psi_c), \quad j = 0, 1, 2, \ldots \tag{5.2}
\]

Then \( \lambda_0 = 1 \) and \( \kappa_1 = g(\psi_c) \), so that \( 0 = \kappa_0 < \kappa_1 < \lambda_0 = 1 \) and, inductively,

\[
0 = \kappa_0 < \cdots < \kappa_j < \kappa_{j+1} < \lambda_j < \kappa_j < \cdots < \lambda_0 = 1. \tag{5.3}
\]

Hence \( \lambda_j \uparrow \lambda_* \) and \( \kappa_j \downarrow \kappa_* \) as \( j \to \infty \), where \( 0 < \kappa_* \leq \lambda_* < 1 \). Also, by continuity, \( \kappa_* = g(\lambda_* \psi_c) \) and \( \lambda_* = g(\kappa_* \phi_c) \), whence, by (4.2),

\[
1 = \lambda_*^2 + 2 \kappa_* \lambda_* \psi_c \quad \text{and} \quad 1 = \lambda_*^2 + 2 \kappa_* \lambda_* \psi_c. \tag{5.4}
\]

Eliminating unity, we find \( \lambda_*/\kappa_* ; \) substituting this, we find \( \kappa_* : \)

\[
\lambda_*/\kappa_* = (\psi_c - \phi_c) + [1 + (\psi_c - \phi_c)^2]^{1/2} \geq 1, \tag{5.5}
\]

\[
\kappa_*^2 = 1 + 2 \psi_c(\psi_c - \phi_c) + 2 \psi_c [1 + (\psi_c - \phi_c)^2]^{1/2}. \tag{5.6}
\]

Thus \( \kappa_*(\gamma_0) \) and \( \lambda_*(\gamma_0) \) are continuous functions. If \( \gamma_0 \to +\infty \), then \( \psi_c \to +\infty \) while \( \phi_c \) has an upper bound; but \( (1 - \lambda_*^2)/\phi_c = (1 - \kappa_*^2)/\psi_c \), whence \( \lambda_* \to 1 \) and \( \gamma_0 \lambda_* \to +\infty \). Also, \( \gamma_0 \kappa_* = \gamma_0 \lambda_* = 0 \) when \( \gamma_0 = 0 \). Previously, we have assumed a nonnegative \( \xi_0 \); hereafter, we can and will choose a nonnegative \( \gamma_0 \) such that

\[
\gamma_0 \kappa_*(\gamma_0) \leq \xi_0 = \gamma_0 \lambda_*(\gamma_0). \tag{5.7}
\]

Given \( y, z \in \mathbb{R}^\infty \), define \( [y, z] = \{ x \in \mathbb{R}^\infty : y \leq x \leq z \} \). If \( 0 \leq k \leq \kappa_* \leq \lambda_* \leq \lambda \leq 1 \), and \( x = (\xi_1, \xi_2, \ldots) \in [k \epsilon, \lambda \epsilon] \), then

\[
\kappa \phi_c \leq \kappa (\gamma_{m-1} + \gamma_{m+1})/2 \gamma_m \leq (\xi_{m-1} + \xi_{m+1})/2 \gamma_m \leq \lambda (\gamma_{m-1} + \gamma_{m+1})/2 \gamma_m \leq \lambda \psi_c. \tag{5.8}
\]
whence $T^jx \in [g(\lambda \psi_c) c, g(\kappa_\phi c) c]$. Therefore, if $j = 0, 1, 2, \ldots$, then, inductively,

$$T([\kappa_j c, \lambda_j c]) \subset [\kappa_{j+1} c, \lambda_{j+1} c] \subset [\kappa_j c, \lambda_j c],$$

$$T([\kappa_{j+1} c, \lambda_{j+1} c]) \subset [\kappa_{j+1} c, \lambda_{j+1} c] \subset [\kappa_{j+1} c, \lambda_{j+1} c].$$

But $0 < Tx$ if $x \in R^\infty$; so $0 < T^2 x < c$, or $T^2 x \in [\kappa_0 c, \lambda_0 c]$. Thus $T^{2j+1} x \in [\kappa_j c, \lambda_j c]$, where $j = 0, 1, 2, \ldots$. Hence $[\kappa_j c, \lambda_j c]$, invariant under $T$, must contain all fixed points of $T$; though $[\kappa_0 c, \lambda_0 c]$, by example (1.7), need contain no iterates $T^j x$ for finite $j$ (since $T^2 x = \kappa_j c$ and $T^{2j+1} = \lambda_j c$). These new concepts yield some sharper results.

**Theorem 5.1.** If $\inf \gamma_m / \sigma^m = 0$, where $\sigma + \sigma^{-1} = 1 + \lambda_*^{-2}$ and $\sigma > 1$, then $T$ has a unique fixed point.

**Proof.** If $x^+, x^-$ are distinct fixed points then $\kappa_* c \leq x^- < x^+ \leq \lambda_* c$. If $\zeta_m = x^+ - x^-$, where $m = 0, 1, 2, \ldots$, then $\zeta_m$ satisfies (4.11), where $\omega_m = 1 + \gamma_m / \zeta_m > 1 + \lambda_*^2 > 2$. But $\zeta_0 = 0$ and $\zeta_1 > 0$. Therefore $\inf \gamma_m / \sigma^m > 0$, because

$$\gamma_{m+1} > \omega_m > 1 + \lambda_*^2 > 2.$$ 

Let $0 < \xi_0 = \eta_0$ and $\kappa_j c \leq x, y \in R^\infty$ (and $\gamma_0 \kappa_* \leq \xi_0 < \gamma_0 \lambda_*$), where $j \geq 0$. Let $0 \leq \beta_0$ and $0 < b = (\beta_1, \beta_2, \ldots) \in R^\infty$ (where $\psi_b$ or $\|y - x\|_b$ may be $+\infty$); then

$$\|Ty - Tx\|_b \leq \psi_b \|g'(\kappa_j \phi c)\| \cdot \|y - x\|_b.$$ 

**Lemma 5.2.** Let $0 < \xi_0 = \eta_0$ and $\kappa_j c \leq x, y \in R^\infty$ (and $\gamma_0 \kappa_* \leq \xi_0 < \gamma_0 \lambda_*$), where $j \geq 0$. Let $0 \leq \beta_0$ and $0 < b = (\beta_1, \beta_2, \ldots) \in R^\infty$ (where $\psi_b$ or $\|y - x\|_b$ may be $+\infty$); then

$$\|Ty - Tx\|_b \leq \psi_b \|g'(\kappa_j \phi c)\| \cdot \|y - x\|_b.$$ 

**Proof.** If $m = 1, 2, \ldots$ then $\kappa_j \phi c \leq (1/\gamma_m) \min(\xi_{m-1} + \xi_m, \eta_{m-1} + \eta_m)$, whence (4.5) shows that

$$\frac{|(Ty)_m - (Tx)_m|}{\beta_m} \leq \frac{|g'(\kappa_j \phi c)|}{2} \left\{ \frac{\beta_m}{\beta_m} \frac{|\eta_{m-1} - \xi_{m-1}|}{\beta_m} + \frac{\beta_{m+1}}{\beta_m} \frac{|\eta_{m+1} - \xi_{m+1}|}{\beta_{m+1}} \right\}.$$ 

**Theorem 5.3.** Let $0 \leq \beta_0$ and $0 < b \in R^\infty$ such that $\psi_b \|g'(\kappa_* \phi c)\| < 1$. Let $0 \leq \xi_0$ and $0 \leq x^0 \in R^\infty$ such that $\|Tx^0 - x^0\|_b < +\infty$. If $K = \{x \in R^\infty : 0 \leq x; \|x - x^0\|_b < +\infty\}$ then $T(K) \subset K$ and $T | K$ has a unique fixed point $x^*$. If $y \in K$ then $\|T^j y - x^*\|_b \to 0$ as $j \to \infty$. 
Proof: If \( x \in K \) then \( \kappa_0 c \leq x, x^0 \), and Lemma 5.2 shows that
\[
\| Tx - x^0 \|_b \leq \| Tx - x^0 - x^0 \|_b + \| Tx^0 - x^0 \|_b \\
\leq \psi_b \| x - x^0 \|_b + \| Tx^0 - x^0 \|_b < +\infty.
\]
(5.13)
But \( Tx \geq 0 \); so \( Tx \in K \). Also, \( \psi_b |g'(\kappa_j, \lambda_j, c)\| < 1 \) for large enough \( j \), and \( K \cap [\kappa_j c, \lambda_j c] \) is a \( T \)-invariant set by (5.9). Hence \( T \), on this set, is a contraction map by (5.11). If \( x^* \) is its unique fixed point, and \( y \in K \), then \( T^{2j+1} y \in K \cap [\kappa_j c, \lambda_j c] \) and \( \lim \| T^{2j+1} y - x^* \|_b = 0 \).

Corollary 5.4. Let \( 0 \leq \beta_0 \) and \( 0 < b \in R^\infty \). Let \( \psi_b \leq 1 \) (i.e., \( b \) is concave) and \( \| \cdot \|_b, \| \cdot \|_c \) be equivalent norms. Then \( T \) has a unique fixed point \( x^* \) in \( R^m \), and \( \lim_j \| T^j x - x^* \|_c = 0 \) for any vector \( x \).

Proof: Here \( \psi_b |g'(\kappa_j, \phi_j)\| < 1 \) because \( \kappa_j \phi_j > 0 \). If \( x^0 = 0 \) then \( 0 < Tx^0 \leq c \); so \( \| Tx^0 - x^0 \|_c \leq 1 \) and \( \| Tx^0 - x^0 \|_b < +\infty \). Thus \( K = \{ x \in R^\infty; 0 \leq x; \| x \|_c \leq +\infty \} \) and \( T \) has a unique fixed point \( x^* \). If \( x \in R^\infty \) then \( 0 < Tx \) and \( T^2 x \in K \), whence \( \lim_j \| T^{j+2} x - x^* \|_c = \lim_j \| T^{j+2} x - x^* \|_b = 0 \).

6. Another Uniqueness Theorem

Although the hypotheses of Theorem 5.1 include the restriction \( \sigma + \sigma^{-1} = 1 + \lambda_x^2 \), actually the value \( \sigma \), by Theorem 2.3, need obey only the condition \( \sigma + \sigma^{-1} < 1 + \lim \inf \frac{\gamma_m^2}{\xi_m} \xi_m^2 \). However, no explicit formula gives this limit, whereas (5.5) and (5.6) determine \( \lambda_x^* \). Here we strengthen (5.1): we assume that \( \lim \gamma_{m+1}/\gamma_m \) exists and
\[
\mu - \lim \gamma_{m+1}/\gamma_m > 0.
\]
(6.1)
(Again, \( \xi_0 \geq 0 \) and \( c > 0 \).) Thereby, extending results of Freud [5] and Nevai [9], we obtain a new uniqueness theorem via our initial remark. If \( x = (\xi_1, \xi_2, \ldots) \in R^\infty \), then \( x(\tau) \) will denote the series \( \sum_{m=1}^{\infty} \xi_m \tau^m \) in the complex variable \( \tau \), and \( 1/s(x) \) will denote the series radius of convergence, whence
\[
s(x) = \lim \sup_m |\xi_m|^{1/m}.
\]
(6.2)
(the Cauchy–Hadamard formula (Goursat and Hedrick [6, pp. 377–378])). The value \( s(x) \), in some sense, measures the "growth rate" of \( (\xi_1, \xi_2, \ldots) \). If also \( y \in R^\infty \) then \( s(x + y) \leq \max(s(x), s(y)) \), and if \( 0 \leq x \leq y \) then \( s(x) \leq s(y) \).
**Lemma 6.1.** (Generalization of Freud [5]). If \( T x = x \), then \( \xi_m/\gamma_m \) has a limit as \( m \to \infty \), and

\[
\theta = \lim_m \xi_m/\gamma_m = (\mu + 1 + \mu^{-1})^{-1/2}.
\]

**Proof.** If \( \theta_- = \lim \inf_m \xi_m/\gamma_m \) and \( \theta_+ = \lim \sup_m \xi_m/\gamma_m \), then \( 0 \leq \theta_- \leq \theta_+ \leq 1 \), since \( 0 < \xi_m/\gamma_m < 1 \). Any positive \( \varepsilon \) determines integer \( n(\varepsilon) \) such that

\[
|\left( \gamma_m^{-1}/\gamma_m \right) - \mu | < \varepsilon, \quad \xi_m/\gamma_m > \theta_+ - \varepsilon,
\]

\[
|\left( \gamma_{m+1}^{-1}/\gamma_m \right) - \mu | < \varepsilon, \quad \xi_m/\gamma_m < \theta_+ + \varepsilon,
\]

whenever \( m \geq n(\varepsilon) \). Accordingly,

\[
1 = \frac{\xi_m}{\gamma_m} \left[ \frac{\xi_m + \gamma_{m+1} \xi_{m+1}}{\gamma_m \gamma_{m+1}} + \frac{\gamma_m \xi_{m-1}}{\gamma_{m-1}} \right]
\]

\[
> \frac{\xi_m}{\gamma_m} \left[ \frac{\xi_m + \mu \xi_{m+1}}{\gamma_m \gamma_{m+1}} + \frac{\xi_{m-1}}{\mu \gamma_{m-1}} \right] - 2\varepsilon
\]

\[
> (\xi_m/\gamma_m)(\xi_m/\gamma_m + \theta_+ (\mu + \mu^{-1}) | - \varepsilon (\mu + 2 + \mu^{-1}),
\]

whenever \( m > n(\varepsilon) \). If \( m \) takes increasing values such that \( \xi_m/\gamma_m \to \theta_+ \), then

\[
1 \geq \theta_+ [\theta_+ + \theta_- (\mu + \mu^{-1})] - \varepsilon (\mu + 2 + \mu^{-1}),
\]

and initially \( \varepsilon \) was arbitrary. Therefore \( \theta_+ [\theta_+ + \theta_- (\mu + \mu^{-1})] \leq 1 \); similarly, \( 1 \leq \theta_- [\theta_- + \theta_+ (\mu + \mu^{-1})] \). Thus \( \theta_+^2 \leq \theta_-^2 \), or \( \theta_- = \theta_+ = \theta \). Finally,

\[
1 = \theta^2 [1 + \mu + \mu^{-1}].
\]

**Lemma 6.2.** Let \( \omega_- = \lim \inf_m \omega_m \) and \( \omega_+ = \lim \sup_m \omega_m \), where \( 2 \leq \omega_1, \omega_2, \ldots \). Let \( \zeta_0 = 0 \) and \( z = (\zeta_1, \zeta_2, \ldots) \neq 0 \), where the \( \zeta_m \) satisfy (2.1). If \( \sigma \) (resp. \( \sigma_+ \)) satisfies the relations \( \sigma > 1 \) and \( \sigma + \sigma^{-1} = \omega_- \) (resp. \( \omega_+ \)), then

\[
\sigma_- \leq s(z) \leq \sigma_+.
\]

**Proof.** This simply restates Theorem 2.3.

**Theorem 6.3.** (Generalization of Nevai [9]). If \( c = (\gamma_1, \gamma_2, \ldots) \) satisfies (6.1) then \( T x = x \) has a unique solution.

**Proof.** Recall Theorem 4.3; suppose nonuniqueness. If \( \xi_m = \xi^+_m - \xi^-_m \), where \( m = 0, 1, 2, \ldots \), then \( \zeta_0 = 0 \) while \( \zeta_1, \zeta_2, \ldots > 0 \), and the \( \zeta_m \) obey (2.1) where \( \omega_m = 1 + \gamma_m^2/\xi^+_m \xi^-_m \). Moreover, \( \lim_m \omega_m = 1 + \theta^{-2} = \mu + 2 + \mu^{-1} \), by Lemma 6.1, because \( x^\pm \) and \( x^\pm \) are fixed points of \( T \). If \( \sigma > 1 \) and
\[ \sigma + \sigma^{-1} = \mu + 2 + \mu^{-1}, \] then \( \mu < \sigma \) by inspection, and \( s(z) = \sigma \) by Lemma 6.2. But \( 0 < x^- \leq x^+ < c \) and \( z = x^+ - x^- \); so \( s(z) \leq s(c) = \mu \), and this is a contradiction.

**Remark.** Theorems 4.3, 5.1, 6.3 and Corollary 5.4 provide different uniqueness criteria, and our motivating recurrences (1.3) require no more. If (1.4) had a unique nonnegative solution when \( \liminf_m \gamma_{m+1}/\gamma_m < +\infty \), then this test would include all these criteria, but this paper attempts no such comprehensive result. However, the following argument shows that unbounded ratios \( \gamma_{m+1}/\gamma_m \) permit multiple nonnegative solutions.

Fix \( \xi_0 \geq 0 \); then consider any real \( \xi \) and use (1.4) to generate \( x = (x_1, x_2, \ldots) \). Either all \( x_m \neq 0 \) and the sequence continues indefinitely, or some \( x_m = 0 \) and the sequence terminates there. Moreover, Corollary 3.4 and Lemma 4.2 show that \( x \) is a nonnegative solution if and only if \( \xi \in [\xi^-_1, \xi^+_1] \). If this interval does not contain \( \xi_1 \), then \( x \) has a first nonpositive element \( \xi_m \). Now equate numerators in (3.8) and equate denominators in (3.8) to define \( q_m(\xi), r_m(\xi) \) for all \( m \):

\[
q_{m+1} = \frac{\gamma_m^2 r_m^2 r_{m-1} - q_m^2 r_{m-1} - q_m r_m q_{m-1}}{r_m^2 r_{m-1}}, \quad m = 1, 2, \ldots \quad (6.8)
\]

Clearly, the factors of each denominator \( r_m \) are simply powers of the preceding numerators \( q_k \). Also, the specified \( \xi_m \) is the quotient \( q_n(\xi_1)/r_n(\xi_1) \). If \( \xi_1 < \xi^+_1 \) (resp. \( \xi^+_1 > \xi_1 \)) then \( [\xi^-_1, \xi^+_1] \) (resp. \( [\xi^+_1, \xi_1] \)) contains a zero of \( q_n(\xi)/r_n(\xi) \) unless it contains a pole, and then it contains a zero of some preceding \( q_k \). Hence the zeros of \( \{q_m; m = 1, 2, \ldots\} \) have \( \xi^-_1 \) and \( \xi^+_1 \) as monotone limits. But \( q_1 \) has the unique root 0, while \( q_2 \) has the positive root \( \gamma_1 g(\xi_0/2\gamma_1) \). Given any positive \( \varepsilon \), choose all \( \gamma_m^2 \) so large that no root of \( q_{m+1} \), via (6.8), exceeds distance \( \varepsilon/2^{m+1} \) from some root of \( r_m^2 r_{m-1} \). The roots of the latter are all roots of some preceding \( q_k \). Therefore, \( \xi^-_1 \neq \xi^+_1 \) when \( \varepsilon \) is small enough.

### 7. Sequence Computation

The recurrence (1.3), by definition, has fixed value \( \xi_0 = 0 \), and Section 8 will show that this recurrence, by Theorem 4.3, has unique nonnegative solution \( x_* = (x^*_1, x^*_2, \ldots) \); also, independent remarks will furnish \( \xi^*_1 \). Theoretically, \( \xi_0 \) and \( \xi^*_1 \), via (1.3), determine all \( \xi^*_m \) for higher \( m \). However, we find here that such forward iteration is an unstable algorithm, and we give a stable one. Hereafter, we fix values \( \gamma_0 = \xi_0 = 0 \) because they simplify the discussion, and we take any positive sequence \( c = (\gamma_1, \gamma_2, \ldots) \) such that (1.4) has unique nonnegative solution \( x_* \). (Later results will need assumption
(6.1).) Further, we choose any \( \xi_t > 0 \) and, from \( \xi_0, \xi_1 \), we define 
\[ x = (\xi_1, \xi_2, ...) \] 
via (1.4). This sequence \( x \) represents a hypothetical computation. Clearly, a program can set \( \xi_0 = 0 \), but it cannot set \( \xi_1 = \xi_1^* \) unless the latter is a machine number—and example (1.3) will have irrational \( \xi_1^* \). Moreover, if the machine \( \xi_1 \) were actually \( \xi_1^* \), still, roundoff error would soon produce an \( m \) such that the computed \( \xi_m \) was not \( \xi_m^* \); so the effective \( \xi_1 \) would not be precisely \( \xi_1^* \).

However, any distinct \( \xi_1 \) and \( \xi_1^* \) yield increasingly different \( \xi_m \) and \( \xi_m^* \). If \( \xi_1 \neq \xi_1^* \) then \( x \neq x^* \), and either some finite \( \xi_m = 0 \), whence \( x \) has no further elements, or else some finite \( \xi_m < 0 \), since \( x^* \) is the only nonnegative solution. If \( \xi_{m+2} \) is the first nonpositive element of \( x \), then \( 0 < \xi_1, ..., \xi_{n+1} \), and \( \xi_m \leq \gamma_m \), by Lemma 3.1, when \( 1 \leq m \leq n \). Now let \( \gamma_1, \gamma_2, ... \) satisfy (5.1), whence (5.4) defines constants \( \kappa_*, \lambda_* \); and let \( \xi_m = (-1)^m(\xi_m^* - \xi_m^*) \) for \( m = 0, 1, 2, ... \). Then \( \xi_0 = 0 \) and \( \xi_1 \neq 0 \), while repeating calculation (4.11) produces again recurrence (2.1), where

\[ \omega_m - 1 + \gamma_m^2/\xi_m \geq 1 + \lambda_*^{-1} \cdot \min\{\gamma_m/\xi_m, m - 1, ..., n\} \geq 1 + \lambda_*^{-1} > 2 \] (7.1)

for \( m = 1, ..., n \). But using Lemma 2.2 gives immediately

\[ \xi_{m+1}/\xi_1 = C_m(\omega_1, ..., \omega_m) \geq C_m(1 + \lambda_*^{-1} \cdot \min\{\gamma_m/\xi_m, m = 1, ..., n\}) \] (7.2)

for \( m = 1, ..., n \). Therefore, the discrepancy \( \xi_m - \xi_m^* \) has alternating sign and exponentially growing magnitude until the value \( \xi_m \) becomes zero or a negative quantity. Clearly, (1.4) has the same instability for backward iteration, because (2.1) has the same properties for decreasing \( m \).

Since we cannot obtain \( \xi_1^*, ..., \xi_n^* \) by an initial-value method, instead we shall compute these numbers via a boundary-value approach. Hereafter, we assume (6.1), and invoking Lemma 6.1, initially we use (6.3) to make the estimates

\[ \xi_m = (\mu + 1 + \mu^{-1})^{-1/2} \gamma_m, \quad m = 1, 2, ..., n + 1. \] (7.3)

Now, fixing this value \( \xi_{n+1} \), we solve (3.1) or, recalling the equivalent problem, we minimize \( F(\xi_1, ..., \xi_m) \). However, this is a trivial matter unless \( n > 1 \); and then, by Lemma 3.1, the domain \( K^0 \) contains the minimum, where

\[ K^0 = \{x \in \mathbb{R}^n; 0 < \xi_m < \gamma_m, m = 1, ..., n\}, \] (7.4)

while also, on this domain, Theorem 3.2 shows that \( F(x) \) is an analytic, strictly convex function. The literature contains many optimization methods, but these facts suggest a Newton-Raphson algorithm. Specifically, taking (7.3) as a first guess, we solve iteratively \( F'(x) = 0 \). Indeed, given any point \( x_{old} \) in \( K^0 \), we compute a successor \( x_{new} = x_{old} - \Delta x \), where (tentatively)

\[ \Delta x \cdot F''(x_{old}) = F'(x_{old}) \] (7.5)
provided $K^0$ contains the resulting $x_{new}$; otherwise some $\alpha \cdot \Delta x$ replaces this $\Delta x$, where $0 < \alpha < 1$. The $n \times n$ Hessian matrix $F''(x)$, on the domain $K^0$, is symmetric, tridiagonal, and diagonally dominant by (3.6). Also, the subdiagonal has all entries unity. Hence the system (7.5), through Gaussian elimination, produces the required $\Delta x$ in $2n - 1$ multiplications/divisions.

Still, our goal is the values $\xi^*_n$. If the estimated $\xi^*_{n+1}$, via this algorithm, determines $\xi_1, \ldots, \xi_n$ with high precision, then the remaining error in $\xi_1, \ldots, \xi_n$ reflects the initial error in $\xi^*_{n+1}$. Indeed, if $\sigma > 1$ and $\sigma + \sigma^{-1} \leq \min(\omega_1, \ldots, \omega_n)$, hence if

$$\sigma + \sigma^{-1} \leq 1 + \lambda^* \cdot \min\{y_m/\xi_m; m = 1, \ldots, n\}$$

(7.6)

and $\zeta_1 \neq 0$, then $\zeta_k/\xi_k < 1/\sigma$ for $k = 1, \ldots, n + 1$. Obviously this is true when $k = 1$, and if it is true when $1 \leq k \leq m$, then

$$\zeta_{m+1}/\xi_m = \omega_m - (\zeta_{m-1}/\xi_m) \geq \sigma + \sigma^{-1} - (\zeta_{m-1}/\xi_m) > \sigma.$$  

(7.7)

Thus $\xi_m/\xi_{n+1} < \sigma^{m-n-1}$, or

$$|\xi_{m} - \xi^*_m| < |\xi_{n+1} - \xi^*_n| \cdot \sigma^{m-n-1}.$$  

(7.8)

Even the poor estimate (7.3) for $\xi^*_n$ should therefore yield accurate values for earlier $\xi_m$. Roughly bounding the error $|\xi_{n+1} - \xi^*_n|$, we merely choose extra large $n$ and keep only those $\xi_m$ where (7.8) implies small enough error $|\xi_{m} - \xi^*_m|$.  

Here $\xi^*_n = (\mu + 1 + \mu^{-1})^{-1/2} \gamma^*_{n+1}$ by (7.3), while $\kappa_\star \gamma^*_{n+1} \leq \xi^*_n \leq \lambda_\star \gamma^*_{n+1}$ by (5.9). Thus

$$|\xi_{n+1} - \xi^*_{n+1}| \leq \gamma^*_{n+1} \cdot \max(|\kappa_\star - (\mu + 1 + \mu^{-1})^{-1/2}|, |\lambda_\star - (\mu + 1 + \mu^{-1})^{-1/2}|).$$  

(7.9)

Further ingenuity may improve this bound. If we can find $y^- = (\eta^-_1, \eta^-_2, \ldots) \in R^\infty$ and $y^+ = (\eta^+_1, \eta^+_2, \ldots) \in R^\infty$ such that

$$y^- \leq T y^+ \leq T y^- \leq y^+,$$  

(7.10)

then we can conclude $T(\{y^-, y^+\}) \subset \{y^-, y^+\}$, hence $x^\star \in \{y^-, y^+\}$, so that

$$\eta^+_n \leq \xi^*_{n+1} \leq \eta^-_{n+1}.$$  

(7.11)

Indeed, $x^\star \in [T^k y^-, T^k y^+]$ for $k = 0, 1, 2, \ldots$, but iterating $T$ may require too much computation. Section 9 uses this device.
Recurrence (1.3) involves a parameter $\rho$ such that $\rho > -1$. Specifically, $\xi_0 = 0$ and $c = c(\rho) = (\gamma_1(\rho), \gamma_2(\rho), \ldots) > 0$, where

$$
\gamma_m(\rho)^2 = \left(\frac{1}{\rho}\right)[2m + \rho - \rho(-1)^m] \quad \text{for } m = 1, 2, \ldots .
$$

(8.1)

Putting $m = 0$ yields $\gamma_0(\rho) = 0$, and this is an appropriate value since it obeys relations (5.7). Also, $\lim_m \gamma_m(\rho)/m = 0$ and $\lim_m \gamma_{m+1}(\rho)/\gamma_m(\rho) = 1$. Hence $0 < \psi_{\xi(\rho)} < +\infty$ and $0 < \kappa_*(\rho) < 1$. Thus (1.3) has a nonnegative solution $x^*$ by Theorem 4.3, and $x^*$ is the only nonnegative solution by Theorem 4.3 or 6.3. Moreover, any two sequences $c(\rho_1), c(\rho_2)$ define equivalent norms $\| \cdot \|_{c(p)}$; and obviously $c(0)$, in particular, is a concave sequence. Therefore, if $y \in R^\infty$ then $\lim_k \| T^k y - x^* \|_{c(p)} = 0$ by Corollary 5.4; and if $x^* = (\xi_1^*, \xi_2^*, \ldots)$ then $\lim_m \xi_m^*/\gamma_m(\rho) = 1/\sqrt{3}$ by Lemma 6.1.

Indeed, we can refine the last statement. Putting $\tau = 1/m$, suppose

$$
(12/m)^{1/2} \xi_m^* = f^+(\tau) = 1 + \alpha_1^+ \tau + \alpha_2^+ \tau^2 + \ldots , \quad m \text{ even},
$$

(8.2)

$$
(12/m)^{1/2} \xi_m^* = f^-(\tau) = 1 + \alpha_1^- \tau + \alpha_2^- \tau^2 + \ldots , \quad m \text{ odd}.
$$

Then (1.3) implies the relations

$$
0 = (-3/f^+(\tau)) + f^+(\tau) + (1 - \tau)^{1/2} f^-(\tau/(1 - \tau))
$$

$$
+ (1 + \tau)^{1/2} f^-(\tau/(1 + \tau)),
$$

(8.3)

$$
0 = (1 + \rho\tau)(-3/f^-(\tau)) + f^-(\tau) + (1 - \tau)^{1/2} f^+(\tau/(1 - \tau))
$$

$$
+ (1 + \tau)^{1/2} f^+(\tau/(1 + \tau)).
$$

(8.4)

The coefficient of each power $\tau^k$ in Eq. (8.3) (resp. Eq. (8.4)) has the form $4\alpha_k^+ + 2\alpha_k^-$ (resp. $4\alpha_k^- + 2\alpha_k^+$) plus some terms in the prior $\alpha_j^+, \alpha_j^-$. Therefore, equating these coefficients to zero yields inductively the values of all $\alpha_k^+, \alpha_k^-$. Computing the first few produces the following series:

$$
(12/m)^{1/2} \xi_m^* = 1 + \tau[-\rho/2 + \tau^2[(\rho^2/4) + (1/24)]
$$

$$
+ \tau^3[(-\rho^3/16) + (-3\rho/8)]
$$

$$
+ \tau^4[(-5\rho^4/128) + (35\rho^2/64) + (-7/576)] + \ldots , \quad m \text{ even};
$$

(8.5)
\[
(12/m)^{1/2} \varepsilon_m^* = 1 + \tau[\rho] + \tau^2[(-\rho^2/8) + (1/24)] \\
+ \tau^3[(-\rho^3/16) + (5\rho/16)] \\
+ \tau^4[(5\rho^4/64) + (-5\rho^2/16) + (-7/576)] + ..., \quad m \text{ odd.}
\]

So far, these expansions have only formal significance, but we show next that (8.5), (8.6) are asymptotic series. If \( x = (\xi_1, \xi_2, \ldots) \in \mathbb{R}^\infty \) and \( 0 < d = (\delta_1, \delta_2, \ldots) \in \mathbb{R}^\infty \), then \( \xi_m = O(\delta_m) \) as \( m \to \infty \) if and only if \( \sup(|\xi_m/\delta_m| : m = 1, 2, \ldots) < +\infty \) (Erdélyi [2, p. 5]), or \( \|x\|_d < +\infty \). In particular, if \( \delta_m = m^\nu \) for \( m > n \), where \( \nu \) is any real number and \( n \) is any positive integer, then \( \xi_m = O(m^\nu) \) as \( m \to \infty \).

**Lemma 8.1.** Any real \( \rho > -1 \) and real \( \nu \leq 0 \) determine an integer \( n > 0 \) and a sequence \( d(\nu) = (\delta_1, \delta_2, \ldots) > 0 \) such that \( \kappa_0 \delta_0 \leq \xi_0 \leq \lambda_0 \delta_0 \), where \( \delta_0 = 0 \), while

\[
\psi_{d(\nu)} \left| g'\left(\kappa_0(\rho) \phi(\rho)\right)\right| < 1,
\]

where \( \psi_{d(\nu)} = \sup\{(\delta_{m-1} + \delta_{m+1})/2\delta_m : m = 1, 2, \ldots\} \), and \( \delta_m = m^\nu \) whenever \( m > n \).

**Proof:** If \( \delta_0 = 0 \) and otherwise \( \delta_m = m^\nu \) then clearly

\[
(\delta_{m-1} + \delta_{m+1})/2\delta_m = (\frac{1}{2})(1 - m^{-1})^\nu + (1 + m^{-1})^\nu \to 1
\]

as \( m \to \infty \). But \( |g'(\kappa_0(\rho) \phi(\rho))| < 1 \); hence (8.8) implies some integer \( n \) such that

\[
|g'(\kappa_0(\rho) \phi(\rho))| \cdot (\delta_{m-1} + \delta_{m+1})/2\delta_m < 1
\]

when \( m > n \). If we define \( \delta_m = m^\nu \) when \( 1 \leq m \leq n \), then we satisfy (8.9) when \( m = 1, 2, \ldots \). The first condition is trivial because \( \xi_0 = 0 \).

**Theorem 8.2.** Expansions (8.5), (8.6) are asymptotic series.

**Proof:** If we truncate (8.5), (8.6) at the power \( \tau^k \) then the resulting polynomials will be nonnegative for small positive \( \tau \), and we can add multiples of the power \( \tau^{k+1} \) so that the resulting \( g^+(\tau) \), \( g^-(\tau) \) will be nonnegative for all positive \( \tau \). If \( h^+(\tau) \geq 0 \), \( h^-(\tau) \geq 0 \), where

\[
0 = (-3/h^+(\tau)) + h^+(\tau) + (1 - \tau)^{1/2} g^-(\tau/(1 - \tau)) \\
+ (1 + \tau)^{1/2} g^-(\tau/(1 + \tau)),
\]

\[
0 = (1 + \rho \tau)(-3/h^-(\tau)) + h^-(\tau) + (1 - \tau)^{1/2} g^+(\tau/(1 - \tau)) \\
+ (1 + \tau)^{1/2} g^+(\tau/(1 + \tau)),
\]

(8.10)
then \( h^+(\tau) \) and \( h^-(\tau) \) have convergent power series for small enough \(|\tau|\),
while \( g^+(\tau) \) and \( g^-(\tau) \) have the same such expansions up to powers \( \tau^k \).
If \( x = (\xi_1, \xi_2, \ldots) \), where \( \xi_m = (m/12)^{1/2} g^+ (1/m) \)
(resp. \( (m/12)^{1/2} g^- (1/m) \)) when \( m \) is even (resp. odd), then \( x \geq 0 \) and \( g^+(m) = (m/12)^{1/2} h^+(1/m) \)
(resp. \( (m/12)^{1/2} h^-(1/m) \)) when \( m \) is even (resp. odd).
Here we exclude components with \( m = 1 \). Hence \( |(Tx)_m - \xi_m| = O(m^{-k-1/2}) \) as \( m \to \infty \).
If \( v = -k - (1/2) \) in Lemma 8.1, then
\[
\|Tx - x\|_{d(v)} < +\infty \text{ for the corresponding } d(v).
\]
But \( T \) has a unique fixed point \( x^* \) by Theorem 4.3, and \( \|x^* - x\|_{d(v)} < +\infty \) by Theorem 5.3. Thus
\( \xi_m - \xi_m = O(m^{-k-1/2}) \) as \( m \to \infty \).

9. Computational Refinements

Freud's [3–5] papers assume the weight function \( w(\tau) = |\tau|^\rho \exp(-|\tau|^\rho) \),
and this function is clearly even; so each orthonormal polynomial \( p_m(\tau) \) is
either even or odd. Indeed, \( p_0(\tau) = \pi_0 \) and \( p_1(\tau) = \pi_1 \tau \), where \( \pi_0, \pi_1 \) are constants and our definition makes them positive. Then the orthonormality
relations (1.1) and standard gamma-function integrals (Abramowitz and
Stegun [1, Eq. (6.1.1)]) yield easily
\[
r/2\pi_0^2 = \Gamma((\rho + 1)/2), \quad r/2\pi_1^2 = \Gamma((\rho + 3)/2).
\]
But Section 1 fixes \( r = 4 \), and system (1.3) assumes \( \xi_m = (\pi_{m-1}/\pi_m)^2 \). Hence
the values \((\pi_{m-1}/\pi_m)^2\) from the polynomials \( p_m \) form the unique nonnegative solution \( x^* \) of Freud's recurrence, and
\[
\xi_0^* = 0, \quad \xi_1^* = \Gamma((\rho + 3)/4)/\Gamma((\rho + 1)/4).
\]
Theoretically, the recurrence now determines all further \( \xi_m^* \); but (7.2) shows
that such forward iteration is an exponentially unstable algorithm, and
experiment confirms this. However, the boundary-value approach of
Section 7 yields accurately the values \( \xi_m^* \) for larger \( m \).

Moreover, Section 8 motivates a further refinement. To obtain an initial
estimate for the computed \( \xi_m \), truncate the series (8.5)–(8.6) at some power \( \tau^k \). Then \( |\xi_m - \xi_m^*| \leq \text{constant} \cdot \tau^{k+1/2} \),
where \( \tau = 1/m \); so this bound is worst when \( m \) is smallest. Let \( \xi_0^*, \ldots, \xi_n^* \) be the desired values, and
\( \varepsilon_m = \gamma \cdot m^{-k-1/2} \), taking \( \gamma \) as the constant such that \( \varepsilon_n \) is the desired
tolerance. Let \( n(j) = \min(n, 2^j) \) where \( j = 0, 1, 2, \ldots \). Then the \( j \)th step of our
refined algorithm accepts \( \xi_{n(0)} + 1, \ldots, \xi_{n(0)} + 1 \) from the truncated series, but it uses
Newton iteration to improve \( \xi_{n(0)} + 1, \ldots, \xi_{n(0)} + 1 \) until the tolerance \( \varepsilon_{n(0)} \) for this step
exceeds the absolute maximum of the first \( n(j) \) residuals. Each step, with
increasing precision, clearly adjusts those values $\xi_m$ which, on the next step, would otherwise need the most correction. Thus each step computes a longer sequence until $n(j)$ reaches $n$, but early steps involve quite small linear systems (7.5).

If $\rho = 0$ then $\gamma_m = \sqrt{m}/2$ by (8.1), whence $\phi_c = 1/\sqrt{2}$ and $\psi_c = 1$ by Section 5; so

$$\kappa_* \simeq 0.52200941, \quad \lambda_* \simeq 0.69683244 \quad (9.3)$$

by (5.5)-(5.6). Moreover, symbolic computation generates the series (Trager [11]):

$$\begin{align*}
  f^+(\tau) &= f^-(\tau) = 1 + \tau^2/2^3 \cdot 3 - 7\tau^4/2^6 \cdot 3^2 + 37\tau^6/2^{10} \cdot 3^2 \\
  &\quad + 92,963\tau^8/2^{15} \cdot 3^4 - 200,039\tau^{10}/2^{16} \cdot 3^5 + 7,394,856,055\tau^{12}/2^{23} \cdot 3^6 \\
  &\quad - 416,852,554,595\tau^{14}/2^{25} \cdot 3^7 + \ldots. \quad (9.4)
\end{align*}$$

Indeed, one can prove inductively that (9.4) contains no odd powers. The structure of Eq. (8.3) and (8.4) implies the nonconvergence of this series, which therefore shows that one cannot strengthen Theorem 8.2. The authors used several terms of this series for $\rho = 0$, and they tried the refined algorithm of this section with $n = 250$. No step needed more than one Newton iteration; some steps required none at all. Thus the numerical linear algebra did not exceed $4n$ multiplications/divisions. To improve further the resulting $\xi_1, \ldots, \xi_m$, the authors took these computed values and applied repeatedly the map $U: x \rightarrow (x + Tx)/2$, that is,

$$(Ux)_m = (\xi_m + (Tx)_m)/2, \quad m = 1, \ldots, n. \quad (9.5)$$

This process has much slower convergence, but its starting-point was almost the exact solution. Hence the numerical values, after six iterations, showed complete stability in 64-bit floating-point arithmetic. Appendix A gives the first 20 terms.

Having thus minimized roundoff, we ignore it, and, to estimate accuracy, we use (7.8). To find $\sigma$ we need (7.6). For a better value we could explicitly substitute $\min\{\gamma_m/\xi_m: m = 1, \ldots, n\}$, but for a faster result we need verify only that $\xi_m \leq \lambda_* \gamma_m$ when $1 \leq m \leq n$. A conservative $\sigma$ then satisfies the equation $\sigma + \sigma^{-1} = 1 + \lambda_*^{-2}$, whence $\sigma \simeq 2.6872901$ and $\sigma^7 \simeq 1012.0498 > 10^3$; so dropping seven terms from the end of the computed sequence improves the accuracy by a factor of at least $10^3$. Even (7.9) yields the crude estimate that $|\xi_{251} - \xi_{251}^*| < 1$, whence $|\xi_m - \xi_m^*| < 10^{-15}$ for $m \leq 216$; but (7.11) gives a better bound when

$$\eta_m^- = ((2m - 1)/24)^{1/2}, \quad \eta_m^+ = ((2m + 1)/24)^{1/2}, \quad m = 1, 2, \ldots. \quad (9.6)$$
(Direct computation checks (7.10) for \( m = 1 \); simple algebra proves it for \( m > 1 \).) Now \( |\xi_{251} - \xi_{251}^*| < 0.00456 \) because \( \xi_{251}^* \) satisfies (7.11), and \( |\xi_m - \xi_m^*| \leq 2 \times 10^{-15} \) when \( m \leq 222 \). Indeed, if \( \xi_{n+1}^* \) is the last value, for any \( n \geq 29 \), while \( \eta_{n+1}^- \leq \xi_{n+1}^- \leq \eta_{n+1}^+ \), then

\[
|\xi_{n+1}^* - \xi_{n+1}^*| \leq |\eta_{n+1}^+ - \eta_{n+1}^-| \leq |\eta_{30}^+ - \eta_{30}^-| < 0.0264,
\]

so that \( |\xi_m - \xi_m^*| < 10^{-14} \) for any \( m \leq n - 28 \).

Another argument improves this estimate. If \( \delta_0 = 0 \) and sequence \( d = (\delta_1, \delta_2, \ldots) \), where \( \delta_1 = \cdots = \delta_g = 1/27 \) and otherwise \( \delta_m = m^{-3/2} \), then Sections 4 and 5 show that

\[
\alpha = \psi_d |g'(\kappa_\star \phi_c)| \leq 2/3.
\]

If \( y = (\eta_1, \eta_2, \ldots) \in R^\infty \), where \( \eta_1, \ldots, \eta_{251} \) are the computed \( \xi_1, \ldots, \xi_{251} \) and otherwise

\[
\eta_m = (m/12)^{1/2} \left[ 1 + (1/24m^2) - (7/576m^4) \right],
\]

then \( \|Ty - y\|_d < +\infty \). Also, \( \kappa_\star c \leq y \) by inspection, whence \( \|T^k y - x^*\|_d \to 0 \) by Theorem 5.3. But Lemma 5.2 implies that

\[
\|y - x^*\|_d \leq \sum_{k=0}^{g-1} \|T^{k+1} y - T^k y\|_d + \|T^g y - x^*\|_d
\]

\[
\leq \sum_{k=0}^{\infty} \alpha^k \|Ty - y\|_d \leq 3 \|Ty - y\|_d;
\]

whence \( |\xi_m - \xi_m^*| \leq 3 \|Ty - y\|_d \cdot m^{-3/2} \) for \( 1 \leq m \leq 251 \). A finite computation yields \( \|Ty - y\|_d \).

Clearly, our \( \xi_m \) should be very accurate when the index \( m \) is small. If we put \( \rho = 0 \) and use (9.2) to find \( \xi_1^* \), then \( \xi_1 \), from the preceding algorithm, has the same value to the last computed digit.

**APPENDIX A**

If \( \rho = 0 \) and \( \xi_0 = 0 \), then the arguments of Section 8 show that system (1.3) has unique nonnegative solution \( x^* \), and the method of Section 9 finds that components \( \xi_{21}^*, \ldots, \xi_{20}^* \) have the following values.
APPENDIX B

If all \( \gamma_m \) have the same value \( \gamma \), then truncations of (1.5) yield solutions of (3.1) where the \( \xi_m \) have varying sign. This appendix, for equal \( \gamma_m \), lists more solutions of this finite system. If any real \( \gamma_1, \ldots, \gamma_n \) and \( \xi_0, \ldots, \xi_{n+1} \) satisfy (3.1), then the constant multiples \( a\gamma_1, \ldots, a\gamma_n \) and \( a\xi_0, \ldots, a\xi_{n+1} \) satisfy (3.1). Hence, taking \( \gamma_1 = \cdots = \gamma_n = 1 \) and fixing \( \xi_0 = 1 = \xi_{n+1} \), we give all real solutions for the smallest few \( n \). We suppress the algebraic details for brevity; the final result suggests the intermediate substitutions.

If \( n = 1 \) and \( \xi_2 = -1 \), then \( (\xi_0, \xi_1, \xi_2) = (1, \eta, -1) \), where \( \eta = \pm 1 \). If \( n = 2 \) and \( \xi_2 = -1 \), then

\[
(\xi_0, \ldots, \xi_3) = (1, 1, -1, -1) \quad \text{or} \quad \left(\frac{1}{3}\right)(2, \eta - 1, \eta + 1, -2),
\]

where \( \eta = \pm \sqrt{3} \). If \( n = 3 \) and \( \xi_4 = 1 \), then

\[
(\xi_0, \ldots, \xi_4) = (1, 1, -1, -1, 1) \quad \text{or} \quad (1, -1, -1, 1, 1)
\]

or

\[
\left(\frac{1}{3}\right)(2, \eta - \eta^{-1}, 2\eta^{-1}, \eta - \eta^{-1}, 2),
\]

where \( \eta^4 + 2\eta^3 - 4\eta^2 - 2\eta - 1 - 0 \). This last equation has just two real roots:

\[
\eta_1 \approx 1.5815460, \quad \eta_2 \approx -3.1120097.
\]

If \( n = 4 \) and \( \xi_5 = 1 \), then

\[
(\xi_0, \ldots, \xi_5) = (1, 1, -1, -1, 1, 1)
\]

or

\[
(1, \eta^{-1}, -1 + \eta, -1 - \eta, -\eta^{-1}, 1)
\]

or

\[
(1, \xi - 2\xi^{-1}, \xi^{-1}, \xi^{-1}, \xi - 2\xi^{-1}, 1).
\]

---

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \xi_{m} )</th>
<th>( m )</th>
<th>( \xi_{m} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.33798912003364232</td>
<td>11</td>
<td>0.95775608488417893</td>
</tr>
<tr>
<td>2</td>
<td>0.40167965976351733</td>
<td>12</td>
<td>1.0002887465597798</td>
</tr>
<tr>
<td>3</td>
<td>0.50510423234482221</td>
<td>13</td>
<td>1.0410891789282268</td>
</tr>
<tr>
<td>4</td>
<td>0.57805815033171129</td>
<td>14</td>
<td>1.0803527252385585</td>
</tr>
<tr>
<td>5</td>
<td>0.64676738204724493</td>
<td>15</td>
<td>1.1182407644978825</td>
</tr>
<tr>
<td>6</td>
<td>0.70786315090515241</td>
<td>16</td>
<td>1.1548882639750164</td>
</tr>
<tr>
<td>7</td>
<td>0.76442312605207728</td>
<td>17</td>
<td>1.1904595014212202</td>
</tr>
<tr>
<td>8</td>
<td>0.81702175201098198</td>
<td>18</td>
<td>1.224902329457462</td>
</tr>
<tr>
<td>9</td>
<td>0.86647036419002228</td>
<td>19</td>
<td>1.2584508557944947</td>
</tr>
<tr>
<td>10</td>
<td>0.91324989944000748</td>
<td>20</td>
<td>1.2911288293490708</td>
</tr>
</tbody>
</table>
where $\eta = \pm 1/\sqrt{2}$ and $\zeta^3 - 4\zeta + 2 = 0$. This last equation has three real roots:

$$
\zeta_1 \simeq 1.6751309, \quad \zeta_2 \simeq 0.53918887, \quad \zeta_3 \simeq -2.2143197. \quad (B.5)
$$

No two sequences with the same $n$ have the same pattern of signs. However, it is unclear whether this is significant.

**ACKNOWLEDGMENTS**

The authors wish to thank P. G. Nevai, whose seminar lecture posed this problem, and whose further correspondence provided important concepts. Also, they wish to thank B. M. Trager, who obtained relation (9.4) using a computer symbol-manipulation system.

**REFERENCES**