Combinatorics and Total Positivity

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It was first observed in (F. Brenti, Mem. Amer. Math. Soc. 413, 1989) that Pólya frequency sequences arise often in combinatorics. In this work we point out that the same is true, more generally, for totally positive matrices and that there is an intimate connection between them and some generalizations of the classical symmetric functions. © 1995 Academic Press, Inc.

1. INTRODUCTION

Total positivity is an important and powerful concept that arises often in various branches of mathematics, statistics, probability, mechanics, economics, and computer science (see, e.g., [24] and the references cited there). It was first observed in [5, 6] that Pólya frequency sequences (see Section 2 for definitions) also arise often in combinatorics. The purpose of the present work is to point out that the same is true, more generally for totally positive matrices, and that there is an intimate connection between them and some generalizations of the classical symmetric functions.

More precisely, in this paper we show that many of the familiar matrices arising in combinatorics, as well as in the theory of symmetric functions and many of their generalizations, have remarkable total positivity properties and that, conversely, any totally positive matrix can be realized as a matrix of generalized complete homogeneous symmetric functions evaluated at nonnegative real numbers. The method that we use to prove these results is completely combinatorial and has its origin in a technique for counting nonintersecting paths in directed graphs first used by Lindström in [26] (although he used it for completely different purposes). This technique was then used by Gessel, Viennot, and others to solve

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various enumerative problems (see, e.g. [18, 19]). In this paper we use some variations and generalizations of it.

The organization of the paper is as follows. In the next section we collect some definitions, notation, and results that will be used throughout the rest of this work. In Section 3 we give a characterization of totally positive matrices in terms of planar digraphs (Theorem 3.1). We then show that any totally positive matrix is a matrix of (suitably defined) generalized complete homogeneous symmetric functions evaluated at nonnegative real numbers (Theorem 3.3) and that any minor of such a matrix is a "generalized flagged Schur function" evaluated at nonnegative real numbers (Theorem 3.5). As consequences of these results we obtain a combinatorial interpretation for any minor of any totally positive matrix as a generating function for certain sets of appropriately weighted column strict tableaux (thus solving a problem originally posed in [6, Section 2.2, p. 10]), and we obtain a simple proof of a generalization of a result of Stembridge concerning immanants of totally positive matrices. In Section 4 we study the total positivity properties of some infinite matrices which depend on three infinite sets of independent variables and on an integer parameter and which include the matrix of super-Schur functions studied in [10]. In Section 5 we apply the results obtained in Section 4 to matrices of combinatorial interest. We obtain total positivity properties for matrices of $q$-binomial coefficients, several types of $(p, q)$-Stirling numbers of both kinds, generalizations of complete homogeneous and elementary symmetric functions, $r$-associated $q$-Stirling numbers of the first kind, Delannoy numbers, $q$-Lah numbers, $q$-Stirling numbers of type $B_n$, and others. Finally, in Section 6 we obtain combinatorial interpretations for the minors of the matrices studied in Section 4 and thus, in particular, for most of the combinatorial matrices considered in Section 5 and for any skew super-Schur function (see Corollary 6.4 and the remarks following Corollary 6.5). These combinatorial interpretations are expressed in terms of "dotted $t$-tableaux," a notion which is equivalent, when $t = 0$ to those of a dotted plane partition, $P'$-tableau, and circled tableau, which were studied in [4, 10, 30, and 37].

2. Notation and Preliminaries

In this section we collect some definitions, notation, and results that will be used in the rest of the paper. We let $P \overset{\text{def}}{=} \{1, 2, 3, \ldots\}$, $N \overset{\text{def}}{=} P \cup \{0\}$, $Q$ be the set of rational numbers, and $R$ be the set of real numbers; for $a \in N$ we let $[a] \overset{\text{def}}{=} \{1, 2, \ldots, a\}$ (where $[0] \overset{\text{def}}{=} \emptyset$). The cardinality of a set $A$ will be denoted by $|A|$, for $r \in N$ we let $(r) \overset{\text{def}}{=} \{T \subseteq A : |T| = r\}$. For $i_1, \ldots, i_r \in P$ we write $\{i_1, i_2, \ldots, i_r\} <$ if $i_1 < i_2 < \cdots < i_r$. For $m, n \in P$ with $m \leq n$ we
let \([m, n] \text{ def } [n] \setminus [m-1]\). Given \(T \subseteq [n]\) there are unique integers \(r \in \mathbb{P}\), and \(1 \leq a_1 \leq a_2 \leq \cdots \leq a_{2r} \leq n\), such that \(T = \bigcup_{i=1}^{\ell} [a_{2i-1}, a_{2i}]\) and \(|a_{2i+1} - a_{2i}| > 1\) for \(i = 1, \ldots, r-1\). We then call the intervals \([a_{2i-1}, a_{2i}]\), \(i = 1, \ldots, r\), the connected components of \(T\). Given a (finite) set \(T\) we denote by \(P(T)\) the set of all (set) partitions of \(T\) (see, e.g., [35, p. 33] for further information about partitions of a set), and by \(S(T)\), the set of all bijections from \(T\) to itself. For \(n \in \mathbb{P}\), we let \(S_n \text{ def } S([n])\). Throughout this work, \(p\) and \(q\) will denote two independent variables. For \(n \in \mathbb{P}\) we let \([n]_p, q \text{ def } p^{n-1} + q p^{n-2} + q^2 p^{n-3} + \cdots + q^n - 1\), \([n]_q \text{ def } [n]_1, q\), and \([k]_{p, q} \text{ def } 0\) if \(k \leq 0\).

We follow [27, Chap. I] for notation and terminology related to partitions and symmetric functions. In particular, given a partition \(\lambda\), we denote by \(\lambda'\) its conjugate, by \(m_i(\lambda)\) the number of parts of \(\lambda\) that are equal to \(i\), for \(i \in \mathbb{P}\), and by \(s_1\) (respectively \(e_1, h_1\)) the Schur (respectively elementary, complete homogeneous) symmetric function associated to \(\lambda\). We will usually identify a partition \(\lambda = (\lambda_1, \ldots, \lambda_r)\) with its diagram \(\{(i, j) \in \mathbb{P} \times \mathbb{P} : 1 \leq i \leq r, 1 \leq j \leq \lambda_i\}\).

We follow [33] for notation and terminology regarding plane partitions. However, we will often need to work with more general objects than skew reverse plane partitions, which we now define. Let \(\lambda = (\lambda_1, \ldots, \lambda_r)\) and \(\mu = (\mu_1, \ldots, \mu_r)\) be two partitions such that \(\lambda \subseteq \mu\). A shifted skew tabloid of shape \(\lambda \setminus \mu\) is an array \(T = (T_{i,j})_{1 \leq i \leq r, \mu_i + i \leq j \leq \lambda_i + i - 1}\) of positive integers (we will sometimes use the notation \(\text{sh}(T) = \lambda \setminus \mu\)). We denote by \(T_i\), the \(i\)th row of \(T\), for \(i = 1, \ldots, r\) and let \(s(T) \text{ def } (T_1, \ldots, T_r, \lambda_1 + 1, \ldots, T_r, \mu_1 + r)\) and call \(s(T)\) the size of \(T\). We also let \(|T| \text{ def } \sum_{(i,j) \in \text{sh}(T)} T_{i,j}\), \(T_e \text{ def } (T_{i,j})_{1 \leq i \leq r, \mu_i + i \leq j \leq \lambda_i + i - 2}\), and \(T_h \text{ def } T_e\). We call \(T\) a shifted skew reverse plane partition if \(T_{i,j} \leq T_{i,j+1}\) and \(T_{i,j} \leq T_{i-1,j+1}\) whenever both sides of the inequality are defined. We say that \(T\) is row strict (respectively, column strict) if \(T_{i,j} < T_{i,j+1}\) (respectively, \(T_{i,j} < T_{i+1,j}\)) whenever both sides of the inequality are defined. If \(\mu = \emptyset\) then we call \(T\) a shifted reverse plane partition of shape \(\lambda\). Note that we do not require the parts of \(\lambda\) to be distinct. Also note that if both \(\mu\) and \(\lambda\) have distinct parts then a shifted skew reverse plane partition of shape \(\lambda \setminus \mu\) is just a skew reverse plane partition of shape \((\lambda_1, \lambda_2 + 1, \ldots, \lambda_r + r - 1)\) \((\mu, \mu_2 + 1, \ldots, \mu_r + r - 1)\). For brevity, we will often use the word tableau in place of “reverse plane partition,” even though the latter is more common.

An infinite (real) matrix \(M = (M_{n,k})_{n,k \in \mathbb{N}}\) (where \(M_{n,k}\) is the entry in the \(n\)th row and \(k\)th column of \(M\)) is said to be totally positive (or, TP, for short) if every minor of \(M\) has nonnegative determinant. An infinite (real) sequence \(\{a_i\}_{i \in \mathbb{N}}\) is said to be a Pólya frequency sequence (or, a PF-sequence, for short) if the infinite matrix \(A \text{ def } (a_{n-k})_{n,k \in \mathbb{N}}\) (where \(a_i \text{ def } 0\) if \(i < 0\)) is...
totally positive. Given $M$, $\{a_i\}_{i \in \mathbb{N}}$, and $A$ as above, and $\{n_1, \ldots, n_r\} \subset \mathbb{N}$ ($r \in \mathbb{P}$) we will find it convenient to let

$$M \left( \begin{array}{c} n_1, \ldots, n_r \\ k_1, \ldots, k_r \end{array} \right) \overset{\text{def}}{=} \det \left[ (M_{n_i,k_j})_{1 \leq i,j \leq r} \right] \quad (1)$$

and

$$\{a_i\}_{i \in \mathbb{N}} \left( \begin{array}{c} n_1, \ldots, n_r \\ k_1, \ldots, k_r \end{array} \right) \overset{\text{def}}{=} A \left( \begin{array}{c} n_1, \ldots, n_r \\ k_1, \ldots, k_r \end{array} \right). \quad (2)$$

Given a matrix $M$ we will denote by $M'$ the transpose of $M$. We will often work with matrices and sequences whose elements are real polynomials over some (usually infinite) set of independent variables $x$. In this case we will say that such a matrix $M = (M_{n,k})_{n,k \in \mathbb{N}}$ (respectively, sequence $\{a_i\}_{i \in \mathbb{N}}$) is $x$-TP (respectively, $x$-PF) if $(1)$ (respectively, $(2)$) is a polynomial with nonnegative coefficients, for all $r \in \mathbb{P}$ and $\{n_1, \ldots, n_r\} \subset \mathbb{N}$. Given a sequence of independent variables $\{x_i\}_{i \in \mathbb{N}}$ and $T \subset \mathbb{N}$, we let $x_T \overset{\text{def}}{=} \prod_{i \in T} x_i$. We also let $0 \overset{\text{def}}{=} (0, 0, 0, \ldots)$ and $1 \overset{\text{def}}{=} (1, 1, 1, \ldots)$.

Let $D = (V, A)$ be a directed graph (or, digraph, for short). We will always assume that $D$ has no loops or multiple edges, so that we can identify the elements of $A$ with ordered pairs $(u, v)$, with $u, v \in V$, $u \neq v$. A path in $D$ is a sequence $\pi = u_1 u_2 \cdots u_n$ of elements of $V$ such that $(u_i, u_{i+1}) \in A$ for $i = 1, \ldots, n - 1$, we then say that $\pi$ goes from $u_1$ to $u_n$. We say that $D$ is locally finite if, for every $u, v \in V$, there are only a finite number of paths from $u$ to $v$. Note that this implies that $D$ is acyclic. We say that $D$ is weighted if there is a function $w: A \rightarrow R$, where $R$ is some commutative $\mathbb{Q}$-algebra. If $R = \mathbb{R}$ and $w((u, v)) \geq 0$ for all $(u, v) \in A$ then we call $D$ a nonnegative digraph. Let $D = (V, A, w)$ be a locally finite, weighted digraph. For a path $\pi = u_0 u_1 \cdots u_k$ in $D$ we let

$$w(\pi) \overset{\text{def}}{=} \prod_{i=1}^{k} w(u_{i-1}, u_i),$$

and, for $u, v \in V$, we let

$$P_D(u, v) \overset{\text{def}}{=} \sum_{\pi} w(\pi),$$

where the sum is over all paths $\pi$ in $D$ going from $u$ to $v$. We adopt the convention that $P_D(u, u) \overset{\text{def}}{=} 1$ for all $u \in V$ (i.e., there is only one path, the empty path, from $u$ to $u$ and its weight is 1). We will usually omit the
subscript $D$ when there is no danger of confusion. Given $\mathbf{u} \overset{\text{def}}{=} (u_1, ..., u_r)$, $\mathbf{v} \overset{\text{def}}{=} (v_1, ..., v_r) \in V^r$, we let

$$N(\mathbf{u}, \mathbf{v}) \overset{\text{def}}{=} \sum_{(\pi_1, ..., \pi_r)} w(\pi_1, ..., \pi_r),$$

where $w(\pi_1, ..., \pi_r) \overset{\text{def}}{=} \prod_{i=1}^r w(\pi_i)$ and where the sum is over all $r$-tuples of paths $(\pi_1, ..., \pi_r)$ from $\mathbf{u}$ to $\mathbf{v}$ (i.e., $\pi_i$ is a path from $u_i$ to $v_i$, for $i = 1, ..., r$) that are non-intersecting (i.e., $\pi_i$ and $\pi_j$ have no vertices in common if $i \neq j$).

We say that $\mathbf{u}$ and $\mathbf{v}$ are compatible if, for every $\sigma \in S_\omega \backslash \{\text{Id}\}$, there are no $r$-tuples of paths from $(u_1, ..., u_r)$ to $(v_{\sigma(1)}, ..., v_{\sigma(r)})$ that are non-intersecting.

The following fundamental result was first proved by Lindström in [26] and has later found numerous applications in enumerative combinatorics (see, e.g., [18, 19, 31, 38]). We refer the reader to [19, Corollary 2; 38, Theorem 1.2; or 26, Lemma 1] for its proof.

**Lemma 2.1.** Let $D = (V, A, w)$ be a locally finite, weighted digraph and $\mathbf{u} \overset{\text{def}}{=} (u_1, ..., u_n), \mathbf{v} \overset{\text{def}}{=} (v_1, ..., v_n) \in V^n$ be compatible. Then

$$N(\mathbf{u}, \mathbf{v}) = \det[ (P_D(u_i, v_j))_{1 \leq i, j \leq n} ].$$

3. **Total Positivity and Lattice Paths**

Lemma 2.1 shows, in particular, that if $D = (V, A, w)$ is a finite, non-negative digraph and $(u_1, ..., u_n), (v_1, ..., v_n) \in V^n$ are compatible, then the matrix

$$(P_D(u_i, v_j))_{1 \leq i, j \leq n}$$

has a nonnegative determinant. It is natural to conjecture that under these assumptions this matrix is actually totally positive. This, however, is false. For example, in the digraph $D$ depicted in Fig. 1 (where we assume that all

![Figure 1](image-url)
the directed edges have weight 1) the triples $u \overset{\text{def}}{=} (u_1, u_2, u_3)$ and $v \overset{\text{def}}{=} (v_1, v_2, v_3)$ are compatible but

$$(P_D(u_i, v_j))_{1 \leq i, j \leq 3} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

is not totally positive, even though $D$ satisfies all the other hypotheses of Lemma 2.1. This suggests the following definition. Let $D = (V, A)$ be a digraph and $u \overset{\text{def}}{=} (u_1, \ldots, u_n)$, $v \overset{\text{def}}{=} (v_1, \ldots, v_n) \in V^n$. We say that $u$ and $v$ are fully compatible if $(u_{i_1}, \ldots, u_{i_r})$ and $(v_{j_1}, \ldots, v_{j_r})$ are compatible for all $\{i_1, \ldots, i_r\} \subset \{j_1, \ldots, j_r\} \subset [n]$ and $r \in [n]$. It is then clear that if $D = (V, A, w)$ is a finite, nonnegative digraph and $(u_1, \ldots, u_n)$, $(v_1, \ldots, v_n) \in V^n$ are fully compatible, then we can apply Lemma 2.1 to every minor of the matrix (3) and thus conclude that it is totally positive. What is surprising, however, is that this actually characterizes totally positive matrices. In fact, we have the following fundamental result. Recall (see, e.g. [24, Chap. 3, Section 3, p. 112]) that a matrix $(a_{i,j})_{1 \leq i, j \leq n}$ is called a Jacobi matrix if $a_{i,j} = 0$ whenever $|i - j| > 1$.

**Theorem 3.1.** Let $U$ be an $n \times n$ (real) matrix. Then $U$ is totally positive if and only if there exists a planar, finite, nonnegative digraph $D = (V, A, w)$ and $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in V^n$ fully compatible, such that

$$U = (P_D(u_i, v_j))_{1 \leq i, j \leq n}.$$ 

**Proof.** One direction of the theorem follows immediately from our definitions and Lemma 2.1, as remarked above. So assume that $U$ is totally positive. Then there follows from Theorem 3.5 of [2] and Theorem 4.1 of [14] that there exist $t, m \in \mathbb{N}$ and matrices $W_1, \ldots, W_{t+m}$ such that $W_i$ is an $n \times n$ upper (respectively, lower) triangular, totally positive, Jacobi matrix for $i = 1, \ldots, t$ (respectively, $i = t+1, \ldots, t+m$) and

$$U = W_1 \cdots W_{t+m}. \quad (4)$$

Let

$$W_i \overset{\text{def}}{=} \begin{bmatrix} a^{(i)}_1 & b^{(i)}_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a^{(i)}_{n-1} & b^{(i)}_{n-1} \\ 0 & 0 & \cdots & 0 & a^{(i)}_n \end{bmatrix} \quad (5)$$
for \( i = 1, \ldots, t \), and

\[
W_i \overset{\text{def}}{=} \begin{bmatrix}
 a_1^{(i)} & 0 & \cdots & 0 & 0 \\
 b_2^{(i)} & a_2^{(i)} & \cdots & 0 & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & \cdots & b_n^{(i)} & a_n^{(i)} 
\end{bmatrix}
\]

(6)

for \( i = t + 1, \ldots, t + m \). Note that, since the \( W_i \)'s are totally positive, 
\( a_{i}^{(j)}, b_{j}^{(j)} \geq 0 \) for all \( i \) and \( j \). Now let \( D(U) \) be the weighted digraph on vertex set 

\[ V \overset{\text{def}}{=} [n] \times [t + m + 1], \]

obtained by putting a directed edge from \((i, j)\), to \((i + 1, j)\) with weight \( a_j^{(i)} \) for \((i, j) \in [n] \times [t + m] \) and a directed edge from \((i, j)\) to \((i + 1, j + 1)\) (respectively, \((i + 1, j - 1)\)) with weight \( b_j^{(i)} \) for \((i, j) \in [n] \times [t] \) (respectively, \((i, j) \in [n] \times [t + 1, t + m] \)). For example, if \( n = 4, m = 2, \) and \( t = 3 \), then we obtain the digraph depicted in Fig. 2.

Now let \( u \overset{\text{def}}{=} ((1, 1), (1, 2), \ldots, (1, n)) \) and \( v \overset{\text{def}}{=} ((t + m + 1, 1), \ldots, (t + m + 1, n)) \).

It is clear that \( D(u) \) is planar and that \( u \) and \( v \) are fully compatible. Furthermore, every directed path in \( D(U) \) from \((1, i)\) to \((t + m + 1, j)\) \((1 \leq i, j \leq n)\) is necessarily of length \( t + m \). By the transfer matrix method (see, e.g., [35, Section 4.7, Theorem 4.7.1]), this implies that \( P_{D(u)}((1, i), (t + m + 1, j)) \) equals the \(((1, i), (t + m + 1, j))\) entry of \( A^{t + m} \),

![Figure 2](image-url)
where $A$ is the adjacency matrix of the weighted digraph $D(U)$. From our construction of $D(U)$ we see that

$$A = \begin{bmatrix} 0 & W_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & W_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & W_{m+1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

(in block matrix notation), where we have given the linear ordering $(1, 1)$, $(1, 2)$, ..., $(1, n)$, $(2, 1)$, ..., $(2, n)$, ..., $(t+m+1, 1)$, ..., $(t+m+1, n)$ to the vertices of $D(U)$. Therefore,

$$A^{m+1} = \begin{bmatrix} 0 & \cdots & 0 & W_1 \cdots W_{m+1} \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

and, hence,

$$(P_{D(U)}((1, i), (t + m + 1, j)))_{1 \leq i, j \leq n} = W_1 \cdots W_{m+1} = U,$$  \hspace{1cm} (7)

as desired. \hfill \Box

The simplicity of the geometry of the digraphs $D(U)$ (cf. Fig. 2) allows us to explicitly describe the generating functions $P_{D(U)}((1, i), (t + m + 1, j))$, $1 \leq i, j \leq n$ and thus the entries of $U$. For this we will find it convenient to embed the digraphs $D(U)$ in a larger digraph and to put independent variables as weights on the edges.

Let $X \overset{\text{def}}{=} (x_{i, j})_{i, j \in \mathbb{N}}$ and $Y \overset{\text{def}}{=} (y_{i, j})_{i, j \in \mathbb{N}}$ be two infinite matrices of independent variables (or of real numbers). We define a weighted digraph $D(X, Y)$ on vertex set $\mathbb{N} \times \mathbb{N}$ by putting a directed edge from $(i,j)$ to $(i+1,j)$ (respectively, $(i,j+1)$) with weight $x_{i+1,j}$ (respectively, $y_{i,j+1}$) for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. A portion of $D(X, Y)$ is shown in Fig. 3.

For $n, k \in \mathbb{N}$ we let

$$h_{n,k}(X, Y) \overset{\text{def}}{=} \sum_{0 = i_0 < i_1 < \cdots < i_n < i_{n+1} = k} x_{1, i_0} \cdots x_{n, i_n} \times y_{0, (i_0, i_1]} y_{1, (i_1, i_2]} \cdots y_{n, (i_n, i_{n+1]}},$$  \hspace{1cm} (8)

where $y_{i, (j, k]} \overset{\text{def}}{=} \prod_{j < s < k} y_{i,s}$ for $i, j, k \in \mathbb{N}$. We also let $h_{n,k}(X, Y) \overset{\text{def}}{=} 0$ if either $n < 0$ or $k < 0$. We call $h_{n,k}(X, Y)$ the generalized complete homogeneous symmetric function of degrees $n$ and $k$. Note that $h_{n,k}(X, Y)$
is a homogeneous polynomial of degree $n$ in the $x_{i,j}$'s and of degree $k$ in the $y_{i,j}$'s. Furthermore, the complete homogeneous symmetric functions are a specialization of the $h_{n,k}(X, Y)$'s since

$$h_{n,k} \left( \begin{bmatrix} x_0 & x_1 & x_2 & \cdots \\ x_0 & x_1 & x_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \right) = h_n(x_0, \ldots, x_k),$$

for all $n, k \in \mathbb{N}$.

The reason for introducing the polynomial $h_{n,k}(X, Y)$ is that they give a simple way to express the path generating function between any two points of the digraph $D(X, Y)$. Given $n, k \in \mathbb{N}$ we let $S_{(n,k)}(X) \overset{\text{def}}{=} (x_{n+i,k+j})_{i,j \in \mathbb{N}}$ and $S_{(n,k)}(X, Y) \overset{\text{def}}{=} (S_{(n,k)}(X), S_{(n,k)}(Y))$.

**Proposition 3.2.** Let $(i, j), (n, k) \in \mathbb{N} \times \mathbb{N}$. Then

$$P_{D(X, Y)}((i, j), (n, k)) = h_{n-i,k-j}(S_{(i,j)}(X, Y)).$$  \hspace{1cm} (9)

**Proof:** We may divide the monomials appearing in the RHS of (8) in two disjoint sets according to whether $i_n = k$ or $i_n < k$. This immediately implies that

$$h_{n,k}(X, Y) = x_{n,k}h_{n-1,k}(X, Y) + y_{n,k}h_{n,k-1}(X, Y)$$
for all \( n, k \in \mathbb{N} \) with \( n + k \in \mathbb{P} \). On the other hand, it is clear from the definition of \( D(X, Y) \) that

\[
P_{D(X, Y)}((0, 0), (n, k)) = x_{n, k} P_{D(X, Y)}((0, 0), (n - 1, k)) + y_{n, k} P_{D(X, Y)}((0, 0), (n, k - 1))
\]

for all \( n, k \in \mathbb{N} \) with \( n + k \in \mathbb{P} \). Since \( P_{D(X, Y)}((0, 0), (0, 0)) = h_{0, 0}(X, Y) = 1 \), this implies that

\[
P_{D(X, Y)}((0, 0), (n, k)) = h_{n, k}(X, Y)
\]

for all \( n, k \in \mathbb{N} \). Now, the definition of \( D(X, Y) \) also implies that

\[
P_{D(X, Y)}((i, j), (n, k)) = P_{D(S_{i, j}(X, Y))}((0, 0), (n - i, k - j))
\]

for all \((i, j), (n, k) \in \mathbb{N} \times \mathbb{N}\). Comparing (10) and (11) yields (9), as desired.

We can now interpret the entries of any totally positive matrix as evaluations of generalized complete homogeneous symmetric functions.

**Theorem 3.3.** Let \( U \) be an \( n \times n \) (real) matrix. Then \( U \) is totally positive if and only if there exist \( m, t \in \mathbb{N} \) and two nonnegative matrices \( A = (a_{i, j})_{i, j \in \mathbb{N}} \) and \( B = (b_{i, j})_{i, j \in \mathbb{N}} \) with finitely many nonzero entries, such that

\[
U = (h_{i + j - i, m + j - i}(S_{(n + 1 - i, i)}(A, B)))_{1 \leq i, j \leq n}.
\]

Furthermore, \( U \) is upper (respectively, lower) triangular if and only if \( m = 0 \) (respectively, \( t = 0 \)).

**Proof.** Suppose first that there exist \( m, t \in \mathbb{N} \) and two matrices \( A, B \), as in the statement of the theorem, such that (12) holds. Then, by Proposition 3.2,

\[
U = (P_{D(A, B)}((n + 1 - i, i), (t + n + 1 - j, m + j)))_{1 \leq i, j \leq n}.
\]

But it is easy to see that \( D(A, B) \) and the \( n \)-tuples \((n, 1), \ldots, (1, n)\) and \(((t + n, m + 1), \ldots, (t + 1, m + n)) \) satisfy all the hypotheses of Theorem 3.1 and, hence, \( U \) is totally positive, as claimed.

Conversely, suppose that \( U \) is totally positive. It then follows from the proof of Theorem 3.1 that there exist \( m, t \in \mathbb{N} \) and \( n \times n \) nonnegative matrices \( W_1 \cdots W_{m+t} \) such that (5), (6), and (7) hold. It is then easy to see that we can use the entries of \( W_1, \ldots, W_{m+t} \) to define two nonnegative
matrices $A = (a_{i,j})_{i,j \in \mathbb{N}}$ and $B = (b_{i,j})_{i,j \in \mathbb{N}}$, with finitely many nonzero entries, such that

$$P_{D(A, B)}((n+1-i, i), (t+n+1-j, m+j)) = P_{D(U)}((1, i), (t+m+1, j))$$

for all $1 \leq i, j \leq n$, where $D(U)$ has the same meaning as in the proof of Theorem 3.1. The thesis then follows from (7), (14), and (9). Finally, it follows from Theorem 4.1 of [14] that if $U$ is upper (respectively, lower) triangular then we can take $m=0$ (respectively, $t=0$) in (7), and this completes the proof.

Note that the preceding theorem explicitly characterizes totally positive matrices. Even though other characterizations of totally positive matrices have appeared before in the literature (see, e.g., [41], where a characterization is given in terms of alternating polytopes), none of these is as simple as the one given above which, in effect, gives an explicit parametrization of the set of all totally positive matrices.

Further evidence of the fact that Theorem 3.3 gives the “right” characterization of totally positive matrices is provided by the comparison with the classical characterization theorem for Pólya frequency sequences obtained by Edrei in 1953 (see, e.g., [17; or 24, Chap. 8, Section 5, Theorem 5.3]). This result, in fact, expresses any element of a Pólya frequency sequence as some “generalized complete homogeneous symmetric function” (more precisely, as a linear combination of super-Schur functions corresponding to one-row shapes) evaluated at nonnegative real numbers (cf. [24, Chap. 8, Section 5; and 10, Section 2, Eq. (6)]).

Besides giving an explicit representation of any totally positive matrix Theorem 3.3 enables us to give a simple combinatorial interpretation of the determinant of any minor of any such matrix. This can be accomplished by analyzing $r$-tuples of nonintersecting paths in the digraph $D(X, Y)$.

Given a weakly increasing sequence $i_0 \leq i_1 \leq \cdots \leq i_{n+1}$ of natural numbers we define its $(X, Y)$-weight to be

$$w_{(X, Y)}(i_0, \ldots, i_{n+1}) \overset{\text{def}}{=} x_{i_1, i_0} x_{i_2, i_1} \cdots x_{i_{n+1}, i_n} y_{i_0, i_1} y_{i_1, i_2} \cdots y_{i_{n+1}, i_n}.$$  

For example, $w_{(X, Y)}(0, 0, 2, 4, 5) = x_{1, 0} x_{3, 0} x_{3, 2} x_{3, 4} y_{2, 1} y_{2, 2} y_{3, 3} y_{3, 4} y_{4, 5}$.

Given two partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$, such that $\mu \subseteq \lambda$ and a shifted skew tabloid $\pi \overset{\text{def}}{=} (\pi_{i,j})_{1 \leq i \leq r, \mu_i + i \leq \lambda_i + i - 1}$ of shape $\lambda \setminus \mu$, we define its $(X, Y)$-weight to be

$$w_{(X, Y)}(\pi) \overset{\text{def}}{=} \prod_{i=1}^{r} w_{S_{\mu_i+1,0}(X, Y)}(\pi_i).$$  

(15)
For example, if \( z_0 \) is

\[
\begin{array}{cccccccc}
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 4 & 4 & 4 & 5 & 5 &  & \\
\end{array}
\]

then its \((X,Y)\)-weight (as a shifted skew tabloid of shape \((10,9,6)\) \(\backslash\) \((3,1,0)\)) is

\[
\begin{align*}
&w_{s_{4,0}(x,y)}(1, 2, 2, 2, 2, 2) \times w_{s_{1,0}(x,y)}(3, 4, 4, 4, 5, 5) \\
&= (x_{[5,9,1,2,4,2]}(x_{[3,2,4,5,3,3]}(x_{[2,4,5,5,4,2]} \times w_{s_{1,0}(x,y)}(3, 4, 4, 4, 5, 5).
\end{align*}
\]

(17)

Note that the \((X,Y)\)-weight of \( z_0 \) depends on \( \lambda \) and \( \mu \) and not just on \( \lambda \backslash \mu \).

Given \((a, b), (c, d) \in N \times N\) and a directed path \( \tau \) in \( D(X, Y) \) from \((a, b)\) to \((c, d)\) we let

\[
M_{\tau}(i) \overset{\text{def}}{=} \max\{j : (i, j) \in \tau\}
\]

for \( a \leq i \leq c \).

**Proposition 3.4.** Let \( r \in P \) and \( \{a_1, \ldots, a_r\} <, \{b_1, \ldots, b_r\} <, \{c_1, \ldots, c_r\} <, \{d_1, \ldots, d_r\} < \subseteq P \). Then there is a bijection between \( r \)-tuples of nonintersecting paths in \( D(X, Y) \) from \((a_1, b_1), \ldots, (a_r, b_r)\) to \((c_1, d_1), \ldots, (c_r, d_r)\) and column strict shifted skew tableaux of shape \((c_1 + 1, \ldots, c_r + 1)\) \(\backslash\) \((a_1 - 1, \ldots, a_r - 1)\) and size \((d_1, \ldots, d_r)\)\(\backslash(b_1, \ldots, b_r)\). Furthermore, if \((\tau_1, \ldots, \tau_r)\) and \( \pi \) correspond under the above bijection then

\[
w_{D(X, Y)}(\tau_1, \ldots, \tau_r) = w_{(x, y)}(\pi).
\]

**Proof.** Let \((\tau_1, \ldots, \tau_r)\) be an \( r \)-tuple of nonintersecting paths in \( D(X, Y) \) from \((a_1, b_1), \ldots, (a_r, b_r)\) to \((c_1, d_1), \ldots, (c_r, d_r)\) (so that \( \tau_i \) goes from \((a_i, b_i)\) to \((c_i, d_i)\), for \( i = 1, \ldots, r \)). Let \( \pi = (\pi_{i,j})_{1 \leq i \leq r, a_i - 1 \leq j \leq c_i} \), where

\[
\pi_{i,j} \overset{\text{def}}{=} M_{\tau_i - j}(i)
\]

(18)

for \( 1 \leq i \leq r, a_i - 1 \leq j \leq c_i \) (where \( M_{a_i - 1}(i) = b_i \) for \( i = 1, \ldots, r \)). For each \( i \in [r] \) we then clearly have that

\[
b_i = \pi_{i, a_i - 1} \leq \pi_{i, a_i + i} \leq \cdots \leq \pi_{i, c_i + i} = d_i.
\]

(19)

Furthermore, for \( i \in [r - 1] \), \( \tau_i \) and \( \tau_{i+1} \) are nonintersecting, if and only if

\[
M_{k - 1}(\tau_{i+1}) > M_k(\tau_i)
\]

(20)
for all \( a_i \leq k \leq c_{i+1} \). Since it is clear from (19) that \( \pi_{i, a_i+i-1} < \pi_{i+1, a_i+i-1} \) and \( \pi_{i, c_{i+1}+i+1} < \pi_{i+1, c_{i+1}+i+1} \), we conclude from (18) and (20) that \( \pi \) is column strict.

Conversely, let \( \pi = (\pi_{i,j})_{1 \leq i \leq r, a_i+i-1 \leq j \leq c_{i}+i} \) be a column-strict tableau satisfying (19). It is clear that, then, for each \( i \in [r] \) there is a unique path \( \pi_i \) from \((a_i, b_i)\) to \((c_i, d_i)\) such that (18) holds for all \( a_i+i \leq j \leq c_i+i \), namely,

\[
\tau_i \overset{\text{def}}{=} \bigcup_{j = a_i}^{c_i} \{(j, k) : \pi_{i, j+i-1} \leq k \leq \pi_{i, j+i}\}.
\]  

(21)

Since \( \pi \) is column strict, this implies that (20) holds for all \( i \in [r-1] \), \( a_i \leq k \leq c_{i+1}+1 \), and this, as remarked above, implies that \( \tau_i \) and \( \tau_{i+1} \) are nonintersecting, for \( i = 1, \ldots, r-1 \).

Finally, if \( \pi \) and \((\tau_1, \ldots, \tau_r)\) correspond under the above bijection then, from (21) and our definitions, we conclude that

\[
\mathcal{W}_{D(X, Y)}(\tau_i) = \prod_{f = a_i}^{c_i} Y_{f, (\pi_{i, j+i-1}, \pi_{i, j+i})} \prod_{f = a_i}^{c_i} x_{j+i-1, \pi_{i, j+i}}
\]

\[
= \mathcal{W}_{S(a_i, 0)(X, Y)}(\pi_i),
\]

for \( i = 1, \ldots, r \) and the thesis follows.  

We illustrate the above proposition with an example. Let \( \pi \) be the shifted skew column strict tableau of shape \((10, 9, 6)\setminus(3, 1, 0)\) given in (16). Then the corresponding triple of nonintersecting paths \((\tau_1, \tau_2, \tau_3)\) in \( D(X, Y) \) from \(((4, 1), (2, 2), (1, 3))\) to \(((9, 2), (8, 3), (5, 5))\) is depicted in Fig. 4. Their weight is

\[
\mathcal{W}_{D(X, Y)}(\tau_1, \tau_2, \tau_3) = (y_{4, 2}x_{[5, 9, 2]})(x_{3, 3}y_{3, 3}x_{[4, 8, 3]})(y_{1, 4}x_{[2, 4]}y_{4, 5}x_{5, 5}),
\]

which is in accordance with (17).
We can now interpret any minor of any totally positive matrix as a “generalized flagged Schur function.” In particular, this gives a combinatorial interpretation to any such minor, thus also providing an answer to a general question first raised in [6; see Section 2.2, p. 10].

**Theorem 3.5.** Let $U$ be an $n \times n$ totally positive matrix and $m, t, A$ and $B$ as in the statement of Theorem 3.3. Let $r \in [n]$ and $\{m_1, ..., m_r\} \subset \{k_1, ..., k_r\} \subset [n]$. Then

$$U\left(\begin{array}{c} m_1, ..., m_r \\ k_1, ..., k_r \end{array}\right) = \sum_{\pi} w_{(A, B)}(\pi),$$

where the sum is over all column strict skew tableaux $\pi$ of shape $(t+n+2-k_1, ..., t+n+1-k_r+r) \setminus (n-m_1, ..., n-m_r+r-1)$ and size $(m+k_1, ..., m+k_r) \setminus (m_1, ..., m_r)$.

**Proof.** By Theorem 3.3 and Proposition 3.2 we have that

$$U = (P_{D(A, B)}((n+1-i, i), (t+n+1-j, m+j)))_{1 \leq i,j \leq n}.$$

Since $D(A, B)$ satisfies the hypotheses of Lemma 2.1, and $u \equiv ((n, 1), ..., (1, n))$, and $v \equiv ((t+n, m+1), ..., (t+1, m+n))$ are fully compatible we conclude that

$$U\left(\begin{array}{c} m_1, ..., m_r \\ k_1, ..., k_r \end{array}\right) = \sum_{(\tau_1, ..., \tau_r)} w_{D(A, B)}(\tau_1, ..., \tau_r),$$

where the sum is over all $r$-tuples $(\tau_1, ..., \tau_r)$ of nonintersecting paths in $D(A, B)$ from $((n+1-m_1, m_1), ..., (n+1-m_r, m_r))$ to $((t+n+1-k_1, m+k_1), ..., (t+n+1-k_r, m+k_r))$. The thesis now follows from Proposition 3.4.

Besides enabling us to give a combinatorial interpretation to any minor of any totally positive matrix, Theorems 3.1 and 3.3 have also other applications to the theory of totally positive matrices. For example, the following fundamental fact (usually proved using the classical Cauchy–Binet identity; see, e.g., [24, Chap. 0, p. 1] is an immediate consequence of Theorem 3.1 (and its proof).

**Corollary 3.6.** Let $U$ and $V$ be two $n \times n$ totally positive matrices. Then $UV$ is totally positive.

Another application is to the computation of immanants of totally positive matrices. Recall (see, e.g., [21, Definition 1.1]) that, given an $n \times n$
matrix \( U \) and a partition \( \lambda \) of \( n \) the \textit{immanant} of \( U \) corresponding to \( \lambda \) is defined by

\[
\operatorname{Imm}_\lambda(U) \overset{\text{def}}{=} \sum_{\sigma \in S_n} \chi_\lambda(\sigma) U_{1, \sigma(1)} \cdots U_{n, \sigma(n)},
\]

where \( \chi_\lambda \) is the irreducible character of \( S_n \) corresponding to the partition \( \lambda \). Recently there has been a revived interest on immanants (especially of totally positive matrices) and several conjectures have been made and proved about them (see, e.g., [21, 22, 39, 40, 23] for further information). Proposition 3.2 enables us to use a combinatorial construction first introduced in [21] to prove the following result.

\textbf{Theorem 3.7.} Let \( n, m, t \in \mathbb{N} \) and

\[
U = (h_{t+i-j, m+j-i}(S_{n+1-i, i}(X, Y)))_{1 \leq i, j \leq n}.
\]

Then

\[
\operatorname{Imm}_\lambda(U) \in \mathbb{N}[X, Y],
\]

for all partitions \( \lambda \) of \( n \).

\textbf{Proof.} By Proposition 3.2 we have that

\[
U = (P_D(X, Y)(u_{i, j}))_{1 \leq i, j \leq n},
\]

where \( u_{i, j} = (n+1-i, i) \) and \( v_{i, j} = (t+n+1-i, m+i) \) for \( i = 1, \ldots, n \). For \( \sigma \in S_n \) let \( \mathcal{C}_\sigma \) be the set of all \( n \)-tuples of paths \( (\tau_1, \ldots, \tau_n) \) such that \( \tau_i \) is a directed path from \( u_i \) to \( v_{\sigma(i)} \) for \( i = 1, \ldots, n \). Then from (22) and (23) we have that

\[
\operatorname{Imm}_\lambda(U) = \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \prod_{i=1}^{n} P_D(X, Y)(u_{i, \sigma(i)})
= \sum_{\sigma \in S_n} \chi_\lambda(\sigma) \sum_{\tau \in \mathcal{C}_\sigma} w_D(X, Y)(\tau)
= \sum_{\tau \in \mathcal{C}} \chi_\lambda(\sigma_\tau) w_D(X, Y)(\tau),
\]

where \( \mathcal{C} \overset{\text{def}}{=} \bigcup_{\sigma \in S_n} \mathcal{C}_\sigma \) and, for \( \tau \in \mathcal{C} \), \( \sigma_\tau \) is the unique element of \( S_n \) such that \( \tau \in \mathcal{C}_{\sigma_\tau} \). Therefore,

\[
\operatorname{Imm}_\lambda(U) = \sum_{\alpha \in \mathcal{A}(X, Y)} \alpha \sum_{\tau \in \mathcal{M}_\alpha} \chi_\lambda(\sigma_\tau) = \sum_{\alpha \in \mathcal{A}(X, Y)} \alpha \chi_\lambda(\sum_{\tau \in \mathcal{M}_\alpha} \sigma_\tau),
\]
where \( \mathcal{M}(X, Y) \) is the set of all monomials (of finite degree) in the variables \( x_{i,j} \) and \( y_{i,j} \), \( i, j \in \mathbb{N} \), and \( \mathcal{A}_\alpha \) is the set of all \( \tau \in \mathcal{C} : w_{D(X, Y)}(\tau) = \alpha \), for \( \alpha \in \mathcal{M}(X, Y) \).

Now fix \( \alpha = \prod_{i,j \geq 0} x_{i,j}^\alpha y_{i,j}^\beta \in \mathcal{M}(X, Y) \), where \( \alpha_{i,j}, \beta_{i,j} \in \mathbb{N} \) for all \( i, j \in \mathbb{N} \).

We claim that

\[
\sum_{\tau \in \mathcal{A}_\alpha} \sigma_\tau = cT_{J_1} \cdots T_{J_r} \tag{25}
\]

for some \( r \in \mathbb{N} \), where \( J_1, \ldots, J_r \) are subintervals of \([n] \), \( c \geq 0 \), and, for any \( J \subseteq [n] \),

\[
T_j \overset{\text{def}}{=} \sum_{\sigma \in S(J)} \sigma
\]

(where we consider \( S(J) \) as a subgroup of \( S_n \) in the obvious way). In fact, let \( \overline{D}_\alpha(X, Y) \) be the directed planar graph obtained from \( D(X, Y) \) by replacing the edge from \((i-1, j)\) (respectively, \((i, j-1)\)) to \((i, j)\) by \( \alpha_{i,j} \) (respectively, \( \beta_{i,j} \)), each having weight \( x_{i,j} \) (respectively, \( y_{i,j} \)), for all \((i, j) \in \mathbb{N} \times \mathbb{N} \). It is then clear that

\[
\sum_{\tau \in \mathcal{A}_\alpha} \sigma_\tau = \prod_{i,j \geq 0} (\alpha_{i,j})! (\beta_{i,j})! \sum_{\tau \in \mathcal{A}_\alpha} \sigma_\tau, \tag{26}
\]

where \( \mathcal{A}_\alpha = \bigcup_{\sigma \in S_n} \mathcal{A}_{\alpha, \sigma} \) and \( \mathcal{A}_{\alpha, \sigma} \) is the set of all \( n \)-tuples of edge-disjoint paths \((\tau_1, \ldots, \tau_n)\) in \( \overline{D}_\alpha(X, Y) \) from \((u_1, \ldots, u_n)\) to \((v_{\sigma(1)}, \ldots, v_{\sigma(n)})\) such that \( w_{D(X, Y)}(\tau) = \alpha \). For \( \tau \in \mathcal{A}_\alpha \) and \( 0 \leq \gamma \leq t + m, \gamma \notin [t + m - 1] \), let \((x_i^{(\gamma)}, y_i^{(\gamma)})\) be the point of intersection of \( \tau \) (considered as a continuous curve) with the line \( x + y = y + n + 1 \), and let \( \sigma_\tau^{(\gamma)} \) be the unique permutation of \( S_n \) such that

\[
x_\sigma^{(\gamma)}(1) < \cdots < x_\sigma^{(\gamma)}(m).
\]

Note that \( \sigma_\tau^{(0)} = \text{Id} \), \( \sigma_\tau^{(t+m)} = \sigma_\tau \), and \( \sigma_\tau^{(a-\delta)} = \sigma_\tau^{(\delta)} \) for all \( a-1 < \gamma < \delta < a \), and \( \gamma \notin [t + m] \). Furthermore, it is clear from the geometry of \( \overline{D}_\alpha(X, Y) \) that if \( a \in [t + m - 1] \) and \( a-1 < \gamma < a \), then

\[
\left( \prod_{i=1}^{a+n} (b_i - b_{i-1})! \right) \sum_{\tau \in \mathcal{A}_\alpha} \sigma_\tau^{(a+1)} = \left( \prod_{i=1}^{a+n} T_{(b_i - b_{i-1})} \right) \sum_{\tau \in \mathcal{A}_\alpha} \sigma_\tau^{(a)} \tag{27}
\]

where \( b_i \overset{\text{def}}{=} \sum_{j=1}^{i-1} (\alpha_{i,a+n+1-i} + \beta_{i,a+n+1-i}) \) for \( i = 0, \ldots, a+n \), and this, by (26), proves (25) as claimed. But it was proved by Greene (see, [22, Theorem 1.3]) that

\[
\chi^2(T_j) \geq 0
\]
for any partition $\lambda$ and any subinterval $J \subseteq [n]$. Thus the thesis follows from (24), (25).

As an immediate consequence of Theorems 3.3 and 3.7 we obtain the following result which was first conjectured, an then proved, by Stembridge (see [39, Conjecture 2.2; 40, Corollary 3.3]).

**Corollary 3.8.** Let $U$ be an $n \times n$ totally positive matrix and $\lambda$ be a partition of $n$. Then $\text{Im}_\lambda(U) \geq 0$.

4. **The Method of Invariant Digraphs**

Even though Theorems 3.1 and 3.3 completely characterize the class of totally positive matrices and have, as we have seen, several interesting and nontrivial consequences, their interest is mainly theoretical and they are difficult to apply in practice. In fact, given a totally positive matrix $U$ it is usually hard to determine the parameters and the weights of the corresponding digraph $D(U)$ constructed in the proof of Theorem 3.1 (or, equivalently, the integers $t, m$ and matrices $A, B$ appearing in the statement of Theorem 3.3). Conversely, given some parameters $t, m, A$ and $B$, the matrix $U$ defined by (12) will usually lack any simple combinatorial meaning, except that given by (13). So the usefulness of Theorems 3.1 and 3.3 as tools for proving the total positivity property of particular matrices is limited.

However, there are other total positivity results that can be deduced from Lemma 2.1 and which, although less powerful than Theorems 3.1 and 3.3, are easier to apply and still powerful enough to yield the total positivity of many interesting matrices arising from algebraic and enumerative combinatorics.

Let $D = (N \times N, A, w)$ be a locally finite, weighted digraph. Recall (see [10, Section 2]) that $D$ is weakly $\gamma$-invariant if

$$P_D((0, m), (n, k)) = P_D((0, 0), (n, k - m))$$

for all $m, n, k \in N$ with $k \geq m$ and that $D$ is $\gamma$-invariant if the map $S_\gamma: D \to D$ defined by

$$S_\gamma((n, k)) \overset{\text{def}}{=} (n, k + 1)$$

for all $(n, k) \in N \times N$ is an isomorphism between $D$ and $S_\gamma(D)$ (as weighted digraphs). The following theorem appears in [10] (see Theorem 2.1) and its proof is an easy consequence of (28) and of Lemma 2.1. We restate it here for completeness and because we will use it repeatedly.
Theorem 4.1. Let $D$ be a locally finite, weighted digraph on vertex set $\mathbb{N} \times \mathbb{N}$. Assume that $D$ is planar and weakly $y$-invariant. Let $\{n_1, \ldots, n_r\} \subseteq \mathbb{N}$, and $n \in \mathbb{N}$. Then

$$(P_D((0,0), (n, k)))_{n,k \in \mathbb{N}} \begin{pmatrix} n_1, \ldots, n_r \\ k_1, \ldots, k_r \end{pmatrix} = N(u_1, v_1)$$

and

$$\{P_D((0,0), (n, k))\}_{k \in \mathbb{N}} \begin{pmatrix} n_1, \ldots, n_r \\ k_1, \ldots, k_r \end{pmatrix} = N(u_2, v_2),$$

where $u_1 = ((0, m-k_1), \ldots, (0, m-k_r))$, $v_1 = ((n_1, m), \ldots, (n_r, m))$, $u_2 = ((0, n_1), \ldots, (0, n_r))$, and $v_2 = ((n, k_1), \ldots, (n, k_r))$.

From this we immediately deduce the following total positivity result.

Corollary 4.2. Let $D$ be a locally finite, nonnegative, digraph on vertex set $\mathbb{N} \times \mathbb{N}$. Assume that $D$ is planar and weakly $y$-invariant. Let $M = (M_{n,k})_{n,k \in \mathbb{N}}$ be the infinite matrix defined by

$$M_{n,k} \overset{\text{def}}{=} P_D((0,0), (n, k))$$

for $(n,k) \in \mathbb{N} \times \mathbb{N}$. Then

(i) $M$ is totally positive;

(ii) every row of $M$ is a PF sequence.

We can now define and study a class of infinite matrices which possesses many remarkable total positivity properties and which includes many familiar matrices arising in enumerative and algebraic combinatorics as well as in the theory of symmetric functions. Let $x \overset{\text{def}}{=} \{x_n\}_{n \in \mathbb{N}}$, $y \overset{\text{def}}{=} \{y_n\}_{n \in \mathbb{N}}$, and $z \overset{\text{def}}{=} \{z_n\}_{n \in \mathbb{N}}$ be three sequences of independent variables. We will adopt the convention that $x_n \overset{\text{def}}{=} y_n \overset{\text{def}}{=} z_n \overset{\text{def}}{=} 0$ if $n < 0$.

Theorem 4.3. Let $t \in \mathbb{N}$. Define a matrix $M = (M_{n,k})_{n,k \in \mathbb{N}}$ by

$$M_{n,k} = z_n M_{n-t,k-1} + y_n M_{n-1-t,k-1} + x_n M_{n-1,k}$$

if $n + k \in \mathbb{P}$ (where $M_{n,k} \overset{\text{def}}{=} 0$ if either $n < 0$ or $k < 0$), and $M_{0,0} \overset{\text{def}}{=} 1$. Then

(i) $M$ is $(x, y, z)$-totally positive;

(ii) every row of $M$ is an $(x, y, z)$-PF sequence.

Proof. We construct a weighted digraph $D(t)$ on $\mathbb{N} \times \mathbb{N}$ as follows. We put an edge from $(n-t, k-1)$ (respectively, $(n-1-t, k-1)$, $(n-1, k)$) to $(n, k)$ with weight $z_n$ (respectively, $y_n$, $x_n$) for all $(n,k)$ for which
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\[(n - t, k - 1)\] (respectively, \((n - 1 - t, k - 1), (n - 1, k)\)) is in \(\mathbb{N} \times \mathbb{N}\). Clearly, \(D(t)\) is planar and \(y\)-invariant, and

\[M_{n,k} = P_{D(t)}((0, 0), (n, k)), \quad (32)\]

so the thesis follows from Theorem 4.1.

In what follows we will always denote by \(M_{r}(x, y, z)\) the matrix defined in the preceding theorem. Note that the columns of \(M_{r}(x, y, z)\) are not, in general \((x, y, z)\)-PF, as simple examples show. Also note that the matrices \(M_{r}(x, y, z)\) include the matrix of super-Schur functions studied in \([10]\), as is shown by the next result, whose proof follows immediately from (32) and Proposition 2.2 of \([10]\). Given a partition \(\lambda\), we denote by \(s_{\lambda}(x/y)\) the super-Schur function corresponding to \(\lambda\) in the variables \(x\) and \(y\) (we refer the reader to \([3, 4, 10, \text{or } 15]\), for the definition and further information about super-Schur functions).

PROPOSITION 4.4. We have that

\[(s_{\lambda}(x_1, \ldots, x_n/y_1, \ldots, y_n))_{n, k \in \mathbb{N}} = M_{0}(1, y, \{0, x_1, x_2, \ldots\}).\]

The matrix \(M_{0}(x, y, z)\) has an additional total positivity property. Given an infinite matrix \(M \equiv (M_{n,k})_{n,k \in \mathbb{N}}\) and \(t \in \mathbb{N}\), we let

\[L_t(M) \equiv (M_{n-kt,k})_{n,k \in \mathbb{N}} \quad (33)\]

(where \(M_{n,k} \equiv 0\) if either \(n < 0\) or \(k < 0\)). So \(L_0(M) = M, L_t(L_r(M)) = L_{t+r}(M)\), and the \(k\)th column of \(L_t(M)\) equals the \(k\)th column of \(M\), preceded by a string of \(kt\) zeros.

THEOREM 4.5. Let \(t \in \mathbb{N}\). Then

\[L_t(M_{0}(x, y, z))\]

is \((x, y, z)\)-totally positive.

Proof. Construct a weighted digraph \(D'(t)\) on \(\mathbb{N} \times \mathbb{N}\) as follows. Put an edge from \((n, k - 1)\) (respectively, \((n - 1, k - t - 1), (n - 1, k - t)\)) to \((n, k)\) with weight \(z_n\) (respectively, \(y_n, x_n\)) for all \((n, k)\) for which \((n, k - 1)\) (respectively, \((n - 1, k - t - 1), (n - 1, k - t)\)) is in \(\mathbb{N} \times \mathbb{N}\). A portion of the digraph \(D'(1)\) is shown in Fig. 5.

Clearly, \(D'(t)\) is planar and \(y\)-invariant, and it follows from (31) and (33) that

\[L_t(M_{0}(x, y, z))' = (P_{D'(t)}((0, 0), (n, k)))_{n,k \in \mathbb{N}}, \quad (34)\]

so the thesis follows from Theorem 4.1.
Note that it is not true, in general, that if an infinite matrix $M$ is totally positive then $L_t(M)$ is totally positive. For example,

$$M = \begin{pmatrix} 1 & 1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 2 & 2 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is totally positive, but $L_1(M)$ is not.

The matrix $L_t(M_0(x, y, z)')$ can also be described directly by a simple linear recurrence analogous to (31).

**Proposition 4.6.** Let $t \in \mathbb{N}$ and $L \overset{\text{def}}{=} L_t(M_0(x, y, z)')$. Then the entries $L_{n,k}$ of $L$ satisfy the recurrence relation.

$$L_{n,k} = x_k L_{n-t,k-1} + y_k L_{n-1-t,k-1} + z_k L_{n-1,k},$$

(35)

for $n + k \in \mathbb{P}$ (where $L_{n,k} \overset{\text{def}}{=} 0$ if either $n < 0$ or $k < 0$), with the initial condition $L_{0,0} \overset{\text{def}}{=} 1$. 

![Fig. 5. The digraph $D'(1)$.](image-url)
There are some simple relations that hold among the matrices $M_t(x, y, z)$ which deserve to be mentioned. Their verification is left to the reader.

**Proposition 4.7.** For $t \in \mathbb{N}$ we have that:

1. $M_t(x, y, 0) = M_{t+1}(x, 0, y)$;
2. $L_1(M_0(x, 0, y)'') = M_0(0, x, y)'$.

We close this section by remarking that other interesting classes of $x$-totally positive matrices can be obtained by “manipulating” the matrices $M_t(x, y, z)$ suitably. We give one such example below.

**Theorem 4.8.** Let \{\(a_n\)\}_{n \in \mathbb{N}}, \{\(b_n\)\}_{n \in \mathbb{N}}, \{\(\alpha_n\)\}_{n \in \mathbb{N}}, \{\(\beta_n\)\}_{n \in \mathbb{N}} be sequences of polynomials in $\mathbb{N}[x]$. Define a matrix $R \overset{\text{def}}{=} (R_{n,k})_{n,k \in \mathbb{N}}$ by

$$R_{n,k} = (a_n + b_n \alpha_k)R_{n-1,k} + b_n \beta_k R_{n-1,k-1}$$

for $n + k \in \mathbb{P}$ (where $a_n = b_n = \alpha_k = \beta_k = R_{n,k} = 0$ if either $n < 0$ or $k < 0$), and $R_{0,0} \overset{\text{def}}{=} 1$. Then $R$ is $x$-totally positive.

**Proof.** Let $M \overset{\text{def}}{=} M_0(\{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}}, 0)$ and $L \overset{\text{def}}{=} M_0(\{\beta_n\}_{n \in \mathbb{N}}, \{\alpha_n\}_{n \in \mathbb{N}})'$. By Theorem 4.3 we know that $L$ and $M$ are $x$-TP. We claim that $R = ML$ (note that $M$ and $L$ are both lower triangular, so that $ML$ is well defined). We will prove this by showing that the entries of $ML$ satisfy the recursion (36), with the same initial condition. In fact, using (31) and (35), we see that

$$(M, L)_{n,k} = \sum_{j=0}^{n} M_{n,j}L_{j,k}$$

$$= \sum_{j=0}^{n-1} M_{n-1,j}(a_n L_{j,k} + b_n L_{j+1,k})$$

$$= a_n (ML)_{n-1,k} + b_n (\alpha_k (ML))_{n-1,k} + \beta_k, (ML)_{n-1,k-1},$$

and, since $(ML)_{0,0} = M_{0,0}L_{0,0} = 1$, the claim follows. But it follows immediately from the Cauchy–Binet formula (see, e.g. [24, p. 1]) that the product of two $x$-TP matrices is again $x$-TP and the thesis follows.

5. **Applications to Combinatorics**

The results obtained in the preceding section have numerous applications to combinatorics and to the theory of symmetric functions. In this section we look in detail at these applications. In many cases we obtain new results; in others we obtain simple an unified proofs of known results.
Our first application is to a natural common generalization of the elementary and complete homogeneous symmetric functions. Let \( t \in \mathbb{N} \) and \( x \stackrel{\text{def}}{=} (x_0, x_1, \ldots) \) be a sequence of independent variables. For \( k \in \mathbb{P} \) and \( n \in \mathbb{N} \) we let

\[
e^{(t)}_k(x_0, \ldots, x_n) \stackrel{\text{def}}{=} \sum_{i_1, \ldots, i_k \in [n]} x_{i_1} \cdots x_{i_k},
\]

where the sum is over all \( i_1, \ldots, i_k \in [n] \) such that \( i_{j+1} - i_j \geq t \) for all \( 0 \leq j \leq k - 1 \) (where \( i_0 \stackrel{\text{def}}{=} 0 \)). We also let \( e^{(t)}_0(x_0, \ldots, x_n) \stackrel{\text{def}}{=} 1 \) for all \( n \in \mathbb{N} \). So, for example, \( e^{(2)}_3(x_0, x_1) = x_3x_5x_7 + x_2x_4x_7 + x_2x_5x_7 \). Also, it is clear that

\[
e^{(t)}_k(x_0, \ldots, x_n) = h_k(x_0, \ldots, x_n)
\]

and

\[
e^{(t)}_k(x_1, \ldots, x_n) = e_k(x_1, \ldots, x_n),
\]

for all \( k, n \in \mathbb{N} \).

**Theorem 5.1.** Let \( t \in \mathbb{N} \), and \( E_t(x) \stackrel{\text{def}}{=} (e^{(t)}_k(x_0, \ldots, x_n))_{n,k \in \mathbb{N}} \). Then

(i) \( E_t(x) \) is \( x \rightarrow TP \);

(ii) every row of \( E_t(x) \) is \( x \rightarrow PF \).

**Proof.** It follows immediately from our definitions that

\[
e^{(t)}_k(x_0, \ldots, x_n) = x_ne^{(t)}_{k-1}(x_0, \ldots, x_{n-1}) + e^{(t)}_k(x_0, \ldots, x_{n-1}),
\]

for all \( n, k \in \mathbb{N} \) with \( n + k \in \mathbb{P} \). Therefore,

\[
E_t(x) = M_t(1, 0, x),
\]

and the thesis follows from Theorem 4.3.

For \( t = 0, 1 \) part (i) of the preceding theorem was first proved by Sagan (see [31, Theorem 5.4]), while part (ii) is then a consequence of the well-known Jacobi–Trudi identity (see, e.g., [27, Chap. I, Section 5, Eqs. (5.4) and (5.5)]).

There is another generalization of the elementary symmetric functions which also possesses interesting total positivity properties. For \( t, k \in \mathbb{P} \) and \( n \in \mathbb{N} \) let

\[
a^{(t)}_k(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \sum_{i_1, \ldots, i_k \in [n]} x_{i_1} \cdots x_{i_k},
\]
where the sum is over all $1 \leq i_1 < \cdots < i_k \leq n$, such that $i_j \equiv j \pmod{t}$ for $j = 1, \ldots, k$. We also let $a_{0}^{(t)}(x_1, \ldots, x_n) \overset{\text{def}}{=} 1$, for all $n \in \mathbb{N}$. So, for example, $a_{2}^{(2)}(x_1, \ldots, x_6) = x_1 x_2 + x_1 x_4 + x_1 x_6 + x_3 x_4 + x_3 x_6 + x_5 x_6$, $a_{k}^{(t)}(x_1, \ldots, x_n) = 0$, unless $n \geq k$, and $a_{k}^{(t)}(x_1, \ldots, x_n) = a_{k}^{(t)}(x_1, \ldots, x_{n-1})$ if $n \not\equiv k \pmod{t}$. Also note that

$$a_{k}^{(1)}(x_1, \ldots, x_n) = e_{k}(x_1, \ldots, x_n)$$

for all $n, k \in \mathbb{N}$.

**Theorem 5.2.** Let $t \in \mathbb{P}$, and $A_{\lambda}(x) \overset{\text{def}}{=} (a_{\lambda-\ell k}^{(t)}(x_1, \ldots, x_n))_{k, n \in \mathbb{N}}$. Then

(i) $A_{\lambda}(x)$ is $x-TP$;

(ii) every row of $A_{\lambda}(x)$ is $x-PF$.

**Proof.** Note that it follows from our definitions that if $n \equiv k \pmod{t}$ and $x_{i_1} \cdots x_{i_k}$ is a monomial appearing on the RHS of (41) with $i_k < n$, then $i_k \equiv n-t \pmod{t}$. Therefore,

$$a_{k}^{(t)}(x_1, \ldots, x_n) = x_n a_{k-1}^{(t)}(x_1, \ldots, x_{n-1}) + a_{k}^{(t)}(x_1, \ldots, x_{n-1}),$$

if $n \equiv k \pmod{t}$. Hence,

$$a_{\lambda-\ell k}^{(t)}(x_1, \ldots, x_n) = x_n a_{\lambda-\ell k}^{(t-\ell k)}(x_1, \ldots, x_{n-1}) + a_{\lambda-\ell k-\ell (k-1)}^{(t)}(x_1, \ldots, x_{n-t}),$$

for all $n, k \in \mathbb{N}$ with $n + k > 0$. This shows that

$$A_{\lambda}(x) = M_{\lambda}(x, 0, 1), \quad (42)$$

and the thesis follows from Theorem 4.3. 

We now concentrate on more concrete combinatorial applications. For $n, k \in \mathbb{N}$ we denote by $S[n, k]_{p, q}$ and $c[n, k]_{p, q}$ the $(p, q)$-Stirling numbers of the second and first kind, respectively. These are defined inductively by letting

$$S_{p, q}[n, k] = p^{k-1} S_{p, q}[n-1, k-1] + [k]_{p, q} S_{p, q}[n-1, k]$$

and

$$c_{p, q}[n, k] = p^{n} c_{p, q}[n-1, k-1] + q[n-1]_{q} c_{p, q}[n-1, k],$$

if $n + k \in \mathbb{P}$ (with the convention that $S_{p, q}[n, k] \overset{\text{def}}{=} c_{p, q}[n, k] \overset{\text{def}}{=} 0$ if either $n < 0$ or $k < 0$), and $S_{p, q}[0, 0] \overset{\text{def}}{=} c_{p, q}[0, 0] \overset{\text{def}}{=} 1$. (We refer the reader to [42, 29] for combinatorial interpretations and further information about these polynomials). It then follows immediately from these definitions and from the definition of $M_{\lambda}(x, y, z)$ (see (31)) that

$$(S_{p, q}[n+1, k+1])_{n, k \in \mathbb{N}} = M_{\emptyset}(0, \{p^n\}_{n \in \mathbb{N}}, \{[n+1]_{p, q}\}_{n \in \mathbb{N}})'$$
and

\[(c_{p,q}[n+1,k+1])_{n,k \in \mathbb{N}} = M_0(\{q^n\}_{n \in \mathbb{N}}, \{p^n\}_{n \in \mathbb{N}}, 0).\]

The next result is, therefore, an immediate consequence of Theorems 4.3 and 4.5.

**COROLLARY 5.3.** Let \( S(p,q) \) be defined as \( (S_{p,q}[n+1,k+1])_{n,k \in \mathbb{N}} \), \( C(p,q) \) be defined as \( (c_{p,q}[n+1,k+1])_{n,k \in \mathbb{N}} \), and \( t \in \mathbb{N} \). Then

1. \( L_t(S(p,q)) \) and \( L_t(C(p,q)) \) are \((p,q)\)-TP;
2. every column of \( S(p,q) \) is \((p,q)\)-PF;
3. every row of \( C(p,q) \) is \((p,q)\)-PF.

We can also obtain easily the main total positivity properties of various matrices of \( q \)-binomial coefficients. For \( n, k \in \mathbb{N} \) we let

\[
\begin{bmatrix} n \\ k \end{bmatrix}_q \overset{\text{def}}{=} \prod_{i=1}^{k} \frac{1 - q^{n-i+1}}{1 - q^i}.
\]

The following result first appeared in [31; see Corollary 5.5 and Section 6].

**COROLLARY 5.4.** Let \( F(q) \) be defined as \( ([n+k])_{n,k \in \mathbb{N}} \) and \( t \in \mathbb{N} \). Then

1. \( L_t(F(q)) \) is \( q \)-TP;
2. every column of \( F(q) \) is \( q \)-PF.

**Proof.** It follows easily from (43) that

\[
\begin{bmatrix} n+k \\ n \end{bmatrix}_q = \begin{bmatrix} n+k-1 \\ n-1 \end{bmatrix}_q + q^n \begin{bmatrix} n+k-1 \\ n \end{bmatrix}_q
\]

for \( n+k \in \mathbb{P} \). Hence,

\[
F(q) = M_0(1,0,\{q^n\}_{n \in \mathbb{N}}),
\]

and the thesis follows from Theorems 4.3 and 4.5.

Note that \( L_t(F(q)) \) is the usual matrix of \( q \)-binomial coefficients \( ([n+k])_{n,k \in \mathbb{N}} \). The fact that, for a fixed \( q \geq 0 \), \( L_1(F(q)) \) and \( L_2(F(q)) \) are totally positive matrices was first proved by Karlin in [24].

There are countless generalizations and analogs of Stirling, \( q \)-Stirling, and \((p,q)\)-Stirling numbers of both kinds existing in the literature, and many of them can be obtained by suitably specializing and combining the matrices \( M_t(x,y,z) \). We give below some of the most interesting such examples.
For \( n, k \in \mathbb{N} \) let \( S[n, k]_{p,q} \), \( \tilde{S}[n, k]_{p,q} \) and \( \tilde{S}[n, k]_{p,q} \) be the \((p,q)\)-Stirling numbers of the second kind defined in Section 6 of [32] (to which we refer the reader for their definition). (Note that what we denote by \( S[n, k]_{p,q} \) is denoted by \( S[n, k]_{p,q} \) in Section 6 of [32]). We then have the following result.

**Theorem 5.5.** Let \( S(p, q) \overset{\text{def}}{=} (S[n + 1, k + 1]_{p,q})_{n,k \in \mathbb{N}} \), \( \tilde{S}(p, q) \overset{\text{def}}{=} (\tilde{S}[n + 1, k + 1]_{p,q})_{n,k \in \mathbb{N}} \), \( \tilde{S}(p, q) \overset{\text{def}}{=} (\tilde{S}[n + 1, k + 1]_{p,q})_{n,k \in \mathbb{N}} \), and \( t \in \mathbb{N} \). Then

(i) \( L_i(S(p, q)) \), \( L_i(\tilde{S}(p, q)) \), and \( L_i(\tilde{S}(p, q)) \) are \((p, q)\) - TP;

(ii) every column of \( S(p, q) \), \( \tilde{S}(p, q) \), and \( \tilde{S}(p, q) \) is \((p, q)\) - PF.

**Proof.** Theorems 6.1 and 6.2 of [32] imply that

\[
S(p, q) = M_{0}(0, 1, \{1 + pq[k]_{p,q}\}_{k \in \mathbb{N}})',
\]

\[
\tilde{S}(p, q) = M_{0}(0, \{qk\}_{k \in \mathbb{N}}, \{[k + 1]_{p,q}\}_{k \in \mathbb{N}})',
\]

and

\[
\tilde{S}(p, q) = M_{0}(0, \{qk\}_{k \in \mathbb{N}}, \{qk + pq[k]_{p,q}\}_{k \in \mathbb{N}})',
\]

so the thesis follows from Theorems 4.3 and 4.5.

For \( n, k \in \mathbb{N} \) let \( S_B(n, k; q) \) be the \(q\)-Stirling number of the second kind of type \( B_n \) defined in Section 3 of [11]. These may also be defined by

\[
S_B(n, k; q) = \sum_{i=k}^{n} \binom{n}{i} q^{n-i} (1 + q)^i S(i, k)
\]

(44)

and have the property that \( S_B(n, k; 0) = S(n, k) \), while \( k! S_B(n, k; 1) \) equals the number of \( k\)-dimensional faces of the Coxeter complex of type \( B_n \) (see [11, Section 3, and [16]], for further details). We then have the following result.

**Theorem 5.6.** Let \( S_B(q) \overset{\text{def}}{=} (S_B(n, k; q))_{n, k \in \mathbb{N}} \) and \( t \in \mathbb{N} \). Then

(i) \( L_i(S_B(q)) \) is \( q\) - TP;

(ii) every column of \( S_B(q) \) is \( q\) - PF.

**Proof.** It follows from (44) (see also Eq. (47) of [11]) that

\[
S_B(n, k; q) = (1 + q)S_B(n - 1, k - 1; q) + (k(1 + q) + q)S_B(n - 1, k; q)
\]

for all \( n, k \in \mathbb{N} \) with \( n + k > 0 \). Hence,

\[
S_B(q) = M_{0}(0, (1 + q)1, \{k(1 + q) + q\}_{k \in \mathbb{N}})',
\]

and the thesis follows from Theorems 4.3 and 4.5.
We now consider $q$-analogues of the $r$-associated Stirling numbers. These, to the best of our knowledge, have never been considered before in the literature.

For $n, k, r \in \mathbb{P}$ let $D_r(n, k)$ be the set of all $\sigma \in S_n$ that have exactly $k$ cycles, each of size $\geq r$, and $D_r(n) \overset{\text{def}}{=} \bigcup_{k \geq 1} D_r(n, k)$. Note that $|D_1(n, k)|$ is just the signless Stirling number of the first kind $c(n, k)$, while $|D_2(n, k)|$ is the number of derangements of $S_n$ having $k$ cycles (these numbers are sometimes called the Jordan numbers; see, e.g., [12]). The numbers $|D_r(n, k)|$ are usually called the signless $r$-associated Stirling numbers of the first kind (see, e.g., [13, p. 257, Ex.7] for further information about these numbers).

Let $\sigma \in D_r(n, k)$. We say that $\sigma$ is written in $r$-normal form if:

(i) each cycle is written with its $r$th smallest element first;

(ii) the cycles are written in increasing order of their first elements.

The $r$-normal representation of $\sigma$ is the word $a_1 \cdots a_n$ obtained from the $r$-normal form of $\sigma$ by erasing all the parentheses. An $r$-inversion of $\sigma$ is an inversion in the $r$-normal representation of $\sigma$ (i.e., a pair $(a_i, a_j)$ such that $a_i > a_j$ and $i < j$). We denote by $\text{inv}_r(\sigma)$ the number of $r$-inversions of $\sigma$. For example, if $\sigma = (316)(789)(245)$, then $(524)(631)(978)$ is its 3-normal form, 524631978 its 3-normal representation, and $\text{inv}_3(\sigma) = 13$. By contrast, its 2-normal form is $(316)(452)(897)$ and, hence, $\text{inv}_2(\sigma) = 10$.

We define the $r$-associated signless $q$-Stirling numbers of the first kind by letting

$$c_r[n, k]_q \overset{\text{def}}{=} \sum_{\sigma \in D_r(n, k)} q^{\text{inv}_r(\sigma)}$$

for $n, k \in \mathbb{P}$ and $c_r[n, 0]_q = c_r[0, n]_q = \delta_{0, n}$ for $n \in \mathbb{N}$. Clearly, $c_1[n, k]_q = c[n, k]_q$, where $c[n, k]_q$ denotes a $q$-Stirling number of the first kind as defined, e.g., in [25]. We then have the following result. Given a (finite) set $T$ and a partition $\lambda$ of $|T|$ we denote by $S(T, \lambda)$ the set of all elements of $S(T)$ of cycle-type $\lambda$.

**Theorem 5.7.** Let $r, n, k \in \mathbb{P}$. Then

$$c_r[n, k]_q = [n - 1]_q c_r[n - 1, k]_q + q^{r - 1} \prod_{i=1}^{r - 1} [n - i]_q c_r[n - r, k - 1]_q$$

(where $c_r[n, k]_q \overset{\text{def}}{=} 0$ if either $n < 0$ or $k < 0$).

**Proof.** Let $\sigma \in D_r(n, k)$ and assume that $\sigma$ is written in $r$-normal form. There are two possible cases.
If \( n \) is in a cycle of size \( \geq r + 1 \) then the permutation obtained by deleting \( n \) from the disjoint cycle decomposition of \( \sigma \), call it \( \tilde{\sigma} \), is in \( D_r(n-1, k) \), and it follows from our definition of the \( r \)-normal form that the \( r \)-normal representation of \( \tilde{\sigma} \) is obtained from that of \( \sigma \) by deleting \( n \). Hence, \( \text{inv}_r(\sigma) = \text{inv}_r(\tilde{\sigma}) + n - 1 - i \), where \( i \) is the number of elements to the left of \( n \) in the \( r \)-normal representation of \( \sigma \). Furthermore, for fixed \( i \in [n-1] \), the map \( \sigma \mapsto \tilde{\sigma} \) is a bijection between those \( \sigma \in D_r(n, k) \) for which \( n \) is in a cycle of size \( \geq r + 1 \) and has \( i \) elements to its left in the \( r \)-normal representation of \( \sigma \), and \( D_r(n-1, k) \).

If \( n \) is in a cycle of size \( r \), call it \( C \), then since \( \sigma \) is in \( r \)-normal form, \( C \) must be the right most cycle of \( \sigma \). Therefore the \( r \)-normal representation of \( \sigma_{[n]-\sigma} \) (respectively, \( \sigma_{[\sigma]} \)) is obtained from that of \( \sigma \) by deleting the last \( r \) (respectively, the first \( n-r \)) elements in the \( r \)-normal representation of \( \sigma \). Hence,

\[
\text{inv}_r(\sigma) = \text{inv}_r(\sigma_{[n]-\sigma}) + \text{inv}_r(\sigma_{[\sigma]}) + \sum_{i=1}^{r-1} \left[ (n-a_i) - (r - i) \right],
\]

where \( \{a_1, ..., a_{r-1}, n\} \) are the elements of \( C \). Furthermore, for fixed \( \mathcal{A} \in \binom{[n]}{r} \), \( n \in \mathcal{A} \), the map \( \sigma \mapsto (\sigma_{[n]-\mathcal{A}}, \sigma_{[\mathcal{A}]} \) gives a bijection between those \( \sigma \in D_r(n, k) \) for which the elements of \( \mathcal{A} \) are a cycle of size \( r \) and \( D_r(n-r, k-1) \times S(\mathcal{A}, (r)) \). Therefore,

\[
c_r[n, k]_q = \sum_{i=1}^{n-1} \sum_{\sigma \in D_r(n-1, k)} q^{\text{inv}_r(\sigma) + n - 1 - i}
+ \sum_{1 \leq a_1 < \cdots < a_{r-1} \leq n-1} \prod_{i=1}^{r-1} q^{n-a_i - r + i} \times \sum_{\sigma \in D_r(n-r, k-1)} q^{\text{inv}_r(\sigma)} \sum_{\tau \in S([r], (r))} q^{\text{inv}_r(\tau)}
= [n-1]_q c_r[n-1, k]_q
+ \sum_{n-r \geq b_1 \geq \cdots \geq b_{r-1} \geq 0} \prod_{i=1}^{r-1} q^{b_i} c_r[n-r, k-1]_q \sum_{\sigma \in S_{r-1}} q^{\text{inv}_r(\sigma) + r - 1}
\]

and the thesis follows from Proposition 1.3.19 and Corollary 1.3.10 of [35]. 

The following result is new even for \( q = 1 \).
COROLLARY 5.8. Let \( r \in \mathbb{P} \) and \( D_r(q) \triangleq (c_r[q][n,k])_{n,k \in \mathbb{N}} \). Then

(i) \( D_r(q) \) is \( q - TP \);

(ii) every row of \( D_r(q) \) is \( q - PF \).

Proof: Theorem 5.7 shows that

\[
D_r(q) = M_r \left( \left\{ [n-1]_q \right\}_{n \in \mathbb{N}}; 0, \left\{ q^{r-1} \prod_{i=1}^{r-1} [n-i]_q \right\}_{n \in \mathbb{N}} \right),
\]

so the thesis follows from Theorem 4.3.

We should mention that other total positivity properties of the matrix \( D_2(1) \) appear in [6] (see Theorem 6.7.2).

Part (ii) of Corollary 5.8 has a nice application to unimodality. Recall that a sequence \( \{a_0, a_1, ..., a_d\} \) (of real numbers) is called log-concave if

\[
a_i > a_{i-1}a_{i+1} \quad \text{for} \quad i = 1, ..., d-1,
\]

and it is said to be unimodal if there exists an index \( 0 \leq j \leq d \) such that \( a_i \leq a_{i+1} \) for \( i = 0, ..., j-1 \) and \( a_i \geq a_{i+1} \) for \( i = j, ..., d-1 \). We say that a polynomial \( \sum_{i=0}^{d} a_i x^i \) is log-concave (respectively, unimodal) if the sequence \( \{a_0, a_1, ..., a_d\} \) has the corresponding property. It is well known that if \( \sum_{i=0}^{d} a_i x^i \) is a polynomial with non-negative coefficients and with only real zeros, then the sequence \( \{a_0, a_1, ..., a_d\} \) is log-concave and unimodal (see, e.g., [13, Theorem B, p. 270]). Unimodal sequences arise often in combinatorics, algebra, and geometry and we refer the reader to [36] for an excellent survey.

The following result is an immediate consequence of part (ii) of Corollary 5.8 and of Theorem 2.2.4 of [6] (see also [24, Chap. 8, Section 3, Corollary 3.1]). For \( \sigma \in S_n \) we denote by \( c(\sigma) \) the number of cycles of \( \sigma \).

COROLLARY 5.9. For \( n, r \in \mathbb{P} \), the polynomial \( \sum_{\sigma \in D_r(n)} x^{c(\sigma)} \) has only real zeros. In particular, it is log-concave and unimodal.

For \( r = 1 \) the preceding result is well known (see, e.g., [35, Proposition 1.3.4]) while for \( r = 2 \) it was first proved by R. Canfield (private communication). Corollary 5.9 is by no means an isolated result; similar results relating unimodality and permutation enumeration appear, e.g., in [7, Section 4; 8, Section 6; 9].

It is natural to expect that the analogues of Theorem 5.7 and Corollary 5.8 will hold for suitable \( q \)-analogues of the \( r \)-associated Stirling numbers of the second kind. While this may be true, we have only been able to obtain the analogue of Theorem 5.7. Since the definition and the result are new and they may very well have applications to total positivity, we present them here. For \( n, k, r \in \mathbb{P} \), let \( \Pi_r([n], k) \) be the set of all \( \pi \in \Pi([n]) \)
that have exactly $k$ blocks, each of size $\geq r$. Given $\pi = \{B_1, \ldots, B_k\} \in \Pi_r([n], k)$ we say that $\pi$ is written in $r$-normal form if $a_{1,r} < a_{2,r} < \cdots < a_{k,r}$, where $\{a_{i,1}, \ldots, a_{i,r}, \ldots\} \leq B_i$ for $i = 1, \ldots, k$. An $r$-inversion of $\pi$ is a pair $(b, a_{i,p})$ such that $b > a_{i,p}$, $1 \leq p \leq r$, and $b \in B_j$ with $j < i$. We denote by $\text{inv}_r(\pi)$ the number of $r$-inversions of $\pi$. For example, if $r = 2$ and $\pi = 15/726/348$ then the 2-normal form of $\pi$ is $348/15/267$ and $\text{inv}_2(\pi) = 9$.

We define the $r$-associated $q$-Stirling numbers of the second kind by letting

$$S_r[n, k]_q \equiv \sum_{\pi \in \Pi_r([n], k)} q^{|\text{inv}_r(\pi)|}$$

for $n, k \in \mathbb{P}$ and $S_r[n, 0]_q = S_r[0, n]_q = \delta_{0,n}$ for all $n \in \mathbb{N}$. Clearly $S_r[n, k]_q = S[n, k]_q$, where $S[n, k]_q$ denotes a $q$-Stirling number of the second kind as defined, e.g. in [28], and $S_r[n, k]_q$ is the $r$-associated Stirling number of the second kind as defined, e.g., in [13, p. 221, Ex. 7]. The next result shows that (45) is a very natural $q$-analogue of the $r$-associated Stirling numbers of the second kind (cf. [13, p. 222, Ex. 7]).

**Theorem 5.10.** Let $r, n, k \in \mathbb{P}$. Then

$$S_r[n, k]_q = [k]_q \cdot S_r[n-1, k]_q + \left[n-1\atop r-1\right] S_r[n-r, k-1]_q$$

(where $S_r[n, k]_q \equiv 0$, if either $n < 0$ or $k < 0$).

**Proof.** Let $\pi = \{B_1, \ldots, B_k\} \in \Pi_r([n], k)$ and assume that $\pi$ is written in $r$-normal form. Let $i \in [k]$ be such that $n \in B_i$. There are two possible cases.

If $|B_i| \geq r+1$ then $\tilde{\pi} \equiv \{B_1, \ldots, B_i \setminus \{n\}, \ldots, B_k\}$ is a partition in $\Pi_r([n-1], k)$ and $\tilde{\pi}$ is still in $r$-normal form, so that $\text{inv}_r(\pi) = \text{inv}_r(\tilde{\pi}) + r(k-i)$. Furthermore, for fixed $i \in [k]$, the map $\pi \mapsto \tilde{\pi}$ is a bijection between those $\pi \in \Pi_r([n], k)$ for which the $i$th block (in the $r$-normal form of $\pi$) has size $\geq r+1$ and contains $n$ and $\Pi_r([n-1], k)$.

If $|B_i| = r$ then (since $\pi$ is in $r$-normal form), $i = k$. Let $\{a_1, \ldots, a_{r-1}, n\} \leq B_k$. Then $\pi' \equiv \{B_1, \ldots, B_{k-1}\}$ is a partition in $\Pi_r([n] \setminus B_k, k-1)$ in $r$-normal form and

$$\text{inv}_r(\pi) = \text{inv}_r(\pi') + \sum_{i=1}^{r-1} [(n-a_i)-(r-i)].$$

Furthermore, for fixed $\{a_1, \ldots, a_{r-1}\} \leq [n-1]$ the map $\pi \mapsto \pi'$ is a bijection between those $\pi$ for which $\{a_1, \ldots, a_{r-1}, n\}$ is a block of $\pi$ and $\Pi_r([n] \setminus \{a_1, \ldots, a_{r-1}, n\}, k-1)$. Therefore,
$$S_q[n,k] = \sum_{i=1}^{k} \sum_{\pi \in \Pi_{t}(n-1),k} q^{\text{inv}_{t}(\pi) + r(k-i)}$$

$$+ \sum_{1 \leq a_1 < \ldots < a_{r-1} \leq n-1} \prod_{i=1}^{r-1} q^{a_i} \sum_{\pi \in \Pi_{t}(n-r),k-1} q^{\text{inv}_{t}(\pi)}$$

$$= \left[k\right]_q \cdot S_q[n-1,k] + S_q[n-r,k-1]$$

$$\times \sum_{n-r \geq b_1 \geq \ldots \geq b_{r-1} \geq 0} \prod_{i=1}^{r-1} q^{b_i},$$

and the result follows from Proposition 1.3.19 of [35].

We now consider some generalizations of the q-Stirling numbers which, to the best of our knowledge, have never been considered before in the literature, even for $q=1$. For $n, k \in \mathbb{P}$, let $\mathcal{C}(n,k) \overset{\text{def}}{=} D_1(n,k)$. Given $\sigma \in \mathcal{C}(n,k)$ we let $S(\sigma) \overset{\text{def}}{=} \{\min(\mathcal{C}_1), \ldots, \min(\mathcal{C}_k)\}$, where $\mathcal{C}_1, \ldots, \mathcal{C}_k$ are the cycles of $\sigma$. For $n, k, t \in \mathbb{P}$ we let $\mathcal{C}^{(t)}(n,k)$ be the set of all $\sigma \in \mathcal{C}(n,k)$ such that all the connected components of $S(\sigma)$ which do not contain $n$ have size which is a multiple of $t$. We then let

$$c^{(t)}[n,k]_q \overset{\text{def}}{=} \sum_{\sigma \in \mathcal{C}^{(t)}(n,k)} q^{\text{inv}(\sigma)}$$

(where $\text{inv}(\sigma) = \text{inv}_1(\sigma)$) and $c^{(t)}[n,0]_q = c^{(t)}[0,n]_q = \delta_{0,n}$ for all $n \in \mathbb{N}$. For example, $\mathcal{C}^{(2)}(5,3) = \{(143)(2)(5), (143)(2)(5), (13)(24)(5), (14)(23)(5), (1)(234)(5), (1)(243)(5)\}$ and $c^{(2)}[5,3]_q = 1 + 2q + 2q^2 + q^3$. Note that $\mathcal{C}^{(1)}(n,k) = \mathcal{C}(n,k)$ and hence,

$$c^{(1)}[n,k]_q = c[n,k]_q$$

for all $n, k \in \mathbb{N}$.

**Proposition 5.11.** Let $t \in \mathbb{P}$. Then the polynomials $c^{(t)}[n,k]_q$ defined by (46) satisfy the recurrence relation

$$c^{(t)}[n,k]_q = \begin{cases} c^{(t)}[n-1,k-1]_q + [n-1]_q c^{(t)}[n-1,k]_q, & \text{if } k \equiv 0 \pmod{t}, \\ c^{(t)}[n-1,k-1]_q, & \text{otherwise}, \end{cases}$$

for $n, k \in \mathbb{P}$.

**Proof.** Note first that if $\sigma \in \mathcal{C}^{(t)}(n,k)$ and $n \in S(\sigma)$ then $\sigma_{|[n-1]} \in \mathcal{C}^{(t)}(n-1,k-1)$ and the map $\sigma \mapsto \sigma_{|[n-1]}$ is a bijection. Therefore,

$$\sum_{\{\sigma \in \mathcal{C}^{(t)}(n,k) : n \in S(\sigma)\}} q^{\text{inv}(\sigma)} = c^{(t)}[n-1,k-1]_q$$

(48)
for all \( n, k \in \mathbb{P} \). On the other hand, if \( \sigma \in \mathcal{E}(t)(n, k) \) and \( n \notin S(\sigma) \) (and this, by our definitions, can only happen if \( k \equiv 0 \pmod{t} \)) then removing \( n \) from \( \sigma \) yields a permutation \( \tilde{\sigma} \in \mathcal{E}(t)(n - 1, k) \). Conversely, given \( \tilde{\sigma} \in \mathcal{E}(t)(n - 1, k) \) we can insert \( n \) in one of the cycles of \( \tilde{\sigma} \) in \( n - 1 \) ways to obtain permutations \( \sigma_1, \ldots, \sigma_{n-1} \in \mathcal{E}(t)(n, k) \) such that \( n \notin S(\sigma_i) \), \( \tilde{\sigma}_i = \sigma_i \), and \( \text{inv}(\sigma_i) = \text{inv}(\sigma) + i - 1 \) for \( i = 1, \ldots, n - 1 \). Therefore,

\[
\sum_{\{\sigma \in \mathcal{E}(t)(n, k) : n \notin S(\sigma)\}} q^{\text{inv}(\sigma)} = \left[n - 1\right]_q c^{(t)}[n - 1, k]_q,
\]

(49)

if \( k \equiv 0 \pmod{t} \), and \( n \in \mathbb{P} \). Summing (48) and (49) we obtain the first equality in (47). Now if \( k \not\equiv 0 \pmod{t} \) and \( \sigma \in \mathcal{E}(t)(n, k) \) then \( |S(\sigma)| \neq 0 \pmod{t} \) and, hence, \( n \in S(\sigma) \). Therefore (48) is equivalent with the second equality in (47), when \( k \not\equiv 0 \pmod{t} \).

The following result gives the main total positivity properties of the polynomials \( c^{(t)}[n, k]_q \) defined by (46).

**Theorem 5.12.** Let \( t \in \mathbb{P} \), and \( C_t(q) = (c^{(t)}[n + 1, (k + 1)t]_q)_{n, k \in \mathbb{N}} \). Then

(i) \( C_t(q) \) is \( q - TP \);

(ii) every row of \( C_t(q) \) is \( q - PF \).

**Proof.** It follows immediately from Proposition 5.11 that

\[
c^{(t)}[n, k]_q = c^{(t)}[n - t, k - t]_q + [n - 1]_q c^{(t)}[n - 1, k]_q,
\]

if \( n \in \mathbb{P} \), and \( k \equiv 0 \pmod{t} \). Therefore,

\[
c^{(t)}[n, kt]_q = c^{(t)}[n - t, (k - 1)t]_q + [n - 1]_q c^{(t)}[n - 1, kt]_q,
\]

for all \( n, k \in \mathbb{P} \). This shows that

\[
C_t(q) = M_t(\{[n]_q\}_{n \in \mathbb{N}}, 0, 1),
\]

(50)

and the thesis follows from Theorem 4.3.

In the case \( t = 1 \), part (ii) of the preceding theorem was first explicitly noted in [31; see Section 6]; part (i) is new even for \( t = 1 \).

An analogous construction can be made for set partitions. For \( n, k \in \mathbb{P} \) let \( \Pi([n], k) \equiv \Pi_1([n], k) \). Given \( \pi = \{B_1, \ldots, B_k\} \in \Pi([n], k) \), we let \( S(\pi) = \{\min(B_1), \ldots, \min(B_k)\} \). We will always assume that \( \pi \) is written in 1-normal form (so that \( \min(B_1) < \cdots < \min(B_k) \)). For \( n, k, t \in \mathbb{P} \), we let \( \Pi^{(t)}([n], k) \) be the set of all \( \pi \in \Pi([n], k) \) such that all the connected
components of $S(\pi)$ which do not contain $n$ have size which is a multiple of $t$. We then let

$$S^{(i)}[n, k]_q \overset{\text{def}}{=} \sum_{\pi \in \Pi^{(i)}(n), \pi(n) = k} q^{\text{inv}(\pi)}, \quad (51)$$

where $\text{inv}(\pi) \overset{\text{def}}{=} \text{inv}_1(\pi)$. For example, $\Pi^{(3)}([4], 3) = \{14/2/3, 1/24/3, 1/2/34\}$ and $S^{(3)}[4, 3]_q = 1 + q + q^2$. We also let $S^{(i)}[n, 0]_q = S^{(i)}[0, n]_q = \delta_{n,t}$ for all $n \in \mathbb{N}$. Note that $\Pi^{(1)}([n], k) = \Pi([n], k)$ and, hence,

$$S^{(1)}[n, k]_q = S[n, k]_q$$

for all $n, k \in \mathbb{N}$.

The proof of the following result is analogous to that of Proposition 5.11 and is therefore omitted.

**Proposition 5.13.** Let $t \in \mathbb{P}$. Then the polynomials $S^{(i)}[n, k]_q$ defined by (51) satisfy the recurrence relation

$$S^{(i)}[n, k]_q = \begin{cases} S^{(i)}[n-1, k-1]_q + [k]_q S^{(i)}[n-1, k]_q, & \text{if } k \equiv 0 \pmod{t}, \\ S^{(i)}[n-1, k-1]_q, & \text{otherwise}, \end{cases} \quad (52)$$

for $n, k \in \mathbb{P}$.

We can now prove the analogue of Theorem 5.12 for the polynomials defined by (51).

**Theorem 5.14.** Let $t \in \mathbb{P}$ and $S_i(q) \overset{\text{def}}{=} (S^{(i)}[n+1, (k+1)t]_q)_{n, k \in \mathbb{N}}$. Then

(i) $S_i(q)$ is $q - TP$;

(ii) every column of $S_i(q)$ is $q - PF$.

**Proof.** It follows immediately from Proposition 5.13 that

$$S^{(i)}[n, k]_q = S^{(i)}[n-t, k-t]_q + [k]_q S^{(i)}[n-1, k]_q,$$

if $n \in \mathbb{P}$, and $k \equiv 0 \pmod{t}$. Therefore,

$$S^{(i)}[n, kt]_q = S^{(i)}[n-t, (k-1)t]_q + [kt]_q S^{(i)}[n-1, kt]_q$$
for all \( n, k \in \mathbb{P} \). This shows that

\[
S_t(q) = L_t(M_0(1, 0, \{(k+1)t\}_{k \in \mathbb{N}}'),
\]

and the thesis follows from Theorems 4.3 and 4.5.

For \( t = 1 \) the preceding result was first proved by Sagan (see [31, Corollary 5.5, and Section 6]).

Besides matrices arising from the enumeration of set partitions and permutations subject to various restrictions and according to various statistics, there are many other matrices of combinatorial interest which can be obtained by specializing and combining the matrices \( M_t(x, y, z) \) and that possess interesting total positivity properties. We shall just mention a few examples.

For \( n, k \in \mathbb{N} \) let \( D(n, k) \) be the Delannoy number. This can be defined by

\[
D(n, k) = \sum_{i=0}^{k} \binom{k}{i} \binom{n+k-i}{k}
\]

(see, e.g., [13, p. 81]). We then have the following result.

**Corollary 5.15.** Let \( D \equiv (D(n, k))_{n, k \in \mathbb{N}} \) and \( t \in \mathbb{N} \). Then

(i) \( L_t(D) \) is totally positive;

(ii) every row of \( D \) is a PF-sequence.

**Proof.** It is well known, (see, e.g., [13, p. 81]) and it is also easy to see from (53) that

\[
D(n, k) = D(n - 1, k) + D(n - 1, k - 1) + D(n, k - 1)
\]

for \( n + k \in \mathbb{P} \) (where \( D(n, k) \) is defined to be zero if either \( k < 0 \) or \( n < 0 \)). Hence,

\[
D = M_0(1, 1, 1)',
\]

and the thesis follows from Theorems 4.3 and 4.5.

Note that Eq. (53) and (54) can be used to define several \( q \)-analogues of the Delannoy numbers for which the \( q \)-analogue of Corollary 5.15 holds. We do not know if any of these \( q \)-analogues have been previously considered in the literature, but we will not pursue this line of investigation here.

For our next application we assume some familiarity with the basic terminology of the theory of hyperplane arrangements as presented, e.g., in [43].
Corollary 5.16. For $k, n \in \mathbb{N}$ let $a(n, k)$ be the number of regions into which $n$ hyperplanes in general position divide $\mathbb{R}^k$ and let $A = (a(n, k))_{n, k \in \mathbb{N}}$. Then

(i) $A$ is totally positive;

(ii) every row of $A$ is a PF-sequence.

Proof. It is well known (see, e.g., [43, Chap. 5, Section E, Eq. (5.7)]) that

$$a(n, k) = \sum_{i=0}^{k} \binom{n}{i}$$

for $n, k \in \mathbb{N}$. It then follows from this (and it is also easy to see directly; see, e.g., [13, p. 72, Ex. 2]) that

$$a(n, k) = a(n - 1, k) + a(n - 1, k - 1)$$

for all $n, k \in \mathbb{P}$ and $a(n, 0) = a(0, n) = 1$ for $n \in \mathbb{N}$. Hence,

$$A = M_0(\mathbf{1}, \mathbf{1}, \{1, 0, 0, \ldots\}),$$

and the thesis follows from Theorem 4.3. 

It would be interesting to obtain a geometric interpretation for the matrices

$$M_0(\mathbf{1}, \mathbf{1}, \{1, \ldots, 1, 0, 0, \ldots\}))$$

for $n \geq 2$.

We define the signless $q$-Lah numbers by

$$L[n, k]_q = \frac{n!}{k!} \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} \binom{n+iq-1}{n}$$

for $n, k \in \mathbb{N}$. For $q \in \mathbb{P}$ the $L[n, k]_q$ were first defined and studied in [1], where they arose from a problem in statistics and where they are called the associated Lah numbers. Using (55) it is easy to see that $L[n, k]_q$ reduces to the absolute value of the ordinary Lah number $L(n, k)$ (as defined, e.g., in [13, p. 135]) when $q = 1$. The following result is new even in the case $q = 1$.

Theorem 5.17. Let $L(q) \overset{\text{def}}{=} (L[n, k]_q)_{n, k \in \mathbb{N}}$; then $L(q)$ is $q$-TP.

Proof. It is proved in [1] that $L[n+1, k]_q = (kq+n) L[n, k]_q + qL[n, k-1]$ for all $n, k \in \mathbb{N}$, with $L[0, 0]_q \overset{\text{def}}{=} 1$. Hence, taking $b_n = 1, \beta_n = q, \alpha_n = nq, a_{n+1} = n$ for $n \in \mathbb{N}$, and $a_0 = 0$ in Theorem 4.8 yields the desired result.
6. Combinatorial Interpretations

The results in Section 4 show not only the \((x, y, z)\)-total positivity of the matrices \(M_r(x, y, z)\) but also give a combinatorial interpretation of their minors in terms of nonintersecting paths in the digraphs \(D(t)\). These combinatorial interpretations, however, are not very explicit. Our goal in this section is to show how one can obtain more explicit combinatorial interpretations. The technique that we use is a generalization of the one used in \([10]\).

Let \(\lambda = (\lambda_1, ..., \lambda_r)\) and \(\mu = (\mu_1, ..., \mu_r)\) \((r \in \mathbb{P})\) be two partitions such that \(\mu \subseteq \lambda\), and \(\alpha, \beta \in \mathbb{Z}\). A shifted skew \((\alpha, \beta)\)-tableau of shape \(\lambda \setminus \mu\) is an array \(\pi = (\pi_{i,j})_{1 \leq i \leq r, \mu_i + i \leq j \leq \lambda_i + i - 1}\) of positive integers such that \(\pi_{i,j} \leq \pi_{i,j+1} + \alpha\) and \(\pi_{i,j} \leq \pi_{i+1,j} + \beta\) whenever both sides of these inequalities are defined. Note that a \((0, 0)\)-tableau is just a reverse plane partition, while a \((0, -1)\)-tableau is a column-strict reverse plane partition.

We will often need to distinguish (i.e., to root) some of the entries of an \((\alpha, \beta)\)-tableau \(\pi\). We will do this by "dotting" (i.e., by putting a dot on top of) these entries. We will then call these entries the dotted entries of \(\pi\), and we will call the resulting array a rooted \((\alpha, \beta)\)-tableau. The next definition is fundamental for this section. Let \(t \in \mathbb{N}\) and \(\lambda, \mu\) be two partitions as above, a shifted skew dotted \(t\)-tableau of shape \(\lambda \setminus \mu\) is a rooted shifted skew \((-t, t)\)-tableau \(\pi = (\pi_{i,j})_{1 \leq i \leq r, \mu_i + i \leq j \leq \lambda_i + i - 1}\) of shape \(\lambda \setminus \mu\) such that:

(i) \(\pi_{i,j} < \pi_{i,j+1} - t\) if \(\pi_{i,j+1}\) is dotted,
(ii) \(\pi_{i,j} < \pi_{i+1,j} + t\) if \(\pi_{i,j}\) is not dotted,

whenever both sides of these inequalities are defined. For example,

\[
\begin{array}{cccc}
1 & 7 & 9 & 12 \\
1 & 4 & 6 & 8 \\
3 & 6 & 8 & 11 & 14
\end{array}
\]

is a shifted skew dotted \(2\)-tableau of shape \((8, 7, 7) \setminus (4, 2, 2)\).

Given two (possibly) dotted integers we will write \(a \doteq b\) to indicate that they are equal as dotted integers and \(a = b\) if they are only equal as integers (so that, for example, \(2 \doteq \hat{2}, 2 = \hat{2}, \hat{2} = \hat{2}\)). We will also write \((a + b)\) instead of the more cumbersome \(\hat{a} + \hat{b}\). Given a sequence \(d \doteq (d_1, d_2, ...)\) of integers and \(t \in \mathbb{N}\), we let \(S_t(d) \doteq (d_{t+1}, d_{t+2}, ...)\), \(S \doteq S_1\), and \(\Sigma_t(d) \doteq \sum_{j=0}^{t-1} S_t(d)\), (so that \(\Sigma_1(d) = d\) and \(\Sigma_0(d) = 0\)).

Given a shifted skew dotted \(t\)-tableau \(\pi\) as above, we let for \(r \in \mathbb{P}\)

\[
m_r(\pi) \doteq |\{(i, j) \in \text{sh}(\pi) : \pi_{i,j} = r\}|,
\]

\[
m_r(\pi) \doteq |\{(i, j) \in \text{sh}(\pi) : \pi_{i,j} \doteq r\}|,
\]
and

\[ m(\pi) \overset{\text{def}}{=} (m_1(\pi), m_2(\pi), ...), \]

\[ \bar{m}(\pi) \overset{\text{def}}{=} (\bar{m}_1(\pi), \bar{m}_2(\pi), ...). \]

We then define the weight of \( \pi \) to be

\[ w(\pi) \overset{\text{def}}{=} x^{s(\pi) - \Sigma_1(m(\pi)) - S_1(\bar{m}(\pi))} y^{m(\pi)} z^{m(\pi) - \bar{m}(\pi)} \]  

(57)

(recall that \( s(\pi) \) is the size of \( \pi \)). For example, if \( \pi \) is the shifted skew dotted 2-tableau given in (56) then

\[ m(\pi) = (2, 0, 1, 1, 0, 2, 1, 2, 1, 0, 1, 1, 1, 0, 0, ...), \]

\[ \bar{m}(\pi) = (2, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1, 0, 0, ...), \]

\[ \Sigma_2(m(\pi)) = (0, 0, 1, 1, 2, 3, 3, 3, 1, 1, 2, 2, 2, 1, 0, 0, ...), \]

and

\[ w(\pi) = x_1 x_2 x_3^2 x_4 x_5 y_3 y_6 y_8 z_5^2 z_7 z_8. \]  

(58)

We are now ready to state and prove one of the main results of this section. Given a directed path \( \tau \) in \( D(i) \) from \((a, b)\) to \((c, d)\), we let

\[ R_\tau(i) \overset{\text{def}}{=} \max\{ j : (j, i) \in \tau \} \]

and

\[ L_\tau(i) \overset{\text{def}}{=} \min\{ j : (j, i) \in \tau \} \]

for \( b \leq i \leq d \).

**Theorem 6.1.** Let \( (a_1, b_1), ..., (a_r, b_r), (c_1, d_1), ..., (c_r, d_r) \in \mathbb{N} \times \mathbb{N} \) be such that \( a_1 \leq \cdots \leq a_r, b_1 \geq \cdots \geq b_r, c_1 \leq \cdots \leq c_r, \) and \( d_1 \geq \cdots \geq d_r \). Then there is a weight-preserving bijection between \( r \)-tuples of nonintersecting paths in \( D(i) \) from \(((a_1, b_1), ..., (a_r, b_r))\) to \(((c_1, d_1), ..., (c_r, d_r))\) and shifted skew dotted \( t \)-tableaux of shape \((d_1 + 2, ..., d_r + 2)\) \((b_1, ..., b_r)\) in which the \( i \)-th row has the largest part \( \overset{\text{def}}{=} (c_i + t + 2) \) and the smallest part \( \overset{\text{def}}{=} (a_i + 1) \) for \( i = 1, ..., r \).

**Proof.** Let \( (\tau_1, ..., \tau_r) \) be a set of nonintersecting paths in \( D(i) \) from \(((a_1, b_1), ..., (a_r, b_r))\) to \(((c_1, d_1), ..., (c_r, d_r))\) (so that \( \tau_i \) goes from \((a_i, b_i)\) to \((c_i, d_i)\) for \( i = 1, ..., r \)). Let

\[ \pi_{i,j} \overset{\text{def}}{=} L_{\tau_i}(j - i) + 1 \]  

(59)
for $1 \leq i \leq r$, $b_i \leq j - i \leq d_i + 1$ (where $L_{\tau_i}(d_i + 1) \overset{\text{def}}{=} c_i + t + 1$ for $i = 1, \ldots, r$).
Now let $\pi \overset{\text{def}}{=} (\pi_{i,j})_{1 \leq i \leq r, b_i \leq i < j \leq d_i + i + 1}$ and dot $\pi_{i,j}$ if and only if either $j = b_i + i$ or $j = d_i + i + 1$ or the step reaching $(L_{\tau_i}(j-i),j-i)$ is a $(t+1,1)$ step. Then

$$\pi_{i,j+1} - \pi_{i,j} = \begin{cases} t+1, & \text{if } \pi_{i,j+1} \text{ is dotted,} \\ t, & \text{otherwise,} \end{cases}$$

for $1 \leq i \leq r, b_i \leq j \leq d_i + i$ and $\pi_{i,b_i+i} = (a_i + 1)^t$, $\pi_{i,d_i+i+1} = (c_i + t + 2)^t$ for $i = 1, \ldots, r$. Furthermore, $\tau_i$ and $\tau_{i+1}$ are nonintersecting if and only if

$$R_{\tau_i}(j) < L_{\tau_{i+1}}(j) \quad (60)$$

for all $b_i \leq j \leq d_i + 1$. But, by our definitions,

$$R_{\tau_i}(j) = \begin{cases} (L_{\tau_i}(j+1) - t - 1, & \text{if } \pi_{i,j+i+1} \text{ is dotted,} \\ (L_{\tau_i}(j+1) - t, & \text{otherwise,} \end{cases}$$

for $b_i \leq j \leq d_i$. Hence, we conclude from (59), (60), and (61) that

$$\pi_{i,j} \overset{\text{def}}{=} \begin{cases} \pi_{i+1,j+1} + t + 1, & \text{if } \pi_{i,j} \text{ is dotted,} \\ \pi_{i+1,j} + t, & \text{otherwise,} \end{cases}$$

for $b_i + i + 1 \leq j \leq d_i + i + 1 + i + 1$. Now, if $b_i > b_{i+1}$, then $\pi_{i,b_i+i} = L_{\tau_i}(b_i) + 1 = a_i + 1 \leq a_{i+1} + 1 < \pi_{i+1,b_i+i} + t + 1$. Furthermore, if $d_i > d_{i+1}$, then $\pi_{i,d_{i+1}+i+2} \leq c_i + t + 2 < c_{i+1} + t + 2 < c_{i+1} + 2t + 3 = \pi_{i+1,d_{i+1}+i+2} + t + 1$ and, if $\pi_{i,d_{i+1}+i+2}$ is not dotted, then $d_{i+1} + i + 2 < d_i + i + 1$ and, hence, $\pi_{i,d_{i+1}+i+2} = \pi_{i,d_{i+1}+i+2} - t - 1 = c_i + 1 < c_{i+1} + 2t + 2 = \pi_{i+1,d_{i+1}+i+2} + t + 1$. Hence, (62) holds also for $j = b_i + i$ (if $b_i > b_{i+1}$) and for $j = d_{i+1} + i + 2$ (if $d_i > d_{i+1}$), and this shows that $\pi$ is a shifted skew dotted $t$-tableau, as desired.

Conversely, let $\pi \overset{\text{def}}{=} (\pi_{i,j})_{1 \leq i \leq r, b_i \leq i \leq j \leq d_i + i + 1}$ be a shifted skew dotted $t$-tableau such that $\pi_{i,b_i+i} = (a_i + 1)^t$ and $\pi_{i,d_i+i+1} = (c_i + t + 2)^t$ for $i = 1, \ldots, r$. Now fix $i \in [r]$ and let

$$R_{i,j} \overset{\text{def}}{=} \begin{cases} \pi_{i,j+i+1} - t - 2, & \text{if } \pi_{i,j+i+1} \text{ is dotted,} \\ \pi_{i,j+i+1} - t - 1, & \text{otherwise,} \end{cases}$$

for $b_i \leq j \leq d_i$, and

$$\tau_i \overset{\text{def}}{=} \bigcup_{j = b_i}^{d_i} \{ (k,j) : \pi_{i,j+i+1} - 1 \leq k \leq R_{i,j} \}. \quad (63)$$
It is then clear that \( \tau_i \) is a path in \( D(t) \) from \((a_i, b_i)\) to \((c_i, d_i)\) for \( i = 1, \ldots, r \). Now, since \( \pi \) is a dotted \( t \)-tableau we know that (62) holds for all \( b_i + i + 1 \leq j \leq d_i + i + 1 \). This implies that

\[
R_{\tau_i}(j) = R_{i,j} < \pi_{i+1,j+i+1} - 1 = L_{\tau_{i+1}}(j)
\]

for all \( b_i \leq j \leq d_i + 1 \), and this, as remarked above, implies that \( \tau_i \) and \( \tau_{i+1} \) are nonintersecting, as desired.

Furthermore, if \((\tau_1, \ldots, \tau_r)\) and \( \pi \) correspond under the above bijection then, by (63), we have that, for \( i \in [r] \),

\[
w(\tau_i) = \frac{1}{y_{c_i + t + 1}} \prod_{f = b_i}^{d_i} \prod_{j \in [t]} x_{[\pi_{i,j+i}, R_{i,j}]} \prod_{j \in [t]} z_{\pi_{i,j-1}} \prod_{j \in [t]} y_{\pi_{i,j-1}}
\]

which is in accordance with (58).

Since we also wish to give a combinatorial interpretation to the minors of the matrices \( L_i(M_0(x, y, z)') \), we will need the analogue of Theorem 6.1

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
(8,6) \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
(9,5) \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
(10,5) \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
(0,4) \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
(0,2) \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
(2,2)
\end{array}
\]
for the digraph $D'(t)$ (cf. the proof of Theorem 4.5). Since $D'(t)$ is the transpose of $D(t)$, Theorem 6.1 implies that we can encode $r$-tuples of non-intersecting paths in $D'(t)$ by shifted skew dotted $t$-tableaux. However, transposition does not preserve the weights of $D(t)$ and $D'(t)$. Therefore, we will need to introduce a second kind of weight on shifted skew dotted $t$-tableaux which will correspond to the weighting that $D(t)$ inherits from $D'(t)$ by transposition.

Let $\pi \overset{\text{def}}{=} (\pi_{i,j})_{1 \leq i \leq r, \mu_i + i \leq \lambda_i + i - 1}$ be a shifted skew dotted $t$-tableau of shape $(\lambda_1, ..., \lambda_r) \setminus (\mu_1, ..., \mu_r)$. For $k \in \mathbb{N}$ we let

$$d_k(\pi) \overset{\text{def}}{=} \{i \in P : (i, i + k) \in \text{sh}(\pi)\},$$

$$d_k(\pi) \overset{\text{def}}{=} \{i \in P : \pi_{i,i+k} \text{ is dotted}\},$$

$$T_k(\pi) \overset{\text{def}}{=} \sum_{i = 1}^{d_k(\pi)} \pi_{i,i+k},$$

and

$$d(\pi) \overset{\text{def}}{=} (d_0(\pi), d_1(\pi), ...),$$

$$d(\pi) \overset{\text{def}}{=} (d_0(\pi), d_1(\pi), ...),$$

$$T(\pi) = (T_0(\pi), T_1(\pi), ...).$$

We then define the transpose weight of $\pi$ to be

$$w'(\pi) \overset{\text{def}}{=} x^{d(\pi) - d(\pi)} y^{d(\pi) - d(\pi)} z^{T(\pi) - d(\pi) - T(\pi)}.$$

For example, if $\pi$ is the shifted skew dotted 2-tableau given in (56) then

$$d(\pi) = (0, 0, 2, 2, 3, 3, 3, 1, 0, 0, ...),$$

$$d(\pi) = (0, 0, 2, 1, 1, 2, 2, 1, 0, 0, ...),$$

$$T(\pi) = (0, 0, 0, 10, 14, 26, 36, 12, 0, 0, ...),$$

and

$$w'(\pi) = x_3 x_5^2 x_6 y_3 y_5^2 z_2^3 z_4^2 z_5^2. \quad (64)$$

We can now state and prove the analogue of Theorem 6.1 for the digraph $D'(t)$.

**Theorem 6.2.** Let $(a_1, b_1), ..., (a_r, b_r), (c_1, d_1), ..., (c_r, d_r) \in \mathbb{N} \times \mathbb{N}$ be such that $a_1 \leq ... \leq a_r, b_1 \geq ... \geq b_r, c_1 \leq ... \leq c_r$, and $d_1 \geq ... \geq d_r$. Then there is a bijection between $r$-tuples of non-intersecting paths in $D'(t)$ from $((a_1, b_1), ..., (a_r, b_r))$ to $((c_1, d_1), ..., (c_r, d_r))$ and shifted skew dotted
t-tableaux of shape \((c_i + 2, \ldots, c_1 + 2)\) \((a_r, \ldots, a_1)\) in which the \(i\)th row has largest part \(=(d_r+1, \ldots, d_i+2)\) and smallest part \(=(b_r+1, \ldots, b_1+1)\), for \(i=1, \ldots, r\). Furthermore, if \((\tau_1, \ldots, \tau_r)\) and \(\pi\) correspond under the above bijection, then \(w(\tau_1, \ldots, \tau_r) = w(\pi)\).

**Proof.** Since \(D'(t)\) is the transpose of \(D(t)\) (as digraphs, not as weighted digraphs), transposition gives a bijection between \(r\)-tuples of nonintersecting paths as in the statement of the theorem and \(r\)-tuples of nonintersecting paths in \(D(t)\) from \(((b_r, a_r), \ldots, (b_1, a_1))\) to \(((d_r, c_r), \ldots, (d_1, c_1))\); hence, the first part of the theorem follows from Theorem 6.1.

Now, if \((\tau_1, \ldots, \tau_r)\) and \(\pi\) correspond under this composite bijection then, by (63), we have that, for \(i=1, \ldots, r\),

\[
\tau_{r+1-i} = \bigcup_{j=a_{r+1-i}}^{c_{r+1-i}} \{ (j, k) : \pi_{i,j+i+1} - 1 \leq k \leq R_{i,j} \},
\]

where

\[
R_{i,j} \overset{\text{def}}{=} \begin{cases} 
\pi_{i,j+i+1} - t - 2, & \text{if } \pi_{i,j+i+1} \text{ is dotted}, \\
\pi_{i,j+i+1} - t - 1, & \text{otherwise},
\end{cases}
\]

for \(a_{r+1-i} \leq j \leq c_{r+1-i}\). Therefore,

\[
w(\tau_{r+1-i}) = \frac{1}{y_{c_{r+1-i}+1}} \prod_{j=a_{r+1-i}}^{c_{r+1-i}} z_j^{\pi_{i,j+i+1}} \prod_{j \in \pi_i} x_{j-i} \prod_{j \in \pi_i} y_{j-i} \prod_{j \in \pi_i} \bigg\{ \prod_{j \in \pi_i} \tau_{j-i-1} \bigg\} \bigg\{ \prod_{j \in \pi_i} \tau_{j-i} \bigg\}
\]

\[
\times \prod_{j \in \pi_i} \frac{z^{\pi_{i,j+i+1}}}{y^{\pi_{i,j+i+1}+1}} \prod_{j \in \pi_i} x_{j-i} \prod_{j \in \pi_i} y_{j-i} \bigg\{ \prod_{j \in \pi_i} \tau_{j-i-1} \bigg\}
\]

where \(\pi_i \overset{\text{def}}{=} \{ j \in [a_{r+1-i} + i + 1, c_{r+1-i} + i + 1] : \pi_{i,j} \text{ is dotted} \}\), and the thesis follows.

We briefly illustrate the preceding theorem with an example. Let \((a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3) = (2, 2, 4, 2, 0, 0, 5, 5, 6, 10, 9, 8)\) and \(\pi\) be the shifted skew dotted 2-tableau given in (56). Then the triple of nonintersecting paths in \(D'(2)\) corresponding to \(\pi\) under the composite bijection described in Theorem 6.2 is the one obtained by transposing the three paths depicted in Fig. 6. The weight of these transposed paths, in \(D'(2)\), is

\[
(2^3 x_3 y_5^3) (y_3 x_4 x_5 z_3^2) (z_4^3 y_5 x_6),
\]

which is in accordance with (64).
Note that the preceding theorem shows that the transpose weight of \( \pi \) is also the weight, defined by (57), of a suitably defined "conjugate" of \( \pi \). However, we have no need for this fact here and we wanted to have an explicit description of this transpose weight; for this reason we have chosen to define it directly.

We are now ready to prove the three main results of this section. Their proofs follow immediately from Theorems 4.1, 6.1, and 6.2, and from Eq. (32) and (34).

**Corollary 6.3.** Let \( t \in \mathbb{N}, \{n_1, \ldots, n_r\} \subset \{k_1, \ldots, k_r\} \subset \mathbb{N}, \text{ and } m \in \mathbb{P}, m > k_r \). Then

\[
M_t(x, y, z) \left( \begin{array}{c} n_1, \ldots, n_r \\ k_1, \ldots, k_r \end{array} \right) = \sum_{\pi} x^\lambda(\pi) - \sum_{\rho} \lambda(\rho) - \sum_{\sigma} \rho(\sigma) x^\mu(\sigma),
\]

where the sum is over all shifted skew dotted \( t \)-tableaux \( \pi \) of shape \((m_2, \ldots, m_r, m - k_1, \ldots, m - k_r)\) in which the \( i \)th row has smallest part \(-1\) and largest part \(= (n_{i} + t + 2)^{'}\) for \( i = 1, \ldots, r \).

**Corollary 6.4.** Let \( t \in \mathbb{N}, \{m_1, \ldots, m_r\} \supset \{k_1, \ldots, k_r\} \supset \mathbb{N}, \text{ and } n \in \mathbb{P}. \) Then

\[
\left\{ M_t(x, y, z)_{\rho, \sigma} \right\}_{k \in \mathbb{N}} \left( \begin{array}{c} m_r, \ldots, m_1 \\ k_r, \ldots, k_1 \end{array} \right)
= \sum_{\pi} x^\lambda(\pi) - \sum_{\rho} \lambda(\rho) - \sum_{\sigma} \rho(\sigma) x^\mu(\sigma),
\]

where the sum is over all skew dotted \( t \)-tableaux \( \pi \) of shape \((k_1 + 2, \ldots, k_r + r + 1, m_1, \ldots, m_r, m + r - 1)\) in which every row has smallest part \(= 1\) and largest part \(= (n + t + 2)^{'}\).

**Corollary 6.5.** Let \( t \in \mathbb{N}, \{n_1, \ldots, n_r\} \supset \{k_1, \ldots, k_r\} \supset \mathbb{N}, \text{ and } m \in \mathbb{P}, m > n_1 \). Then

\[
L_t(M_0(x, y, z)) \left( \begin{array}{c} n_r, \ldots, n_1 \\ k_r, \ldots, k_1 \end{array} \right)
= \sum_{\pi} x^\lambda(\pi) - \lambda(\pi) - \sum_{\sigma} \rho(\sigma) x^\mu(\sigma),
\]

where the sum is over all shifted dotted \( t \)-tableaux \( \pi \) of shape \((k_1 + 2, \ldots, k_r + 2)\) in which the \( i \)th row has largest part \(= (m + t + 2)^{'}\) and smallest part \(= (m - n_i + 1)^{'}\) for \( i = 1, \ldots, r \).
Note that by Proposition 4.4, Corollary 6.4 gives, when \( t = 0, x = 1, \) and \( z_0 = 0, \) a combinatorial interpretation for the super-Schur function (in the variables \( z_1, ..., z_n/y_1, ..., y_n \)) corresponding to the skew shape \( (m-m_r+1, ..., m-m_1+r) \setminus (m-k_r+1, ..., m-k_1+r) \) (where \( m \geq \max\{m_1, k_1\} \)).

This combinatorial interpretation is different from (and simpler than) the one given in Theorem 3.4 of [10] and reduces, when \( \{k_1, ..., k_r\} > = \{r, ..., 2, 1\} \), to the one given in [34] (see Eq. (11)). Also note that in this case Corollary 6.5 reduces to Theorem 3.3 of [10].

Using the preceding three results it is easy to give combinatorial interpretations to the minors of almost all of the matrices considered in the preceding section. For example, consider the minors associated to the matrix \( E_t(x) \) defined in Theorem 5.1. The next two results follow from Corollaries 6.3 and 6.4 and from (40).

**Corollary 6.6.** Let \( t \in \mathbb{N}, \{n_1, ..., n_r\} <, \{k_1, ..., k_r\} < \subseteq \mathbb{N}, \) and \( m \in \mathbb{P}, m > k_r. \) Then

\[
E_t(x) \binom{n_1, ..., n_r}{k_1, ..., k_r} = \sum_\pi x^{m(\pi)},
\]

where the sum is over all shifted skew \((-t, t-1)\)-tableaux \( \pi \) of shape \((m+2, ..., m+2) \setminus (m-k_1, ..., m-k_r) \) and size \((n_1+t+1, ..., n_r+t+1) \setminus (1').

**Corollary 6.7.** Let \( t \in \mathbb{N}, \{m_1, ..., m_r\} >, \{k_1, ..., k_r\} > \subseteq \mathbb{N}, \) and \( n \in \mathbb{P}. \) Then

\[
\{e_k^{(t)}(x_0, ..., x_n)\} = \sum_\pi x^{m(\pi)},
\]

where the sum is over all skew \((-t, t-1)\)-tableaux \( \pi \) of shape \((k_1, ..., k_r + r - 1) \setminus (m_1, ..., m_r + r - 1) \) with entries \( \leq n+1 \) and \( \geq t+1. \)

It is easy to see that, when \( t = 0, 1, \) the preceding corollaries reduce to Corollaries 5.1, 5.2, 5.3, and 5.4 of [10]. In particular, Corollary 6.7 reduces to the Jacobi–Trudi identity (see, e.g., [27, Chap. I, Section 5, Eqs. (5.4) and (5.5)]).

In a similar way, combining Corollaries 6.3, 6.4, and 6.5 with the various specializations of the matrices \( M_t(x, y, z) \) and \( L_t(M_0(x, y, z)') \) considered in Section 5, it is not hard to obtain explicit combinatorial interpretations for the minors of all these specializations. It is then possible to use appropriate bijections to get “natural” combinatorial interpretations (in the sense of [10, Section 5]) analogous to those appearing in Sections 5–8 of [10]. We leave this task to the interested reader.
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