Multiple solutions to a three-point boundary value problem for higher-order ordinary differential equations

Zengji Du a,*, Wenbin Liub, Xiaojie Lin a

a School of Mathematical Sciences, Xuzhou Normal University, Xuzhou, Jiangsu 221116, PR China
b Department of Mathematics, China University of Mining and Technology, Xuzhou, Jiangsu 221008, PR China

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Abstract

In this paper, we provide sufficient conditions for the existence of at least three solutions to a three-point boundary value problem for higher-order ordinary differential equations. The nonlinear term \( f \) in the differential equation under consideration may depend on higher-order derivatives of arbitrary order and this is where the main novelty of this work lies. By applying the two pairs of upper and lower solutions method of Henderson and Thompson, as well as degree theory, the existence of at least three solutions of the problem is given.

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1. Introduction

Consider the \( n \)th order ordinary differential equation

\[
-u^{(n)}(t) + f(t, u(t), u'(t), \ldots, u^{(n-1)}(t)) = 0, \quad t \in (0, 1),
\]

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* Corresponding author.

E-mail addresses: duzengji@163.com (Z. Du), wbliu@163.com (W. Liu), linxiaojie1973@163.com (X. Lin).
together with the boundary conditions

\[ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) - \xi u^{(n-2)}(\eta) = 0, \tag{2} \]

where \( \eta \in (0, 1) \), \( \xi > 0 \) are two constants, satisfying \( 0 < \xi \eta < 1 \).

Boundary value problems for ordinary differential equations play a very important role in both theory and applications. They are used to describe a large number of physical, biological and chemical phenomena. The works of Love [19], Prescott [22], and Timoshenko [25] on elasticity, the monographs by Mansfield [21] and Soedel [24] on deformation of structures, the studying of plasma physics and electrical potential in an isolated neutral atom of Agarwal and O’Regan [1], and the work of Dulácska [10] on the effects of soil settlement are rich sources of such applications.

Higher-order boundary value problems were discussed in many papers in recent years, for instance, see [4–8,11–13,16,18,20,23] and references therein. However, Anuradha et al. [4], Bai and Wang [5], Baxley and Houmand [6], Du et al. [8], Graef et al. [11,12], Ma [20] all studied the situation when the nonlinear term \( f \) only depended on \( t \) and \( u \), but did not involve the higher-order derivative. Shi and Chen [23] considered the nonlinear term \( f \) depending on \( t \), \( u \) and even order derivative, but did not involve the odd order derivative. Under the resonance case, sufficient conditions for the existence of solutions to \( n \)th order boundary value problems were investigated in [8,16,18]. In this article, we discuss the existence of multiple solutions (at least three) to the nonresonance problem (i.e. \( 0 < \xi \eta < 1 \)) where the differential equation with the nonlinear term \( f \) may depend on higher-order derivatives.

The methods used in our work follow similar lines to those established by Henderson and Thompson [15], i.e. the method of two pairs of upper and lower solutions. By applying this method, Du et al. [9] and Khan and Webb [17] discussed second order three-point boundary value problems respectively, which generalized the two-point boundary value problems considered in [15]. Agarwal, Thompson and Tisdell [3] used this method and degree theory to establish existence results for multiple solutions to a second order differential equation with nonlinear two-point boundary conditions, which apply to many different types of boundary conditions including those of Dirichlet, Neumann, periodic and Sturm–Liouville, and complement the results in [15]. Moreover, in [3] a very nice problem from chemical reactor theory was considered and the existence of three distinct solutions was proven. This type of physical application naturally motivates our study.

In this paper, we assume that there exist two pairs of upper and lower solutions for problem (1), (2) where the nonlinear \( f \) satisfies a Nagumo growth condition with respect to higher-order derivatives. We use the upper and lower solutions to modify \( f \) and establish \textit{a priori} bounds on solutions of the modified problem. Then we use topological degree theory to discuss the existence of multiple solutions for problem (1), (2).

For further works on multiple solutions to differential equations, we refer the reader to [1–3, 6,9,11,13–15,20,23].

2. Preliminary results

For \( x \in C^{n-1}[0, 1] \), we denote the norm \( \|x\|_\infty = \max\{|x(t)|: t \in [0, 1]\} \), and \( \|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \ldots, \|x^{(n-1)}\|_\infty\} \).

The following lower and upper solutions are used to obtain \textit{a priori} bounds on solutions to Eqs. (1), (2).
\textbf{Definition 1.} We call \( \alpha \) a lower solution for problem (1), (2), if \( \alpha \in C^{n,1}([0,1]), \)
\[ \alpha^{(n)}(t) + f(t, \alpha(t), \alpha'(t), \ldots, \alpha^{(n-1)}(t)) \geq 0, \quad 0 < t < 1, \]
and
\[ \alpha^{(i)}(0) \leq 0, \quad i = 0, 1, \ldots, n-2, \quad \alpha^{(n-2)}(1) - \xi \alpha^{(n-2)}(\eta) \leq 0. \]  
Similarly, we call \( \beta \) an upper solution for problem (1), (2), if \( \beta \in C^{n,1}([0,1]), \)
\[ \beta^{(n)}(t) + f(t, \beta(t), \beta'(t), \ldots, \beta^{(n-1)}(t)) \leq 0, \quad 0 < t < 1, \]
and
\[ \beta^{(i)} \geq 0, \quad i = 0, 1, \ldots, n-2, \quad \beta^{(n-2)}(1) - \xi \beta^{(n-2)}(\eta) \geq 0. \]
We say \( \alpha \) (\( \beta \)) is a strict lower solution (strict upper solution) for problem (1), (2) if the above inequality (3) (or (5)) is strict for \( t \in (0,1). \)

\textbf{Remark 1.} Let \( f : [0,1] \times R^n \to R \) be continuous and \( u \) be a solution of (1), (2), if \( \alpha(\beta) \) is a strict lower solution (strict upper solution) for (1), (2) with \( \alpha^{(n-2)} \leq u^{(n-2)} \leq \beta^{(n-2)}, \) then \( \alpha^{(n-2)} < u^{(n-2)} < \beta^{(n-2)} \) on \( (0,1). \)

\textbf{Definition 2.} Let \( \alpha \) be a lower solution and \( \beta \) an upper solution for problem (1), (2) satisfying \( \alpha \leq \beta \) and \( \alpha^{(i)} \leq \beta^{(i)} \) (\( i = 1, 2, \ldots, n-2 \)) on \([0,1]. \) We say that \( f \) satisfies a Nagumo condition with respect to \( \alpha \) and \( \beta, \) if there exists a function \( \Phi \in C([0,\infty); (0, +\infty)) \) such that
\[ |f(t, x_1, x_2, \ldots, x_n)| \leq \Phi(|x_n|), \]
for all \((t, x_1, x_2, \ldots, x_n) \in [0,1] \times [\alpha(t), \beta(t)] \times [\alpha'(t), \beta'(t)] \times \cdots \times [\alpha^{(n-2)}(t), \beta^{(n-2)}(t)] \times R, \) and
\[ \int_0^\infty \frac{s}{\Phi(s)} \, ds = \infty. \]  

To obtain a solution of boundary value problem (1), (2), we need a mapping whose kernel \( G(t, s) \) is the Green’s function of problem \(-u^{(n)} = 0, \) with boundary conditions (2). From [2], it is clear that
\[ g(t, s) = \frac{\partial^{n-2} G(t, s)}{\partial t^{n-2}}, \]
is the Green’s function of the problem \(-u'' = 0, u(0) = 0, u(1) - \xi u(\eta) = 0, \) which is given by the following lemma.

\textbf{Lemma 1.} (See [12].) Let \( g(t, s) \) be the Green’s function for the problem \(-u''(t) = 0 \) with boundary condition \( u(0) = 0, u(1) - \xi u(\eta) = 0. \) Then
\[ g(t, s) = \begin{cases} 
1 & s \in [0, \eta], \quad s \leq t, \\
\frac{s(1-t) - \xi(\eta-t)}{(1-s) - \xi(\eta-s)} & t \leq s; \\
\frac{s(1-t) + \xi \eta(t-s)}{(t-1)s} & s \leq t, \quad t \leq s; 
\end{cases} \]
and \( g(t, s) \geq 0, (t, s) \in [0,1] \times [0,1], \) \( \text{mes}\{t, s\} \in [0,1] \times [0,1]: \) \( g(t, s) = 0 \) = 0.
We shall require the following simple property from degree theory for the proof of our main theorem.

**Lemma 2 (Additivity of degree).** If \( \Omega = \Omega^1 \cup \Omega^2 \cup \Omega^3 \), where \( \Omega^i \) are open, bounded sets and pairwise disjoint, then

\[
\deg(f, \Omega, 0) = \deg(f, \Omega^1, 0) + \deg(f, \Omega^2, 0) + \deg(f, \Omega^3, 0),
\]

provided the degree in (11) is defined.

### 3. Existence of multiple solutions

The result in this section will guarantee the existence of at least three solutions to problem (1), (2).

**Theorem 1.** Assume that

(A1) There exist two strict lower and upper solutions \( \alpha_1, \alpha_2 \) and \( \beta_1, \beta_2 \) of (1), (2), satisfying

\[
\alpha^{(i)}_1 \leq \alpha^{(i)}_2 \leq \beta^{(i)}_2, \quad \alpha^{(i)}_1 \leq \beta^{(i)}_1 \leq \beta^{(i)}_2, \quad \alpha^{(i)}_2 \leq \beta^{(i)}_1, \quad i = 0, 1, \ldots, n-2;
\]

(A2) Let \( f(t, x_1, x_2, \ldots, x_n) : [0, 1] \times R^n \to R \) be a continuous function and nondecreasing with respect to \( x_1, x_2, \ldots, x_{n-2} \), for \( (t, x_1, x_2, \ldots, x_n) \in [0, 1] \times [\alpha(t), \beta(t)] \times R^{n-1} \);

(A3) \( f \) satisfies Nagumo condition with respect to \( \alpha_1 \) and \( \beta_2 \).

Then problem (1), (2) has at least three solutions \( u_1, u_2 \) and \( u_3 \) satisfying

\[
\alpha^{(i)}_j \leq u^{(i)}_j \leq \beta^{(i)}_j, \quad i = 0, 1, \ldots, n-2, \quad j = 1, 2, \quad \text{on } [0, 1],
\]

\[
u^{(i)}_3 \neq \beta^{(i)}_1 \quad \text{and} \quad u^{(i)}_3 \neq \alpha^{(i)}_2, \quad i = 0, 1, \ldots, n-2, \quad \text{on } [0, 1].
\]

**Proof.** From assumption (A3), we can choose \( C > 0 \), such that

\[
\int_{\lambda}^{C} \frac{s}{\Phi(s)} \, ds > \lambda,
\]

where \( \lambda = \max_{t \in [0, 1]} \beta^{(n-2)}_2(t) - \min_{t \in [0, 1]} \alpha^{(n-2)}_1(t) \). Let

\[
L = \max \left\{ \| \alpha_1^{(n-1)} \|_\infty, \| \beta_2^{(n-1)} \|_\infty, C, 2\lambda \right\}.
\]

We define the auxiliary functions \( f_1, \ldots, f_{n-1} \) and \( F : [0, 1] \times R^n \to R \) as

\[
f_1(t, x_1, x_2, \ldots, x_n) = \begin{cases} f(t, \beta_2, x_2, \ldots, x_n), & x_1 > \beta_2(t), \ t \in [0, 1], \\ f(t, x_1, x_2, \ldots, x_n), & \alpha_1(t) \leq x_1 \leq \beta_2(t), \ t \in [0, 1], \\ f(t, \alpha_1, x_2, \ldots, x_n), & x_1 < \alpha_1(t), \ t \in [0, 1], \end{cases}
\]
\[ f_2(t, x_1, x_2, \ldots, x_n) = \begin{cases} f_1(t, x_1, \beta'_2, x_3, \ldots, x_n), & x_2 > \beta'_2(t), \ t \in [0, 1], \\ f_1(t, x_1, x_2, x_3, \ldots, x_n), & \alpha'_1(t) \leq x_2 \leq \beta'_2(t), \ t \in [0, 1], \\ f_1(t, x_1, \alpha'_1, x_3, \ldots, x_n), & x_2 < \alpha'_1(t), \ t \in [0, 1], \end{cases} \] (16)

\[
\vdots
\]

\[ f_{n-1}(t, x_1, x_2, \ldots, x_n) = \begin{cases} f_{n-2}(t, x_1, \ldots, x_{n-2}, \beta^{(n-2)}_2, x_n), & x_{n-1} > \beta^{(n-2)}_2(t), \ t \in [0, 1], \\ f_{n-2}(t, x_1, \ldots, x_{n-2}, x_{n-1}, x_n), & \alpha^{(n-2)}_1 \leq x_{n-1} \leq \beta^{(n-2)}_2, \ t \in [0, 1], \\ f_{n-2}(t, x_1, \ldots, x_{n-2}, \alpha^{(n-2)}_1, x_n), & x_{n-1} < \alpha^{(n-2)}_1(t), \ t \in [0, 1], \end{cases} \] (17)

and

\[ F(t, x_1, x_2, \ldots, x_n) = \begin{cases} f_{n-1}(t, x_1, \ldots, x_{n-1}, L), & x_n > L, \ t \in [0, 1], \\ f_{n-1}(t, x_1, \ldots, x_{n-1}, x_n), & |x_n| \leq L, \ t \in [0, 1], \\ f_{n-1}(t, x_1, \ldots, x_{n-1}, -L), & x_n < -L, \ t \in [0, 1]. \end{cases} \] (18)

Thus \( F \) is a continuous function on \([0, 1] \times \mathbb{R}^n\), satisfying

\[
|F(t, x_1, x_2, \ldots, x_n)| \leq M, \quad \text{for} \ (t, x_1, x_2, \ldots, x_n) \in [0, 1] \times \mathbb{R}^n,
\] (19)

where \( M \) is a constant and satisfies \( M > \max\{\|\alpha_1\|_{\infty}, \|\beta_2\|_{\infty}\} \).

Consider the modified problem

\[ u^{(n)}(t) + F(t, u, u', \ldots, u^{(n-1)}) = 0, \quad t \in (0, 1), \] (20)

with the boundary conditions (2).

To finish the proof from the definition of \( F \), it suffices to show that problem (20) with (2) has at least three solutions \( u_1, u_2 \) and \( u_3 \) satisfying

\[
\alpha^{(i)}_j(t) \leq u^{(i)}_j(t) \leq \beta^{(i)}_j(t), \quad |u^{(n-1)}_j(t)| \leq L, \\
t \in [0, 1], \ i = 0, 1, \ldots, n-2, \ j = 1, 2, 3,
\] (21)

since \( F = f \) in the region. We divide the proof into two steps.

**Step 1.** Suppose that the problem (20) with (2) has a solution \( u \), then \( u \) satisfies (21), moreover, \( u \) is a solution of problem (1), (2).

We first show that \( \alpha^{(n-2)}_1 \leq u^{(n-2)} \leq \beta^{(n-2)}_2 \) on \([0, 1]\). We only need to show \( u^{(n-2)} \leq \beta^{(n-2)}_2 \) on \([0, 1]\). Similarly, we can prove \( \alpha^{(n-2)}_1 \leq u^{(n-2)} \) on \([0, 1]\), hence we omit it. If \( u^{(n-2)} \leq \beta^{(n-2)}_2 \) on \([0, 1]\) is not true, then there exists \( t \in [0, 1] \) with \( u^{(n-2)}(t) > \beta^{(n-2)}_2(t) \). Set

\[
\omega(t) := u^{(n-2)}(t) - \beta^{(n-2)}_2(t).
\]

Then \( \omega(t_0) = \max\{u^{(n-2)}(t) - \beta^{(n-2)}_2(t): t \in [0, 1]\} > 0 \) for some \( t_0 \in [0, 1] \).

**Case (I).** If \( t_0 = 0 \), then \( u^{(n-2)}(0) > \beta^{(n-2)}_2(0) \). From (6), we have the contradiction \( \beta^{(n-2)}_2(0) \geq 0 = u^{(n-2)}(0) \).
Case (II). If \( t_0 \in (0, 1) \), we have \( \omega(t_0) > 0 \), \( \omega'(t_0) = 0 \), and \( \omega''(t_0) \leq 0 \). But on the other hand,

\[
\omega''(t_0) = -F(t_0, u(t_0), u'(t_0), \ldots, u^{(n-1)}(t_0)) - \beta_2^{(n)}(t_0)
\]

\[
= -f_{n-1}(t_0, u(t_0), u'(t_0), \ldots, u^{(n-2)}(t_0), \beta_2^{(n-1)}(t_0)) - \beta_2^{(n)}(t_0)
\]

\[
= -f_{n-2}(t_0, u(t_0), u'(t_0), \ldots, u^{(n-3)}(t_0), \beta_2^{(n-2)}(t_0), \beta_2^{(n-1)}(t_0)) - \beta_2^{(n)}(t_0).
\]  

Subcase (i). If \( u^{(n-3)}(t_0) > \beta_2^{(n-3)}(t_0) \), from the above inequality (22), one has

\[
\omega''(t_0) = -f_{n-3}(t_0, u(t_0), u'(t_0), \ldots, u^{(n-4)}(t_0), \beta_2^{(n-3)}(t_0), \ldots, \beta_2^{(n-1)}(t_0)) - \beta_2^{(n)}(t_0).
\]  

If \( u^{(n-4)}(t_0) > \beta_2^{(n-4)}(t_0) \), from the inequality (23), we have

\[
\omega''(t_0) = -f_{n-4}(t_0, u(t_0), u'(t_0), \ldots, u^{(n-5)}(t_0), \beta_2^{(n-4)}(t_0), \ldots, \beta_2^{(n-1)}(t_0)) - \beta_2^{(n)}(t_0).
\]  

If \( u^{(n-4)}(t_0) \leq \beta_2^{(n-4)}(t_0) \), from the inequality (23) and (A2), we have

\[
\omega''(t_0) \geq -f_{n-4}(t_0, u(t_0), u'(t_0), \ldots, u^{(n-5)}(t_0), \beta_2^{(n-4)}(t_0), \ldots, \beta_2^{(n-1)}(t_0)) - \beta_2^{(n)}(t_0).
\]  

In view of (24) and (25), either \( u^{(n-4)}(t_0) > \beta_2^{(n-4)}(t_0) \) or \( u^{(n-4)}(t_0) \leq \beta_2^{(n-4)}(t_0) \), we always could have the following inequality:

\[
\omega''(t_0) \geq -f_{n-4}(t_0, u(t_0), u'(t_0), \ldots, u^{(n-5)}(t_0), \beta_2^{(n-4)}(t_0), \ldots, \beta_2^{(n-1)}(t_0)) - \beta_2^{(n)}(t_0).
\]  

Similar to the above argument, we could discuss the following two cases \( u^{(i)}(t_0) > \beta_2^{(i)}(t_0) \), or \( u^{(i)}(t_0) \leq \beta_2^{(i)}(t_0) \), \( i = 0, 1, \ldots, n-5 \), and have the following inequality

\[
\omega''(t_0) \geq -f(t_0, \beta_2(t_0), \beta_2^{(1)}(t_0), \ldots, \beta_2^{(n-1)}(t_0)) - \beta_2^{(n)}(t_0) > 0,
\]  

which contradicts \( \omega''(t_0) \leq 0 \).

Subcase (ii). If \( u^{(n-3)}(t_0) \leq \beta_2^{(n-3)}(t_0) \), from the inequality (22) and (A2), one has

\[
\omega''(t_0) = -f_{n-3}(t_0, u(t_0), u'(t_0), \ldots, u^{(n-3)}(t_0), \beta_2^{(n-2)}(t_0), \beta_2^{(n-1)}(t_0)) - \beta_2^{(n)}(t_0)
\]

\[
\geq -f_{n-3}(t_0, u(t_0), u'(t_0), \ldots, u^{(n-4)}(t_0), \beta_2^{(n-3)}(t_0), \ldots, \beta_2^{(n-1)}(t_0)) - \beta_2^{(n)}(t_0).
\]  

Similar to the argument in Subcase (i), we could obtain the contradiction (27).

Case (III). If \( t_0 = 1 \), then

\[
\omega(1) > 0.
\]  

From (6), we have \( \omega(0) \leq 0 \), thus there exists \( \sigma \in [0, 1) \) such that

\[
\omega(\sigma) = 0 \quad \text{and} \quad \omega(t) > 0, \quad \text{for all} \ t \in (\sigma, 1].
\]
If \( \sigma \in (\eta, 1) \), then there exists \( t_1 \in (0, \sigma) \) such that \( \omega(t_1) = \max \{ \omega(t) : t \in [0, \sigma] \} \). From (2), (6) and (29), we have
\[
\omega(t_1) \geq \omega(\eta) = u^{(n-2)}(\eta) - \beta_2^{(n-2)}(\eta)
\geq \frac{1}{\xi} [u^{(n-2)}(1) - \beta_2^{(n-2)}(1)]
= \frac{1}{\xi} \omega(1) > 0.
\]
Moreover, \( \omega'(t_1) = 0 \) and \( \omega''(t_1) \leq 0 \). Similar to Case (II), we have the same contradiction.

If \( \sigma \in (0, \eta) \), then for all \( t \in [\sigma, 1] \), we have that \( \omega(t) > 0 \). We consider the following two subcases: \( \omega'(t) \geq 0, \ t \in [\sigma, 1] \) or there exists some \( t_2 \in (\sigma, 1) \), such that \( \omega(t_2) > 0, \ \omega'(t_2) = 0, \ \omega''(t_2) \leq 0. \)

For the first case \( \omega'(t) \geq 0, \ t \in [\sigma, 1] \), similar to Case (II), we have
\[\omega''(t) > 0\]
or
\[\omega(t) > 0, \ \omega''(t) > 0, \ \text{for all} \ t \in (\sigma, 1),\]
which implies that the graph of \( \omega \) is concave upward on \( (\sigma, 1) \), and so
\[\frac{\omega(\eta)}{\eta} < \frac{\omega(1)}{1}.
\]
On the other hand, we have
\[\omega(1) = u^{(n-2)}(1) - \beta_2^{(n-2)}(1) \leq \xi [u^{(n-2)}(\eta) - \beta_2^{(n-2)}(\eta)] = \xi \omega(\eta),\]
from \( 0 < \xi \eta < 1 \), we obtain
\[\frac{\omega(\eta)}{\eta} \geq \frac{\omega(1)}{1},\]
which is a contradiction.

For the second case, similar to the argument of Case (II), we could have contradiction.

Thus we show \( u^{(n-2)} \leq \beta_2^{(n-2)} \) on \([0, 1]\), and show
\[
\alpha_1^{(n-2)} \leq u^{(n-2)} \leq \beta_2^{(n-2)} \quad \text{on} \quad [0, 1]. \tag{31}
\]

By integrating the inequality (31) on \([0, t]\), according to (4) and (6), we obtain \( \alpha_1^{(i)} \leq u^{(i)} \leq \beta_2^{(i)} \) on \([0, 1]\), \( i = 0, 1, \ldots, n-3 \).

Now we show that \( |u^{(n-1)}| \leq L \) on \([0, 1]\). If the assertion is not true, without loss of the generality, we suppose that there exists \( t \in [0, 1] \), satisfying \( u^{(n-1)}(t) > L \). Let \( t_3 \) be the point where \( u^{(n-1)}(t) - L \) attains its positive maximum over \([0, 1]\). From mean value theorem and \( \alpha_1^{(n-2)} \leq u^{(n-2)} \leq \beta_2^{(n-2)} \) on \([0, 1]\), there exists \( \theta \in (0, 1) \), such that
\[u^{(n-1)}(\theta) = u^{(n-2)}(1) - u^{(n-2)}(0) \leq \beta_2^{(n-2)}(1) - \alpha_1^{(n-2)}(1) \leq \lambda < L.
\]
Since \( u^{(n-1)} \in C[0, 1] \), then there exists interval \([t_4, t_5] \subseteq [0, 1] \) (or \([t_5, t_4] \subseteq [0, 1] \), such that
\[u^{(n-1)}(t_4) = \lambda, \quad u^{(n-1)}(t_5) = L, \quad \lambda < u^{(n-1)}(t) < L, \ t \in (t_4, t_5). \tag{32}\]
From (7), we obtain
\[ |u^{(n)}(t)| = |F(t, u, u', \ldots, u^{(n-1)})| = |f(t, u, u', \ldots, u^{(n-1)})| \leq \Phi(|u^{(n-1)}|), \]
for \( t \in (t_4, t_5). \)

Then
\[ \left| \int_{t_4}^{t_5} \frac{u^{(n-1)}(t)u^{(n)}(t)}{\Phi(u^{(n-1)}(t))} \, dt \right| \leq \left| \int_{t_4}^{t_5} u^{(n-1)}(t) \, dt \right| \leq \lambda. \] (33)

In view of (14) and (32), we have
\[ \left| \int_{t_4}^{t_5} \frac{u^{(n-1)}(t)u^{(n)}(t)}{\Phi(u'(t))} \, dt \right| = \left| \int_{\lambda}^{L} \frac{s}{\Phi(s)} \, ds \right| > \lambda. \] (34)

Then (33) contradicts (34). So that \(|u^{(n-1)}| \leq L\) on \([0, 1]\). Thus \(u\) is the required solution.

**Step 2.** We show that problem (20) with (2) has at least three solutions \(u_1, u_2\) and \(u_3\).

Let
\[ \Omega = \{ u \in C^{n-1}[0, 1]: \|u\| < PM + L \}, \]
where \( P > \max\{\max_{t \in [0, 1]} \int_{0}^{1} |G(t, s)| \, ds, 1\} \), \( G(t, s) \) is Green’s function the problem (1), (2).

Define \( S : C[0, 1] \rightarrow C^{n-1}[0, 1] \) by
\[ (S\phi)(t) = \int_{0}^{1} G(t, s)\phi(s) \, ds, \]
for all \( \phi \in C[0, 1] \) and \( t \in [0, 1] \). It is clear that \( S \) is completely continuous.

Define \( H : C^{n-1}[0, 1] \rightarrow C[0, 1] \) as
\[ H(\phi)(t) = F(t, \phi(t), \phi'(t), \ldots, \phi^{n-1}(t)). \]

Then \( u \in C^{n-1}[0, 1] \) is a solution of (20) with (2) if and only if \((I - SH)(u) = 0\). For \( u \in \overline{\Omega}\), from (19), we have
\[ SH(x) = \int_{0}^{1} G(t, s) F(s, u(s), u'(s), \ldots, u^{n-1}(s)) \, ds \leq M \int_{0}^{1} G(t, s) \, ds < PM < PM + L. \]

Clearly \( SH(\overline{\Omega}) \subset \Omega \) and \( SH \) is completely continuous. Then we have
\[ \deg(I - SH, \Omega, 0) = \deg(I, \Omega, 0) = 1. \]

Let
\[ \Omega_{\alpha_2} = \{ u \in \Omega: u^{(n-2)} > \alpha_2^{(n-2)} \text{ on } (0, 1) \}, \]
\[ \Omega^{\beta_1} = \{ u \in \Omega: u^{(n-2)} < \beta_1^{(n-2)} \text{ on } (0, 1) \}. \]
Since \( \alpha^{(n-2)}_2 \neq \beta^{(n-2)}_1 \), \( \alpha^{(n-2)}_2 \geq \alpha^{(n-2)}_1 > -L \), and \( \beta^{(n-2)}_1 \leq \beta^{(n-2)}_2 < L \), it follows that

\[
\Omega_{\alpha_2} \neq \emptyset \neq \Omega_{\beta_1}, \quad \overline{\Omega}_{\alpha_2} \cap \overline{\Omega}_{\beta_1} = \emptyset, \quad \Omega \setminus \left\{ \overline{\Omega}_{\alpha_2} \cup \overline{\Omega}_{\beta_1} \right\} \neq \emptyset.
\]

According to (A1) and Remark 1, there is no solution on \( \partial \Omega_{\alpha_2} \cup \partial \Omega_{\beta_1} \). From Lemma 2, one has

\[
\deg(I - SH, \Omega, 0) = \deg(I - SH, \Omega \setminus \left\{ \overline{\Omega}_{\alpha_2} \cup \overline{\Omega}_{\beta_1} \right\}, 0) + \deg(I - SH, \Omega_{\beta_1}, 0)
\]

If we show that

\[
\deg(I - SH, \Omega_{\beta_1}, 0) = \deg(I - SH, \Omega_{\alpha_2}, 0) = 1,
\]

then

\[
\deg(I - SH, \Omega \setminus \left\{ \overline{\Omega}_{\alpha_2} \cup \overline{\Omega}_{\beta_1} \right\}, 0) = -1,
\]

and hence there are solutions in \( \Omega_{\alpha_2}, \Omega_{\beta_1} \) and \( \Omega \setminus \left\{ \overline{\Omega}_{\alpha_2} \cup \overline{\Omega}_{\beta_1} \right\} \), respectively.

We show that \( \deg(I - SH, \Omega_{\alpha_2}, 0) = 1 \). The proof that \( \deg(I - SH, \Omega_{\beta_1}, 0) = 1 \) is the same and hence omitted. Similar to the definitions of \( f \), we define

\[
f^*_1(t, x_1, x_2, \ldots, x_n) = \begin{cases} f(t, \beta_2, x_2, \ldots, x_n), & x_1 > \beta_2(t), \ t \in [0, 1], \\ f(t, x_1, x_2, \ldots, x_n), & \alpha_2(t) \leq x_1 \leq \beta_2(t), \ t \in [0, 1], \\ f(t, x_1, x_2, \ldots, x_n), & x_1 < \alpha_2(t), \ t \in [0, 1], \\ & \vdots \\ f^*_{n-1}(t, x_1, x_2, \ldots, x_n) \\ f^*_n(t, x_1, x_2, \ldots, x_n) \end{cases}
\]

Now from \( I - SH|_{\Omega_{\alpha_2}} \), we define its extension \( I - SH^* : \overline{\Omega} \rightarrow C^{n-1}[0, 1] \), as follows.

\[
F^*(t, x_1, x_2, \ldots, x_n) = \begin{cases} f^*_{n-1}(t, x_1, \ldots, x_{n-1}, L), & x_n > L, \ t \in [0, 1], \\ f^*_{n-1}(t, x_1, \ldots, x_{n-1}, x_n) & |x_n| \leq L, \ t \in [0, 1], \\ f^*_{n-1}(t, x_1, \ldots, x_{n-1}, -L), & x_n < -L, \ t \in [0, 1]. \\ & \vdots \\ f^*_n(t, x_1, x_2, \ldots, x_n) \\ F^*(t, x_1, x_2, \ldots, x_n) \end{cases}
\]

Thus \( F^* \) is a continuous function on \( [0, 1] \times R^n \) and satisfies

\[
|F^*(t, x_1, x_2, \ldots, x_n)| \leq M,
\]

for all \( (t, x_1, x_2, \ldots, x_n) \in [0, 1] \times R^n \), where \( M \) is given in (19).

Define \( H^* : C^{n-1}[0, 1] \rightarrow C[0, 1] \) as follows

\[
H^*(\phi)(t) = F^*(t, \phi(t), \phi'(t), \ldots, \phi^{n-1}(t)).
\]

Then \( u \in C^{n-1}[0, 1] \) is a solution of \( (I - SH^*)(u) = 0 \) if and only if \( u \) is a solution of
\[ u^{(n)}(t) + F^*(t, u, u', \ldots, u^{(n-1)}) = 0, \quad t \in (0, 1), \quad (35) \]

with (2). Similar to the above argument, it follows that \( u \) is a solution of (35) with (2) only if \( u \in \Omega_{\alpha_2} \). Thus

\[
\deg(I - SH^*, \Omega \setminus \overline{\Omega}_{\alpha_2}) = 0.
\]

Similarly, we show that \( SH^*(\Omega) \subset \Omega \). Then we have

\[
\deg(I - SH^*, \Omega, 0) = 1.
\]

Thus

\[
\deg(I - SH, \Omega_{\alpha_2}, 0) = \deg(I - SH^*, \Omega_{\alpha_2}, 0)
\]

\[
= \deg(I - SH^*, \Omega \setminus \overline{\Omega}_{\alpha_2}, 0) + \deg(I - SH^*, \Omega_{\alpha_2}, 0)
\]

\[
= \deg(I - SH^*, \Omega, 0) = 1.
\]

Therefore there are three solutions for problem (1), (2). Then the proof is finished. \( \square \)

We now present an example to illustrate that the assumptions of Theorem 1 can easily be verified.

**Example 1.** We are concerned with the following third order boundary value problem:

\[
u'''(t) + h(u') + g(u'') = 0, \quad t \in (0, 1), \quad (36)\]

\[u(0) = u'(0) = u'(1) - \frac{1}{2} u'(\frac{1}{3}) = 0, \quad (37)\]

where \( f(t, u(t), u'(t), u''(t)) = h(u') + g(u'') \), \( \xi = \frac{1}{2}, \eta = \frac{1}{3} \), such that \( 0 < \xi \eta < 1 \).

We suppose that \( g, h : \mathbb{R} \rightarrow \mathbb{R} \) are continuous, \( h \) is nondecreasing and near the origin \( g(0) = 0, h(0) < 0, h(-a) > 0 \), and \( h(b) < 0 \) for some \( a > 0 \) and \( b > 0 \) (which may be large), while near the origin, they behave as follows:

\[
\begin{align*}
&\begin{cases}
  h(x) \geq -c, & \text{for } 0 \leq x \leq \frac{c}{8}, \\
  h(x) > 2c, & \text{for } \frac{c}{8} \leq x \leq \frac{c}{4},
\end{cases} \\
&\begin{cases}
  g(y) > 3c, & \text{for } y \in \left[-c, -\frac{c}{\sqrt{2}}\right] \cup \left[\frac{c}{\sqrt{2}}, c\right], \\
  g(y) \geq 0, & \text{for } y \in \left[-\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right],
\end{cases}
\end{align*}
\]

for small \( c \). Moreover, we assume \( |g(y)| \leq c_1 + c_2|y|^p, 1 \leq p \leq 2 \). Take

\[
\alpha_1(t) = -at, \quad \alpha_2(t) = \frac{c}{6} t^2 (3 - 2t), \quad t \in [0, 1].
\]

Then we have \( \alpha_1, \alpha_2 \in C^{3,1}([0, 1]) \) satisfying the boundary conditions:

\[
\begin{align*}
\alpha_1(0) &= 0, \quad \alpha_1'(0) = -a < 0, \quad \alpha_1'(1) - \frac{1}{2} \alpha_1'(\frac{1}{3}) = -\frac{a}{2} < 0, \\
\alpha_2(0) &= 0, \quad \alpha_2'(0) = 0, \quad \alpha_2'(1) - \frac{1}{2} \alpha_2'(\frac{1}{3}) = -\frac{c}{9} < 0.
\end{align*}
\]
Moreover, for every $t \in (0, 1)$, we have
\[
\alpha''(t) + f(t, \alpha(t), \alpha'(t), \alpha''(t)) = h(-a) + g(0) > 0,
\alpha''(t) + f(t, \alpha(t), \alpha'(t), \alpha''(t)) = -2c + h(ct(1-t)) + g(c(1-2t)) > 0.
\]
Thus, $\alpha_1$ and $\alpha_2$ are strict lower solutions of problem (36), (37).

Now we take
\[
\beta_1(t) = 0, \quad \beta_2(t) = bt, \quad t \in [0, 1].
\]
Then $\beta_1, \beta_2 \in C^{3,1}([0, 1])$ satisfying the boundary conditions:
\[
\beta_1(0) = \beta_1'(0) = 0, \quad \beta_1'(1) - \frac{1}{2} \beta_1'(\frac{1}{3}) = 0,
\beta_2(0) = 0, \quad \beta_2'(0) = b > 0, \quad \beta_2'(1) - \frac{1}{2} \beta_2'(\frac{1}{3}) = \frac{b}{2} > 0.
\]
Moreover, for every $t \in (0, 1)$, we have
\[
\beta_1''(t) + f(t, \beta_1(t), \beta_1'(t), \beta_1''(t)) = h(0) + g(0) < 0,
\beta_2''(t) + f(t, \beta_2(t), \beta_2'(t), \beta_2''(t)) = h(b) + g(0) < 0.
\]
Thus, $\beta_1$ and $\beta_2$ are strict upper solutions of problem (36), (37). Further, we note that
\[
\alpha_1(t) \leq \alpha_2(t) \leq \beta_2(t), \quad \alpha_1(t) \leq \beta_1(t) \leq \beta_2(t), \quad \alpha_2(t) \neq \beta_1(t) \quad \text{on } [0, 1].
\]
Moreover, for every $(t, u, u') \in [0, 1] \times [-a, b] \times [-a, b]$, we have
\[
|f(t, u, u', u'')| \leq |g(u'')| + |h(u')| \leq m + |g(u'')| \leq c_1 + c_2|u''|^p + m = \Phi(|u''|),
\]
where $1 \leq p \leq 2$ and $m = \max(|h(u')|: u' \in [-a, b])$. Since
\[
\int_0^\infty \frac{s}{\Phi(s)} \, ds = \int_0^\infty \frac{s}{c_1 + c_2 s^p + m} \, ds = \infty.
\]
It follows that $f$ satisfies a Nagumo condition. Hence all the conditions of Theorem 1 are satisfied, and so the problem has at least three solutions satisfying
\[
\alpha_1(t) \leq u_i(t) \leq \alpha_2(t), \quad 0 \leq t \leq 1, \quad i = 1, 2, 3.
\]

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