Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Computational and Applied Mathematics 174 (2005) 213–225
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Stable *hp* mixed finite elements based on the Hellinger–Reissner principle

Manil Suri*,[1]

Department of Mathematics and Statistics, University of Maryland, Baltimore County, 1000, Hilltop Circle, Baltimore, MD 21250, USA

Received 4 February 2004; received in revised form 13 April 2004

Abstract

In the Hellinger–Reissner formulation for linear elasticity, both the displacement u and the stress σ are taken as unknowns, giving rise to a saddle point problem. We present new pairings of quadrilateral 'trunk' finite element spaces for this method and prove stability (and optimality) in terms of both h and p. The effect of mesh shape regularity on the stability constant is explicitly tracked. Our results provide a theoretical basis for recent numerical experiments (in the context of a mixed p formulation for viscoelasticity) that showed these spaces worked well computationally.

MSC: 65N30

Keywords: h version; p version; hp version; Elasticity; Mixed finite elements; Babuska–Brezzi condition

1. Introduction

The Hellinger–Reissner (HR) principle gives a mixed variational formulation for problems in elasticity by casting them as first-order systems with both the stress σ and the displacement u as unknowns. It is sometimes referred to as the primal mixed method, to differentiate it from the dual mixed method (see e.g. [3]). In the latter method, integration by parts is carried out to shift the derivatives from the strains $\varepsilon(u)$ to the stresses σ so that the components of σ are $H(\mathrm{div})$ functions,

* Tel.: +1-410-455-2311; fax: +1-410-455-1066.
E-mail address: suri@math.umbc.edu (M. Suri).
[1] The work of this author was supported in part by the National Science Foundation under Grant DMS-0074160.

0377-0427/$ - see front matter
doi:10.1016/j.cam.2004.04.008

while those of u are L_2 functions.[2] In contrast, the HR formulation[3] leads to L_2 components for σ and H^1 components for u.

The above formulations are particularly useful when applied to problems such as plasticity and viscoelasticity, where the displacements and stresses are generally approximated separately since the stresses cannot be straightforwardly recovered from the displacements. In this paper, we are motivated by a recent solution method for viscoelastic problems using the p finite element method on quadrilateral meshes [9,10]. Here, the HR principle turns out to be the most natural method to use, facilitating easy discretization of the time dependence by the backward Euler method, and ensuring that the approximate stresses, being in L_2, have no continuity constraints that need to be imposed. At each discrete time step t_k, this method requires the solution of a mixed finite element method for a linear elasticity problem based on the HR principle. Therefore, one criterion that is crucial is that the underlying finite element spaces be *stable* in the sense of Babuška–Brezzi.

This is the main goal of our paper—the construction of stable pairs of spaces on quadrilateral elements for the HR mixed formulation. Since [9,10] deals with the p version, our main objective is stability with respect to p, the polynomial degree used—but our proof also leads to stability in terms of h, the mesh spacing. For triangles, the construction of such spaces turns out to be straightforward. If one use continuous piecewise polynomials of degree p for the displacement space V_N and discontinuous piecewise polynomials of degree $p-1$ for the stress space Σ_N, then it is immediate to see that the following inclusion holds:

$$\varepsilon(V_N) \subset \Sigma_N. \tag{1.1}$$

This easily leads to the inf–sup condition in terms of both h and p (Theorem 2.1).

The situation is less clear for meshes consisting of parallelograms. Here, one needs to take the same degree for both Σ_N and V_N for the Q_p ('product') and Q'_p ('trunk' or 'serendipity') spaces that are usually implemented in finite element packages to ensure that (1.1) holds. However, this combination is not optimal in terms of approximation order, since the ideal combination would use one lower polynomial degree for the L_2 stress approximation.

The computational results in [9,10] were tested using the p version with Q'_p (and also Q_p) spaces in the commercial code Stress Check. The experimental results showed that with Q'_p spaces, the combination of degree p for V_N and degree $p-1$ for Σ_N was stable in terms of p. We prove this analytically in this paper, establishing stability (and hence optimality) in both p and h (when $p>2$) for spaces over parallelograms. Since this combination violates condition (1.1), new techniques of proof are needed. An interesting feature of our proof is that we track the effect of the shape regularity of the elements on the inf–sup constant.

We also prove that for the above spaces, a local version of the inf–sup constant is zero over each element when $p \leqslant 2$ (which is why our proof does not extend to the h version for the cases $p=1,2$). For Q_p elements, the $p/p-1$ combination also gives a zero local inf–sup constant for

[2] We use standard Sobolev Space notation, with $H^k(\Omega)$ denoting the space of k square integrable functions over a domain Ω, $L_2(\Omega)=H^0(\Omega)$, and $H(\mathrm{div},\Omega)$ denoting $L_2(\Omega)$ functions whose divergence is also in $L_2(\Omega)$. $(\cdot,\cdot)_{k,\Omega}$ and $\|\cdot\|_{k,\Omega}$ will denote the usual $H^k(\Omega)$ inner product and norm, respectively (with the same notation being used for vectors and tensors as well). The symbol Ω will be dropped from the above notations when the domain is understood.

[3] We remark that in the literature, the dual mixed method is also sometimes referred to as arising from the Hellinger–Reissner principle, but here, we will use HR to signify only the primal mixed formulation.

any $p \geqslant 1$. This agrees well with the results in [9], where it was found computationally that the p version does not work well in practice for this choice. (Moreover, the inf–sup constant was shown numerically to be zero in [9] for this choice.) This suggests that for Q_p elements, one should use the same degree for Σ_N and V_N. When this is done, (1.1) will again hold, proving stability.

Analogs of our results also hold for the primal mixed formulation of the Poisson equation, as described in Section 5. We also discuss here the extension of our stability results to curvilinear elements.

2. The Hellinger–Reissner formulation

We consider the elasticity problem on a domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega = \Gamma = \Gamma_D \cup \Gamma_N$ consisting of disjoint portions $\Gamma_D \neq \emptyset$ and Γ_N: Find u satisfying

$$-\operatorname{div} \sigma = f, \quad \sigma = E\varepsilon(u) \text{ in } \Omega,$$

$$u = 0 \text{ on } \Gamma_D, \ \sigma n = g \text{ on } \Gamma_N.$$

Here, as usual, $\varepsilon(u)$ is the symmetric gradient of u, given by

$$(\varepsilon(u))_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

E is a symmetric nonsingular matrix of elastic constants and σ is the (symmetric) stress.

The so-called Hellinger–Reissner principle is given by: Find $(\sigma, u) \in \Sigma \times V = \mathbb{S}((L_2(\Omega))^4) \times (H^1_D(\Omega))^2$ satisfying, for all $(\tau, v) \in \Sigma \times V$,

$$(E^{-1}\sigma, \tau)_0 - (\tau, \varepsilon(u))_0 = 0, \tag{2.1}$$

$$-(\sigma, \varepsilon(v))_0 = -(f, v)_0 + \int_{\Gamma_D} g \cdot v \, \mathrm{d}x. \tag{2.2}$$

Here, $\mathbb{S}((X)^4)$ denotes the set of symmetric 2×2 matrices with components in X and $H^1_D(\Omega) = \{u \in H^1(\Omega), u = 0 \text{ on } \Gamma_D\}$.

Finite element approximations can now be obtained by choosing finite-dimensional subspaces $\Sigma_N \times V_N \subset \Sigma \times V$ and finding $(\sigma^N, u^N) \in \Sigma_N \times V_N$ satisfying (2.1) and (2.2) for all $(\tau, v) \in \Sigma_N \times V_N$. As mentioned in the introduction, these subspaces should be chosen so that the usual inf–sup or Babuška–Brezzi conditions are satisfied. The first of these conditions requires the following:

$$(E^{-1}\tau, \tau)_0 \geqslant \alpha \|\tau\|^2_\Sigma \quad \forall \tau \in X_N, \tag{2.3}$$

where the constant α is independent of τ and the discretization parameter N, and

$$X_N = \{\tau \in \Sigma_N, \ (\tau, \varepsilon(v))_0 = 0 \quad \forall v \in V_N\}.$$

This coercivity condition is trivially satisfied for *all* of Σ whenever the matrix E^{-1} is positive definite. We assume this is true, and exclude the case of plane strain conditions with incompressible (or nearly incompressible) materials. (The latter case can cause Poisson ratio locking similar to that observed in the standard finite element method—see e.g. [2]).

The second condition requires that the following be true:

$$\sup_{\tau \in \Sigma_N} \frac{(\tau, \varepsilon(u))_0}{\|\tau\|_\Sigma} \geqslant \beta \|u\|_V \quad \forall u \in V_N \tag{2.4}$$

with $\beta > 0$ (the 'inf–sup constant') independent of N. To this end, we have the following theorem (noted, e.g. in [3]).

Theorem 2.1. *Let the family* $\{\Sigma_N, V_N\}$ *satisfy* (1.1) *for all* N. *Then* (2.4) *holds with* β *independent of* N.

Proof. The proof follows immediately by taking $\tau = \varepsilon(u)$ on each element and using Korn's inequality. $\square$

As we shall see in the sequel, (1.1) is sufficient, but not necessary for (2.4) to hold.

3. Stability on a single element

Let $\hat{K} = (-1,1)^2$ be the reference quadrilateral and $\hat{T} = \{(\hat{x}_1,\hat{x}_2) | 0 < \hat{x}_1 < 1, 0 < \hat{x}_2 < 1 - \hat{x}_1\}$ the reference triangle. Then any parallelogram K (respectively, triangle T) with coordinates denoted by (x_1,x_2) can be represented as the image of $\hat{K}$ (respectively, $\hat{T}$) under an affine mapping $\mathscr{F}$ given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathscr{F}\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = A\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{3.1}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 invertible matrix satisfying

$$\det A = ad - bc = \kappa \neq 0.$$

In this section, we will prove a condition that leads to a local version of (2.4) for three combinations of polynomial spaces defined on an arbitrary element $S = K$ or T. We will denote, for $M \subset \mathbb{R}^2$, $P_p(M)$ to be the space of polynomials of total degree p on M and $Q_{p,q}(M)$ to be the space of polynomials of degree p in x_1 and q in x_2. Then $Q_{p,p}(K) = Q_p(K)$ is the usual product space on K. Also, $Q'_p(K) = \operatorname{span}\{P_p(K), x_1^p x_2, x_1 x_2^p\}$ is the trunk or serendipity space. The three combinations we consider are the following:

(S1) $\Sigma_p = \mathbb{S}((Q'_{p-1}(K))^4)$, $\quad V_p = (Q'_p(K))^2$, $\quad p > 2$.
(S2) $\Sigma_p = \mathbb{S}((P_{p-1}(T))^4)$, $\quad V_p = (P_p(T))^2$, $\quad p \geqslant 1$.
(S3) $\Sigma_p = \mathbb{S}((Q_p(K))^4)$, $\quad V_p = (Q_p(K))^2$, $\quad p \geqslant 1$.

Cases (S2) and (S3) are included here only for completeness, since the meshes in the *hp* version often need elements such as triangles as well.

Associated with each element S, we will define a shape regularity parameter δ_S by

$$\delta_S = \frac{h_S}{\rho_S}, \tag{3.2}$$

where h_S is the diameter of $\bar{S}$ (the closure of S) and ρ_S is the diameter of the largest circle that can be inscribed within $\bar{S}$. Then the following is the main theorem of this section.

Theorem 3.1. *Let* (Σ_p, V_p) *be given by* (S1), (S2) *or* (S3), *with* $S=K$ *or* T *denoting the respective underlying element, which is the image of* $\hat{K}$ *or* $\hat{T}$ *under an affine mapping* $\mathscr{F}$ *with shape regularity parameter* δ_S. *Then given* $w \in V_p$, *there exists* $\sigma \in \Sigma_p$ *such that*

$$(\sigma, \varepsilon(w))_{0,S} \geqslant \frac{1}{(1+\gamma p^{-1}\sqrt{\delta_S})^2}\|\varepsilon(w)\|_{0,S}^2 \quad \text{and} \quad \|\sigma\|_{0,S} \leqslant \|\varepsilon(w)\|_{0,S}, \tag{3.3}$$

with γ *a constant independent of* w, p.

Noting that combinations (S2) and (S3) satisfy (1.1), we see that Theorem 3.1 follows similarly to Theorem 2.1 for these cases, by taking $\sigma = \varepsilon(w)$, which gives (3.3) with $\gamma = 0$. The proof for (S1) is more involved, and needs a sequence of lemmas, which we now prove. Unless otherwise stated, Σ_p, V_p will refer to the spaces in (S1) defined on the element K. Also, $\hat{\Sigma}_p, \hat{V}_p$ will refer to the same spaces defined on the reference square $\hat{K}$.

For any function $f(x, y)$ defined on K, we will denote its image on $\hat{K}$ by $\hat{f} = f \circ \mathscr{F}$ and vice versa. Also, we will denote the Jacobian of the mapping $\mathscr{F}$ by J. (For our affine mappings, $J = A$, and $\det J = \det A = \kappa$, a constant.) Then the following lemma follows by the change of variables formula for integrals.

Lemma 3.2. *Given* $w \in V_p$, $\sigma \in \Sigma_p$, *with corresponding images* $\hat{w} = w \circ \mathscr{F}$, $\hat{\sigma} = \sigma \circ \mathscr{F}$ *on* $\hat{K} = \mathscr{F}^{-1}(K)$, *the following relations hold*:

$$\|\sigma\|_{0,K} = \|\hat{\sigma}\sqrt{|\det J|}\|_{0,\hat{K}}, \quad \|\varepsilon(w)\|_{0,K} = \|\hat{\varepsilon}(w)\sqrt{|\det J|}\|_{0,\hat{K}}, \tag{3.4}$$

$$(\sigma, \varepsilon(w))_{0,K} = (\hat{\sigma}, \hat{\varepsilon}(\hat{w})|\det J|)_{0,\hat{K}}, \tag{3.5}$$

where, using summation notation for repeated indices

$$(\hat{\varepsilon}(\hat{w}))_{ij} = \frac{1}{2}\left(\frac{\partial \hat{w}_i}{\partial \hat{x}_k}\frac{\partial \hat{x}_k}{\partial x_j} + \frac{\partial \hat{w}_j}{\partial \hat{x}_k}\frac{\partial \hat{x}_k}{\partial x_i}\right). \tag{3.6}$$

Let L_p denote the Legendre polynomial of degree p. Then any $\hat{w} = (\hat{w}_1, \hat{w}_2)^T \in \hat{V}_p$ may be written as (we use the equivalent notation $(x, y) = (\hat{x}_1, \hat{x}_2)$ for convenience)

$$\begin{aligned}\hat{w}_1 &= \sum_{0 \leqslant i+j \leqslant p} a_{ij}L_i(x)L_j(y) + a_{1p}L_1(x)L_p(y) + a_{p1}L_p(x)L_1(y)\\ &= A + A^1 + A^2,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\hat{w}_2 &= \sum_{0 \leqslant i+j \leqslant p} b_{ij}L_i(x)L_j(y) + b_{1p}L_1(x)L_p(y) + b_{p1}L_p(x)L_1(y)\\ &= B + B^1 + B^2.\end{aligned} \tag{3.8}$$

Moreover, by (3.6), we have for the case that the mapping $\mathscr{F}$ is given by (3.1),

$$(\hat{\varepsilon}(\hat{w}))_{11} = (d(A_x + A_x^1 + A_x^2) - c(A_y + A_y^1 + A_y^2))/\kappa, \tag{3.9}$$

$$(\hat{\varepsilon}(\hat{w}))_{22} = (-b(B_x + B_x^1 + B_x^2) + a(B_y + B_y^1 + B_y^2))/\kappa, \tag{3.10}$$

$$\begin{aligned}(\hat{\varepsilon}(\hat{w}))_{12} = (\hat{\varepsilon}(\hat{w}))_{21} = (&-b(A_x + A_x^1 + A_x^2) + a(A_y + A_y^1 + A_y^2) \\ &+ d(B_x + B_x^1 + B_x^2) - c(B_y + B_y^1 + B_y^2))/2\kappa.\end{aligned} \tag{3.11}$$

Let us define the usual L_2 projection from $L_2(\hat{K})$ onto $Q'_{p-1}(\hat{K})$ by

$$(\Pi_p(\hat{u}), \hat{s})_{0,\hat{K}} = (\hat{u}, \hat{s})_{0,\hat{K}} \quad \forall \hat{s} \in Q'_{p-1}(\hat{K}). \tag{3.12}$$

Then we have the following lemma.

Lemma 3.3. *Let* A, A^1, A^2 *be as in* (3.7). *Then*

$$\Pi_p(A_x) = A_x, \quad \Pi_p(A_y) = A_y, \tag{3.13}$$

$$\Pi_p(A_x^1) = 0, \quad \Pi_p(A_y^1) = A_y^1, \tag{3.14}$$

$$\Pi_p(A_x^2) = A_x^2, \quad \Pi_p(A_y^2) = 0. \tag{3.15}$$

Similar equations hold for B, B^1, B^2 *as in* (3.8).

Proof. First, it is easy to see that A_x, A_y, A_y^1, A_x^2 all lie in $Q'_{p-1}(\hat{K})$, so that the corresponding equalities follow. Next, we note that

$$A_x^1 = a_{1p} L_p(y).$$

For $\hat{s}(x, y) \in Q'_{p-1}(\hat{K})$, $\hat{s}(x_0, y)$ is a polynomial in y of degree no larger than $p-1$ for each fixed x_0. Hence $(3.14)_1$ follows by the orthogonality of Legendre polynomials. Eq. $(3.15)_2$ follows similarly. □

Suppose now that we are given a $w \in V_p$. Then its image $\hat{w} \in \hat{V}_p$ may be written as (3.7) and (3.8). We then define $\hat{\sigma} \in \hat{\Sigma}_p$ by

$$\hat{\sigma} = \Pi_p(\hat{\varepsilon}(\hat{w})), \tag{3.16}$$

where the projection Π_p is understood to be made component-wise. We then have the following result.

Lemma 3.4. *For* $w \in V_p$, *with* $\hat{w}$ *given by* (3.7) and (3.8), *let* $\hat{\sigma} \in \hat{\Sigma}_p$ *be defined by* (3.16). *Then for* $p > 2$,

$$\|\kappa \hat{\sigma}_{11}\|_{0,\hat{K}}^2 \geqslant K p^2 (c^2 \|A_x^1\|_{0,\hat{K}}^2 + d^2 \|A_y^2\|_{0,\hat{K}}^2), \tag{3.17}$$

$$\|\kappa \hat{\sigma}_{22}\|_{0,\hat{K}}^2 \geqslant K p^2 (a^2 \|B_x^1\|_{0,\hat{K}}^2 + b^2 \|B_y^2\|_{0,\hat{K}}^2), \tag{3.18}$$

$$\|\kappa \hat{\sigma}_{12}\|_{0,\hat{K}}^2 = \|\kappa \hat{\sigma}_{21}\|_{0,\hat{K}}^2 \geqslant K p^2 (\|a A_x^1 - c B_x^1\|_{0,\hat{K}}^2 + \| - b A_y^2 + d B_y^2\|_{0,\hat{K}}^2), \tag{3.19}$$

where K *is a constant independent of* p.

Proof. Let us first prove (3.17). Using (3.9), (3.16) and Lemma 3.3, we see that

$$\begin{aligned}\kappa\hat{\sigma}_{11} &= d(A_x + A_x^2) - c(A_y + A_y^1)\\ &= d\left(\sum_{0\leqslant i+j\leqslant p} a_{ij}L_i'(x)L_j(y) + a_{p1}L_p'(x)L_1(y)\right)\\ &\quad - c\left(\sum_{0\leqslant i+j\leqslant p} a_{ij}L_i(x)L_j'(y) + a_{1p}L_1(x)L_p'(y)\right).\end{aligned} \tag{3.20}$$

We now note that (Eq. (A.4.4) of [8])

$$L_p'(t) = n_pL_{p-1}(t) + L_{p-2}'(t),$$

where $n_p = 2p-1$, so that

$$a_{p1}L_p'(x)L_1(y) = n_pa_{p1}L_{p-1}(x)L_1(y) + a_{p1}L_{p-2}'(x)L_1(y),$$

$$a_{1p}L_1(x)L_p'(y) = n_pa_{1p}L_1(x)L_{p-1}(y) + a_{1p}L_1(x)L_{p-2}'(y).$$

Substituting in (3.20), we may group together terms in $P_{p-1}(\hat{K})$ and write $\hat{\sigma}_{11}$ as

$$\begin{aligned}\kappa\hat{\sigma}_{11} &= P + dn_pa_{p1}L_{p-1}(x)L_1(y) - cn_pa_{1p}L_1(x)L_{p-1}(y)\\ &= P + P_1 + P_2.\end{aligned} \tag{3.21}$$

Consider the term $P_1 = dn_pa_{p1}L_{p-1}(x)L_1(y)$ in (3.21). Clearly, this is $L_2(\hat{K})$ orthogonal to the last term P_2 in (3.21) for $p \neq 2$. Moreover, the presence of $L_{p-1}(x)$ makes it orthogonal to every term in P except $a_{p0}L_p'(x)L_0(y)$ and $a_{p-1,1}L_{p-1}(x)L_1'(y)$. But both these latter terms have the factor $L_0(y)$ in them, which makes P_1 orthogonal to them as well. Similarly, P_2 is orthogonal to every other term in (3.21). Hence,

$$\begin{aligned}\|\kappa\hat{\sigma}_{11}\|_{0,\hat{K}}^2 &= \|P\|_{0,\hat{K}}^2 + \|P_1\|_{0,\hat{K}}^2 + \|P_2\|_{0,\hat{K}}^2\\ &= \|P\|_{0,\hat{K}}^2 + d^2n_p^2a_{p1}^2\|L_{p-1}\|_{0,\hat{K}}^2\|L_1\|_{0,\hat{K}}^2 + c^2n_p^2a_{1p}^2\|L_1\|_{0,\hat{K}}^2\|L_{p-1}\|_{0,\hat{K}}^2\\ &\geqslant (d^2a_{p1}^2 + c^2a_{1p}^2)(2p-1)^2\left(\frac{2}{3}\right)\left(\frac{2}{2p-1}\right)\\ &= \tfrac{4}{3}(2p-1)(d^2a_{p1}^2 + c^2a_{1p}^2).\end{aligned} \tag{3.22}$$

Next, from (3.7) we have

$$A_x^1 = a_{1p}L_1'(x)L_p(y) = a_{1p}L_0(x)L_p(y)$$

so that

$$\|A_x^1\|_{0,\hat{K}}^2 = a_{1p}^2(2)\left(\frac{2}{2p+1}\right) = \frac{4a_{1p}^2}{2p+1}. \tag{3.23}$$

Similarly,

$$\|A_y^2\|_{0,\hat{K}}^2 = \frac{4a_{p1}^2}{2p+1}, \tag{3.24}$$

so that (3.17) follows from (3.22)–(3.24). Eq. (3.18) can be similarly established.

To prove (3.19), we note that $\hat{\sigma}_{12}$ may be orthogonally decomposed similarly to (3.21), leading to the following analog of (3.22):

$$\|\kappa\hat{\sigma}_{12}\|_{0,\hat{K}}^2 = \tfrac{4}{3}(2p-1)((-ba_{p1}+db_{p1})^2 + (aa_{1p}-cb_{1p})^2). \tag{3.25}$$

Then (3.22) follows by noting that

$$\|-bA_y^2 + dB_y^2\|_{0,\hat{K}}^2 = \|-ba_{p1}L_p(x)L_0(y) + db_{p1}L_p(x)L_0(y)\|_{0,\hat{K}}^2 = \frac{4(-ba_{p1}+db_{p1})^2}{2p+1} \tag{3.26}$$

with a similar formula holding for $\|aA_x^1 - cB_x^1\|_{0,\hat{K}}^2$. $\square$

Corollary 3.4.1. *For $\hat{w}$ given by* (3.7) *and* (3.8) *and $\hat{\sigma}$ defined by* (3.16) *as in the above lemma, we have, for $p > 2$ and C a constant independent of p,*

$$\|\kappa\hat{\sigma}\|_{0,\hat{K}}^2 \geqslant Cp^2((a^2+c^2)(\|A_x^1\|_{0,\hat{K}}^2 + \|B_x^1\|_{0,\hat{K}}^2) + (b^2+d^2)(\|A_y^2\|_{0,\hat{K}}^2 + \|B_y^2\|_{0,\hat{K}}^2)).$$

Proof. Using the identity

$$(cA_x^1)^2 + (aB_x^1)^2 + (aA_x^1 - cB_x^1)^2 = (aA_x^1)^2 + (cB_x^1)^2 + (cA_x^1 - aB_x^1)^2,$$

it is easy to show that

$$c^2\|A_x^1\|_{0,\hat{K}}^2 + a^2\|B_x^1\|_{0,\hat{K}}^2 + \|aA_x^1 - cB_x^1\|_{0,\hat{K}}^2 \geqslant \tfrac{1}{2}(a^2+c^2)(\|A_x^1\|_{0,\hat{K}}^2 + \|B_x^1\|_{0,\hat{K}}^2).$$

The corollary then can be obtained using Lemma 3.4. $\square$

We now relate the shape regularity parameter δ_K to the coefficients a, b, c, d in the mapping $\mathscr{F}$ given in (3.1).

Lemma 3.5.

$$\max\left\{\frac{b^2+d^2}{a^2+c^2}, \frac{a^2+c^2}{b^2+d^2}\right\} \leqslant \delta_K^2.$$

Proof. Consider the points $\hat{A}_1 = (-1, 0)$ and $\hat{A}_2 = (1, 0)$, which are the midpoints of two opposite sides of $\hat{K}$. Their images under $\mathscr{F}$ are points lying in $\bar{K}$ given by $A_1 = (-a + b_1, -c + b_2)$ and $A_2 = (a + b_1, c + b_2)$, respectively, so that the Euclidean norm of A_1A_2 is given by

$$\|A_1A_2\| = 2\sqrt{a^2+c^2}.$$

Since both A_1 and A_2 lie in $\bar{K}$, we get

$$2\sqrt{a^2+c^2} \leqslant h_K.$$

Similarly, considering B_1 and B_2, the images of $\hat{B}_1 = (0, -1)$ and $\hat{B}_2 = (0, 1)$, respectively, we get

$$\|B_1B_2\| = 2\sqrt{b^2+d^2} \leqslant h_K.$$

We now observe that A_1, A_2 and B_1, B_2 are the midpoints of the opposite sides of the parallelogram K, so that

$$\rho_K \leqslant \min\{\|A_1A_2\|, \|B_1B_2\|\}.$$

The lemma follows easily. $\square$

Lemma 3.6. *For $w \in V_p$ and $\hat{\sigma} \in \hat{\Sigma}_p$ defined by* (3.16),

$$\|\hat{\sigma}\|_{0,\hat{K}} \geqslant \frac{1}{(1+\gamma p^{-1}\sqrt{\delta_K})} \|\hat{\varepsilon}(\hat{w})\|_{0,\hat{K}}$$

for a constant γ independent of p.

Proof. We note that using (3.9)–(3.11) and Lemma 3.3,

$$\kappa\hat{\varepsilon}(\hat{w}) = \kappa\hat{\sigma} + \begin{bmatrix} dA_x^1 - cA_y^2 & \frac{1}{2}(-bA_x^1 + aA_y^2 + dB_x^1 - cB_y^2) \\ \frac{1}{2}(-bA_x^1 + aA_y^2 + dB_x^1 - cB_y^2) & -bB_x^1 + aB_y^2 \end{bmatrix},$$

so that

$$\begin{aligned} \|\kappa\hat{\varepsilon}(\hat{w})\|_{0,\hat{K}} &\leqslant \|\kappa\hat{\sigma}\|_{0,\hat{K}} + (|b|+|d|)(\|A_x^1\|_{0,\hat{K}} + \|B_x^1\|_{0,\hat{K}}) \\ &\quad + (|a|+|c|)(\|A_y^2\|_{0,\hat{K}} + \|B_y^2\|_{0,\hat{K}}) \\ &\leqslant \left(1 + \frac{\gamma}{p}\left(\max\left\{\frac{b^2+d^2}{a^2+c^2}, \frac{a^2+c^2}{b^2+d^2}\right\}\right)^{1/2}\right)\|\kappa\hat{\sigma}\|_{0,\hat{K}} \end{aligned}$$

using Corollary 3.4.1. The result follows using Lemma 3.5. $\square$

Remark 3.1. An analogous but simpler proof to that of Lemma 3.6 above can be used to show that for either a scalar or vector w with components in $Q_p'(\hat{K})$,

$$\|\Pi_p \nabla w\|_{0,\hat{K}} \geqslant C\|\nabla w\|_{0,\hat{K}},$$

for C a constant independent of p.

Proof of Theorem 3.1. Noting that $\det J$ is a constant, we see from Lemma 3.2, (3.16) and Lemma 3.6 that

$$\begin{aligned} (\sigma, \varepsilon(w))_{0,K} &= |\kappa|(\hat{\sigma}, \hat{\varepsilon}(\hat{w}))_{0,\hat{K}} = |\kappa|\|\hat{\sigma}\|_{0,\hat{K}}^2 \\ &\geqslant \frac{|\kappa|}{(1+\gamma p^{-1}\sqrt{\delta_K})^2}\|\hat{\varepsilon}(\hat{w})\|_{0,\hat{K}}^2 = \frac{1}{(1+\gamma p^{-1}\sqrt{\delta_K})^2}\|\varepsilon(w)\|_{0,K}^2. \end{aligned}$$

Moreover, since $\hat{\sigma}$ is the L_2 projection of $\hat{\varepsilon}(\hat{w})$, we have, using Lemma 3.2 again that

$$\|\sigma\|_{0,K} = \sqrt{\kappa}\|\hat{\sigma}\|_{0,\hat{K}} \leqslant \sqrt{\kappa}\|\hat{\varepsilon}(\hat{w})\|_{0,\hat{K}} = \|\varepsilon(w)\|_{0,K}. \qquad \square$$

Let us now show that Theorem 3.1 cannot hold for certain pairs of local subspaces, i.e. that the inf–sup constant is zero in a local sense.

Theorem 3.7. *Let $K=\hat{K}$, the reference quadrilateral. Let (Σ_p, V_p) be given by* (S1) *with* $p=1,2$ *or by* $\Sigma_p=\mathbb{S}((Q_{p-1}(\hat{K}))^4), V_p=(Q_p(\hat{K}))^2, p\geqslant 1$. *Then there exists* $w\in V_p$ *for which*

$$(\sigma, \varepsilon(w))_{0,\hat{K}}=0 \quad \text{for all } \sigma\in\Sigma_p. \tag{3.27}$$

Proof. First, consider

$$w=\begin{bmatrix} L_1(x)L_2(y) \\ -L_2(x)L_1(y)\end{bmatrix}\in(Q_2'(\hat{K}))^2.$$

Then

$$\varepsilon(w)=\begin{bmatrix} L_0(x)L_2(y) & 0 \\ 0 & -L_2(x)L_0(y)\end{bmatrix}$$

which is orthogonal to all $\sigma\in\mathbb{S}((Q_1'(\hat{K}))^4)$, proving (3.27) for this case.

Next, define for $p\geqslant 1$,

$$w=\begin{bmatrix} L_p(x)L_p(y) \\ 0\end{bmatrix}\in(Q_p(\hat{K}))^2.$$

Then

$$\varepsilon(w)=\begin{bmatrix} L_p'(x)L_p(y) & \frac{1}{2}L_p(x)L_p'(y) \\ \frac{1}{2}L_p(x)L_p'(y) & 0\end{bmatrix}$$

so that once again (3.27) holds by orthogonality for all $\sigma\in\mathbb{S}((Q_{p-1}(\hat{K}))^4)$. Noting that $Q_p'(\hat{K})=Q_p(\hat{K})$ for $p=1$ (and $p=0$) completes the proof for all cases. □

Theorem 3.7 shows that if the p version is used on a single element with the choice $\Sigma_p=\mathbb{S}((Q_{p-1}(\hat{K}))^4), V_p=(Q_p(\hat{K}))^2$, then for the case $\Gamma_D=\emptyset$ (with appropriate filtration of rigid body modes), the method would have a zero inf–sup constant. In [9], it has been computationally shown that this choice gives a zero inf–sup constant even when $\Gamma_D\neq\emptyset$ and more than a single element is used.

4. Global *hp* stability and optimality

We assume we are given a sequence of meshes $\{\mathcal{T}_N\}$, consisting of triangles and parallelograms, and parameterized by N. For instance, these could be meshes with geometric refinement around points of singularities such as corners (see e.g. [6,8]). We also assume that there exist affine maps $\mathcal{F}_S$ which map the reference element K (respectively, T) onto $S\in\mathcal{T}_N$ if S is a parallelogram (respectively, triangle). Then the spaces Σ_N, V_N are defined by

$$\Sigma_N=\{\sigma\in\Sigma, \sigma_{ij}|_S\in\Sigma_{ps}\}, \quad V_N=\{v\in V, v_i|_S\in V_{ps}\}. \tag{4.1}$$

Here, for each S, the corresponding pair of spaces (Σ_{p_S}, V_{p_S}) is chosen to be one of (S1), (S2) or (S3), so that Theorem 3.1 is satisfied.

We assume the meshes $\{\mathscr{T}_N\}$ are uniformly shape regular, i.e.

$$\delta = \sup_N \max_{S \in \mathscr{T}_N} \delta_S < \infty. \tag{4.2}$$

Then we have the following stability theorem.

Theorem 4.1. *Let the meshes $\{\mathscr{T}_N\}$ be uniformly shape regular. Let $\{\Sigma_N, V_N\}$ be as in* (4.1). *Then* (2.4) *holds with β independent of N.*

Proof. Let $u \in V_N$. Using Theorem 3.1, define, for each $S \in \mathscr{T}_N, \sigma|_S \in \Sigma_{p_S}$ such that (3.3) holds. Using (4.2) and the fact that $p_S \geqslant 1$, this gives a $\sigma \in \Sigma_N$ which satisfies

$$(\sigma, \varepsilon(u))_{0,\Omega} \geqslant \frac{1}{(1+\gamma\sqrt{\delta})^2} \|\varepsilon(u)\|_{0,\Omega}^2 \text{ and } \|\sigma\|_{0,\Omega} \leqslant \|\varepsilon(u)\|_{0,\Omega}.$$

Eq. (2.4) can now be established by using Korn's inequality. □

Remark 4.1. We see that the stability constant β more precisely satisfies

$$\beta \geqslant (1 + \gamma p_N^{-1} \sqrt{\delta})^{-2},$$

where p_N is the minimum polynomial degree used over the elements in $\{\mathscr{T}_N\}$. For cases (S2) and (S3), $\gamma = 0$, so the method is stable with $\beta = 1$ even when very thin elements are used (e.g. at the boundary, to capture boundary layers). For (S1), the deterioration due to thin elements is still not serious in the p/hp versions, where p_N can be expected to become large (particularly in accordance with the prescription for capturing boundary layers, see e.g. [7]). In fact, $\beta \to 1$ as $p_N \to \infty$ for (S1).

The above theorem, together with the coercivity of the form $(E^{-1}\sigma, \tau)_0$ in (2.1) immediately gives the following optimality result by the usual theory of mixed methods (see e.g. [3,5]).

Theorem 4.2. *Let (σ, u) be the exact solution of* (2.1), (2.2) and $\{(\sigma^N, u^N)\}$ *a sequence of finite element approximations using the spaces $\{\Sigma_N, V_N\}$ described above. Then there exists a constant C independent of N such that*

$$\|\sigma - \sigma^N\|_{0,\Omega} + \|u - u^N\|_{1,\Omega} \leqslant C \left\{ \inf_{\tau \in \Sigma_N} \|\sigma - \tau\|_{0,\Omega} + \inf_{v \in V_N} \|u - v\|_{1,\Omega} \right\}. \tag{4.3}$$

Theorem 4.2 shows that optimal h and p convergence rates can be obtained when using the above elements. Moreover, exponential hp convergence can be obtained by using meshes that are properly refined around points of singularities [6,8]. (These often combine quadrilaterals and triangles, which is why we have carried through the case of triangular elements.) For some computational p version results obtained using these methods, both in the context of linear elasticity and an application to a viscoelasticity problem, we refer to [9,10].

5. Some extensions

In this section, we outline some extensions of our results, remarking only briefly on the arguments involved.

We first note that our method of proof carries over to the analogous primal method for the Poisson equation [3]: Find $(\sigma,u)\in\Sigma\times V=(L_2(\Omega))^2\times H^1_D(\Omega)$ satisfying, for all $(\tau,v)\in\Sigma\times V$,

$$(\sigma,\tau)_0-(\tau,\nabla u)_0=0, \tag{5.1}$$

$$-(\sigma,\nabla v)_0=-(f,v)_0+\int_{\Gamma_D} gv\,\mathrm{d}x. \tag{5.2}$$

Once again, the analogs of subspaces (S1)–(S3) lead to stability and optimality results similar to Theorems 4.1 and 4.2.

In [5] it is noted that the combination

$$\Sigma_p=Q_{p-1,p}(K)\times Q_{p,p-1}(K),\quad V_p=Q_p(K)$$

also satisfies the analogous inclusion property for this Poisson case, and is hence immediately stable. An analogous modification can be made for the components σ_{11} and σ_{22} in the HR case, though σ_{12} and σ_{21} would still have to belong to Q_p.

We also note that in [5], the case of curved elements for problems (5.1) and (5.2) is treated. The idea is based on the following definition of Σ_N, V_N which turns out to be equivalent to the one analogous to (4.1) when the mappings $\mathscr{F}_S$ are affine:

$$\Sigma_N=\{\sigma\in\Sigma, J_{\mathscr{F}_S}D\mathscr{F}_S^{-1}(\sigma|_S\circ\mathscr{F}_S)\in\hat{\Sigma}_{ps}\}, \tag{5.3}$$

$$V_N=\{v\in V, v|_S\circ\mathscr{F}_S\in\hat{V}_{ps}\}. \tag{5.4}$$

Here $J_{\mathscr{F}_S}$ is the Jacobian of the mapping $\mathscr{F}_S$ and $D\mathscr{F}_S^{-1}$ is the derivative of its inverse. The spaces $\hat{\Sigma}_{ps}, \hat{V}_{ps}$ contain the same choice of polynomials as Σ_{ps}, V_{ps}, but on the reference element.

When $\mathscr{F}_S$ is not affine, choosing polynomials in $\hat{\Sigma}_{ps}, \hat{V}_{ps}$ will lead to nonpolynomial basis functions in Σ_{ps}, V_{ps}. The stability proof can still be carried through, however, as indicated in [5] for the choice (S2) (and by extension (S3)). The idea is as follows. Given $w\in V_N$ we let $\hat{w}=w|_S\circ\mathscr{F}_S$ denote the image of $w|_S$ on the reference element $\hat{S}$. We now define

$$\tilde{\sigma}=\nabla\hat{w}, \tag{5.5}$$

which is clearly a polynomial on $\hat{S}$. Then we define $\sigma|_S$ to be the Piola transform of the polynomial $\tilde{\sigma}$,

$$\sigma|_S=\frac{1}{J_{\mathscr{F}_S}}D\mathscr{F}_S\tilde{\sigma}\circ\mathscr{F}_S^{-1}. \tag{5.6}$$

It may be seen by (5.3) that this elementwise definition yields a $\sigma\in\Sigma_N$. Moreover, a property of the Piola transform (see e.g. [5]), gives the key relation

$$(\sigma,\nabla w)_{0,S}=(\tilde{\sigma},\nabla\hat{w})_{0,\hat{S}}. \tag{5.7}$$

Eqs. (5.5) and (5.7) then yield an analog of Theorem 3.1 (and hence Theorem 4.1) provided suitable assumptions on the mappings and the mesh are satisfied.

This idea can easily be extended to the case (S1) as well. We simply modify (5.5) to

$$\tilde{\sigma} = \Pi_{p_S} \nabla \hat{w}.$$

Then defining $\sigma|_S$ by (5.6) again, the relation (5.7) will hold as before. This gives, by Remark 3.1,

$$(\sigma, \nabla w)_{0,S} = \|\Pi_{p_s} \nabla \hat{w}\|^2_{0,\hat{S}} \geqslant C \|\nabla \hat{w}\|^2_{0,\hat{S}},$$

from which the analog of Theorem 3.1 follows (again with suitable assumptions on $\mathscr{F}_S$).

We remark that a somewhat related Piola mapping idea was used to construct stable curvilinear elements in [4] for the dual elasticity formulation. Requirements on the mappings for h and p stability are discussed in that reference, from which analogs can be formulated for the HR problem as well. Let us point out, however, that even when the stability is not an issue, there can be a degradation of the h approximability properties of the spaces when general quadrilateral elements (rather than rectangles or parallelograms) are used. See [1] for more details.

For the HR formulation, the above idea (used component-wise) will once more lead to stability over curved (S1) elements. However, using spaces of form (5.3) will now lead to a nonsymmetric σ, necessitating the use of a Σ_N which allows the possibility of $\sigma_{12} \neq \sigma_{21}$. (This is why, for the symmetric spaces considered here, we were not able to use the Piola transform to give a simpler proof of local stability in Section 3.) We remark that in [9,10], the (S1) spaces used were the usual mapped symmetric spaces, defined by

$$\Sigma_N = \{\sigma \in \Sigma, \sigma|_S \circ \mathscr{F}_S \in \hat{\Sigma}_{p_S}\}. \tag{5.8}$$

These were shown to work well numerically for the p version, so that the Piola-transformed non-symmetric spaces (of form (5.3)) may not be needed in practice. Similar numerical results were observed for the dual formulation in (see [4]), where it was shown that a certain class of curvilinear spaces of form (5.8) worked as well as those of form (5.3) (both for the p and h versions).

References

[1] D. Arnold, D. Boffi, R. Falk, Approximation by quadrilateral finite elements, Math. Comp. 71 (2002) 909–922.

[2] I. Babuška, M. Suri, Locking effects in the finite element approximation of elasticity problems, Numer. Math. 62 (1992) 439–463.

[3] D. Braess, Finite Elements, Cambridge University Press, Cambridge, 2001.

[4] L. Chilton, M. Suri, On the construction of stable curvilinear p version elements for mixed formulations of elasticity and Stokes flow, Numer. Math. 86 (2000) 29–48.

[5] J. Roberts, J.-M. Thomas, Mixed and hybrid methods, In: P.G. Ciarlet, J.L. Lions (Eds.), Handbook of Numerical Analysis, Vol. II, Finite Element Methods (Part 1), Elsevier Science Publishers B.V, North-Holland, Amsterdam, 1991, pp. 523–639.

[6] C. Schwab, *p*- and *hp*-Finite Element Methods, Clarendon Press, Oxford, 1998.

[7] C. Schwab, M. Suri, C. Xenophontos, The *hp* finite element method for problems in mechanics with boundary layers, Comput. Methods Appl. Mech. Engrg. 157 (1998) 311–333.

[8] B. Szabó, I. Babuška, Finite Element Analysis, Wiley, New York, 1991.

[9] L. Zhang, Solution of viscoelastic problems by the mixed p-version of the finite element method. Ph.D. Thesis, Washington University, St. Louis, MO, December 2000.

[10] L. Zhang, B. Szabó, Solution of viscoelastic problems by the mixed p-version of the finite element method, Internat. J. Numer. Methods Engrg., submitted for publication.