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Hecke Operators and an Identity for the Dedekind Sums

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If $h, k \in Z$, $k > 0$, the Dedekind sum is given by

$$s(h, k) = \sum_{\mu=1}^{k} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right),$$

with

$$\begin{aligned} ((x)) &= x - [x] - \tfrac{1}{2}, & x \notin Z, \\ &= 0, & x \in Z. \end{aligned}$$

The Hecke operators T_n for the full modular group $SL(2, Z)$ are applied to log $\eta(\tau)$ to derive the identities ($n \in Z^+$)

$$\sum_{\substack{ad=n \\ d>0}} \sum_{b \pmod d} s(ah + bk, dk) = \sigma(n)\, s(h, k), \tag{*}$$

where $(h, k) = 1$, $k > 0$ and $\sigma(n)$ is the sum of the positive divisors of n. Petersson had earlier proved (*) under the additional assumption $k \equiv 0$, $h \equiv 1 \pmod n$. Dedekind himself proved (*) when n is prime.

1. INTRODUCTION

If $h, k \in Z$ with $k > 0$, the *Dedekind sum* $s(h, k)$ is defined by

$$s(h, k) = \sum_{\mu=1}^{k} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right),$$

where for real x,

$$\begin{aligned} ((x)) &= x - [x] - \tfrac{1}{2}, & x \notin Z, \\ &= 0, & x \in Z. \end{aligned}$$

These sums have been studied extensively since 1892, the year of their introduction by Dedekind in his commentary on Riemann [3], yet they continue

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to draw the serious attention of a good many mathematicians. Detailed expositions of their many fascinating properties can be found in [6, 7]. Two simple properties we shall require are

$$\begin{aligned} &\text{(a)}\quad s(h+k,k)=s(h,k) \\ &\text{(b)}\quad s(qh,qk)=s(h,k), \qquad q\in Z^{+}; \end{aligned} \tag{1.1}$$

(a) is immediate from the definition of $s(h, k)$, while (b) occurs as formula (23) of [3] and as Theorem 1 in [7].

In a recent letter to Emil Grosswald, Hans Petersson stated (without proof)[1] the following identity, which appears to be new:

$$\sum_{\substack{ad=n\\ d>0}} \sum_{b \pmod d} s(ah+bk, dk) = \sigma(n)\, s(h,k), \tag{1.2}$$

where k and h are relatively prime integers with $k > 0$, $k \equiv 0$, $h \equiv 1 \pmod n$ and $\sigma(n)$ is the sum of the positive divisors of the positive integer n. On the other hand, Dedekind's original work contains the following similar identity [3, formula (28)] (see also [7, Theorem 2]):

$$s(ph,k) + \sum_{m=0}^{p-1} s(h+mk, ph) = (p+1)\, s(h,k), \tag{1.3}$$

where p is a prime, but with no congruence condition imposed upon the integers k and h. It is a simple matter to verify that (1.2) reduces to (1.3) when n is a prime p. As Grosswald pointed out to me when he received the letter, the summation conditions in (1.2) suggest the Hecke operators T_n for the modular group. (Definition to be given in Section 2.)

My purpose in this note is to explicate the connection with the Hecke operators sufficiently to obtain a proof of (1.2), but without the congruence conditions on k and h, that is, for arbitrary relatively prime integers k and h, with $k > 0$. (The assumption $(h, k) = 1$ can also be removed by application of (1.1b) to (1.2).) There is undoubtedly an identity similar to (1.3) satisfied by sums recently introduced by Berndt in connection with the transformation formulas of $\log \theta(\tau)$ [2, p. 339].

2. Idea of the Proof

The proof depends upon the occurrence of the Dedekind sums in the transformation formulas ((2.3) below) of $\log \eta(\tau)$, the link which led Dedekind

[1] See, however, note 3.

to introduce them initially. Here $\eta(\tau)$ is the well-known modular form of weight $\frac{1}{2}$, defined for $\operatorname{Im} \tau > 0$ by

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}). \tag{2.1}$$

Since $\eta(\tau)$ has no zeros in $\operatorname{Im} \tau > 0$, it has a single-valued logarithm, uniquely defined by

$$\log \eta(\tau) = \frac{\pi i\tau}{12} - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} e^{2\pi i m n\tau}, \qquad \operatorname{Im} \tau > 0. \tag{2.2}$$

The transformation property central to the theory of $\log \eta(\tau)$ is

$$\log \eta \left(\frac{\alpha\tau + \beta}{k\tau + h}\right) = \log \eta(\tau) + \frac{1}{2} \log(-i(k\tau + h)) + \frac{\pi i}{12k} (h + \alpha) - i\pi\, s(h, k), \tag{2.3}$$

for $\alpha, \beta, k, h \in Z$ such that $\alpha h - \beta k = 1$ and $k > 0$. (In particular, (2.3) requires that $(h, k) = 1$.)

It is a familiar fact that (2.3) may be used to derive rather simply the "reciprocity formula"

$$s(h, k) + s(k, h) = -\frac{1}{4} + \frac{1}{12}\left(\frac{h}{k} + \frac{1}{hk} + \frac{k}{h}\right), \quad \text{for } h > 0, \; k > 0, \; (h, k) = 1, \tag{2.4}$$

as Dedekind in fact did [3, formula (19)]. (Chapter 2 of [6] is devoted to various proofs of (2.4) that do not depend upon (2.3).) In much the same way (1.2) will follow from (2.3) if we can show in addition that $\log \eta(\tau)$ behaves decently under application of the Hecke operator T_n, defined for $n \in Z^+$ by[2]

$$T_n f(\tau) = \sum_{\substack{ad=n \\ d>0}} \sum_{b \pmod d} f\left(\frac{a\tau + b}{d}\right). \tag{2.5}$$

(See [1, 4, 5] for a discussion of the operators T_n.) In fact, Dedekind himself observed that

$$T_p \log \eta(\tau) = (p + 1) \log \eta(\tau) + \frac{(p - 1)\pi i}{24}, \tag{2.6}$$

[2] If $f(\tau + 1) = f(\tau)$, then $T_n f$ does not depend upon the choice of residue system $b \pmod d$. However, since $\log \eta$ is not quite periodic ($\log \eta(\tau + 1) = \log \eta(\tau) + \pi i/12$, as one sees from (2.2)), $T_n \log \eta$ is independent of the choice only up to an additive constant. Here, in application of T_n to $\log \eta$, $b \pmod d$ always means $0 \leqslant b \leqslant d - 1$.

for p a prime [3, formula (27)] and used this to derive (1.3), not by applying (2.3) directly, but rather from the boundary behavior of $\log \eta(\tau)$ near rational points. From (2.6) it follows by the multiplicative properties of the T_n that $\log \eta(\tau)$ is "virtually" an eigenfunction of all the T_n, with eigenvalues $\sigma(n)$, in the sense that

$$T_n \log \eta(\tau) = \sigma(n) \log \eta(\tau) + K_n , \qquad n \in Z^+, \tag{2.7}$$

where K_n is a constant.[3]

In section 3 we give the proof of (2.6). In Section 4 we combine (2.7) with (2.3) to derive (1.2).

3. Proof of (2.6)

When $n = p$, a prime, (2.5) simplifies to $T_p f(\tau) = f(p\tau) + \sum_{b=0}^{p-1} f((\tau + b)/p)$, and from (2.2) we have

$$\begin{aligned} T_p \log \eta(\tau) = \frac{\pi i p \tau}{12} + \sum_{b=0}^{p-1} \frac{\pi i}{12p} (\tau + b) - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} e^{2\pi i m n p \tau} \\ - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} e^{2\pi i m n \tau/p} \sum_{b=0}^{p-1} e^{2\pi i m n b/p}. \end{aligned} \tag{3.1}$$

Now,

$$\begin{aligned} \frac{\pi i p \tau}{12} + \sum_{b=0}^{p-1} \frac{\pi i}{12p} (\tau + b) &= \frac{\pi i p \tau}{12} + \frac{\pi i \tau}{12} + \frac{\pi i}{12p} \frac{(p-1)p}{2} \\ &= (p+1) \frac{\pi i \tau}{12} + \frac{\pi i}{24} (p-1). \end{aligned} \tag{3.2}$$

Further,

$$\begin{aligned} \sum_{b=0}^{p-1} e^{2\pi i m n b/p} &= 0, \qquad \text{if} \quad p \nmid mn, \\ &= p, \qquad \text{if} \quad p \mid mn, \end{aligned}$$

[3] In a letter I received from Professor Petersson in response to my description of the contents of this note, he pointed out that (2.7) is proved, with an explicit constant K_n, in his paper "Über die Primformen der Hauptkongruenzgruppen," *Abh. Math. Sem. Univ. Hamburg* **38** (1972), 21–22 (especially formula (2.12)). He goes on to say, "of course, I derived my formula on Dedekind sums [i.e., (1.2)] from the above identities on $\log \eta$." It is worth noting that T_n is multiplicative in n, but not totally multiplicative. Indeed, for $n, m \in Z^+$, $T_m T_n = \sum_{d \mid (m,n)} d T_{mn/d^2}$, when these operators are applied to a periodic function f. Since $\log \eta$ is not actually periodic, this multiplicative relation must be modified by an additive constant when the operators are applied to $\log \eta$. That is, (*) $T_m T_n \log \eta(\tau) = \sum_{d \mid (m,n)} d T_{mn/d^2} \log \eta(\tau) + \gamma(m, n)$, where γ is independent of τ. That $\sigma(n)$ satisfies the same multiplicative identity $\sigma(m)\sigma(n) = \sum_{d \mid (m,n)} d\sigma(mn/d^2)$ makes possible the derivation of (2.7) from (*) and (2.6).

so that the second double sum in (3.1) equals

$$\sum_{\substack{n=1 \\ p|mn}}^{\infty} \sum_{m=1}^{\infty} \frac{p}{m} e^{2\pi i m n \tau/p} = \sum_{\substack{m,n \\ p|m}} + \sum_{\substack{m,n \\ p|n}} - \sum_{\substack{m,n \\ p|m, p|n}}, \tag{3.3}$$

with an abbreviation of notation. We have for the three sums in (3.3),

$$\sum_{\substack{m,n \\ p|m}} \frac{p}{m} e^{2\pi i m n \tau/p} = \sum_{m',n} \frac{1}{m'} e^{2\pi i m' n \tau},$$

$$\sum_{\substack{m,n \\ p|n}} \frac{p}{m} e^{2\pi i m n \tau/p} = p \sum_{m,n'} \frac{1}{m} e^{2\pi i m n' \tau}$$

and

$$\sum_{\substack{m,n \\ p|m, p|n}} \frac{p}{m} e^{2\pi i m n \tau/p} = \sum_{m',n'} \frac{1}{m'} e^{2\pi i m' n' p \tau},$$

where we have put $m = m'p$, $n = n'p$ when appropriate. From (3.1), (3.2), and (3.3) we now conclude that

$$\begin{aligned} T_p \log \eta(\tau) &= \frac{\pi i}{24}(p-1) + (p+1)\frac{\pi i \tau}{12} - \sum_{m,n} \frac{1}{m} e^{2\pi i m n p \tau} \\ &\quad - \sum_{m,n} \frac{1}{m} e^{2\pi i m n \tau} - p \sum_{m,n} \frac{1}{m} e^{2\pi i m n \tau} + \sum_{m,n} \frac{1}{m} e^{2\pi i m n p \tau} \\ &= \frac{\pi i}{24}(p-1) + (p+1)\frac{\pi i \tau}{12} - (p+1) \sum_{m,n} \frac{1}{m} e^{2\pi i m n \tau} \\ &= (p+1) \log \eta(\tau) + \frac{\pi i}{24}(p-1). \end{aligned}$$

This proves (2.6) and therefore (2.7) as well.

4. Proof of (1.2)

From (2.7) it follows that $T_n \log \eta(\tau)$ and $\sigma(n) \log \eta(\tau)$ have identical transformation properties under modular substitutions $\tau \to (\alpha\tau + \beta)/(k\tau + h)$, $\operatorname{Im} \tau > 0$ ($\alpha, \beta, k, h \in Z$, $k > 0$, and $\alpha h - \beta k = 1$). From (2.3) we have

$$\begin{aligned} &\sigma(n) \log \eta\left(\frac{\alpha\tau + \beta}{k\tau + h}\right) - \sigma(n) \log \eta(\tau) \\ &\quad = \frac{\sigma(n)}{2} \log(-i(k\tau + h)) + \frac{\pi i \sigma(n)}{12k}(h + \alpha) - \pi i \sigma(n)\, s(h, k). \end{aligned} \tag{4.1}$$

It remains to find a suitable form of the transformation formula for $T_n \log \eta(\tau)$. This can be done by applying (2.3) together with the fact that for $V = \left(\begin{smallmatrix} \alpha & \beta \\ k & h \end{smallmatrix}\right)$, $M = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with integer entries such that $\alpha h - \beta k = 1$, $ad = n$, $d > 0$, and $0 \leqslant b \leqslant d - 1$, there exist $V' = \left(\begin{smallmatrix} \alpha' & \beta' \\ k' & h' \end{smallmatrix}\right)$, $M' = \left(\begin{smallmatrix} a' & b' \\ 0 & d' \end{smallmatrix}\right)$, again with integer entries and $\alpha' h' - \beta' k' = 1$, $a'd' = n$, $d' > 0$, $0 \leqslant b' \leqslant d' - 1$, such that $MV = V'M'$. Furthermore, as the $M = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ run through all matrices subject to the conditions $ad = n$, $d > 0$, $0 \leqslant b \leqslant d - 1$, the M' do as well. (See [1, pp. 122–125], for example.) Straightforward calculation shows that

$$\alpha' = \frac{d'}{n}(a\alpha + bk), \qquad k' = \frac{dd'k}{n}, \qquad h' = \frac{a'\,dh - b'\,dk}{n} \tag{4.2}$$

and consequently,

$$(k'M'\tau + h')\,d' = (k\tau + h)d. \tag{4.3}$$

By (2.5),

$$T_n \log \eta(\tau) = \sum_{\substack{ad=n \\ d>0}} \sum_{b (\mathrm{mod}\, d)} \log \eta\left(\frac{a\tau + b}{d}\right).$$

In a typical term replace τ by $V\tau = (\alpha\tau + \beta)/(k\tau + h)$, $\alpha h - \beta k = 1$, $k > 0$. This gives, by (2.3), (4.2), and (4.3),

$$\begin{aligned}
\log(MV\tau) &= \log \eta(V'M'\tau) = \log\left(\frac{\alpha' M'\tau + \beta'}{k'M'\tau + h'}\right) \\
&= \log \eta(M'\tau) + \frac{1}{2}\log(-i(k'M'\tau + h')) \\
&\quad + \frac{\pi i}{12k'}(h' + \alpha') - i\pi\, s(h', k') \\
&= \log \eta(M'\tau) + \frac{1}{2}\log(-i(k\tau + h)\, d/d') \\
&\quad + \frac{\pi i n}{12dd'k}\left(\frac{a'\,dh - b'\,dk}{n} + \frac{a\,d'\alpha + b\,d'k}{n}\right) - i\pi\, s(h', k').
\end{aligned}$$

Thus,

$$\begin{aligned}
\log \eta(MV\tau) - \log \eta(M'\tau) &= \frac{1}{2}\log(-i(k\tau + h)) + \frac{1}{2}(\log d - \log d') \\
&\quad + \frac{\pi i}{12}\left(\frac{a'h}{d'k} + \frac{a\alpha}{dk} + \frac{b}{d} - \frac{b'}{d'}\right) - i\pi\, s(h', k').
\end{aligned} \tag{4.4}$$

Now sum over the set of matrices M. Since the M' are simply a permutation of the M and since the matrices M are $\sigma(n)$ in number, we have

$$\sum \log \eta(M'\tau) = T_n \log \eta(\tau), \qquad \frac{1}{2} \sum (\log d - \log d') = 0,$$

$$\frac{\pi i}{12} \sum \left(\frac{b}{d} - \frac{b'}{d'}\right) = 0, \qquad \frac{\pi i}{12} \sum \frac{a'h}{d'k} = \frac{\pi i}{12} \sum \frac{ah}{dk},$$

where the range of each summation is $ad = n$, $d > 0$, $0 \leqslant b \leqslant d - 1$. It then follows from (4.4) that

$$\begin{aligned} &T_n \log \eta(V\tau) - T_n \log \eta(\tau) \\ &\quad = \frac{\sigma(n)}{2} \log(-i(k\tau + h)) + \frac{\pi i(h + \alpha)}{12k} \sum \frac{a}{d} - \pi i \sum s(h', k'). \end{aligned} \tag{4.5}$$

Now

$$\sum \frac{a}{d} = \sum_{\substack{ad=n \\ d>0}} \sum_{b (\mathrm{mod}\ d)} \frac{a}{d} = \sum_{\substack{ad=n \\ d>0}} a = \sigma(n)$$

and

$$\begin{aligned} \sum s(h', k') &= \sum s\left(\frac{a'\, dh - b'\, dk}{n}, \frac{dd'k}{n}\right) = \sum s(a'\, dh - b'\, dk, dd'k) \\ &= \sum s(a'h - b'k, d'k) = \sum s(ah - bk, dk) \\ &= \sum s(ah + (d - b)\, k, dk) = \sum s(ah + bk, dk), \end{aligned}$$

where we have used (1.1). Then, (4.5) takes the form

$$\begin{aligned} &T_n \log \eta(V\tau) - T_n \log \eta(\tau) \\ &\quad = \frac{\sigma(n)}{2} \log(-i(k\tau + h)) + \frac{\pi i \sigma(n)}{12k} (h + \alpha) - \pi i \sum s(ah + bk, dk). \end{aligned} \tag{4.6}$$

Application of (2.7) yields (1.2) through comparison of (4.6) with (4.1).

References

1. T. Apostol, "Modular Functions and Dirichlet Series in Number Theory," Springer-Verlag, New York, 1976.
2. B. Berndt, Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan, *J. Reine Angew. Math.* **303** (1978), 332–365.
3. R. Dedekind, "Gesammelte mathematische Werke" (R. Fricke, E. Noether, and O. Ore Eds.), Vol. 1, pp. 159–173, Friedrich Vieweg und Sohn, Braunschweig, 1930.

4. R. C. GUNNING, "Lectures on Modular Forms," Annals of Mathematics Studies, No. 48, Princeton Univ. Press, Princeton, N.J., 1962.
5. J. LEHNER, "A Short Course in Automorphic Functions," Holt, Rinehart & Winston, New York, 1966.
6. H. RADEMACHER AND E. GROSSWALD, "Dedekind Sums," Carus Mathematical Monographs, No. 16, Math. Assoc. Amer., Washington, D.C., 1972.
7. H. RADEMACHER AND A. WHITEMAN, Theorems on Dedekind sums, *Amer. J. Math.* **63** (1941), 377–407.