Tate cohomology and Gorensteinness for triangulated categories

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Abstract

Motivated by the classical structure of Tate cohomology, we develop and study a Tate cohomology theory in a triangulated category \( \mathcal{C} \). Let \( \mathcal{E} \) be a proper class of triangles. By using \( \mathcal{E} \)-projective, as well as \( \mathcal{E} \)-injective objects, we give two alternative approaches to this theory that, in general, are not equivalent. So, in the second part of the paper, we study triangulated categories in which these two theories are equivalent. This leads us to study the categories in which all objects have finite \( \mathcal{E} \)-G-projective as well as finite \( \mathcal{E} \)-G-injective dimension. These categories will be called \( \mathcal{E} \)-Gorenstein triangulated categories. We give a characterization of these categories in terms of the finiteness of two invariants: \( \mathcal{E} \)-sylp\( \mathcal{C} \), the supremum of the \( \mathcal{E} \)-injective dimension of \( \mathcal{E} \)-projective objects of \( \mathcal{C} \) and \( \mathcal{E} \)-spli\( \mathcal{C} \), the supremum of the \( \mathcal{E} \)-projective dimension of \( \mathcal{E} \)-injective objects of \( \mathcal{C} \), where finiteness of each of these invariants for a category implies the finiteness of the other. Finally, we show that over \( \mathcal{E} \)-Gorenstein triangulated categories, the class of objects of finite \( \mathcal{E} \)-projective dimension and the class of \( \mathcal{E} \)-G-injective objects form an \( \mathcal{E} \)-complete cotorsion theory.

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1. Introduction

Let \( \mathcal{C} \) be a triangulated category with triangulation \( \Delta \). Beligiannis (in [Be]) developed a homological algebra in \( \mathcal{C} \) which parallels the homological algebra in an exact category in the sense of Quillen. He did this by specifying a class of triangles \( \mathcal{E} \subseteq \Delta \) which is closed under translations and satisfies the analogous formal properties of a proper class of short exact sequences. Such a class of triangles is called a proper class of triangles. By fixing a proper class of triangles \( \mathcal{E} \), he defined projective and injective (co)resolutions and hence projective and injective dimensions.

In an attempt to extend the theory, in [AS] we introduced and studied Gorenstein projective and Gorenstein injective objects, and hence Gorenstein projective and Gorenstein injective dimensions of objects. They are defined by modifying what Enochs and Jenda have done in abelian categories [EJ]. Instead of looking at projective resolutions, one looks at resolutions by objects which can be resolved by projectives in both the negative and the positive directions. These objects are called \( \mathcal{E} \)-Gprojective objects. It is shown that the subcategory of \( \mathcal{E} \)-Gprojective objects of \( \mathcal{C} \), \( \mathcal{GP}(\mathcal{E}) \), is full, additive, closed under isomorphisms, direct summands and \( \Sigma \)-stable, i.e. \( \Sigma(\mathcal{GP}(\mathcal{E})) = \mathcal{GP}(\mathcal{E}) \). Moreover there is an inclusion of categories \( \mathcal{P}(\mathcal{E}) \subseteq \mathcal{GP}(\mathcal{E}) \) where \( \mathcal{P}(\mathcal{E}) \) denotes the full subcategory of \( \mathcal{E} \)-projective objects of \( \mathcal{C} \). Dually \( \mathcal{E} \)-Ginjective objects are defined and it is shown that the full subcategory of \( \mathcal{E} \)-Ginjective objects, denoted \( \mathcal{GI}(\mathcal{E}) \), has properties dual to those of \( \mathcal{GP}(\mathcal{E}) \).

In this paper we first attempt to develop a homology theory in a triangulated category \( \mathcal{C} \) that is motivated by the properties of Tate–Farrell cohomology in the category of groups. This will be done for objects of finite \( \mathcal{E} \)-Gorenstein projective dimension. We show that this theory not only shares basic properties with ordinary cohomology, but also enjoys some distinctive features.

Our next aim is to study triangulated categories in which all objects has finite \( \mathcal{E} \)-Gprojective as well as finite \( \mathcal{E} \)-Ginjective dimension. We call them \( \mathcal{E} \)-Gorenstein triangulated categories. Their definition is motivated by the structure of module categories over Gorenstein rings. They exists naturally. For example, if the stable category of a triangulated category \( \mathcal{C} \) with enough \( \mathcal{E} \)-projectives modulo the full subcategory \( \mathcal{P}(\mathcal{E}) \) of \( \mathcal{E} \)-projective objects is triangulated, then \( \mathcal{C} \) is \( \mathcal{E} \)-Gorenstein.

We now outline the results of the paper. In Section 2 we summarize some preliminaries and basic facts about triangulated categories which will be used throughout the paper. At the end of this section, we study the \( \mathcal{E} \)-derived functors of the bifunctor \( \mathcal{C}(\cdot, \cdot) \). For any proper class of triangles with enough projectives or in any situation where one can derive functors like \( \mathcal{C}(\cdot, B) \), there exists a natural map from the space \( \mathcal{P}(A, B) \) to the zeroth relative Ext group but in distinction from the situation with the usual \( \text{Ext}^0 \) in abelian categories, it is rarely invertible. In fact its kernel and image play a fundamental rôles in the theory since they control the space of phantom maps, see [Be, 4.22]. Keeping this fact in mind, we treat some situations in which the zeroth ‘Ext’ group is isomorphic to the ‘Hom’ space. We will need these observations to prove our main results in the next section.

In Section 3, using the notion of complete \( \mathcal{E} \)-projective resolutions, we introduce and study a cohomology theory, the so-called \( \mathcal{E} \)-Tate cohomology, denoted \( \hat{\text{Ext}}^*_\mathcal{P}(\cdot, \cdot) \). Its structure is motivated by the construction of Tate cohomology groups in the homological theory of module categories. It is easily seen to be a covariant functor of the second argu-
We prove that it is also a contravariant functor in the first variable. As in the case of modules, we show that this theory is rigid, in the sense that the vanishing of any one of these functors implies the vanishing of all of them. And then we show that vanishing characterizes objects of finite $\mathcal{E}$-projective dimension. There is another version of Tate cohomology theory, denoted $\hat{\mathcal{E}}\text{Ext}^*_I(-,-)$, using complete $\mathcal{E}$-injective resolutions. We only review the definitions and properties of this theory without going into details.

In the last section, we use the techniques developed in the earlier sections of the paper to study $\mathcal{E}$-Gorenstein triangulated categories. To any triangulated category $C$, we associate two invariants: $\text{silp} C$, the supremum of the $\mathcal{E}$-injective dimension of $\mathcal{E}$-projective objects of $C$ and $\text{spli} C$, the supremum of the $\mathcal{E}$-projective dimension of $\mathcal{E}$-injective objects of $C$. These invariants are motivated by Gedrich and Gruenberg’s invariants of a ring, $\text{silp} R$ and $\text{spli} R$, see [GG]. We show that over $\mathcal{E}$-Gorenstein triangulated categories these invariants are finite. In fact, if for a category $C$ we know that either $\mathcal{E}$-projective dimension or that the $\mathcal{E}$-injective dimension of all objects are finite, we may deduce that $\text{spli} C$ and $\text{silp} C$ both are finite and are equal. We show that the converse also hold, if we have an extra assumption. Finally, it is shown that over $\mathcal{E}$-Gorenstein triangulated categories, $(\tilde{P}(\mathcal{E}), \mathcal{G}I(\mathcal{E}))$ form an $\mathcal{E}$-complete cotorsion theory, where $\tilde{P}(\mathcal{E})$ denotes the full subcategory of $\mathcal{C}$ whose objects has finite $\mathcal{E}$-projective dimension. By a cotorsion theory we mean a pair $(D, \mathcal{E})$ of classes of objects of $\mathcal{C}$ each of which is the orthogonal complement of the other with respect to the Ext functors.

As usual the composition of morphisms $f:X \to Y$ and $g:Y \to Z$ in a given category $K$ is denoted by $fg$.

### 2. Notations, definitions and preliminary results

In this section we recall basic definitions and properties of triangulated categories used throughout the paper. For the triangulated and derived categories the reader is referred to the original article of Verdier [Ve] and Hartshorne’s notes [Ha], and further to excellent modern accounts: Gelfand and Manin’s book [GM] and Neeman’s book on triangulated categories [N]. For terminology we shall follow [Be] and [AS].

Throughout the paper we fix a triangulated category $C = (C, \Sigma, \Delta)$, where $C$ is an additive category, $\Sigma$ is the suspension functor, i.e. an autoequivalence of $C$, and $\Delta$ is the triangulation. In [Be, 2.1], some equivalent formulations for the Octahedral axiom are given. We use these equivalent conditions instead of the Octahedral axiom, when it is more convenient.

A triangle $(T): A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is called split if it is isomorphic to the triangle

$$A \xrightarrow{(1, 0)} A \oplus C \xrightarrow{0} C \xrightarrow{0} \Sigma A.$$  

It is easy to see that $(T)$ is split if and only if $f$ has a retraction or $g$ has a section or $h = 0$. The full subcategory of $\Delta$ consisting of the split triangles will be denoted by $\Delta_0$. The following seems to be well known.
Lemma 2.1. If a triangle \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \) is split, then \( B \cong A \oplus C \).

Following definitions are quoted verbatim from [Be, 2.2]. A class of triangles \( \mathcal{E} \) is closed under base change if for any triangle \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \mathcal{E} \) and any morphism \( \varepsilon : E \to C \) as in (i) of Proposition 2.1 of [Be], the triangle \( A \xrightarrow{f} G \xrightarrow{g'} E \xrightarrow{h'} \Sigma A \) belongs to \( \mathcal{E} \). Dually a class of triangles is closed under cobarbase change if for any triangle \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \mathcal{E} \) and any morphism \( \alpha : A \to D \) as in (ii) of Proposition 2.1 of [Be], the triangle \( D \xrightarrow{f'} F \xrightarrow{g'} C \xrightarrow{h'} \Sigma D \) belongs to \( \mathcal{E} \). A class of triangles is closed under suspension if for any triangle \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \mathcal{E} \) and any integer \( i \in \mathbb{Z} \), the triangle \( \Sigma^i A \xrightarrow{(-1)^i \Sigma^i f} \Sigma^i B \xrightarrow{(-1)^i \Sigma^i g} \Sigma^i C \xrightarrow{(-1)^i \Sigma^i h} \Sigma^{i+1} A \) is in \( \mathcal{E} \). A class of triangles \( \mathcal{E} \) is called saturated if in the situation of base change in Proposition 2.1 of [Be], whenever the third vertical and the second horizontal triangles are in \( \mathcal{E} \), then the triangle \( A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \mathcal{E} \).

Definition 2.2. (see [Be, 2.2]) A full subcategory \( \mathcal{E} \subseteq \text{Diag} (\mathcal{C}, \Sigma) \) is called a proper class of triangles if the following conditions hold.

(i) \( \mathcal{E} \) is closed under isomorphisms, finite coproducts and \( \Delta_0 \subseteq \mathcal{E} \subseteq \Delta \).
(ii) \( \mathcal{E} \) is closed under suspensions and is saturated.
(iii) \( \mathcal{E} \) is closed under base and cobarbase change.

It is known that \( \Delta_0 \) and the class of all triangles \( \Delta \) in \( \mathcal{E} \) are proper classes of triangles. If \( \{ \mathcal{E}_i : i \in I \} \) is a family of proper classes of triangles, then \( \bigcap_{i \in I} \mathcal{E}_i \) is a proper class of triangles. If \( \{ \mathcal{E}_i : i \in I \} \) is an increasing chain of such triangles, then \( \bigcup_{i \in I} \mathcal{E}_i \) is proper. Moreover, if \( U : \mathcal{C} \to \mathcal{D} \) is an exact functor of triangulated categories and \( \mathcal{E} \) is a proper class of triangles in \( \mathcal{D} \), then \( U^{-1} (\mathcal{E}) \) is a proper class of triangles in \( \mathcal{C} \). The family of proper classes of triangles in \( \mathcal{C} \) is a big poset with \( 0 \) (the class \( \Delta_0 \)) and \( 1 \) (the class \( \Delta \)), defining \( \mathcal{E}_1 \preceq \mathcal{E}_2 \iff \mathcal{E}_1 \subseteq \mathcal{E}_2 \).

In general in any triangulated category one gets, in a natural fashion, a proper class of triangles induced by a homological or cohomological functor. More precisely, let \( F : \mathcal{C} \to \mathcal{U} \) be a (co)homological functor from \( \mathcal{C} \) to an abelian category \( \mathcal{U} \). Then we can get a proper class of triangles \( \mathcal{E}(F) \) in \( \mathcal{C} \) to be the class of all triangles \( A \to B \to C \to \Sigma A \) such that for all \( i \in \mathbb{Z} \), the induced sequence \( 0 \to F^i (A) \to F^i (B) \to F^i (C) \to 0 \) is exact in \( \mathcal{U} \), where \( F^i = F \Sigma^i \), see [Be, 2.3].

Throughout the paper we fix a proper class of triangles \( \mathcal{E} \) in the triangulated category \( \mathcal{C} \).

Definition 2.3. (see [Be, 4.1]) An object \( P \in \mathcal{C} \) (respectively \( I \in \mathcal{C} \)) is called \( \mathcal{E} \)-projective (respectively \( \mathcal{E} \)-injective) if for any triangle \( A \to B \to C \to \Sigma A \) in \( \mathcal{E} \), the induced sequence

\[
0 \to \mathcal{C}(P, A) \to \mathcal{C}(P, B) \to \mathcal{C}(P, C) \to 0,
\]

(respectively \( 0 \to \mathcal{C}(C, I) \to \mathcal{C}(B, I) \to \mathcal{C}(A, I) \to 0 \))

is exact in the category \( \text{Ab} \) of abelian groups.
The symbol $P(E)$ (respectively $I(E)$) denotes the full subcategory of $E$-projective (respectively $E$-injective) objects of $C$. It follows easily from the definition that the subcategories $P(E)$ and $I(E)$ are full, additive, closed under isomorphisms, direct summands and $\Sigma$-stable.

$C$ is said to have enough $E$-projectives if for any object $A \in C$ there exists a triangle $K \to P \to A \to \Sigma K$ in $E$ with $P \in P(E)$. Dually we say that $C$ has enough $E$-injectives if for any object $A \in C$ there exists a triangle $A \to I \to L \to \Sigma A$ in $E$ with $I \in I(E)$.

In general it is not so easy to find a proper class $E$ of triangles in a triangulated category having enough $E$-projectives or $E$-injectives. Here we list some nontrivial examples which are of great interest. We thank Apostolos Beligiannis for these examples.

(1) Take a compactly generated triangulated category $C$. Then the class of pure triangles (which is induced by the compact objects) is proper and $C$ has enough projectives and enough injectives, see [Be, Section 11].

(2) Take $C$ to be the unbounded homotopy category of complexes of objects from a Grothendieck category. Then the so-called Cartan–Eilenberg injective complexes (or homotopically injective complexes) form the relative injective objects for a proper class of triangles in $C$. If the Grothendieck category has enough projectives, then the dual holds, see [Be, Sections 12.4 and 12.5].

The example in (2) in some sense gives the absolute homological algebra. This is in contrast to (1) which gives the pure version. The following is a generalization of (1).

(3) Let $C$ be a triangulated category which admits infinite coproducts, and let $\mathcal{X}$ be a full subcategory of $C$ which is closed under shifts and contains only a set of isomorphism classes of objects. Then $\mathcal{X}$ induces a proper class $E$ of triangles in $C$ and $C$ has enough $E$-projectives; in addition $C$ has enough $E$-injectives in case $\mathcal{X}$ generates $C$ and in case $\mathcal{X}$ consists of compact objects, see [Be, Section 8].

(4) One gets further examples by suitably generalizing the situation in example (2), i.e. starting not with injectives or projectives in a Grothendieck category, but with a full subcategory which is the additive closure of a set of objects.

Let $K \to P \to A \to \Sigma K$ be a triangle in $E$ with $P$ $E$-projective. Then $K$ is called a first $E$-syzygy of $A$. An $n$th $E$-syzygy of $A$ is defined as usual by induction. By Schanuel’s lemma [Be, 4.4] any two $E$-syzygies of $A$ are isomorphic modulo $E$-projectives.

The $E$-projective dimension $E$-$\text{pd} A$ of an object $A \in C$ is defined inductively. When $A = 0$, put $E$-$\text{pd} A = -1$. If $A \in P(E)$ then define $E$-$\text{pd} A = 0$. Next by induction, for an integer $n > 0$, put $E$-$\text{pd} A \leq n$ if there exists a triangle $K \to P \to A \to \Sigma K$ in $C$ with $P \in P(E)$ and $E$-$\text{pd} K \leq n - 1$. We define $E$-$\text{pd} A = n$ if $E$-$\text{pd} A \leq n$ and $E$-$\text{pd} A \not\leq n - 1$. If $E$-$\text{pd} A \neq n$, for all $n \geq 0$, we set $E$-$\text{pd} A = \infty$.

Similarly one can define the $E$-injective dimension of an object. We let $\tilde{P}(E)$ (respectively $\tilde{I}(E)$) denote the full subcategory of $C$ whose objects are of finite $E$-projective (respectively $E$-injective) dimension.
Definition 2.4. An $\mathcal{E}$-exact complex $X$ is a diagram

$$
\cdots \to X_1 \overset{d_1}{\longrightarrow} X_0 \overset{d_0}{\longrightarrow} X_{-1} \overset{d_{-1}}{\longrightarrow} X_{-2} \to \cdots
$$

in $\mathcal{C}$, such that for each integer $n$, there exist triangles

$$
K_{n+1} \overset{g_n}{\longrightarrow} X_n \overset{f_n}{\longrightarrow} K_n \overset{h_n}{\longrightarrow} \Sigma K_{n+1}
$$

in $\mathcal{E}$ and a differential is defined as $d_n = f_n g_{n-1}$, for any $n$.

Definition 2.5. A triangle $A \to B \to C \to \Sigma A$ in $\mathcal{E}$, is called $C(\mathcal{P}(\mathcal{E}))$-exact (respectively $C(\mathcal{I}(\mathcal{E}))$-exact), if for any $Q \in \mathcal{P}(\mathcal{E})$ (respectively $I \in \mathcal{I}(\mathcal{E})$), the induced complex

$$
0 \to C(C, Q) \to C(B, Q) \to C(A, Q) \to 0
$$

(respectively $0 \to C(I, A) \to C(I, B) \to C(I, C) \to 0$)

is exact in $\mathcal{A}b$.

Definition 2.6. A complete $\mathcal{P}(\mathcal{E})$-exact (respectively $\mathcal{I}(\mathcal{E})$-exact) complex $X$ is a diagram

$$
X : \cdots \to X_1 \overset{d_1}{\longrightarrow} X_0 \overset{d_0}{\longrightarrow} X_{-1} \to \cdots
$$

in $\mathcal{C}$ such that for all integer $n$, there exists $C(\mathcal{P}(\mathcal{E}))$-exact (respectively $C(\mathcal{I}(\mathcal{E}))$-exact) triangles

$$
K_{n+1} \overset{g_n}{\longrightarrow} X_n \overset{f_n}{\longrightarrow} K_n \overset{h_n}{\longrightarrow} \Sigma K_{n+1}
$$

in $\mathcal{E}$ where a differential $d_n$, for any integer $n$, is defined as $d_n = f_n g_{n-1}$.

Definition 2.7. A complete $\mathcal{E}$-projective resolution is a complete $\mathcal{P}(\mathcal{E})$-exact complex

$$
P : \cdots \to P_1 \overset{d_1}{\longrightarrow} P_0 \overset{d_0}{\longrightarrow} P_{-1} \to \cdots
$$

in $\mathcal{C}$ such that $P_n$, for any integer $n$, is $\mathcal{E}$-projective. Similarly a complete $\mathcal{E}$-injective coresolution is a complete $\mathcal{I}(\mathcal{E})$-exact complex

$$
I : \cdots \to I_1 \overset{d_1}{\longrightarrow} I_0 \overset{d_0}{\longrightarrow} I_{-1} \to \cdots
$$

in $\mathcal{C}$ such that $I_n$, for any integer $n$, is $\mathcal{E}$-injective.
Definition 2.8. (see [AS, 3.6]) Let $P$ be a complete $E$-projective resolution in $C$. So for any integer $n$, there exists a triangle

$$K_{n+1} \xrightarrow{g_n} P_n \xrightarrow{f_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}$$

in $E$. The objects $K_n$ for any integer $n$, are called $E$-$G$-projectives.

We denote by $GP(E)$ the full subcategory of $E$-$G$-projective objects of $C$. It follows directly from the definition that the category $GP(E)$ is full, additive and closed under isomorphisms. Every $E$-projective object is $E$-$G$-projective. In particular, there is an inclusion of categories $P(E) \subseteq GP(E)$, see [AS, 3.7].

Similarly, let $I$ denote a complete $E$-injective coresolution in $C$. By definition, for any integer $n$, there exists a triangle

$$K_{n+1} \xrightarrow{g_n} I_n \xrightarrow{f_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}$$

in $E$. The objects $K_n$ for any integer $n$, are called $E$-$G$-injectives. It is easy to see that $GI(E)$, the subcategory of $E$-$G$-injective objects is full, additive, closed under isomorphisms, direct summands and $\Sigma$-stable, i.e. $\Sigma(GI(E)) = GI(E)$. Moreover there is an inclusion of categories $I(E) \subseteq GI(E)$ where $I(E)$ denotes the full subcategory of $E$-injective objects of $C$.

Throughout the paper, we use freely dual results of [AS] for $E$-$G$-injective objects, when it is necessary.

Let $A$ be an object of $C$. We assign a homological invariant to $A$, denoted $E$-$G$-$pd$ $A$, that will be called $E$-$G$-Gorenstein projective dimension of $A$. When $A = 0$ put $E$-$G$-$pd$ $A = -1$. If $A \in GP(E)$, then we define $E$-$G$-$pd$ $A = 0$. Let $n > 0$. We define $E$-$G$-$pd$ $A = n$ if $E$-$G$-$pd$ $A \not\leq n - 1$ and there exists a triangle $K \xrightarrow{} P \xrightarrow{} A \xrightarrow{} \Sigma K$ in $E$ such that $P$ is $E$-projective and $E$-$G$-$pd$ $K \leq n - 1$. Finally if $E$-$G$-$pd$ $A \not= n$, for all $n \geq 0$, we set $E$-$G$-$pd$ $A = \infty$. Since $P(E) \subseteq GP(E)$, $E$-$G$-$pd$ $A \leq E$-$pd$ $A$ for all objects $A$ of $C$. Dually one defines $E$-$G$-Gorenstein injective dimension of $A$. This is denoted $E$-$G$-$id$ $A$.

We let $GP(E)$ (respectively $GI(E)$) denote the full subcategory of $C$ whose objects have finite $E$-$G$-projective (respectively $E$-$G$-injective) dimension.

Let us here recall the notion of (co)generating subcategories from [Be, 4.14].

Definition 2.9. A full subcategory $\mathcal{X}$ of $C$ is called a generating subcategory of $C$, if $\mathcal{X}$ is $\Sigma$-stable and for all $A \in C$, $C(\mathcal{X}, A) = 0$ implies that $A = 0$. Dually a full subcategory $\mathcal{Y}$ of $C$ is called a cogenerating subcategory of $C$, if $\mathcal{Y}$ is $\Sigma$-stable and for all $B \in C$, $C(B, \mathcal{Y}) = 0$ implies that $B = 0$.

We remark that in some natural examples the fact that the $E$-projectives form a generating subcategory implies that the $E$-injectives form a cogenerating subcategory. Usually these examples arise in triangulated categories in which the Brown representability theorem holds (this is the case for purity in compactly generated triangulated categories).

2.10. $E$-Derived functors. Let $C$ have enough $E$-projectives. An $E$-projective resolution of $A \in C$ is an $E$-exact complex $P \rightarrow A$ such that $P_n \in P(E)$, for all $n \geq 0$. Using standard
arguments in homological algebra, one can prove a version of the comparison lemma for $\mathcal{E}$-projective resolutions. It follows that any two $\mathcal{E}$-projective resolution of an object are homotopically equivalent. So they can be used to compute derived functors. Let $B \in \mathcal{C}$. For any integer $n \geq 0$, the $\mathcal{E}$-extension functor $\text{Ext}_{\mathcal{I}(\mathcal{E})}^n(A, B)$ is defined in [Be, p. 287] to be the $n$th right $\mathcal{E}$-derived functor of the functor $\mathcal{C}(A, B)$, that is

$$\text{Ext}_{\mathcal{I}(\mathcal{E})}^n(A, B) := \mathcal{R}_{\mathcal{E}}^n\mathcal{C}(A, B).$$

Now assume that $\mathcal{C}$ has enough $\mathcal{E}$-injectives. An $\mathcal{E}$-injective coresolution of $B \in \mathcal{C}$ is an $\mathcal{E}$-exact complex $B \to I$ such that $I^n \in \mathcal{I}(\mathcal{E})$, for all $n \geq 0$. Again one can prove easily that any two $\mathcal{E}$-injective coresolutions of an object are homotopically equivalent, so they can be used to compute derived functors. For any object $A \in \mathcal{C}$ and any integer $n \geq 0$, the $\mathcal{E}$-extension functor $\text{Ext}_{\mathcal{I}(\mathcal{E})}^n(A, )$ is defined to be the $n$th right $\mathcal{E}$-derived functor of the functor $\mathcal{C}(A, )$. But it is not difficult to see that the constructions via $\mathcal{E}$-projectives and $\mathcal{E}$-injectives are equivalent. In fact the usual proof, using the double complex arising from a deleted $\mathcal{E}$-projective resolution of $A$ and a deleted $\mathcal{E}$-injective coresolution of $B$, with the necessary modifications, works.

So for any objects $A$ and $B$ of $\mathcal{C}$, where $\mathcal{C}$ has enough $\mathcal{E}$-injectives and enough $\mathcal{E}$-projectives, and for any integer $n$, we write

$$\text{Ext}_{\mathcal{I}(\mathcal{E})}^n(A, B) := \text{Ext}_{\mathcal{I}(\mathcal{E})}^n(A, B) \cong \text{Ext}_{\mathcal{P}(\mathcal{E})}^n(A, B).$$

Using classical methods in homological algebra, one can see that, for any triangle in $\mathcal{E}$, long exact sequences of ‘$\text{Ext}$’ functors exists in both arguments.

For any objects $A$ and $B$ of $\mathcal{C}$, there is always a natural map from the space $\mathcal{C}(A, B)$ to $\text{Ext}_{\mathcal{C}}^0(A, B)$. It is clear that for any $\mathcal{E}$-projective object $P$ and any object $B$ of $\mathcal{C}$, $\text{Ext}_{\mathcal{C}}^0(P, B) \cong \mathcal{C}(P, B)$. On the other hand, for an $\mathcal{E}$-injective object $I$ and any object $A$ of $\mathcal{C}$, we always have $\text{Ext}_{\mathcal{C}}^0(A, I) \cong \mathcal{C}(A, I)$. In our next result we treat situations in which the zeroth ‘$\text{Ext}$’ is isomorphic to the ‘Hom’ space.

**Lemma 2.11.** Let $G$ be an $\mathcal{E}$-$G$projective object. Then for any $\mathcal{E}$-projective object $P$ of $\mathcal{C}$, $\text{Ext}_{\mathcal{C}}^0(G, P) \cong \mathcal{C}(G, P)$. Dually let $G$ be an $\mathcal{E}$-$G$injective object. Then for any $\mathcal{E}$-injective object $I$ of $\mathcal{C}$, $\text{Ext}_{\mathcal{C}}^0(I, G) \cong \mathcal{C}(I, G)$.

**Proof.** Let $P$ be an $\mathcal{E}$-projective object. Let $Q$ be a complete $\mathcal{E}$-projective resolution of $G$. Then $\mathcal{C}(Q, P)$ is an exact complex, and this implies that

$$0 \to \mathcal{C}(G, P) \to \mathcal{C}(Q_0, P) \to \mathcal{C}(Q_1, P) \to \cdots$$

is exact, whence

$$\mathcal{C}(G, P) \cong H^0(0 \to \mathcal{C}(Q_0, P) \to \mathcal{C}(Q_1, P) \to \cdots) = \text{Ext}_{\mathcal{C}}^0(G, P).$$

The second assertion follows similarly. □
Definition 2.12. Given a set $\mathcal{X}$ of objects of $\mathcal{C}$, we consider the following orthogonal classes:

$$\mathcal{X}^\perp = \{ B \in \mathcal{C} \mid \text{Ext}_E^1(X, B) = 0, \text{ for any } X \in \mathcal{X} \},$$

$$\mathcal{X} = \{ A \in \mathcal{C} \mid \text{Ext}_E^1(A, X) = 0, \text{ for any } X \in \mathcal{X} \}.$$ 

Proposition 2.13. $\mathcal{G}(\mathcal{E}) \subseteq \hat{\mathcal{P}}(\mathcal{E})^\perp$ and $\mathcal{G}(\mathcal{E}) \subseteq \perp \hat{\mathcal{I}}(\mathcal{E}).$

Proof. Let $E \in \mathcal{G}(\mathcal{E})$. So by definition there exist triangles $E^i \to I^i \to E^{i-1} \to \Sigma E^i$, $i \geq 0$, in $\mathcal{E}$, in which $I^i$ is $\mathcal{E}$-injective. Here $E^{-1} = E$. Let $X \in \hat{\mathcal{P}}(\mathcal{E})$. Using the long exact sequence of $\text{Ext}$ groups we get the isomorphism of abelian groups

$$\text{Ext}_E^1(X, E) \cong \text{Ext}_E^2(X, E^0) \cong \cdots \cong \text{Ext}_E^{i+1}(X, E^{i-2}) \cong \cdots.$$ 

Since $\mathcal{E}$-pd $X < \infty$, we get the result. The second part is just the dual of the first part, so we omit its proof. $\square$

Remark 2.14. Using the same argument as in the proof of [AS, 3.19], it follows from [AS, 3.18], that when $\mathcal{C}$ has enough $\mathcal{E}$-projectives and $A \in \mathcal{C}$ is an object of finite $\mathcal{E}$-$\mathcal{G}$projective dimension, then

$$\mathcal{E}$-pd $A = \sup \{ i \in \mathbb{N}_0 : \text{there exists } P \in \mathcal{P}(\mathcal{E}) \text{ such that } \text{Ext}_E^i(A, P) \neq 0 \}.$$ 

One can use similar argument to deduce that when $\mathcal{C}$ has enough $\mathcal{E}$-injectives and $B \in \mathcal{C}$ is an object of finite $\mathcal{E}$-$\mathcal{G}$injective dimension, then

$$\mathcal{E}$-Gid $B = \sup \{ i \in \mathbb{N}_0 : \text{there exists } I \in \mathcal{I}(\mathcal{E}) \text{ such that } \text{Ext}_E^i(I, B) \neq 0 \}.$$ 

Our last result in this section, follows from Remark 2.14.

Proposition 2.15. Let $\mathcal{C}$ have enough $\mathcal{E}$-projectives. Then

$$\mathcal{G}(\mathcal{E}) = \hat{\mathcal{G}}(\mathcal{E}) \cap \perp \hat{\mathcal{P}}(\mathcal{E}) = \hat{\mathcal{G}}(\mathcal{E}) \cap \perp \mathcal{P}(\mathcal{E}).$$

Dually if $\mathcal{C}$ has enough $\mathcal{E}$-injectives, then

$$\mathcal{G}(\mathcal{E}) = \hat{\mathcal{G}}(\mathcal{E}) \cap \hat{\mathcal{I}}(\mathcal{E}) \cap \mathcal{I}(\mathcal{E})^\perp = \hat{\mathcal{G}}(\mathcal{E}) \cap \mathcal{I}(\mathcal{E})^\perp.$$

3. $\mathcal{E}$-Tate cohomology

We begin this section by the following construction.

Construction 3.1. Let $A$ be an object of $\mathcal{C}$ of finite $\mathcal{E}$-$\mathcal{G}$projective dimension. Let $\mathcal{E}$-$\mathcal{G}$ pd $A = n$ and consider an $\mathcal{E}$-projective resolution $P \to A$ of $A$,

$$P : \cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0.$$
By [AS, 3.16], \( K_n \) in the triangle \( K_n \to P_{n-1} \to K_{n-1} \to \Sigma K_n \) is the first \( \mathcal{E} \)-syzygy of the above resolution that lies in \( \mathcal{GP}(\mathcal{E}) \). So there exists a complete \( \mathcal{E} \)-projective resolution

\[
Q : \cdots \to Q_{n+1} \to Q_n \to Q_{n-1} \to \cdots
\]

with triangle \( K_n \to Q_{n-1} \to K'_{n-1} \to \Sigma K_n \) in \( \mathcal{E} \).

The \( \mathcal{E} \)-projective resolution \( P \) of \( A \) and the complete \( \mathcal{E} \)-projective resolution \( Q \) of \( K_n \) can be put together in a commutative diagram

\[
\begin{array}{ccccccccc}
& \cdots & \to P_n & \to Q_{n-1} & \to \cdots & \to Q_1 & \to Q_0 & \to Q_{-1} & \to \cdots \\
\| & v_{n-1} & \downarrow & & v_1 & \downarrow & v_0 & \downarrow & v_{-1} \\
& \cdots & \to P_n & \to P_{n-1} & \to \cdots & \to P_1 & \to P_0 & \to 0 & \to \cdots
\end{array}
\]

Since the upper row is \( \mathcal{C}(\mathcal{P}(\mathcal{E})) \)-exact, the vertical maps can be constructed inductively, started from \( K_n \), see [AS, 4.5]. Such a construction will be called an \( \mathcal{E} \)-complete resolution of \( A \). We shall denote it by \( Q \xrightarrow{\nu} P \xrightarrow{\pi} A \).

Let \( A \in \mathcal{C} \) be an object of finite \( \mathcal{E} \)-G-projective dimension. By the above construction, \( A \) admits an \( \mathcal{E} \)-complete resolution \( Q \xrightarrow{\nu} P \xrightarrow{\pi} A \).

For each \( n \in \mathbb{Z} \) and each object \( B \) define an \( \mathcal{E} \)-Tate cohomology group by the equality

\[
\widehat{\mathcal{E}xt}^n_P(A, B) = H^n \mathcal{C}(Q, B).
\]

These groups come equipped with comparison morphisms

\[
\hat{\mathcal{E}}\mathcal{xt}^n_P(A, B) : \mathcal{E}\mathcal{xt}^n_{\mathcal{E}}(A, B) \to \widehat{\mathcal{E}xt}^n_P(A, B)
\]

given by

\[
H^n \mathcal{C}(\nu, B) : H^n \mathcal{C}(P, B) \to H^n \mathcal{C}(Q, B),
\]

where \( \nu : Q \to P \) is a morphism of \( \mathcal{E} \)-exact complexes.

First of all we should show that the assignment \((A, B) \mapsto \widehat{\mathcal{E}xt}^n_P(A, B)\) defines a functor

\[
\widehat{\mathcal{E}xt}^n_P : \mathcal{GP}(\mathcal{E})^{\text{op}} \times \mathcal{C} \to \mathcal{Ab}
\]

and the maps \( \hat{\mathcal{E}}\mathcal{xt}^n_P(A, B) \) yields a morphism of functors \( \hat{\mathcal{E}}\mathcal{xt}^n_P : \mathcal{E}\mathcal{xt}^n_{\mathcal{E}} \to \widehat{\mathcal{E}xt}^n_P \) such that both \( \hat{\mathcal{E}}\mathcal{xt}^n_P \) and \( \hat{\mathcal{E}}\mathcal{xt}^n_P \) are independent of the choice of resolutions and liftings. This follows from the following lemma.

**Lemma 3.2.** Let \( Q \xrightarrow{\nu} P \xrightarrow{\pi} A \) and \( Q' \xrightarrow{\nu'} P' \xrightarrow{\pi'} A' \) be \( \mathcal{E} \)-complete resolutions of \( A \) and \( A' \), respectively. For each morphism \( \mu : A \to A' \) there exists a morphism \( \tilde{\mu} \), unique up
to homotopy, making the right-hand square of the diagram

\[
\begin{array}{ccc}
Q & \xrightarrow{\nu} & P \\
\downarrow{\hat{\mu}} & & \downarrow{\mu} \\
Q' & \xrightarrow{\nu'} & P'
\end{array}
\]

commutative, and for each choice of \(\bar{\mu}\), there exists a morphism \(\hat{\mu}\), unique up to homotopy, making the left-hand square commute up to homotopy. In particular, if \(\mu = 1_A\), then \(\bar{\mu}\) and \(\hat{\mu}\) are homotopy equivalences.

**Proof.** Since \(P \xrightarrow{\pi} A\) and \(P' \xrightarrow{\pi'} A'\) are \(\mathcal{E}\)-projective resolutions of \(A\) and \(A'\), respectively, by [Be, Section 4] we know that \(\bar{\mu} : P \to P'\) exists, is unique up to homotopy and is homotopy equivalence whenever \(\mu = 1_A\). So we just need to discuss the existence and uniqueness of \(\hat{\mu}\). Set

\[g := \max\{\mathcal{E}\cdot\text{G pd } A, \mathcal{E}\cdot\text{G pd } A'\}.
\]

Since for \(i \geq g\), \(Q_i = P_i\) and \(Q'_{i} = P'_{i}\), we set \(\hat{\mu}_i = \bar{\mu}_i\) for these \(i\)'s. So let \(i < g\). We construct \(\hat{\mu}_i\) using a reverse induction, starting from \(K_g\) and \(K'_g\), where for \(i \in \mathbb{N}\), \(K_i\) (respectively \(K'_i\)) is the \(i\)th \(\mathcal{E}\)-syzygy of \(A\) (respectively \(A'\)). Using the map \(\bar{\mu}_{g-1} : P_{g-1} \to P'_{g-1}\), we get a morphism \(\hat{\mu}_g : K_g \to K'_g\). Now consider the diagram

\[
\begin{array}{ccccccc}
K_g & \rightarrow & Q_{g-1} & \rightarrow & Q_{g-2} & \rightarrow & Q_{g-3} & \rightarrow & \cdots \\
\downarrow{\hat{\mu}_g} & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K'_g & \rightarrow & Q'_{g-1} & \rightarrow & Q'_{g-2} & \rightarrow & Q'_{g-3} & \rightarrow & \cdots
\end{array}
\]

Since \(Q\) (respectively \(Q'\)) is a complete \(\mathcal{E}\)-projective resolution, the rows are \(\mathcal{C}(, P(\mathcal{E}))\)-exact. Hence, starting from \(\hat{\mu}_g : K_g \to K'_g\), we can construct morphisms \(\hat{\mu}_i : Q_i \to Q'_{i}\), \(i \leq g - 1\), that makes the relevant squares commute. So we get a chain map \(\hat{\mu} : Q \to Q'\). The chain map \((\hat{\mu}_i)_{i \geq g}\) is a lifting of \(\hat{\mu}_g : K_g \to K'_g\) to the \(\mathcal{E}\)-projective resolutions of \(K_g\) and \(K'_g\) and so is unique up to homotopy. Now, consider the chain map \((\hat{\mu}_i)_{i < g}\). Since rows of the above diagram are \(\mathcal{C}(, P(\mathcal{E}))\)-exact, an argument similar to that concerning \(\mathcal{E}\)-injective coresolutions, can be applied to show that the chain map \((\hat{\mu}_i)_{i < g}\) is also unique up to homotopy. Putting these together we deduce that \(\hat{\mu} : Q \to Q'\) is unique up to homotopy.

In order to complete the proof, it remains to show that the square

\[
\begin{array}{ccc}
Q & \xrightarrow{\nu} & P \\
\downarrow{\hat{\mu}} & & \downarrow{\hat{\mu}} \\
Q' & \xrightarrow{\nu'} & P'
\end{array}
\]
commute up to homotopy. To this end, for any integer \( i \), we have to construct morphisms 
\[ s_i : Q_i \to P'_{i+1} \]
such that 
\[ \hat{\mu}_i v'_i - v_i \hat{\mu}_i = s_i \pi'_{i+1} + r_i s_{i-1}, \]
where \( r_i : Q_i \to Q_{i-1} \). Since \( P_i = 0 \) for \( i < 0 \), we set \( s_i = 0 \) for negative \( i \). Assume that \( i \geq 0 \) and we have already constructed \( s_{i-1} \). A simple diagram chase shows that 
\[ (\hat{\mu}_i v'_i - v_i \hat{\mu}_i - r_i s_{i-1}) \pi'_i = 0. \]
So there exists \( s_i : Q_i \to P'_{i+1} \) such that 
\[ s_i \pi'_{i+1} = \hat{\mu}_i v'_i - v_i \hat{\mu}_i - r_i s_{i-1}. \]
This completes the inductive step. If \( \mu = 1_A \), one should reverse the rôle of \( A \) and \( A' \) to get a morphism 
\[ \hat{\mu} : Q' \to Q \]
inducing \( 1_A \). So \( \hat{\mu} : Q' \to Q' \)
induces \( 1_A \), and hence is homotopic to \( 1_Q \).
By symmetry, \( \hat{\mu} \hat{\mu}' \) is homotopic to \( 1_Q \). So \( \hat{\mu} \) is a homotopy equivalence.

**Lemma 3.3.** Let \( Q \) be a complete \( \mathcal{E} \)-projective resolution.

(i) For each \( A \in \mathcal{P}(\mathcal{E}) \), \( C(Q, A) \) is exact.
(ii) For each \( B \in \mathcal{I}(\mathcal{E}) \), \( C(Q, B) \) is exact.

**Proof.**

(i) Let \( A \in \mathcal{P}(\mathcal{E}) \). For an arbitrary integer \( i \), the \( i \)th cohomology of the complex \( C(Q, A) \) is \( \mathcal{E}xt^1_\mathcal{E}(K_{i-1}, A) \), where \( K_{i-1} \) is obtained from the triangle \( K_i \to Q_{i-1} \to K_{i-1} \to \Sigma K_i \) and hence is \( \mathcal{E} \)-\( \mathcal{G} \)-projective. But by [AS, 3.20], \( \mathcal{E}xt^1_\mathcal{E}(K_{i-1}, A) = 0 \). So the complex \( C(Q, A) \) is exact in degree \( i \). This follows the result.

(ii) Let \( \mathcal{E} \)-id \( B = n \). For some fixed integer \( i \), we show that complex \( C(Q, B) \) is exact in degree \( i \). The truncated \( \mathcal{E} \)-exact complex

\[ \cdots \to Q_{i+1} \to Q_i \to Q_{i-1} \to \cdots \to Q_{i-n} \to Q_{i-(n+1)} \to K_{i-(n+1)} \to 0, \]
is in fact an \( \mathcal{E} \)-projective resolution of \( K_{i-(n+1)} \). So the cohomology of \( C(Q, B) \) in the \( i \)th degree is equal to \( \mathcal{E}xt^1_\mathcal{E}(K_{i-(n+1)}, B) \). This group is zero, since \( \mathcal{E} \)-id \( B = n \). Hence complex \( C(Q, B) \) is exact in degree \( i \). Since \( i \) was arbitrary, the result follows.

Module version of the following theorem can be found in [AM].

**Theorem 3.4.**

(1) Let \( A \in \mathcal{G}\mathcal{P}(\mathcal{E}) \). For any integer \( g \) the following are equivalent.
(i) \( \mathcal{E} \)-\( \mathcal{G} \)-pd \( A \) \( \leq g \).
(ii) \( \mathcal{E}xt^n_\mathcal{P}(A, B) : \mathcal{E}xt^n_\mathcal{P}(A, B) \to \mathcal{E}xt^n_\mathcal{P}(A, B) \) is bijective for all \( n > g \) and all \( B \in \mathcal{C} \).
(2) If \( \mathcal{E} \)-pd \( A < \infty \), then \( \mathcal{E}xt^n_\mathcal{P}(A, B) = 0 \), for all \( n \in \mathbb{Z} \).
(3) If \( \mathcal{E} \)-pd \( A < \infty \), then \( \mathcal{E}xt^n_\mathcal{P}(A, B) = 0 \), for all \( n \in \mathbb{Z} \).
(4) If \( \mathcal{E} \)-id \( B < \infty \), then \( \mathcal{E}xt^n_\mathcal{P}(A, B) = 0 \), for all \( n \in \mathbb{Z} \).
Proof. (1) (i) $\Rightarrow$ (ii) Since $E$-$\text{Gpd} A \leq g$, $A$ has a complete resolution $Q \xrightarrow{v} P \xrightarrow{\pi} A$ such that $v_n$ is bijective for all $n \geq g$. So $\hat{E}xt^n_P(A, \ )$ is an isomorphism for all $n > E$-$\text{Gpd} A$. 

(ii) $\Rightarrow$ (i) If $\hat{E}xt^n_P(A, \ )$ is bijective, then in particular for any object $P$ of finite $E$-projective dimension and for all $n > g$, $E$-$\text{Ext}^n_P(A, P) = \hat{E}xt^n_P(A, P)$. But since every complete $E$-projective resolution is $C(\ , P(E))$-exact, the latter group is zero. So $E$-$\text{Ext}^n_P(A, P) = 0$, for all $n > g$ and all object $P$ with $E$-$\text{pd} P < \infty$. Therefore by [AS, 3.20], $E$-$\text{Gpd} A \leq g$.

(2) Since $E$-$\text{pd} A = p < \infty$, in Construction 3.1 choose an $E$-projective resolution $P \xrightarrow{\pi} A$ of length $p$. Set $g = p + 1$ and resolve $\Omega^p P$ with $Q = 0$. So $0 \rightarrow P \rightarrow A$ is a complete $E$-projective resolution of $A$. Hence $\hat{E}xt^n_P(A, \ ) = 0$.

(3) This follows from Lemma 3.3(i).

(4) This follows from Lemma 3.3(ii). \qed

The next theorem establish the existence of a long exact sequence of Tate cohomology, associated to a triangle, in the second argument.

**Theorem 3.5.** Let $A \in \hat{G}P(\mathcal{E})$ and consider the triangle $B = B \xrightarrow{\beta} B' \xrightarrow{\beta'} B'' \xrightarrow{\beta''} \Sigma B$ in $E$. Then there exists natural in $A$ and $B$ homomorphisms $\hat{\partial}^n(A, B)$, such that the following sequence is exact in $\mathcal{A}b$.

\[
\cdots \rightarrow \hat{E}xt^n_P(A, B) \xrightarrow{\hat{E}xt^n_P(A, \beta)} \hat{E}xt^n_P(A, B') \xrightarrow{\hat{E}xt^n_P(A, \beta')} \hat{E}xt^n_P(A, B'') \rightarrow \hat{E}xt^{n+1}_P(A, B) \rightarrow \cdots
\]

Moreover the connecting maps $\hat{\partial}^n(A, B)$ satisfy

\[
\hat{E}xt^n_P(A, B'') \hat{\partial}^n(A, B) = \partial^n(A, B) \hat{E}xt^{n+1}_P(A, B)
\]

for all $n \in \mathbb{Z}$.

**Proof.** Choose a complete resolution $Q \xrightarrow{v} P \xrightarrow{\pi} A$ of $A$ to get the commutative diagram

\[
\begin{array}{cccccc}
0 & \xrightarrow{} & C(P, B) & \xrightarrow{} & C(P, B') & \xrightarrow{} & C(P, B'') & \xrightarrow{} & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \xrightarrow{} & C(Q, B) & \xrightarrow{} & C(Q, B') & \xrightarrow{} & C(Q, B'') & \xrightarrow{} & 0
\end{array}
\]

of complexes, in $\mathcal{A}b$. The rows are exact, because $P_n$ and $Q_n$, for any integer $n$ are $E$-projective. The homology exact sequence of the bottom row is the desired long exact sequence. The commutativity of the diagram proves the last claim. Naturality in $B$ is clear. Naturality in $A$ follows from Lemma 3.2. \qed

To get a long exact sequence in the first argument, we need a version of the Horseshoe Lemma for complete resolutions.
Lemma 3.6. Let $\mathbf{A}: A \xrightarrow{\alpha} A' \xrightarrow{\alpha'} A'' \rightarrow \Sigma A$ be a triangle in $\mathcal{E}$ such that $\mathcal{E}$-$\mathcal{G}$pd $A$ and $\mathcal{E}$-$\mathcal{G}$pd $A''$ both are finite. Let $Q \xrightarrow{\nu} P \xrightarrow{\pi} A$ and $Q' \xrightarrow{\nu'} P' \xrightarrow{\pi'} A'$ be $\mathcal{E}$-complete resolutions of $A$ and $A'$, respectively. Then there is a commutative diagram

whose rows are triangles in $\mathcal{E}$ and columns are $\mathcal{E}$-complete resolutions.

Proof. By [Be, 4.11], we have the two lower rows of the diagram. Let $n = \max\{\mathcal{E}$-$\mathcal{G}$pd $A$, $\mathcal{E}$-$\mathcal{G}$pd $A''\}$. So we have a triangle $K_{n+1} \rightarrow K'_{n+1} \rightarrow K''_{n+1} \rightarrow \Sigma K_{n+1}$ in $\mathcal{E}$, where $K_i$ (respectively $K'_i$, $K''_i$) denotes the $i$th $\mathcal{E}$-syzygy of $A$ (respectively $A'$, $A''$) obtained from $P$ (respectively $P'$, $P''$). By our assumption $K_{n+1}$ and $K''_{n+1}$ are $\mathcal{E}$-$\mathcal{G}$projective. So by [AS, 3.11], $K'_{n+1}$ is also $\mathcal{E}$-$\mathcal{G}$projective. Now an argument similar to that in the proof of [AS, 3.11] can be applied to construct $Q'$ so that we get the upper row of the diagram. □

Proposition 3.7. For each triangle $\mathbf{A}: A \xrightarrow{\alpha} A' \xrightarrow{\alpha'} A'' \rightarrow \Sigma A$ in $\mathcal{E}$ of objects of finite $\mathcal{E}$-$\mathcal{G}$projective dimension and each object $B$ of $\mathcal{C}$, there exists natural in $\mathbf{A}$ and $B$ homomorphisms $\hat{\partial}^n(\mathbf{A}, B)$, making the following sequence of cohomology groups exact:

and the connecting maps $\hat{\partial}^n(\mathbf{A}, B)$ satisfy

$\hat{\partial}^n(\mathbf{A}, B)\hat{\partial}^n(\mathbf{A}, B) = \partial^n(\mathbf{A}, B)\hat{\partial}^{n+1}(A'', B)$.

Proof. Since all triangles in two upper rows of the diagram of Lemma 3.6 are split triangles in $\mathcal{E}$, applying the functor $\mathcal{C}(\mathbf{ }, B)$ on the two upper rows of the diagram, yields a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & C(P'', B) & \longrightarrow & C(P', B) & \longrightarrow & C(P, B) & \longrightarrow & 0 \\
\downarrow C(\nu'', B) & & \downarrow C(\nu', B) & & \downarrow C(\nu, B) & & & \\
0 & \longrightarrow & C(Q'', B) & \longrightarrow & C(Q', B) & \longrightarrow & C(Q, B) & \longrightarrow & 0
\end{array}
\]
of abelian groups with exact rows. The homology exact sequence of the lower row gives the desired long exact sequence of $\mathcal{E}$-Tate cohomology groups. Commutativity of the diagram gives the desired formula. Naturality in $B$ is clear. Naturality in $A$ follows, by a routine check, from Lemma 3.2. □

$\mathcal{E}$-Tate cohomology is rigid. This can be seen from the following theorem.

**Theorem 3.8.** Let $\mathcal{P}(\mathcal{E})$ be a generating subcategory of $\mathcal{C}$. Then for any object $A \in \overline{\mathcal{G}} \mathcal{P}(\mathcal{E})$, the following are equivalent.

(i) $\mathcal{E}$-pd $A < \infty$.

(ii) $\widehat{\text{Ext}}^n_{\mathcal{E}}(A, \ ) = 0$, for some $n \in \mathbb{Z}$.

(iii) $\widehat{\text{Ext}}^n_{\mathcal{E}}(A, \ ) = 0$, for all $n \in \mathbb{Z}$.

(iv) $\widehat{\text{Ext}}^n_{\mathcal{E}}(\ , A) = 0$, for some $n \in \mathbb{Z}$.

(v) $\widehat{\text{Ext}}^n_{\mathcal{E}}(\ , A) = 0$, for all $n \in \mathbb{Z}$.

(vi) $\widehat{\text{Ext}}^0_{\mathcal{E}}(A, A) = 0$.

**Proof.** The fact that (i) implies (ii)–(vi) follows from Theorem 3.4(2) and (3).

(vi) $\Rightarrow$ (i) Let $\mathcal{E}$-$\mathcal{G}$-pd $A = g$ and consider an $\mathcal{E}$-projective resolution $\mathcal{P} \xrightarrow{\pi} A$ of $A$. Hence for any integer $i \geq 0$, there exists triangle $K_i \xrightarrow{h_i} P_{i-1} \xrightarrow{f_i} K_{i-1} \xrightarrow{\Sigma} K_i$ in $\mathcal{E}$. Here $K_0 = A$. Since $P_i$ for any integer $i$ is $\mathcal{E}$-projective, by Theorem 3.4(3), $\widehat{\text{Ext}}^n_{\mathcal{P}}(\ , P_i) = 0$, for all $n \in \mathbb{Z}$. Hence by applying the functor $\widehat{\text{Ext}}_{\mathcal{P}}(A, \ )$ on the above triangles we get

$$\widehat{\text{Ext}}^0_{\mathcal{E}}(A, A) \cong \widehat{\text{Ext}}^{g+1}_{\mathcal{E}}(A, K_{g+1}).$$

In view of Theorem 3.4(1), $\text{Ext}^{g+1}_{\mathcal{E}}(A, K_{g+1}) \cong \text{Ext}^{g+1}_{\mathcal{E}}(A, K_{g+1})$. So $\text{Ext}^{g+1}_{\mathcal{E}}(A, K_{g+1}) = 0$. But $\text{Ext}^{g+1}_{\mathcal{E}}(A, K_{g+1})$ is the cohomology of the complex $C(P_g, K_{g+1}) \to C(P_{g+1}, K_{g+1}) \to C(P_{g+2}, K_{g+1})$. So the morphism $f_{g+1} : P_{g+1} \to K_{g+1}$ belongs to the image of map $C(P_g, K_{g+1}) \to C(P_{g+1}, K_{g+1})$. So there exists $\alpha : P_g \to K_{g+1}$ such that $d_{g+1} \alpha = f_{g+1}$. Hence $f_{g+1} h_g \alpha = f_{g+1}$. Let $P \in \mathcal{P}(\mathcal{E})$ and apply the functor $C(P, \ )$ to this equality. Since $C(P, f_{g+1})$ is surjective, it follows that $C(P, h_g) \circ C(P, \alpha) = 1_{C(P, K_{g+1})}$. So $C(P, P_g) \cong C(P, K_g \oplus K_{g+1})$. Hence $P \cong K_g \oplus K_{g+1}$, because $\mathcal{P}(\mathcal{E})$ generate $C$. So $K_{g+1}$ as a direct summand of $P_g$ is $\mathcal{E}$-projective. Therefore $\mathcal{E}$-pd $A < \infty$.

(ii) $\Rightarrow$ (i) It is clear that without loss of generality we may assume that $n > 0$. Now the isomorphism $\widehat{\text{Ext}}^n_{\mathcal{P}}(A, K_{n-1}) \cong \widehat{\text{Ext}}^0_{\mathcal{P}}(K_{n-1}, K_{n-1})$ implies the result.

(iv) $\Rightarrow$ (i) We prove this by a simple induction on $\mathcal{E}$-$\mathcal{G}$ pd $A = g$. Assume that $g = 0$. So there exists an $\mathcal{E}$-exact sequence

$$0 \to A \to P^0 \to P^1 \to P^2 \to \cdots.$$
in which \(P^i\), for any integer \(i\), is \(E\)-projective. Consider the triangles

\[
K^i \to P^{i+1} \to K^{i+1} \to \Sigma K^i \quad (i \geq -1).
\]

Here \(K^{-1} = A\). By applying the functor \(\widehat{\text{Ext}}_P(\cdot, A)\) on these triangles, we get the isomorphism

\[
\widehat{\text{Ext}}^0_P(A, A) \cong \widehat{\text{Ext}}^n_P(K^{n-1}, A).
\]

Hence in view of our assumption, \(\widehat{\text{Ext}}^0_P(A, A) = 0\) and thus \(E\)-pd \(A < \infty\). Now assume that \(g > 0\) and consider the triangle

\[
K_1 \to P_0 \to A \to \Sigma K_1,
\]

in which \(P_0 \in \mathcal{P}(E)\) and \(E\)-pd \(K_1 = g - 1\). Our assumption implies that \(\widehat{\text{Ext}}^n_P(\cdot, K_1) = 0\). Hence by induction assumption, \(E\)-pd \(K_1 < \infty\). Therefore \(E\)-pd \(A < \infty\). This completes the inductive step and hence the proof of this implication.

The rest implications are trivial.

There is another \(E\)-Tate cohomology theory based on \(E\)-Ginjective objects. Since the arguments and proofs are similar (or rather dual) to the above mentioned Tate cohomology, we just review the idea without going into details. In the next section, we explore a sort of balance between these two cohomological functors.

Let \(B\) be an object of finite \(E\)-Ginjective dimension. Similar to the Construction 3.1, one can construct an \(E\)-complete coresolution, \(B \xrightarrow{i} I \xrightarrow{v} E\), in which \(I\) is an \(E\)-injective coresolution of \(B\) and \(E\) is a complete \(E\)-injective coresolution.

Now choose an \(E\)-complete coresolution \(B \xrightarrow{i} I \xrightarrow{v} E\) of \(B\). For each \(n \in \mathbb{Z}\) and each \(A \in \mathcal{C}\), define an \(E\)-Tate cohomology group by the equality

\[
\widehat{\text{Ext}}^n_L(A, B) := \mathcal{H}^n_C(A, E).
\]

By using the argument dual to Lemma 3.2, one can show that \(\widehat{\text{Ext}}^n_L\) and \(\hat{\varepsilon}^n_L(\cdot, \cdot) : \mathcal{E}\text{xt}^n_L(\cdot, \cdot) \to \widehat{\text{Ext}}^n_L(\cdot, \cdot)\) are independent of the choice of coresolutions and liftings, where \(\hat{\varepsilon}^n_L(\cdot, \cdot) : \mathcal{E}\text{xt}^n_L(\cdot, \cdot) \to \widehat{\text{Ext}}^n_L(\cdot, \cdot)\). Moreover, we have the following result.

**Theorem 3.9.**

1. Let \(B \in \mathcal{G}\mathcal{L}(E)\). For any integer \(g\) the following are equivalent.
   1. \(E\)-Gid \(B \leq g\).
   2. \(\hat{\varepsilon}^n_L(A, B) : \mathcal{E}\text{xt}^n_L(A, B) \to \widehat{\text{Ext}}^n_L(A, B)\) is bijective for all \(n > g\) and all \(A \in \mathcal{C}\).
2. If \(E\)-id \(B < \infty\), then \(\widehat{\text{Ext}}^n_L(\cdot, B) = 0\), for all \(n \in \mathbb{Z}\).
3. If \(E\)-id \(B < \infty\), then \(\widehat{\text{Ext}}^n_L(B, \cdot) = 0\), for all \(n \in \mathbb{Z}\).
4. If \(E\)-pd \(A < \infty\), then \(\widehat{\text{Ext}}^n_L(A, B) = 0\), for all \(n \in \mathbb{Z}\).
Proposition 3.10.

(a) Let $B \in \mathcal{GI}(E)$ and $A \to A' \to A'' \to \Sigma A$ be a triangle in $E$. Then there is a long exact sequence of $E$-Tate cohomology groups

$$
\cdots \to \mathcal{E}xt^n_T(A'', B) \to \mathcal{E}xt^n_T(A', B) \to \mathcal{E}xt^n_T(A, B) \\
\to \mathcal{E}xt^{n+1}_T(A'', B) \to \cdots
$$

(b) Let $B \to B' \to B'' \to \Sigma B$ be a triangle in $\mathcal{E}$ of objects of finite $E$-Ginjective dimension. Then for any object $A \in C$, there is a long exact sequence of $E$-Tate cohomology groups

$$
\cdots \to \mathcal{E}xt^n_T(A, B) \to \mathcal{E}xt^n_T(A, B') \to \mathcal{E}xt^n_T(A, B'') \\
\to \mathcal{E}xt^{n+1}_T(A, B) \to \cdots
$$

In both cases the connecting homomorphisms are natural in $A$ and $B$ homomorphisms.

We also have the following rigidity theorem for this version of $E$-Tate cohomology groups.

Theorem 3.11. Assume that $I(E)$ is a cogenerating subcategory of $C$. Then for any object $B \in \mathcal{GI}(E)$, the following are equivalent.

(i) $E$-id $B < \infty$.
(ii) $\mathcal{E}xt^n_T(, B) = 0$, for some $n \in \mathbb{Z}$.
(iii) $\mathcal{E}xt^n_T(, B) = 0$, for all $n \in \mathbb{Z}$.
(iv) $\mathcal{E}xt^n_T(B, ) = 0$, for some $n \in \mathbb{Z}$.
(v) $\mathcal{E}xt^n_T(B, ) = 0$, for all $n \in \mathbb{Z}$.
(vi) $\mathcal{E}xt^0_T(B, B) = 0$.

4. $\mathcal{E}$-Gorenstein triangulated categories

In this section, we study triangulated categories, for them every object has finite $E$-$\mathcal{G}$projective and $E$-$\mathcal{G}$injective dimension. Throughout let $C$ be a triangulated category having enough $E$-injectives and enough $E$-projectives.

We begin by assigning the following two invariants to a triangulated category.

Definition 4.1. Let $C$ be a triangulated category. We assign two invariants to $C$ as follows:

$$
\mathcal{E}$-silp$C = \sup \{ E \text{-id} P \mid P \in \mathcal{P}(E) \}$
$$
$$
\mathcal{E}$-spli$C = \sup \{ E \text{-pd} I \mid I \in \mathcal{I}(E) \}$.


These invariants are motivated by Gedrich and Gruenberg’s invariants of a ring, \( \text{silp} R \) and \( \text{spli} R \), see [GG].

**Proposition 4.2.** If for a category \( C \) both \( \mathcal{E}\text{-silp} C \) and \( \mathcal{E}\text{-spli} C \) are finite, then they are equal.

**Proof.** Set \( \mathcal{E}\text{-silp} C = t \) and \( \mathcal{E}\text{-spli} C = s \). So there exists \( I \in \mathcal{I}(\mathcal{E}) \) such that \( \mathcal{E}\text{-pd} I = s \). Therefore, using classical argument, one may deduce that there exists \( P \in \mathcal{P}(\mathcal{E}) \) such that \( \operatorname{Ext}^t_{\mathcal{E}}(I, P) \neq 0 \). This implies that \( \mathcal{E}\text{-id} P \geq s \), which in turn implies that \( \mathcal{E}\text{-silp} C = t \geq s \). Similar argument gives \( s \geq t \). The result hence follows. \( \square \)

**Theorem 4.3.** Let \( \mathcal{I}(\mathcal{E}) \) be a cogenerating subcategory of \( C \). Let \( B \in C \) be such that both \( \mathcal{E}\text{-G id} B \) and \( \mathcal{E}\text{-pd} B \) are finite. Then \( \mathcal{E}\text{-id} B \) is finite. Dually let \( \mathcal{P}(\mathcal{E}) \) be a generating subcategory of \( C \) and let \( A \in C \) be such that both \( \mathcal{E}\text{-id} A \) and \( \mathcal{E}\text{-G pd} A \) are finite. Then \( \mathcal{E}\text{-pd} A \) is finite.

**Proof.** Assume that \( \mathcal{E}\text{-G id} B \) and \( \mathcal{E}\text{-pd} B \) both are finite. By Theorem 3.11, \( \mathcal{E}\text{-id} B \) is finite if and only if \( \widehat{\operatorname{Ext}}^t_\mathcal{E}(B, B) = 0 \). Let \( P \to B \) be an \( \mathcal{E}\text{-projective} \) resolution of \( B \) and consider its relevant triangles \( L_i+1 \to P_i \to L_i \to \Sigma L_{i+1} \) with \( L_0 = B \). It is easy to see that \( \widehat{\operatorname{Ext}}^0_\mathcal{E}(B, B) \cong \widehat{\operatorname{Ext}}^t_\mathcal{E}(L_1, B) \). But for \( i > \mathcal{E}\text{-G id} B \), \( \widehat{\operatorname{Ext}}^i_\mathcal{E}(L_i, B) \cong \operatorname{Ext}^i_\mathcal{E}(L_i, B) \). The finiteness of the \( \mathcal{E}\text{-projective} \) dimension of \( B \) implies that \( \mathcal{E}\text{-pd} L_i < \infty \), and so for \( i > \mathcal{E}\text{-pd} L_i \), \( \operatorname{Ext}^i_\mathcal{E}(L_i, B) = 0 \). The result hence follows. Second assertion follows dually. \( \square \)

**Proposition 4.4.** Let \( \mathcal{P}(\mathcal{E}) \) be a generating and \( \mathcal{I}(\mathcal{E}) \) be a cogenerating subcategory of \( C \). Assume that any object of \( C \) has \( \mathcal{E}\text{-G projective} \) dimension less than or equal to \( n \), for some positive integer \( n \). Then \( \mathcal{E}\text{-silp} C = \mathcal{E}\text{-spli} C \leq n \).

**Proof.** The fact that \( \mathcal{E}\text{-spli} C < \infty \) follows from the above theorem. Now assume that \( P \) is an \( \mathcal{E}\text{-projective} \) object. Since for any object \( A \), \( \mathcal{E}\text{-G pd} A \leq n \), it follows from Theorem 3.4(1) that \( \operatorname{Ext}^i_\mathcal{E}(A, P) \cong \widehat{\operatorname{Ext}}^i_\mathcal{E}(A, P) \), for all \( i > n \). But, it follows from Theorem 3.8 that \( \widehat{\operatorname{Ext}}^i_\mathcal{E}(A, P) = 0 \) for all integer \( i \) and all \( \mathcal{E}\text{-projective} \) object \( P \). So \( \operatorname{Ext}^i_\mathcal{E}(A, P) = 0 \), for all \( i > n \). Now since \( \mathcal{I}(\mathcal{E}) \) is a cogenerating subcategory of \( C \) dual of [Be, 4.17], implies that \( \mathcal{E}\text{-id} P \leq n \). So \( \text{silp} C \leq n \). Their equivalence follows from Proposition 4.2. \( \square \)

Similar argument can be applied to prove the following.

**Proposition 4.5.** Let \( \mathcal{P}(\mathcal{E}) \) be a generating and \( \mathcal{I}(\mathcal{E}) \) be a cogenerating subcategory of \( C \). Assume that any object of \( C \) has \( \mathcal{E}\text{-G injective} \) dimension less than or equal to \( n \), for some positive integer \( n \). Then \( \mathcal{E}\text{-silp} C = \mathcal{E}\text{-spli} C < \infty \).

**Definition 4.6.** Let \( n \) be a nonnegative integer. We say that \( C \) is an \( \mathcal{E}\text{-n-Gorenstein} \) (or sometimes \( \mathcal{E}\text{-Gorenstein} \)) triangulated category, if any object \( A \) of \( C \) has both \( \mathcal{E}\text{-G injective} \) and \( \mathcal{E}\text{-G projective} \) dimension less than or equal to \( n \).

It follows from the definition that any \( \mathcal{E}\text{-Gorenstein} \) triangulated category has enough \( \mathcal{E}\text{-injective} \) and enough \( \mathcal{E}\text{-projective} \) objects.
Example 4.7. Let $\mathcal{C}$ be a triangulated category and let $\mathcal{E}$ be a proper class of triangles in $\mathcal{C}$. Let $\mathcal{C}/\mathcal{P}(\mathcal{E})$ denote the stable category of $\mathcal{C}$ modulo the full subcategory $\mathcal{P}(\mathcal{E})$ of $\mathcal{E}$-projective objects. In [Be, Section 7], it is shown that $\mathcal{C}/\mathcal{P}(\mathcal{E})$ carries in a natural way a left triangulated structure which is not necessarily triangulated: it is triangulated if and only if $\mathcal{P}(\mathcal{E}) = \mathcal{I}(\mathcal{E})$, see [Be, 7.2]. But in this case, it is easy to see that every object of $\mathcal{C}$ is $\mathcal{E}$-Gorenstein projective as well as $\mathcal{E}$-Gorenstein injective. In fact, for any object $A$ one could paste the usual $\mathcal{E}$-projective resolution of $A$ with its $\mathcal{E}$-injective coresolution and verify that the resulting sequence is a complete $\mathcal{E}$-projective resolution, as well as complete $\mathcal{E}$-injective coresolution, of $A$. So when left triangulated category $\mathcal{C}/\mathcal{P}(\mathcal{E})$ is triangulated, $\mathcal{C}$ is an $\mathcal{E}$-0-Gorenstein triangulated category.

Remark 4.8. Let $\mathcal{P}(\mathcal{E})$ be a generating and $\mathcal{I}(\mathcal{E})$ be a cogenerating subcategory of $\mathcal{C}$. Assume that $\mathcal{C}$ is an $\mathcal{E}$-$n$-Gorenstein triangulated category. Then $\mathcal{E}$-projective dimension of an object is finite if and only if its $\mathcal{E}$-injective dimension is finite, i.e. $\mathcal{P}(\mathcal{E}) = \mathcal{I}(\mathcal{E})$. Moreover, we have

$$\sup\{\mathcal{E}\text{-pd } A \mid A \in \mathcal{P}(\mathcal{E})\} \leq n \quad \text{and} \quad \sup\{\mathcal{E}\text{-id } B \mid B \in \mathcal{I}(\mathcal{E})\} \leq n.$$ 

In particular, $\text{spli} \mathcal{C} = \text{silp} \mathcal{C} \leq n$.

Let $Q$ be an $\mathcal{E}$-complete projective resolution. We know from [AS, 3.5] that $Q$ is $\mathcal{C}(\cdot, \mathcal{P}(\mathcal{E}))$-exact. In the following we show that over an $\mathcal{E}$-Gorenstein triangulated category, every $\mathcal{E}$-exact complex $T$ of $\mathcal{E}$-projective objects is an $\mathcal{E}$-complete projective resolution.

Lemma 4.9. Let $\mathcal{P}(\mathcal{E})$ be a generating and $\mathcal{I}(\mathcal{E})$ be a cogenerating subcategory of $\mathcal{C}$. Assume that $\mathcal{C}$ is an $\mathcal{E}$-$n$-Gorenstein triangulated category. Then any $\mathcal{E}$-exact complex of $\mathcal{E}$-projective objects is a complete $\mathcal{E}$-projective resolution. Dually any $\mathcal{E}$-exact complex of $\mathcal{E}$-injective objects is a complete $\mathcal{E}$-injective coresolution.

Proof. Let $T$ be an $\mathcal{E}$-exact complex of $\mathcal{E}$-projective objects. We show that for any integer $d$, $K_d$, in the relevant triangle $K_{d+1} \to T_d \to K_d \to \Sigma K_{d+1}$ is $\mathcal{E}$-projective. Let $P \in \mathcal{P}(\mathcal{E})$ be an arbitrary $\mathcal{E}$-projective object of $\mathcal{C}$. Since $\mathcal{C}$ is $\mathcal{E}$-$n$-Gorenstein, $\mathcal{E}$-$\text{id } P = t \leq n$. Applying the homological functor $\mathcal{C}(\cdot, P)$ on triangles $K_{i+1} \to T_i \to K_i \to \Sigma K_{i+1}$, $i \leq d$, in view of the fact that $T_i$ for any integer $i$, is $\mathcal{E}$-projective, implies that $\mathcal{E}\text{xt}_{\mathcal{C}}^i(K_d, P) \cong \mathcal{E}\text{xt}_{\mathcal{C}}^{i+1}(K_{d-1}, P)$. So $\mathcal{E}\text{xt}_{\mathcal{C}}^i(K_d, P) = 0$ which implies that $K_d$ is $\mathcal{E}$-$\mathcal{G}$projective. Now Lemma 2.11, can be applied to deduce that these triangles are $\mathcal{C}(\cdot, \mathcal{P}(\mathcal{E}))$-exact. So $T$ is an $\mathcal{E}$-complete resolution.

The next theorem explain the connection between $\mathcal{E}$-Gorenstein triangulated categories and the invariants $\mathcal{E}$-silp $\mathcal{C}$ and $\mathcal{E}$-spli $\mathcal{C}$. The proof of the implication (ii) $\Rightarrow$ (i) is a slight modification of [GG, 4.1].

Theorem 4.10. Let $\mathcal{P}(\mathcal{E})$ be a generating and $\mathcal{I}(\mathcal{E})$ be a cogenerating subcategory of $\mathcal{C}$. For any nonnegative integer $n$, the following are equivalent.
(i) $C$ is an $E$-$n$-Gorenstein triangulated category.

(ii) $E$-$silpC = E$-$spliC \leq n$ and every $E$-exact complex $T$ of $E$-projective (respectively $E$-injective) objects is a complete $E$-projective resolution (respectively complete $E$-injective coresolution).

**Proof.** (i) $\Rightarrow$ (ii) This follows from Propositions 4.4 and the above lemma.

(ii) $\Rightarrow$ (i) Set $n = E$-$silpC = E$-$spliC$ and let $A$ be an arbitrary object of $C$. Let

$$0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \cdots$$

be an $E$-injective coresolution of $A$ and consider the relevant triangles $K^i \rightarrow I^i \rightarrow K^{i+1} \rightarrow \Sigma K^i$, $i \geq 0$, in $E$. Here $K^0 = A$. Now, by [Be, 4.11], for each of these triangles consider an $E$-projective resolution

$$
\begin{array}{cccc}
P(K^i) & \rightarrow & P(I^i) & \rightarrow & P(K^{i+1}) & \rightarrow & \Sigma(P(K^i)) \\
K^i & \rightarrow & I^i & \rightarrow & K^{i+1} & \rightarrow & \Sigma(K^i)
\end{array}
$$

We may choose the $E$-projective resolutions so that they can be pasted together to construct a commutative diagram of objects and morphisms

$$
\begin{array}{cccc}
0 & \rightarrow & P(A) & \rightarrow & P(I^0) & \rightarrow & P(I^1) & \rightarrow & P(I^2) & \rightarrow & \cdots \\
0 & \rightarrow & A & \rightarrow & I^0 & \rightarrow & I^1 & \rightarrow & I^2 & \rightarrow & \cdots
\end{array}
$$

where the rows and the columns are $E$-exact complexes. Let $\Omega_jX$, for any integer $j$ and any object $X$, denote the $j$th $E$-syzygy of $X$ in an $E$-projective resolution of $X$. For each $i, j \geq 0$, consider the diagram

$$
\begin{array}{cccc}
\Omega_{j+1}K^i & \rightarrow & \Omega_{j+1}I^i & \rightarrow & \Omega_{j+1}K^{i+1} & \rightarrow & \Sigma(\Omega_{j+1}K^i) \\
P_j(K^i) & \rightarrow & P_j(I^i) & \rightarrow & P_j(K^{i+1}) & \rightarrow & \Sigma(P_j(K^i)) \\
\Omega_jK^i & \rightarrow & \Omega_jI^i & \rightarrow & \Omega_jK^{i+1} & \rightarrow & \Sigma(\Omega_jK^i) \\
\Sigma(\Omega_{j+1}K^i) & \rightarrow & \Sigma(\Omega_{j+1}I^i) & \rightarrow & \Sigma(\Omega_{j+1}K^{i+1}) & \rightarrow & \Sigma^2(\Omega_{j+1}K^i)
\end{array}
$$
We show that, by induction on \( j \), we may assume that the first row of the above diagram is in \( \mathcal{E} \). Case \( j = 0 \), is trivial. So assume inductively that \( j > 0 \). Since the second and third rows and all columns are in \( \mathcal{E} \), they are \( \mathcal{C}(\mathcal{P}(\mathcal{E}), \cdot) \)-exact. Now a simple diagram chase shows that the first row is also \( \mathcal{C}(\mathcal{P}(\mathcal{E}), \cdot) \)-exact. So by [Be, 4.2(iii)], it is in \( \mathcal{E} \). Therefore we get the following commutative diagram, in which all rows (and all columns) are \( \mathcal{E} \)-exact sequences:

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 \\
\downarrow & & & & & & & \\
0 & \Omega_n A & \Omega_n I^0 & \Omega_n I^1 & \Omega_n I^2 & \cdots \\
\downarrow & & & & & & & \\
0 & P_{n-1}(A) & P_{n-1}(I^0) & P_{n-1}(I^1) & P_{n-1}(I^2) & \cdots \\
\downarrow & & & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\downarrow & & & & & & & \\
0 & P_0(A) & P_0(I^0) & P_0(I^1) & P_0(I^2) & \cdots \\
\downarrow & & & & & & & \\
0 & A & I^0 & I^1 & I^2 & \cdots \\
\downarrow & & & & & & & \\
0 & 0 & 0 & 0 & 0
\end{array}
\]

Since \( \mathcal{E} \)-split \( \mathcal{C} = n \), for any integer \( i \), \( \Omega_n I^i \) is \( \mathcal{E} \)-projective. Hence we obtain an \( \mathcal{E} \)-exact complex of \( \mathcal{E} \)-projective objects

\[
\cdots \to P_{n+1} \to P_n \to \Omega_n I^0 \to \Omega_n I^1 \to \cdots .
\]

By assumption it is a complete \( \mathcal{E} \)-projective resolution. Since \( \Omega_n A \) is an \( \mathcal{E} \)-syzygy of the above resolution, it is \( \mathcal{E} \)-G-projective, and hence \( \mathcal{E} \)-G\( \text{pd} \) \( A \leq n \). Similar argument shows that \( \mathcal{E} \)-Ginjective dimension of any object is finite and so \( \mathcal{C} \) is \( \mathcal{E} \)-Gorenstein. \( \square \)

**Theorem 4.11.** Let \( \mathcal{C} \) be an \( \mathcal{E} \)-\( n \)-Gorenstein triangulated category. Then for any objects \( A \) and \( B \) of \( \mathcal{C} \) and any integer \( i \in \mathbb{Z} \),

\[
\widehat{\mathcal{E}xt}_P^i(A, B) \cong \widehat{\mathcal{E}xt}_\mathcal{F}^i(A, B).
\]
Proof. Since both $\mathcal{E}$-$\mathcal{G}$-pd $A$ and $\mathcal{E}$-$\mathcal{G}$-id $B$ are less than or equal to $n$, by Theorem 3.4(1) and Theorem 3.9(1), for all $i > n$, $\mathcal{E}xt^i_P(A, B) \cong \mathcal{E}xt^i_K(A, B)$, because both groups are isomorphic to the absolute $\mathcal{E}xt_{\mathcal{C}}$. Let $i \in \mathbb{Z}$ be an arbitrary integer. Consider the triangle $K_0 \to P_0 \to B \to \Sigma K$ in $\mathcal{E}$, where $P_0 \in \mathcal{P}(\mathcal{E})$. This triangle exists because $\mathcal{C}$ has enough $\mathcal{E}$-projectives. Since $P_0 \in \mathcal{P}(\mathcal{E})$, by Theorem 3.8, $\mathcal{E}xt^j_P(\ , P_0) = 0$, for all $j \in \mathbb{Z}$. So the long exact sequence of Theorem 3.5 implies that for all integer $j$, $\mathcal{E}xt^j_P(A, B) \cong \mathcal{E}xt^{j+1}_K(A, K_0)$. By continuing in this way, we are able to increase the superscript of $\mathcal{E}xt^j_P$ up to $n + 1$. On the other hand, since $\mathcal{E}$-$\text{silp} \mathcal{C} < \infty$, $\mathcal{E}$-$\text{id} P_0 < \infty$ and so by Theorem 3.11, $\mathcal{E}xt^j_P(\ , P_0) = 0$ for all $j \in \mathbb{Z}$. Hence the long exact sequence of Proposition 3.10, gives the isomorphism $\mathcal{E}xt^j_P(A, B) \cong \mathcal{E}xt^{j+1}_K(A, K_0)$. Again we can increase the powers of $\mathcal{E}xt^j_P$ up to $n + 1$. The result hence follows. □

Definition 4.12. A pair of classes of objects $(A, B)$ is a cotorsion theory provided that $A = A^\perp B$ and $B = A^\perp$. A cotorsion theory $(A, B)$ is called $\mathcal{E}$-complete provided that for each object $X$ there is a triangle $X \to B \to A \to \Sigma X$ in $\mathcal{E}$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$ (or, equivalently, provided that for each object $Y \in \mathcal{C}$ there is a triangle $B \to A \to Y \to \Sigma B$ in $\mathcal{E}$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$).

Cotorsion theories were invented by L. Salce [S] in the category of abelian groups, and were rediscovered by E.E. Enochs and coauthors in the 1990’s. Given a class $\mathcal{X}$ of objects of $\mathcal{C}$ the pairs

$$(\updownarrow \mathcal{X}, (\updownarrow \mathcal{X})^\perp) \quad \text{and} \quad (\updownarrow (\mathcal{X}^\perp), \mathcal{X}^\perp)$$

are cotorsion theories.

Theorem 4.13. Let $\mathcal{C}$ be an $\mathcal{E}$-$n$-Gorenstein triangulated category. Then $(\mathcal{P}(\mathcal{E}), \mathcal{G}I(\mathcal{E}))$ form an $\mathcal{E}$-complete cotorsion theory.

Proof. First we prove that $\mathcal{P}(\mathcal{E}) = (\mathcal{G}I(\mathcal{E}))$. The inclusion $\mathcal{P}(\mathcal{E}) \subseteq (\mathcal{G}I(\mathcal{E}))$ follows from Proposition 2.13. Now let $A \in \mathcal{C}$ be such that $\mathcal{E}xt^1_{\mathcal{E}}(A, G) = 0$ for all $G \in \mathcal{G}I(\mathcal{E})$. Let

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0,$$

be an $\mathcal{E}$-projective resolution of $A$. Consider the relevant triangles $K_{i+1} \to P_i \to K_i \to \Sigma K_{i+1} \ (i \geq 0)$. Since $\mathcal{C}$ is $\mathcal{E}$-$n$-Gorenstein, $\mathcal{E}$-$\text{pd} A \leq n$. Hence $K_n$ is $\mathcal{E}$-$\text{G}$-projective. The triangles $K_{i+1} \to P_i \to K_i \to \Sigma K_{i+1} \ (i \geq 0)$, in view of the fact that $\mathcal{E}$-$\text{id} P_i < \infty$, imply that $\mathcal{E}xt^1_{\mathcal{E}}(A, \ ) \cong \mathcal{E}xt^{-n+1}_{\mathcal{E}}(K_n, \ )$. On the other hand, by Theorem 3.9(1), for any $\mathcal{E}$-$\text{G}$-injective object $G$ and any integer $i > 0$, $\mathcal{E}xt^i_{\mathcal{E}}(A, G) \cong \mathcal{E}xt^i_{\mathcal{E}}(A, G)$. So in view of our assumption, $\mathcal{E}xt^{-n+1}_{\mathcal{E}}(K_n, G) = 0$. Now consider an $\mathcal{E}$-$\text{G}$-injective coresolution of $K_n$. The $\mathcal{E}$-Gorensteinness of $\mathcal{C}$ implies that $\mathcal{E}$-$\text{G}$-id $K_n < n$. So its $n$th $\mathcal{E}$-cosyzygy $L^n$ is $\mathcal{E}$-$\text{G}$-injective. The relevant triangles in the $\mathcal{E}$-injective coresolution of $K_n$ imply that $\mathcal{E}xt^{-n+1}_{\mathcal{E}}(K_n, \ ) \cong$
Therefore, for any $E$-injective object $G$, $\hat{\text{Ext}}^1(T, L^n, G) = 0$. Since $L^n$ is $E$-injective, we have triangle $H \to E \to L^n \to \Sigma H$ in $E$ in which $E$ is $E$-injective and $H$ is $E$-injective. Using this triangle we get $\hat{\text{Ext}}^0(T, L^n, E) \cong \hat{\text{Ext}}^1(T, L^n, H)$. But since $H$ is $E$-injective the latter group is zero. So Theorem 3.11 can be applied to show that $L^n$ has finite $E$-injective dimension. Hence $E$-id $K_n < \infty$. Therefore we can use Theorem 4.3 to deduce that $K_n$ is $E$-projective. So $E$-pd $A < \infty$. The second equality $\tilde{\mathcal{P}}(E)^{\perp} = \mathcal{G}I(E)$ follows from Proposition 2.15, in view of the facts that $E$-Gorensteinness of $C$ implies that any object of $C$ belongs to $\mathcal{G}I$ and also $\tilde{I}(E) = \tilde{\mathcal{P}}(E)$.

$E$-completeness of this cotorsion theory follows from a construction dual to the [AS, Construction 4.5].

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\section*{References}