Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®
European Journal of Combinatorics 25 (2004) 141–144
European Journal of Combinatorics
www.elsevier.com/locate/ejc

Letter to the Editor

Comments on "Supersolvable frame-matroid and graphic-lift lattices" by T. Zaslavsky [European Journal of Combinatorics 22 (2001) 119]☆

Lori Koban

Department of Mathematics, Binghamton University, Binghamton, 13902-6000 NY, USA

Received 30 January 2003; accepted 6 June 2003

Abstract

To characterize supersolvable frame matroids, Zaslavsky, in his 2001 paper "Supersolvable frame-matroid and graphic-lift lattices", utilizes a classification of their modular copoints. Unfortunately, one type of modular copoint is missing in the classification. We correct the modular copoint theorem and show that the supersolvability theorem is not affected, while giving a simpler proof.

Supersolvability is a property of matroids that provides information about their characteristic polynomials. Though there is no classification theorem for supersolvable matroids in general, Zaslavsky utilizes his theory of biased graphs to prove such a theorem for frame matroids, one example being the linear dependence matroid of a subset of a classical root system. Employed in the proof of Zaslavsky's supersolvability theorem is a classification of the modular copoints of frame matroids. Unfortunately, a case is missing in the classification. Here we present the missing modular copoint and explain why the supersolvability theorem is not affected, while giving a simpler proof.

We assume familiarity with [3]. All references in this paper are to [3] unless indicated otherwise. In particular, Theorems 2.1 and 2.2 are from [3].

We begin by correcting Theorem 2.1, which classifies the modular copoints of a frame matroid. Frame matroids have two kinds of copoint: a maximal balanced edge set A, and

☆ PII of original article: S0029-8018(00)00004-4

E-mail address: fern@math.binghamton.edu (L. Koban).

0195-6698/$ - see front matter © 2003 Elsevier Ltd. All rights reserved.
doi:10.1016/S0195-6698(03)00122-7

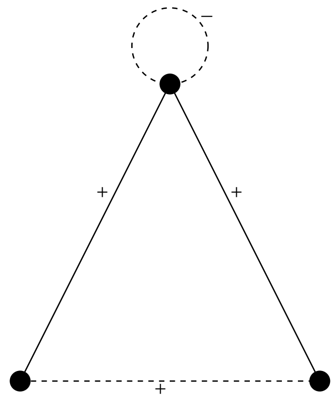

Fig. 1. A modular copoint of this biased graph consists of the dashed edges. This example shows why Case (1.5) of Theorem 2.1′ is necessary.

an unbalanced edge set of the form

$$A = E : Y^c \cup A_Y \text{ where } \emptyset \subset Y \subset V, b(\Omega : Y^c) = 0, (Y, A_Y) \text{ is connected, and } A_Y \text{ is a maximal balanced edge set in } \Omega : Y. \quad (*)$$

The modular copoints of the first type appear in Theorem 2.1′(2–6); those of the second type appear in Cases (1) and (1.5).

Theorem 2.1′. *Let Ω be a connected, simply biased graph. A subset $A \subset E$ is a modular copoint of $G(\Omega)$ if and only if Ω and A are of one of the following types.*

(1) *Ω is a one-point amalgamation $\Omega_1 \cup_v \Omega_2$, where Ω_2 is balanced, $E_1 \neq \emptyset$, and v is a bias-simplicial vertex in Ω_1; and $A = (E_1 : \{v\}^c) \cup E_2$. ($\Omega_2 = \{v\}$ is possible.)*

(1.5) *Ω is a one-point amalgamation $\Omega_1 \cup_v \Omega_2$, where Ω_2 is balanced, $E_2 \neq \emptyset$, and v is a bias-simplicial vertex in Ω_2; and $A = (E_2 : \{v\}^c) \cup E_1$. ($\Omega_1 = \{v\}$ is possible.)*

(2–6) *As in [3, Theorem 2.1(2–6)].*

Case (1) of this theorem is identical to Theorem 2.1(1) except for the necessary additional requirement that $E_1 \neq \emptyset$. More important is the addition of Case (1.5). Fig. 1 shows a biased graph with a modular copoint different from Cases (1) and (2–6), which fits into Case (1.5).

Proof. The proof of sufficiency is straightforward.

We prove necessity. If A is a maximal balanced edge set, then Ω and A are as in one of (2–6) (see [3]).

Otherwise A is as in (∗). For A to intersect every line of type M_2, all links with one vertex in Y and the other vertex in Y^c must be incident on some node, call it v.

Suppose $v \in Y$. Then Ω and A are as in Case (1) with $\Omega_1 = \Omega : (Y^c \cup \{v\})$ and $\Omega_2 = \Omega : Y$ less the unbalanced edge at v if it exists. The proof can be found in [3, top of p. 125], where Ω_2, Ω_1, and p correspond to our Ω_1, Ω_2, and v respectively.

Now suppose $v \in Y^c$. We will show that Ω and A are as in Case (1.5) with $\Omega_1 = \Omega : Y^c$ and $\Omega_2 = \Omega : (Y \cup \{v\})$ less the unbalanced edges at v if it exists. To avoid duplicating the previous case, we assume the existence of non-parallel links e and f, each having one vertex in Y and the other at v. We must prove that Ω_2 is balanced and that v is bias simplicial in Ω_2. Using the techniques in [3, top of p. 125], we can show that $E(\Omega : Y) = A_Y$, which is balanced. Now let C be an unbalanced circle in Ω_2 of length greater than 2. Then C contains v, and the modularity of A forces C to be part of a theta graph with two balanced circles (one being a triangle on v), a contradiction. Now we need to show that Ω_2 contains no unbalanced digons. Suppose eg is an unbalanced digon. This forces the existence of links k and l such that ekf and glf are lines of type $\langle C_3 \rangle$. But $E(\Omega : Y)$ is balanced, so kl is a balanced digon, which is a contradiction. Thus Ω_2 is balanced. It follows that v is a bias-simplicial vertex in Ω_2. □

The proof given in [3] of Theorem 2.2, the classification of supersolvable frame matroids, consists of six arguments, one for each type of modular copoint in Theorem 2.1. This proof is incomplete because of the existence of Case (1.5). We could prove the theorem by making an argument for this type of modular copoint, but instead we provide a proof that is conceptually simpler. We will show that if $G(\Omega)$ is supersolvable, then we can almost always find a bias-simplicial vertex v, and hence a modular copoint of the form $E(\Omega : \{v\}^c)$. This idea is formalized in Lemma 1, which explains why the addition of Case (1.5) does not necessitate corrections to Theorem 2.2. The heart of Lemma 1 is the first case; the second case contains the exceptions.

Lemma 1. *If Ω and A are as in Theorem 2.1′ and $G(\Omega) \mid A$ is supersolvable, then either*

(1) *Ω has a bias-simplicial vertex v, hence a modular copoint $A' = E(\Omega : \{v\}^c)$, for which $G(\Omega) \mid A'$ is supersolvable, or*

(2) (a) *$\Omega = (mK_2, \emptyset)$ for some $m \geq 2$, or*
  (b) *$\Omega = \langle \pm K_3 \rangle$, or*
  (c) *Ω and A are as in [3, Theorem 2.1(5)].*

Proof. Assume Ω and A are in Case (1). If $\Omega_2 = \{v\}$, we are done, so assume Ω_2 contains an edge. Since Ω_2 is balanced, $G(\Omega) = G(\Omega_1) \oplus G(\Omega_2)$. Thus the supersolvability of $G(\Omega)$ implies the supersolvability of $G(\Omega_1)$ and $G(\Omega_2)$. Consequently, $\|\Omega_2\|$ is a chordal graph. By Dirac's lemma (see [1, Lemma 4.2]), $\|\Omega_2\|$ has a simplicial vertex different from v, call it u. Since u is also a bias-simplicial vertex in Ω, we let $A' = E(\Omega : \{u\}^c)$. Because $\Omega_2 : \{u\}^c$ is balanced, $G(\Omega) \mid A' = G(\Omega_1) \oplus G(\Omega_2 : \{u\}^c)$; and because $\|\Omega_2 : \{u\}^c\|$ is chordal, we conclude that $G(\Omega) \mid A'$ is supersolvable.

The proof of Case (1.5) is similar to that of Case (1).

Before proceeding with the other cases, we prove by induction on the order of Ω that if Ω is as in Theorem 2.1(2) and Γ is chordal, then Ω has a b.s.v.o. Assume that Ω has order $n \geq 3$, and suppose $U \neq V$. If Γ is complete, then any $u \in V \backslash U$ is a bias-simplicial vertex in Ω. By induction, $\Omega : \{u\}^c$ has a b.s.v.o., so adding u to the end of the ordering yields a b.s.v.o. for Ω. If Γ is not complete, then by Dirac's lemma there is a simplicial vertex of Γ that is not in U. This vertex is a bias-simplicial vertex in Ω, so we can proceed as above. If $U = V$, u can be any vertex.

Now assume Ω and A are as in Case (2). Since $G(\Omega) \mid A$ is supersolvable, we know that Γ is chordal. So Ω has a b.s.v.o., call it $v_1, \ldots, v_n$. Let $A' = E(\Omega : \{v_n\}^c)$. Since $v_1, \ldots, v_{n-1}$ is a b.s.v.o. for $\Omega \mid A'$, we conclude that $G(\Omega) \mid A'$ is supersolvable.

We omit the proofs of Cases (3–6) because they are similar to the proof of Case (2). There are a few exceptional cases where a bias-simplicial vertex may not be found. These exceptions appear in Lemma 1(2). □

Proof of Theorem 2.2. As in [3], we assume that Ω is connected and unbalanced, and we omit the proof of sufficiency. To prove necessity, we assume that Ω has a modular copoint A such that $G(\Omega) \mid A$ is supersolvable. By applying Lemma 1, we know that Ω is as in Theorem 2.2(ii) or else there exists A' as in Lemma 1(1). We pursue the last possibility by induction on the order of Ω. Assume Ω has order $n \geq 3$. By induction, either $\Omega \mid A'$ has a b.s.v.o. or $\Omega \mid A'$ is a simplicial extension of one of the base graphs specified in Theorem 2.2(ii). In the former case, add v to the end of the ordering to get a b.s.v.o. for Ω. In the latter case, v cannot support an unbalanced edge or unbalanced digon, so it is a simplicial vertex. Thus Ω is also a simplicial extension of one of the base graphs in Theorem 2.2(ii). □

The other results in [3] (including the graphic-lift results in Section 3) are correct, except that Corollary 4.5(b) requires a minor change.

Corollary 4.5(b)′. *Let Γ be a graph of finite order. $G(\Gamma, \emptyset)$ has a modular coatom if and only if Γ has a leaf whose degree is one or whose neighbor is filled, or Γ has a component Γ' where $\Gamma' = mK_2$ with $m \geq 2$, or Γ' is a unicycle.*

This paper is the result of my study of [2], where Yoon purports to characterize the supersolvable frame matroids that arise from signed graphs. His result, therefore, ought to be a special case of Theorem 2.2. While comparing the results of Zaslavsky and Yoon, I discovered both the incompleteness of Theorem 2.1 and that Yoon's supersolvability result is incorrect.

References

[1] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, Academic Press, New York, 1980.
[2] Young-Jin Yoon, A characterization of supersolvable signed graphs, Commun. Korean Math. Soc. 12 (4) (1997) 1069–1073.
[3] T. Zaslavzky, Supersolvable frame-matroid and graphic-lift lattices, European J. Combin. 22 (2001) 119–133.