Tridiagonal pairs and the q-tetrahedron algebra

Darren Funk-Neubauer *

Department of Mathematics and Physics, Colorado State University – Pueblo, 2200 Bonforte Boulevard, Pueblo, CO 81001, USA

ARTICLE INFO

ABSTRACT

The q-tetrahedron algebra $\boxtimes_q$ was recently introduced and has been studied in connection with tridiagonal pairs. In this paper we further develop this connection. Let $\mathbb{K}$ denote an algebraically closed field and let q denote a nonzero scalar in $\mathbb{K}$ that is not a root of unity. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension and let A, A^* denote a tridiagonal pair on V. Let $\{\theta_i\}_{i=0}^{d}$, (resp. $\{\theta_i^*\}_{i=0}^{d}$) denote a standard ordering of the eigenvalues of A (resp. A^*). Ito and Terwilliger have shown that when $\theta_i = q^{2i-d}$ and $\theta_i^* = q^{d-2i} (0 \leqslant i \leqslant d)$ there exists an irreducible $\boxtimes_q$-module structure on V such that the $\boxtimes_q$ generators x_{01}, x_{23} act as A, A^* respectively. In this paper we examine the case in which there exists a nonzero scalar c in $\mathbb{K}$ such that $\theta_i = q^{2i-d}$ and $\theta_i^* = q^{2i-d} + cq^{d-2i}$ for $0 \leqslant i \leqslant d$. In this case we associate to A, A^* a polynomial P in one variable and prove the following theorem as our main result.

Theorem. *The following are equivalent:*

(i) *There exists a $\boxtimes_q$-module structure on V such that x_{01} acts as A and $x_{30} + cx_{23}$ acts as A^*, where x_{01}, x_{30}, x_{23} are standard generators for $\boxtimes_q$.*
(ii) $P(q^{2d-2}(q - q^{-1})^{-2}) \neq 0.$

Suppose (i) and (ii) hold. Then the $\boxtimes_q$-module structure on V is unique and irreducible.

© 2009 Elsevier Inc. All rights reserved.

* Tel.: +1 719 549 2693; fax: +1 719 549 2962.
 E-mail address: darren.funkneubauer@colostate-pueblo.edu

1. Tridiagonal pairs

We begin by recalling the notion of a *tridiagonal pair* [16]. Let $\mathbb{K}$ denote a field and let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $A : V \to V$ denote a linear transformation and let W denote a subspace of V. We say W is an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that

$$W = \{v \in V | Av = \theta v\}.$$

In this case, we call θ an *eigenvalue* of A. We say A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1.1 [16, Definition 1.1]. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *tridiagonal pair* on V, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions:

  (i) Each of A, A^* is diagonalizable.
 (ii) There exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \quad (0 \leqslant i \leqslant d), \tag{1}$$

  where $V_{-1} = 0, V_{d+1} = 0$.
(iii) There exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leqslant i \leqslant \delta), \tag{2}$$

  where $V_{-1}^* = 0, V_{\delta+1}^* = 0$.
(iv) There does not exist a subspace W of V such that $AW \subseteq W, A^* W \subseteq W, W \neq 0, W \neq V$.

Note 1.2. According to a common notational convention A^* denotes the conjugate-transpose of A. We are not using this convention. For a tridiagonal pair A, A^* the linear transformations A and A^* are arbitrary subject to (i)–(iv) above.

Tridiagonal pairs originally arose in algebraic combinatorics through the study of a combinatorial object called a *P*- and *Q*-polynomial association scheme [16]. Since then they have appeared in many other areas of mathematics. For instance, examples of tridiagonal pairs appear in representation theory [2,6,8,12,21,23], the study of orthogonal polynomials and special functions [39,43,47], the theory of partially ordered sets [38,42], and statistical mechanics [4,15,40]. The tridiagonal pairs for which the V_i, V_i^* all have dimension 1 are called *Leonard pairs*. The Leonard pairs are classified and correspond to a family of orthogonal polynomials consisting of the *q*-Racah polynomials and related polynomials in the Askey scheme [39,43]. Currently there is no classification of tridiagonal pairs; this paper is largely motivated by the search for this classification. For further information on tridiagonal pairs and Leonard pairs see [1,3,5,9,10,13,18,22,24–37,41, 44–46,48–50].

We now recall a few basic facts about tridiagonal pairs. Let A, A^* denote a tridiagonal pair on V and let d, δ be as in Definition 1.1(ii) and (iii). By [16, Lemma 4.5] we have $d = \delta$; we call this common value the *diameter* of A, A^*. An ordering of the eigenspaces of A (resp. A^*) will be called *standard* whenever it satisfies (1) (resp. (2)). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^{d}$ denote a standard ordering of the eigenspaces of A. Then the ordering $\{V_{d-i}\}_{i=0}^{d}$ is standard and no other ordering is standard. A similar result holds for the eigenspaces of A^*. An ordering of the eigenvalues of A (resp. A^*) will be called *standard* whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard. Let $\{\theta_i\}_{i=0}^{d}$ (resp. $\{\theta_i^*\}_{i=0}^{d}$) denote a standard ordering of the eigenvalues of A (resp. A^*). The θ_i, θ_i^* both satisfy a three term recurrence relation that has been solved in closed form [16, Theorem 11.1, Theorem 11.2]. The following two special cases will be of interest to us. For $0 \neq q \in \mathbb{K}$

we call A, A^* *q-geometric* whenever $\theta_i = q^{2i-d}$ and $\theta_i^* = q^{d-2i}$ for $0 \leqslant i \leqslant d$. For a study of this case see [2,3,21,23]. For $0 \neq q \in \mathbb{K}$ we call A, A^* *q-mixed* whenever there exists $0 \neq c \in \mathbb{K}$ such that $\theta_i = q^{2i-d}$ and $\theta_i^* = q^{2i-d} + cq^{d-2i}$ for $0 \leqslant i \leqslant d$. The main result of this paper concerns the q-mixed tridiagonal pairs. These two cases are of interest because of their connection to the q-tetrahedron algebra $\boxtimes_q$. We discuss this connection in the next section.

2. The *q*-tetrahedron algebra

The q-tetrahedron algebra $\boxtimes_q$ was introduced in [20] as part of the continuing investigation of tridiagonal pairs. It is closely related to a number of well known algebras including the quantum group $U_q(\mathfrak{sl}_2)$ [20, Proposition 7.4], the $U_q(\mathfrak{sl}_2)$ loop algebra [20, Proposition 8.3], and positive part of $U_q(\widehat{\mathfrak{sl}}_2)$ [20, Proposition 9.4]. The finite dimensional irreducible $\boxtimes_q$-modules are described in [20]. For further information on $\boxtimes_q$ see [19]. We note that $\boxtimes_q$ is a q-analogue of the tetrahedron algebra $\boxtimes$ [7,11,14,15,17].

We now fix some notation and then recall the definition of $\boxtimes_q$.

Throughout the rest of this paper $\mathbb{K}$ will denote an algebraically closed field. We fix a nonzero scalar $q \in \mathbb{K}$ that is not a root of 1. For an integer $n \geqslant 0$ we define

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{3}$$

We let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4.

Definition 2.1 [20, Definition 6.1]. Let $\boxtimes_q$ denote the unital associative $\mathbb{K}$-algebra that has generators

$$\{x_{ij} \mid i, j \in \mathbb{Z}_4, \ j - i = 1 \text{ or } j - i = 2\}$$

and the following relations:

(i) For $i, j \in \mathbb{Z}_4$ such that $j - i = 2$,

$$x_{ij} x_{ji} = 1.$$

(ii) For $h, i, j \in \mathbb{Z}_4$ such that the pair $(i - h, j - i)$ is one of $(1, 1), (1, 2), (2, 1)$,

$$\frac{q x_{ij} x_{hi} - q^{-1} x_{hi} x_{ij}}{q - q^{-1}} = 1. \tag{4}$$

(iii) For $h, i, j, k \in \mathbb{Z}_4$ such that $i - h = j - i = k - j = 1$,

$$x_{hi}^3 x_{jk} - [3] x_{hi}^2 x_{jk} x_{hi} + [3] x_{hi} x_{jk} x_{hi}^2 - x_{jk} x_{hi}^3 = 0. \tag{5}$$

We call $\boxtimes_q$ the *q-tetrahedron algebra*. We refer to the x_{ij} as the *standard generators* for $\boxtimes_q$.

Remark 2.2. Eq. (5) is called the *cubic q-Serre relation*.

We now recall a few basic facts about $\boxtimes_q$-modules. Let V denote a finite dimensional irreducible $\boxtimes_q$-module. By [20, Theorem 12.3] each generator x_{ij} of $\boxtimes_q$ is diagonalizable on V. Moreover, there exist an integer $d \geqslant 0$ and a scalar $\varepsilon \in \{1, -1\}$ such that for each generator x_{ij} the set of distinct eigenvalues of x_{ij} on V is $\{\varepsilon q^{2n-d} \mid 0 \leqslant n \leqslant d\}$. We call ε the *type* of V.

We now discuss the connection between q-geometric tridiagonal pairs and finite dimensional irreducible $\boxtimes_q$-modules. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let A, A^* denote a q-geometric tridiagonal pair on V. Then there exists an irreducible $\boxtimes_q$-module structure on V of type 1 such that A acts as x_{01} and A^* acts as x_{23}. Conversely, let V denote a finite dimensional irreducible $\boxtimes_q$-module of type 1. Then the generators x_{01}, x_{23} act on V as a q-geometric tridiagonal pair [47, Theorem 34.14].

Inspired by the above result we consider the connection between q-mixed tridiagonal pairs and finite dimensional irreducible $\boxtimes_q$-modules. In the next section we give a detailed description of the situation we wish to consider.

3. Assumptions and motivation

Throughout the rest of the paper we will be concerned with the following situation.

Assumption 3.1. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension and let A, A^* denote a tridiagonal pair on V. Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) denote a standard ordering of the eigenspaces of A (resp. A^*). For $0 \leqslant i \leqslant d$, let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with V_i (resp. V_i^*). We assume there exists a nonzero $c \in \mathbb{K}$ such that $\theta_i = q^{2i-d}$ and $\theta_i^* = q^{2i-d} + cq^{d-2i}$ for $0 \leqslant i \leqslant d$.

To motivate our main result we make some comments.

Lemma 3.2 [16, Theorems 10.1, 11.1]. *With reference to Assumption* 3.1 *we have*

(i) $A^3 A^* - [3]A^2 A^* A + [3]A A^* A^2 - A^* A^3 = 0$,
(ii) $A^{*3}A - [3]A^{*2}AA^* + [3]A^*AA^{*2} - AA^{*3} + c(q^2 - q^{-2})^2(A^*A - AA^*) = 0$.

Using (4) and (5) it can be shown that for $0 \neq c \in \mathbb{K}$ the elements x_{01} and $x_{30} + cx_{23}$ of $\boxtimes_q$ satisfy the relations in Lemma 3.2. Given this, it is natural to ask the following question. With reference to Assumption 3.1 when does there exist an irreducible $\boxtimes_q$-module structure on V of type 1 such that A acts as x_{01} and A^* acts as $x_{30} + cx_{23}$? In this paper we answer this question; our main result is Theorem 5.4. In the next section we establish some notation needed to state our main result.

4. A split decomposition and its raising/lowering maps

We now recall the notion of a split decomposition of a tridiagonal pair and its corresponding raising and lowering maps.

Definition 4.1. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *decomposition* of V we mean a sequence $\{U_i\}_{i=0}^d$ consisting of nonzero subspaces of V such that $V = \sum_{i=0}^d U_i$ (direct sum). For notational convenience we set $U_{-1} := 0, U_{d+1} := 0$.

Referring to Assumption 3.1 the sequences $\{V_i\}_{i=0}^d$ and $\{V_i^*\}_{i=0}^d$ are both decompositions of V. We now mention another decomposition of interest.

Lemma 4.2 [16, Theorem 4.6]. *With reference to Assumption* 3.1, *for* $0 \leqslant i \leqslant d$ *define*

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d).$$

Then $\{U_i\}_{i=0}^d$ *is a decomposition of* V. *Moreover, for* $0 \leqslant i \leqslant d$

$$(A^* - \theta_i^* I)U_i \subseteq U_{i-1}, \quad (A - \theta_i I)U_i \subseteq U_{i+1}, \tag{6}$$

$$U_0 + \cdots + U_i = V_0^* + \cdots + V_i^*, \ U_i + \cdots + U_d = V_i + \cdots + V_d. \tag{7}$$

We call $\{U_i\}_{i=0}^d$ *the* split decomposition *of* V *corresponding to the given orderings* $\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d$.

Definition 4.3 [16, Definition 5.2]. With reference to Assumption 3.1 and Lemma 4.2 we define the following. For $0 \leqslant i \leqslant d$ we define a linear transformation $F_i : V \to V$ by

$$(F_i - I)U_i = 0,$$
$$F_i U_j = 0 \quad \text{if } j \neq i, \quad (0 \leqslant j \leqslant d).$$

In other words, F_i is the projection map from V onto U_i. We call F_i the *ith projection map corresponding to* $\{U_i\}_{i=0}^d$.

Definition 4.4 [16, Definition 6.1]. With reference to Assumption 3.1 and Definition 4.3 we define

$$R = A - \sum_{h=0}^d \theta_h F_h, \quad L = A^* - \sum_{h=0}^d \theta_h^* F_h. \tag{8}$$

Combining (6) and (8) we have $RU_i \subseteq U_{i+1}$ and $LU_i \subseteq U_{i-1}$ for $0 \leqslant i \leqslant d$. We call R (resp. L) the *raising (resp. lowering) map corresponding to* $\{U_i\}_{i=0}^d$.

5. The main theorem

In this section we state our main result. We begin with a few comments.

Lemma 5.1 [27, Theorem 1.3]. *With reference to Assumption 3.1 and Lemma 4.2 we have* $\dim(U_0) = 1$.

Definition 5.2. With reference to Definition 4.4 and Lemma 5.1 we find that for $0 \leqslant i \leqslant d$ U_0 is contained in an eigenspace for $L^i R^i$; let ζ_i denote the corresponding eigenvalue.

With reference to (3) for an integer $n \geqslant 0$ we define

$$[n]! = [n][n-1] \cdots [1]. \tag{9}$$

We interpret $[0]! = 1$.

Definition 5.3. With reference to Assumption 3.1 and Definition 5.2 we define a polynomial $P \in \mathbb{K}[\lambda]$ (λ indeterminate) by

$$P = \sum_{i=0}^d \frac{q^{i(1-i)} \zeta_i \lambda^i}{[i]!^2}.$$

We now state our main result.

Theorem 5.4. *With reference to Assumption 3.1 the following are equivalent*:

(i) *There exists a* $\boxtimes_q$-*module structure on V such that x_{01} acts as A and $x_{30} + c x_{23}$ acts as A^*.*
(ii) $P(q^{2d-2}(q - q^{-1})^{-2}) \neq 0$ *where P is from Definition 5.3.*

Suppose (i) *and* (ii) *hold. Then the* $\boxtimes_q$-*module structure on V is unique, irreducible, and has type* 1.

6. An outline of the proof of Theorem 5.4

Our proof of Theorem 5.4 will consume the remainder of the paper from Section 7 to Section 18. Here we sketch an overview of the argument.

We adopt Assumption 3.1. The main idea used in proving Theorem 5.4 is the following. We modify the linear transformation $A^* : V \to V$ to produce a new linear transformation $\widetilde{A}^* : V \to V$ and we show that $A, \widetilde{A}^*$ is a q-geometric tridiagonal pair on V if and only if $P(q^{2d-2}(q - q^{-1})^{-2}) \neq 0$. Then

we apply [23, Theorem 2.7, 21, Theorem 10.4] to $A, \widetilde{A}^*$ to produce the $\boxtimes_q$-module structure on V as in Theorem 5.4. The plan for the paper is as follows. In Sections 7 and 8 we present some lemmas and definitions which will be used as tools throughout the remainder of the paper. In Section 9 we define the linear transformation $\widetilde{A}^* : V \to V$. We show that $\widetilde{A}^*$ is diagonalizable on V and the set of distinct eigenvalues of $\widetilde{A}^*$ on V is $\{q^{d-2i} | 0 \leqslant i \leqslant d\}$. In Section 10 we show that A and $\widetilde{A}^*$ satisfy Definition 1.1(ii) and (iii). Sections 11–17 are devoted to showing that A and $\widetilde{A}^*$ satisfy Definition 1.1(iv) if and only if $P(q^{2d-2}(q - q^{-1})^{-2}) \neq 0$. We note that the arguments given in Sections 11–17 are a modification of the arguments from [23, Sections 7–12]. In Section 18 we show how to use [23, Theorem 2.7, 21, Theorem 10.4] applied to the q-geometric tridiagonal pair $A, \widetilde{A}^*$ to produce the $\boxtimes_q$-module structure on V as in Theorem 5.4.

7. Some more raising/lowering maps

We now present another split decomposition for the tridiagonal pair A, A^* and its corresponding raising and lowering maps.

Definition 7.1. With reference to Assumption 3.1 and Lemma 4.2 let $\{W_i\}_{i=0}^d$ denote the split decomposition of V corresponding to the orderings $\{V_{d-i}\}_{i=0}^d, \{V_i^*\}_{i=0}^d$. With reference to Definition 4.3 for $0 \leqslant i \leqslant d$, let G_i denote the ith projection map corresponding to $\{W_i\}_{i=0}^d$. With reference to Definition 4.4 let r (resp. l) denote the raising (resp. lowering) map corresponding to $\{W_i\}_{i=0}^d$.

We make the following three remarks in order to emphasize the similarities and differences between the two split decompositions $\{U_i\}_{i=0}^d$ and $\{W_i\}_{i=0}^d$.

Remark 7.2. With reference to Definition 7.1 we emphasize the following. For $0 \leqslant i \leqslant d$ we have

$$W_i = (V_0^* + \cdots + V_i^*) \cap (V_0 + \cdots + V_{d-i}).$$

Moreover, for $0 \leqslant i \leqslant d$

$$(A^* - \theta_i^* I)W_i \subseteq W_{i-1}, \ (A - \theta_{d-i} I)W_i \subseteq W_{i+1}, \tag{10}$$
$$W_0 + \cdots + W_i = V_0^* + \cdots + V_i^*, \ W_i + \cdots + W_d = V_0 + \cdots + V_{d-i}. \tag{11}$$

Remark 7.3. With reference to Definition 7.1 we emphasize that for $0 \leqslant i \leqslant d$

$$(G_i - I)W_i = 0,$$
$$G_i W_j = 0 \ \text{ if } j \neq i, \ \ (0 \leqslant j \leqslant d).$$

Remark 7.4. With reference to Definition 7.1 we emphasize that

$$r = A - \sum_{h=0}^d \theta_{d-h} G_h, \ \ l = A^* - \sum_{h=0}^d \theta_h^* G_h. \tag{12}$$

Moreover, for $0 \leqslant i \leqslant d, rW_i \subseteq W_{i+1}$ and $lW_i \subseteq W_{i-1}$.

8. Some linear algebra

In this section we state some linear algebraic results that will be useful throughout the paper.

We use the following notation. Let V denote a finite dimensional vector space over $\mathbb{K}$ and let $X : V \to V$ denote a linear transformation. For $\theta \in \mathbb{K}$ we define

$$V_X(\theta) = \{v \in V | Xv = \theta v\}.$$

Observe that θ is an eigenvalue of X if and only if $V_X(\theta) \neq 0$, and in this case $V_X(\theta)$ is the corresponding eigenspace.

Lemma 8.1 [20, Lemma 11.2]. *Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $X : V \to V$ and $Y : V \to V$ denote linear transformations. Then for all nonzero $\theta \in \mathbb{K}$ the following are equivalent*:

(i) *The expression $qXY - q^{-1}YX - (q - q^{-1})I$ vanishes on $V_X(\theta)$.*
(ii) *$(Y - \theta^{-1}I)V_X(\theta) \subseteq V_X(q^{-2}\theta)$.*

Lemma 8.2 [20, Lemma 11.3]. *Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $X : V \to V$ and $Y : V \to V$ denote linear transformations. Then for all nonzero $\theta \in \mathbb{K}$ the following are equivalent*:

(i) *The expression $qXY - q^{-1}YX - (q - q^{-1})I$ vanishes on $V_Y(\theta)$.*
(ii) *$(X - \theta^{-1}I)V_Y(\theta) \subseteq V_Y(q^2\theta)$.*

Lemma 8.3. *Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $X : V \to V$ and $Y : V \to V$ denote linear transformations. Fix a nonzero $c \in \mathbb{K}$. Then for all nonzero $\theta \in \mathbb{K}$ the following are equivalent*:

(i) *The expression $qXY - q^{-1}YX - (q - q^{-1})(X^2 + cI)$ vanishes on $V_X(\theta)$.*
(ii) *$(Y - \theta I - c\theta^{-1}I)V_X(\theta) \subseteq V_X(q^{-2}\theta)$.*

Proof. For $v \in V_X(\theta)$ we have

$$(qXY - q^{-1}YX - (q - q^{-1})(X^2 + cI))v = q(X - q^{-2}\theta I)(Y - \theta I - c\theta^{-1}I)v$$

and the result follows. $\square$

Lemma 8.4 [20, Lemma 11.4]. *Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $X : V \to V$ and $Y : V \to V$ denote linear transformations such that*

$$\frac{qXY - q^{-1}YX}{q - q^{-1}} = I.$$

Then for all nonzero $\theta \in \mathbb{K}$,

$$\sum_{n=0}^{\infty} V_X(q^{-2n}\theta) = \sum_{n=0}^{\infty} V_Y\left(q^{2n}\theta^{-1}\right). \tag{13}$$

9. The linear transformations B and $\widetilde{A}^*$

Definition 9.1. With reference to Assumption 3.1 and Definition 7.1 let $B : V \to V$ denote the linear transformation such that for $0 \leqslant i \leqslant d$, W_i is an eigenspace of B with eigenvalue q^{2i-d}.

Lemma 9.2. *With reference to Assumption 3.1 and Definition 9.1 we have*

$$\frac{qAB - q^{-1}BA}{q - q^{-1}} = I, \tag{14}$$

$$\frac{qBA^* - q^{-1}A^*B}{q - q^{-1}} = B^2 + cI. \tag{15}$$

Proof. Recall that $\{W_i\}_{i=0}^d$ is a decomposition of V. By (10) $(A - q^{d-2i}I)W_i \subseteq W_{i+1}$ for $0 \leqslant i \leqslant d$. Using this and Lemma 8.2 we obtain (14). By (10) $(A^* - q^{2i-d} - cq^{d-2i}I)W_i \subseteq W_{i-1}$ for $0 \leqslant i \leqslant d$. Using this and Lemma 8.3 we obtain (15). $\square$

Definition 9.3. With reference to Assumption 3.1 and Definition 9.1 let $\widetilde{A}^* : V \to V$ denote the following linear transformation:

$$\widetilde{A}^* = c^{-1}(A^* - B).$$

Lemma 9.4. *With reference to Definitions 7.1 and 9.3 we have*

$$(\widetilde{A}^* - q^{d-2i}I)W_i \subseteq W_{i-1} \quad (0 \leqslant i \leqslant d).$$

Proof. Let i be given. Recall that W_i is an eigenspace for B with eigenvalue q^{2i-d}. We have

$$\begin{aligned}(\widetilde{A}^* - q^{d-2i}I)W_i &= c^{-1}(A^* - q^{2i-d}I - cq^{d-2i}I)W_i \quad \text{(by Definition 9.3)} \\ &\subseteq W_{i-1} \quad \text{(by (10))}. \quad \square\end{aligned}$$

Lemma 9.5. *With reference to Definitions 9.1 and 9.3 we have*

$$\frac{qB\widetilde{A}^* - q^{-1}\widetilde{A}^*B}{q - q^{-1}} = I. \tag{16}$$

Proof. Immediate from Lemmas 8.1 and 9.4. $\square$

Lemma 9.6. *With reference to Definition 9.3 the following holds. $\widetilde{A}^*$ is diagonalizable with eigenvalues $\{q^{d-2i}\}_{i=0}^d$. Moreover, for $0 \leqslant i \leqslant d$, the dimension of the eigenspace of $\widetilde{A}^*$ associated with q^{d-2i} is equal to the dimension of W_i.*

Proof. We start by displaying the eigenvalues of $\widetilde{A}^*$. Notice that the scalars $q^{d-2i} (0 \leqslant i \leqslant d)$ are distinct since q is not a root of unity. Using Lemma 9.4 we see that, with respect to an appropriate basis for V, $\widetilde{A}^*$ is represented by a upper triangular matrix that has diagonal entries $q^d, q^{d-2}, \ldots, q^{-d}$, with q^{d-2i} appearing $\dim(W_i)$ times for $0 \leqslant i \leqslant d$. Hence for $0 \leqslant i \leqslant d$ q^{d-2i} is a root of the characteristic polynomial of $\widetilde{A}^*$ with multiplicity $\dim(W_i)$. It remains to show that $\widetilde{A}^*$ is diagonalizable. To do this we show that the minimal polynomial of $\widetilde{A}^*$ has distinct roots. Recall that $\{W_i\}_{i=0}^d$ is a decomposition of V. Using Lemma 9.4 we find that $\prod_{i=0}^d (\widetilde{A}^* - q^{d-2i}I)V = 0$. By this and since $q^{d-2i} (0 \leqslant i \leqslant d)$ are distinct we see that the minimal polynomial of $\widetilde{A}^*$ has distinct roots. We conclude that $\widetilde{A}^*$ is diagonalizable and the result follows. $\square$

Definition 9.7. With reference to Definition 9.3 and Lemma 9.6, for $0 \leqslant i \leqslant d$ we let $\widetilde{V}_i^*$ denote the eigenspace for $\widetilde{A}^*$ with eigenvalue q^{d-2i}. For notational convenience we set $\widetilde{V}_{-1}^* := 0, \widetilde{V}_{d+1}^* := 0$. We observe that $\{\widetilde{V}_i^*\}_{i=0}^d$ is a decomposition of V.

10. The linear transformations $A, \widetilde{A}^*$ satisfy the cubic q-Serre relations

Lemma 10.1. *With reference to Assumption 3.1 and Definition 9.3 we have*

(i) $A^3\widetilde{A}^* - [3]A^2\widetilde{A}^*A + [3]A\widetilde{A}^*A^2 - \widetilde{A}^*A^3 = 0$,
(ii) $\widetilde{A}^{*3}A - [3]\widetilde{A}^{*2}A\widetilde{A}^* + [3]\widetilde{A}^*A\widetilde{A}^{*2} - A\widetilde{A}^{*3} = 0$.

Proof. By Definition 9.3 we have $A^* = c\widetilde{A}^* + B$. Substitute this into Lemma 3.2(i) and (ii) and simplify the result using (14) and (16). □

Lemma 10.2. *With reference to Assumption* 3.1, *Definition* 9.3, *and Definition* 9.7 *we have*

(i) $\widetilde{A}^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$, $0 \leqslant i \leqslant d$,
(ii) $A\widetilde{V}_i^* \subseteq \widetilde{V}_{i-1}^* + \widetilde{V}_i^* + \widetilde{V}_{i+1}^*$, $0 \leqslant i \leqslant d$.

Proof. Immediate from Lemma 10.1 and [20, Lemma 11.1]. □

Remark 10.3. Recall that in order to prove Theorem 5.4 we need to show that $A, \widetilde{A}^*$ is a q-geometric tridiagonal pair on V if and only if $P(q^{2d-2}(q - q^{-1})^{-2}) \neq 0$ (see Theorem 17.2). Combining Assumption 3.1, Lemma 9.6, and Lemma 10.2 we have that $A, \widetilde{A}^*$ satisfy Definition 1.1(i)–(iii). Sections 11–17 are devoted to showing that $A, \widetilde{A}^*$ satisfy Definition 1.1(iv) if and only if $P(q^{2d-2}(q - q^{-1})^{-2}) \neq 0$ (see Theorem 17.1).

11. The linear transformation K

Definition 11.1. With reference to Assumption 3.1 and Lemma 4.2 let $K : V \to V$ denote the linear transformation such that for $0 \leqslant i \leqslant d$, U_i is an eigenspace of K with eigenvalue q^{2i-d}.

Remark 11.2. Combining (6) and Definition 11.1 we have

$$(A - K)U_i \subseteq U_{i+1} \quad (0 \leqslant i \leqslant d), \tag{17}$$

$$(A^* - K - cK^{-1})U_i \subseteq U_{i-1} \quad (0 \leqslant i \leqslant d). \tag{18}$$

The goal for the remainder of this section is to prove a number of relations between the linear transformations A, A^*, B, K, K^{-1} which will be used in Section 13.

Lemma 11.3. *With reference to Assumption* 3.1 *and Definition* 11.1 *we have*

$$\frac{qK^{-1}A - q^{-1}AK^{-1}}{q - q^{-1}} = I, \tag{19}$$

$$\frac{qKA^* - q^{-1}A^*K}{q - q^{-1}} = K^2 + cI. \tag{20}$$

Proof. Recall that $\{U_i\}_{i=0}^d$ is a decomposition of V. Combining Definition 11.1, (17), and Lemma 8.1 we obtain (19). Combining Definition 11.1, (18), and Lemma 8.3 we obtain (20). □

Lemma 11.4. *With reference to Lemma* 4.2, *Definitions* 9.1, *and* 11.1 *we have*

$$(B - K)U_i \subseteq U_{i-1} \quad (0 \leqslant i \leqslant d), \tag{21}$$

$$(A^* - B - cK^{-1})U_i \subseteq U_{i-1} \quad (0 \leqslant i \leqslant d). \tag{22}$$

Proof. First we show (21). Using Lemma 8.1 and (14) we have

$$(B - q^{d-2i}I)V_i \subseteq V_{i-1} \quad (0 \leqslant i \leqslant d). \tag{23}$$

We have

$$
\begin{aligned}
(B - K)U_i &= (B - q^{2i-d}I)U_i \quad \text{(by Definition 11.1)} \\
&\subseteq (B - q^{2i-d}I)(U_0 + \cdots + U_i) \\
&= (B - q^{2i-d}I)(W_0 + \cdots + W_i) \quad \text{(by (7), (11))} \\
&\subseteq W_0 + \cdots + W_{i-1} \quad \text{(by Definition 9.1)} \\
&= U_0 + \cdots + U_{i-1} \quad \text{(by (7), (11))}
\end{aligned}
$$

and also

$$
\begin{aligned}
(B - K)U_i &= (B - q^{2i-d}I)U_i \quad \text{(by Definition 11.1)} \\
&\subseteq (B - q^{2i-d}I)(U_i + \cdots + U_d) \\
&= (B - q^{2i-d}I)(V_i + \cdots + V_d) \quad \text{(by (7))} \\
&\subseteq V_{i-1} + \cdots + V_d \quad \text{(by (23))} \\
&= U_{i-1} + \cdots + U_d \quad \text{(by (7)).}
\end{aligned}
$$

Using this and since $\{U_i\}_{i=0}^d$ is a decomposition of V we have (21). Combining (18) and (21) we obtain (22). $\square$

Lemma 11.5. *With reference to Definitions* 9.1 *and* 11.1 *we have*

$$
\frac{qBK^{-1} - q^{-1}K^{-1}B}{q - q^{-1}} = I. \tag{24}
$$

Proof. Recall that $\{U_i\}_{i=0}^d$ is a decomposition of V. Combining Lemma 8.2 and (21) we obtain (24). $\square$

Lemma 11.6. *With reference to Assumption* 3.1, *Definitions* 9.1 *and* 11.1 *for an integer $j \geqslant 1$ we have*

$$
\frac{q^j(B-K)^j(A^* - K - cK^{-1}) - q^{-j}(A^* - K - cK^{-1})(B-K)^j}{q^j - q^{-j}} = (B-K)^{j+1}, \tag{25}
$$

$$
\frac{q^j(A-K)^j(B-K) - q^{-j}(B-K)(A-K)^j}{q^j - q^{-j}} = -(q^{2-2j}K^2 - I)(A-K)^{j-1}. \tag{26}
$$

Proof. First we show (25) by induction on j. Multiplying out the left hand side of (25) with $j = 1$ and simplifying the result using (15), (20), and (24) we obtain the right hand side of (25) with $j = 1$. To prove (25) for $j \geqslant 2$ note that (25) is equivalent to

$$
(B-K)^j(A^* - K - cK^{-1}) = q^{-2j}(A^* - K - cK^{-1})(B-K)^j + q^{-j}(q^j - q^{-j})(B-K)^{j+1}.
$$

This is shown by a routine induction argument using the $j = 1$ case. We now show (26) by induction on j. Multiplying out the left hand side of (26) with $j = 1$ and simplifying the result using (14), (19), and (24) we obtain the right hand side of (26) with $j = 1$. Note that (19) is equivalent to

$$
(A-K)K = q^{-2}K(A-K). \tag{27}
$$

To prove (26) for $j \geqslant 2$ note that (26) is equivalent to

$$
\begin{aligned}
(B-K)(A-K)^j = \; &q^{2j}(A-K)^j(B-K) \\
&+ q^{2-j}(q^j - q^{-j})(K^2 - q^{2j-2}I)(A-K)^{j-1}.
\end{aligned}
$$

This is shown by a routine induction argument using the $j = 1$ case and (27). $\square$

12. The projections $E_i, \widetilde{E}_i^*$

Definition 12.1. With reference to Assumption 3.1 and Definition 9.7 we define the following linear transformations.

(i) For $0 \leqslant i \leqslant d$, we let E_i denote the linear transformation on V satisfying both

$(E_i - I)V_i = 0,$
$E_i V_j = 0 \quad \text{if } j \neq i \quad (0 \leqslant j \leqslant d).$

(ii) For $0 \leqslant i \leqslant d$, we let $\widetilde{E}_i^*$ denote the linear transformation on V satisfying both

$(\widetilde{E}_i^* - I)\widetilde{V}_i^* = 0,$
$\widetilde{E}_i^* \widetilde{V}_j^* = 0 \quad \text{if } j \neq i, \quad (0 \leqslant j \leqslant d).$

In other words, E_i (resp. $\widetilde{E}_i^*$) is the projection map from V onto V_i (resp. $\widetilde{V}_i^*$).

Lemma 12.2. *With reference to Assumption* 3.1, *Definitions* 9.3, *and* 12.1, *for* $0 \leqslant i \leqslant d$ *we have*

$$E_i = \prod_{\substack{0 \leqslant j \leqslant d \\ j \neq i}} \frac{A - q^{2j-d}I}{q^{2i-d} - q^{2j-d}}, \tag{28}$$

$$\widetilde{E}_i^* = \prod_{\substack{0 \leqslant j \leqslant d \\ j \neq i}} \frac{\widetilde{A}^* - q^{d-2j}I}{q^{d-2i} - q^{d-2j}}. \tag{29}$$

Proof. Concerning (28), let E_i' denote the expression on the right in that line. Using Assumption 3.1 we find $(E_i' - I)V_i = 0$ and $E_i' V_j = 0$ $(0 \leqslant j \leqslant d, \ j \neq i)$. By this and Definition 12.1(i) we find $E_i = E_i'$. We have now proved (28). The proof of (29) is similar. $\quad\square$

Lemma 12.3. *With reference to Assumption* 3.1, *Remark* 7.3, *and Definition* 12.1(*i*) *the following holds for* $0 \leqslant i \leqslant d$: *The linear transformations*

$W_{d-i} \to V_i \quad V_i \to W_{d-i}$
$w \to E_i w \quad v \to G_{d-i} v$

are bijections, and moreover, they are inverses.

Proof. It suffices to show $G_{d-i}E_i - I$ vanishes on W_{d-i} and $E_i G_{d-i} - I$ vanishes on V_i. We will use the following notation. Recall by (11) that for $0 \leqslant j \leqslant d, W_{d-j} + \cdots + W_d = V_0 + \cdots + V_j$; let Z_j denote this common sum. We set $Z_{-1} = 0$. By the construction $Z_i = W_{d-i} + Z_{i-1}$ (direct sum) and $Z_i = V_i + Z_{i-1}$ (direct sum). Also $(I - G_{d-i})Z_i = Z_{i-1}$ and $(I - E_i)Z_i = Z_{i-1}$. We now show $G_{d-i}E_i - I$ vanishes on W_{d-i}. Pick $w \in W_{d-i}$. Using $G_{d-i}E_i - I = (G_{d-i} - I)E_i + E_i - I$ and our preliminary comments we routinely find $(G_{d-i}E_i - I)w \in Z_{i-1}$. But $(G_{d-i}E_i - I)w \in W_{d-i}$ by construction and $W_{d-i} \cap Z_{i-1} = 0$ so $(G_{d-i}E_i - I)w = 0$. We now show $E_i G_{d-i} - I$ vanishes on V_i. Pick $v \in V_i$. Using $E_i G_{d-i} - I = (E_i - I)G_{d-i} + G_{d-i} - I$ and our preliminary comments we routinely find $(E_i G_{d-i} - I)v \in Z_{i-1}$. But $(E_i G_{d-i} - I)v \in V_i$ by construction and $V_i \cap Z_{i-1} = 0$ so $(E_i G_{d-i} - I)v = 0$. We have now shown $G_{d-i}E_i - I$ vanishes on W_{d-i} and $E_i G_{d-i} - I$ vanishes on V_i. Consequently the given maps are inverses. Each of these maps has an inverse and is therefore a bijection. $\quad\square$

13. How $\widetilde{E}_0^*, E_d, P$ are related

The goal of this section is to prove the following theorem which will be used in the proof of Theorem 17.1.

Theorem 13.1. *With reference to Assumption* 3.1, *Lemma* 4.2, *Definitions* 5.3 *and* 12.1, *for* $u \in U_0$ *we have*

$$\widetilde{E}_0^* E_d u = c^{-d} q^{2d(1-d)} P(q^{2d-2}(q - q^{-1})^{-2}) u. \tag{30}$$

Before we prove Theorem 13.1 we develop some notation and prove some preliminary lemmas. With reference to (9) for integers n, m with $n \geqslant 0$ and $0 \leqslant m \leqslant n$ we define

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}. \tag{31}$$

Lemma 13.2. *For integers* n, m *with* $n \geqslant 1$ *and* $0 \leqslant m \leqslant n - 1$ *we have*

$$\begin{bmatrix} n-1 \\ m \end{bmatrix} + q^n \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = q^m \begin{bmatrix} n \\ m \end{bmatrix}, \tag{32}$$

$$\begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{-n} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = q^{-m} \begin{bmatrix} n \\ m \end{bmatrix}. \tag{33}$$

Proof. Immediate from (3), (9) and (31). $\quad\square$

The following two lemmas provide key formulas to be used in the proof of Theorem 13.1.

Lemma 13.3. *With reference to Assumption* 3.1, *Definitions* 9.1 *and* 11.1 *for an integer* $i \geqslant 0$ *we have*

$$(A^* - B - cK^{-1})^i = \sum_{j=0}^{i} (-1)^j q^{j-ji} \begin{bmatrix} i \\ j \end{bmatrix} (A^* - K - cK^{-1})^{i-j}(B - K)^j. \tag{34}$$

Proof. We prove (34) by induction on i. For $i = 0$ both sides of (34) equal I. Now let $i \geqslant 1$. Abbreviate $\Delta = A^* - K - cK^{-1}$ and $\Gamma = B - K$. We have

$$
\begin{aligned}
(A^* - B - cK^{-1})^i &= (A^* - B - cK^{-1})^{i-1}(\Delta - \Gamma) \\
&= \sum_{j=0}^{i-1} (-1)^j q^{2j-ji} \begin{bmatrix} i-1 \\ j \end{bmatrix} \Delta^{i-j-1}(\Gamma^j \Delta - \Gamma^{j+1}) \quad \text{(by induction)} \\
&= \sum_{j=0}^{i-1} (-1)^j q^{-ji} \begin{bmatrix} i-1 \\ j \end{bmatrix} \Delta^{i-j-1}(\Delta \Gamma^j - \Gamma^{j+1}) \quad \text{(by (25))} \\
&= \sum_{j=0}^{i-1} (-1)^j q^{-ji} \begin{bmatrix} i-1 \\ j \end{bmatrix} \Delta^{i-j} \Gamma^j \\
&\quad + \sum_{j=1}^{i} (-1)^j q^{i-ji} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} \Delta^{i-j} \Gamma^j \\
&= \sum_{j=0}^{i} (-1)^j q^{j-ji} \begin{bmatrix} i \\ j \end{bmatrix} \Delta^{i-j} \Gamma^j \quad \text{(by (32)).} \quad\square
\end{aligned}
$$

Lemma 13.4. *Fix an integer* $i \geqslant 1$. *With reference to Definition* 11.1, *for integers* $\mu, \nu \geqslant 0$, *define a polynomial* $f_{\mu,\nu} \in \mathbb{K}[K^2]$ *by* $f_{\mu,\nu} = \prod_{s=0}^{\mu-1}(K^2 - q^{2i-2s-2\nu}I)$. *With reference to Assumption* 3.1 *and Definition* 9.1 *for* $1 \leqslant j \leqslant i$ *we have*

$$(B - K)^j (A - K)^i = \sum_{h=0}^{j} q^{M_{h,i,j}} N_{h,i} \begin{bmatrix} j \\ h \end{bmatrix} f_{h,j} (A - K)^{i-h} (B - K)^{j-h} \tag{35}$$

where $M_{h,i,j} = (h/2)(3h - 1) + hj - 3hi + 2ij$ and $N_{h,i} = \begin{bmatrix} i \\ h \end{bmatrix} [h]! (q - q^{-1})^h$.

Proof. Before we prove (35) we have a comment. Observe that (24) is equivalent to

$$(B - K)K = q^2 K(B - K). \tag{36}$$

To prove (35) we let i be given and use induction on j. For $j = 1$ (35) is equivalent to (26). Now let $j \geqslant 2$. Abbreviate $\Delta = A - K$ and $\Gamma = B - K$. We have

$$
\begin{aligned}
\Gamma^j \Delta^i &= \Gamma \Gamma^{j-1} \Delta^i \\
&= \Gamma \sum_{h=0}^{j-1} q^{M_{h,i,j-1}} N_{h,i} \begin{bmatrix} j-1 \\ h \end{bmatrix} f_{h,j-1} \Delta^{i-h} \Gamma^{j-h-1} \quad \text{(by induction)} \\
&= \sum_{h=0}^{j-1} q^{M_{h,i,j}} q^{-h-2i} N_{h,i} \begin{bmatrix} j-1 \\ h \end{bmatrix} q^{4h} f_{h,j+1} \Gamma \Delta^{i-h} \Gamma^{j-h-1} \quad \text{(by (36))} \\
&= \sum_{h=0}^{j-1} q^{M_{h,i,j}} q^{3h-2i} N_{h,i} \begin{bmatrix} j-1 \\ h \end{bmatrix} f_{h,j+1} q^{2i-2h} \Delta^{i-h} \Gamma^{j-h} \quad \text{(by (26))} \\
&\quad + \sum_{h=0}^{j-1} q^{M_{h+1,i,j}} q^{h+1-j} N_{h+1,i} \begin{bmatrix} j-1 \\ h \end{bmatrix} f_{h,j+1} (K^2 - q^{2i-2h-2} I) \Delta^{i-h-1} \Gamma^{j-h-1} \\
&= \sum_{h=0}^{j-1} q^{M_{h,i,j}} q^h N_{h,i} \begin{bmatrix} j-1 \\ h \end{bmatrix} f_{h-1,j+1} (K^2 - q^{2i-2j-2h} I) \Delta^{i-h} \Gamma^{j-h} \\
&\quad + \sum_{h=1}^{j} q^{M_{h,i,j}} q^{h-j} N_{h,i} \begin{bmatrix} j-1 \\ h-1 \end{bmatrix} f_{h-1,j+1} (K^2 - q^{2i-2h} I) \Delta^{i-h} \Gamma^{j-h} \\
&= \sum_{h=0}^{j} q^{M_{h,i,j}} N_{h,i} \begin{bmatrix} j \\ h \end{bmatrix} f_{h-1,j+1} (K^2 - q^{2i-2j} I) \Delta^{i-h} \Gamma^{j-h} \quad \text{(by (32), (33))} \\
&= \sum_{h=0}^{j} q^{M_{h,i,j}} N_{h,i} \begin{bmatrix} j \\ h \end{bmatrix} f_{h,j} \Delta^{i-h} \Gamma^{j-h}. \quad \square
\end{aligned}
$$

We are now ready to prove Theorem 13.1.

Proof of Theorem 13.1. Let $u \in U_0$. Using Definition 9.3 and Lemma 12.2 we have

$$\widetilde{E}_0^* E_d u = c^{-d} q^{d-d^2} (q - q^{-1})^{-2d} [d]!^{-2} \prod_{j=1}^{d} (A^* - B - cq^{d-2j} I) \prod_{j=0}^{d-1} (A - q^{2j-d} I) u. \tag{37}$$

Applying Definition 11.1, (17), and (22) to (37) we have

$$\widetilde{E}_0^* E_d u = c^{-d} q^{d-d^2} (q - q^{-1})^{-2d} [d]!^{-2} (A^* - B - cK^{-1})^d (A - K)^d u. \tag{38}$$

We now express the right hand side of (38) in terms of the maps R, L from Definition 4.4. By (21) we have $(B - K)u = 0$. Using Definition 4.3, (8), and Definition 11.1 we have $R = A - K$ and $L = A^* - K - cK^{-1}$. Using Lemma 13.3, Lemma 13.4, and the previous two sentences we have

$$(A^* - B - cK^{-1})^d (A - K)^d u = \sum_{j=0}^{d} C_j \, L^{d-j} \prod_{s=0}^{j-1} (K^2 - q^{2d-2s-2j}I) \, R^{d-j}u \tag{39}$$

where $C_j = (-1)^j \, q^{(j/2)(5j+1)-2dj} \begin{bmatrix} d \\ j \end{bmatrix}^2 [j]! \, (q - q^{-1})^j$.

Using Definition 4.4 we have $R^{d-j}u \in U_{d-j}$ for $0 \leqslant j \leqslant d$. So by Definition 11.1 $(K^2 - q^{2d-4j}I)R^{d-j}u = 0$. Using this on the right hand side of (39) and simplifying the result we have

$$(A^* - B - cK^{-1})^d (A - K)^d u = \sum_{j=0}^{d} q^{j-j^2} [d]!^2 \, [d-j]!^{-2} \, (q - q^{-1})^{2j} L^{d-j} R^{d-j}u. \tag{40}$$

Changing the index of summation in (40) by letting $j = d - t$, using Definition 5.2, and simplifying the result we have

$$(A^* - B - cK^{-1})^d (A - K)^d u$$
$$= q^{d-d^2} [d]!^2 \, (q - q^{-1})^{2d} \sum_{t=0}^{d} q^{t(1-t)} [t]!^{-2} \, (q^{2d-2} \, (q - q^{-1})^{-2})^t \, \zeta_t u.$$

Combining the previous line with (38) and using Definition 5.3 we obtain (30). □

14. The raising/lowering maps revisited

In this section we prove a number of relations between r, l from Definition 7.1 and B from Definition 9.1. These relations will help to motivate the next section.

Lemma 14.1. *With reference to Assumption* 3.1, *Definitions* 7.1, *and* 9.1 *we have*

(i) $r = A - B^{-1}$,
(ii) $l = A^* - B - cB^{-1}$.

Proof. (i) Recall by Assumption 3.1 that $\theta_i = q^{2i-d}$ for $0 \leqslant i \leqslant d$. Using Remark 7.3 and Definition 9.1 we find $\sum_{i=0}^{d} \theta_{d-i} G_i = B^{-1}$. Using this and (12) we obtain the desired result.
(ii) Similar to (i). □

Lemma 14.2. *With reference to Definitions* 7.1 *and* 9.1 *we have*

(i) $Br = q^2 rB$,
(ii) $Bl = q^{-2} lB$.

Proof. (i) Recall by Definition 7.1 that $\{W_i\}_{i=0}^{d}$ is a decomposition of V. So it suffices to show $Br - q^2 rB$ vanishes on W_i for $0 \leqslant i \leqslant d$. Let i be given and let $w \in W_i$. Using Remark 7.4 and Definition 9.1 we find rw is an eigenvector for B with eigenvalue q^{2i+2-d}. From this we find $(Br - q^2 rB)w = 0$ and the result follows.
(ii) Similar to (i). □

Lemma 14.3. *With reference to Definitions* 7.1 *and* 9.1 *we have*

(i) $r^3 l - [3]r^2 lr + [3]rlr^2 - lr^3 = q^{-4} \, (q - q^{-1})^3 \, [3]! \, r^2 B^{-2}$,
(ii) $rl^3 - [3]lrl^2 + [3]l^2rl - l^3r = q^{-4} \, (q - q^{-1})^3 \, [3]! \, B^{-2}l^2$.

Proof. By Lemma 14.1 we have $A = r + B^{-1}$ and $A^* = l + B + cB^{-1}$. Substituting these into Lemma 3.2(i) and (ii) and simplifying using Lemma 14.2 we obtain the desired result. $\square$

15. The algebra $A_q(\alpha)$

Motivated by Lemmas 14.2 and 14.3 we define an algebra $A_q(\alpha)$. We find a spanning set for $A_q(\alpha)$ that will be used in the proof of Lemma 16.6 in the next section.

Definition 15.1. Fix a scalar $\alpha \in \mathbb{K}$. Let $A_q(\alpha)$ denote the unital associative $\mathbb{K}$-algebra defined by generators x, y, z, z^{-1} subject to the relations

$$zz^{-1} = 1 = z^{-1}z, \tag{41}$$

$$zx = q^2 xz, \tag{42}$$

$$zy = q^{-2} yz, \tag{43}$$

$$x^3 y - [3]x^2 yx + [3]xyx^2 - yx^3 = \alpha\, x^2 z^{-2}, \tag{44}$$

$$xy^3 - [3]yxy^2 + [3]y^2 xy - y^3 x = \alpha\, z^{-2} y^2. \tag{45}$$

Note 15.2. In the case $\alpha = 0$ we note that the algebra $A_q(0)$ is the algebra $\mathcal{B}$ from [6, Definition 1.10]. The algebra $A_q(\alpha)$ is a special case of a more general algebra called the augmented tridiagonal algebra developed by T. Ito and P. Terwilliger.

Before we display a spanning set for $A_q(\alpha)$ we have a number of preliminary comments.

For the moment we view x, y as formal symbols and let F denote the free unital associative $\mathbb{K}$-algebra on x, y.

Definition 15.3. By a *word in F* we mean an element of F of the form $a_1 a_2 \cdots a_n$ where n is a nonnegative integer and $a_i \in \{x, y\}$ for $1 \leqslant i \leqslant n$. We call n the *length* of $a_1 a_2 \cdots a_n$. We interpret the word of length 0 as the identity element of F. We say this word is *trivial*. Observe $F = \sum_{n=0}^{\infty} F_n$ (direct sum) where F_n denotes the subspace of F spanned by all the words of length n. Moreover, $F_n F_m = F_{n+m}$.

Definition 15.4. Let $a_1 a_2 \cdots a_n$ denote a word in F. Observe there exists a unique sequence $(i_1, i_2, \ldots, i_s)$ of positive integers such that $a_1 a_2 \cdots a_n$ is one of $x^{i_1} y^{i_2} x^{i_3} \cdots y^{i_s}$ or $x^{i_1} y^{i_2} x^{i_3} \cdots x^{i_s}$ or $y^{i_1} x^{i_2} y^{i_3} \cdots x^{i_s}$ or $y^{i_1} x^{i_2} y^{i_3} \cdots y^{i_s}$. We call the sequence $(i_1, i_2, \ldots, i_s)$ the *signature* of $a_1 a_2 \cdots a_n$.

Example 15.5. *Each of the words* $yx^2 y^2 x, xy^2 x^2 y$ *has signature* $(1, 2, 2, 1)$.

Definition 15.6. Let $a_1 a_2 \cdots a_n$ denote a word in F and let $(i_1, i_2, \ldots, i_s)$ denote the corresponding signature. We say $a_1 a_2 \cdots a_n$ is *reducible* whenever there exists an integer $\eta (2 \leqslant \eta \leqslant s - 1)$ such that $i_{\eta-1} \geqslant i_\eta < i_{\eta+1}$. We say a word in F is *irreducible* whenever it is not reducible.

Example 15.7. *A word in F of length less than 4 is irreducible. The only reducible words in F of length 4 are* xyx^2 *and* yxy^2.

In the following lemma we give a necessary and sufficient condition for a given nontrivial word in F to be irreducible.

Lemma 15.8. *Let $a_1 a_2 \cdots a_n$ denote a nontrivial word in F and let $(i_1, i_2, \ldots, i_s)$ denote the corresponding signature. Then the following are equivalent*:

(i) *The word $a_1 a_2 \cdots a_n$ is irreducible.*
(ii) *There exists an integer $t (1 \leqslant t \leqslant s)$ such that*

$$i_1 < i_2 < \cdots < i_t \geqslant i_{t+1} \geqslant i_{t+2} \geqslant \cdots \geqslant i_{s-1} \geqslant i_s.$$

Proof. Immediate from Definition 15.6. $\square$

For the moment we view x, y, z, z^{-1} as formal symbols and let $\mathcal{F}$ denote the free unital associative $\mathbb{K}$-algebra on x, y, z, z^{-1}. We identify F with the subalgebra of $\mathcal{F}$ generated by x, y.

We now view $A_q(\alpha)$ as a vector space over $\mathbb{K}$ and display a spanning set.

Theorem 15.9. *Let $\pi : \mathcal{F} \to A_q(\alpha)$ denote the canonical quotient map. Consider the following elements in $\mathcal{F}$:*

$$wz^j, \quad w \text{ is an irreducible word in } F, \quad j \in \mathbb{Z}.$$

Then $A_q(\alpha)$ is spanned by the images of the above elements under π.

To prove Theorem 15.9 we will need the following two lemmas and definition.

Lemma 15.10. *Let Ω denote the subspace of F spanned by all the irreducible words. Let Λ denote the two sided ideal of F generated by*

$$x^3 y - [3]x^2 yx + [3]xyx^2 - yx^3, \tag{46}$$
$$xy^3 - [3]yxy^2 + [3]y^2 xy - y^3 x. \tag{47}$$

For an integer $n \geqslant 0$ let $\Omega_n = \Omega \cap F_n$ and $\Lambda_n = \Lambda \cap F_n$. Then the following (i)–(iv) hold:

(i) $F = \Omega + \Lambda$ *(direct sum)*,
(ii) $\Omega = \sum_{n=0}^{\infty} \Omega_n$ *(direct sum)*,
(iii) $\Lambda = \sum_{n=0}^{\infty} \Lambda_n$ *(direct sum)*,
(iv) $F_n = \Omega_n + \Lambda_n$ *(direct sum) $0 \leqslant n < \infty$.*

Proof. (i) View the $\mathbb{K}$-algebra F/Λ as a vector space over $\mathbb{K}$. By [18, Theorem 2.29] F/Λ has a basis consisting of the images of the irreducible words in F under the canonical quotient map $F \to F/\Lambda$. The result follows immediately from this.

(ii) The words in F form a basis for F.

(iii) The generators (46) and (47) of Λ are in F_4.

(iv) Recall $F = \sum_{n=0}^{\infty} F_n$ (direct sum). Combining this with (i)–(iii) above we obtain the desired result. $\square$

Lemma 15.11. *We have $\Lambda_n = 0$ for $n \leqslant 3$. Also*

$$\pi(\Lambda_n) \subseteq \pi(F_{n-2} z^{-2}) \quad n \geqslant 4 \tag{48}$$

where $\pi : \mathcal{F} \to A_q(\alpha)$ is the canonical quotient map.

Proof. The first assertion follows since the generators (46) and (47) of Λ are in F_4. For $n \geqslant 4$ we have by construction that

$$\Lambda_n = \sum_{i,j} F_i \, (x^3 y - [3]x^2 yx + [3]xyx^2 - yx^3) \, F_j$$
$$+ \sum_{i,j} F_i \, (xy^3 - [3]yxy^2 + [3]y^2 xy - y^3 x) \, F_j$$

where each sum is over all nonnegative integers i, j such that $i + j = n - 4$. Applying π and using (42)–(45) we have

$$\pi(\Lambda_n) = \sum_{i,j} \pi(F_i x^2 F_j z^{-2}) + \sum_{i,j} \pi(F_i y^2 F_j z^{-2}) \qquad (49)$$

where each sum is over all nonnegative integers i, j such that $i + j = n - 4$. For all such i, j we have $F_i x^2 F_j \subseteq F_{n-2}$ and $F_i y^2 F_j \subseteq F_{n-2}$. Simplifying (49) using this we obtain (48). $\square$

Definition 15.12. By a *word in* $\mathcal{F}$ we mean an element of $\mathcal{F}$ of the form $a_1 a_2 \cdots a_n$ where n is a nonnegative integer and $a_i \in \{x, y, z, z^{-1}\}$ for $1 \leqslant i \leqslant n$. By the (x, y)-*length* of $a_1 a_2 \cdots a_n$ we mean the number of x's plus the number of y's in $a_1 a_2 \cdots a_n$.

We are now ready to prove Theorem 15.9.

Proof of Theorem 15.9. Abbreviate

$$S = \text{Span}\{\pi(wz^j) \mid w \text{ is an irreducible word in } F \text{ and } j \in \mathbb{Z}\}.$$

We show $S = A_q(\alpha)$. Since $\mathcal{F}$ is spanned by its words and since $\pi : \mathcal{F} \to A_q(\alpha)$ is surjective it suffices to show that S contains the image under π of every word in $\mathcal{F}$. By a *counterexample* we mean a word in $\mathcal{F}$ whose image under π is not contained in S. We assume there exists a counterexample and obtain a contradiction. Among all counterexamples let v denote a counterexample with minimal (x, y)-length. Let t denote the (x, y)-length of v. Using (41)–(43) we may assume without loss that $v = v' z^j$ where v' is a word in F_t and $j \in \mathbb{Z}$. Recall every word in F of length less than 4 is irreducible. By construction v' is reducible and so $t \geqslant 4$. By Lemma 15.10(iv) there exists $\varpi \in \Omega_t$ and $\lambda \in \Lambda_t$ such that $v' = \varpi + \lambda$. Now $v = \varpi z^j + \lambda z^j$ so

$$\pi(v) = \pi(\varpi z^j) + \pi(\lambda z^j). \qquad (50)$$

By construction ϖ is a linear combination of irreducible words so $\pi(\varpi z^j) \in S$. We now show $\pi(\lambda z^j) \in S$. By Lemma 15.11 and since $\lambda \in \Lambda_t$ we have $\pi(\lambda z^j) \in \pi(F_{t-2} z^{j-2})$. Every word in $F_{t-2} z^{j-2}$ has (x, y)-length $t - 2$ and is therefore not a counterexample by the minimality assumption. Hence the image under π of every word in $F_{t-2} z^{j-2}$ is contained in S. Since $F_{t-2} z^{j-2}$ is spanned by its words we have $\pi(F_{t-2} z^{j-2}) \subseteq S$. Therefore $\pi(\lambda z^j) \in S$. We have now shown $\pi(\varpi z^j) \in S$ and $\pi(\lambda z^j) \in S$ so $\pi(v) \in S$ by (50). This is a contradiction and the result follows. $\square$

16. A result concerning $(A, \widetilde{A}^*)$-submodules of V

Referring to Assumption 3.1 and Definition 9.3 let W denote an irreducible $(A, \widetilde{A}^*)$-submodule of V. The goal of this section is to prove $V_d \subseteq W$ (see Lemma 16.7). This fact will be used in the proof of Theorem 17.1.

We note that the arguments given in this section are a modification of the arguments from [23, Section 11].

Definition 16.1. With reference to Assumption 3.1 and Definition 9.3 let W denote an irreducible $(A, \widetilde{A}^*)$-submodule of V. Observe that W is the direct sum of the nonzero spaces among $\{E_i W\}_{i=0}^d$ where E_i is from Definition 12.1(i). We define

$$t = \max\{i \mid 0 \leqslant i \leqslant d, E_i W \neq 0\}.$$

We call t the *endpoint* of W.

Lemma 16.2. *With reference to Assumption 3.1 and Definition 9.3 let W denote an irreducible $(A, \widetilde{A}^*)$-submodule of V and let t denote the endpoint of W. Then $\dim(E_t W) = 1$.*

Proof. By construction W is an irreducible $(A, \widetilde{A}^*)$-module. Using this, Lemma 9.6, and Lemma 10.2 we find that $A|_W, \widetilde{A}^*|_W$ is a q-geometric tridiagonal pair on W. Let s denote the diameter of $A|_W, \widetilde{A}^*|_W$.

Note that $\{E_{s-i}W\}_{i=s-t}^{2s-t}$ is a standard ordering of the eigenspaces of $A|_W$. Applying [23, Theorem 9.1] to $A|_W, \tilde{A}^*|_W$ we find $\dim(E_t W) = 1$. $\square$

Lemma 16.3. *With reference to Assumption* 3.1 *and Definition* 9.3 *let W denote an irreducible $(A, \tilde{A}^*)$-submodule of V and let t denote the endpoint of W. With reference to Definition* 12.1(*i*) *and Definition* 7.1 *pick $v \in E_t W$ and write $u = G_{d-t}v$. Then $lu = 0$ where l is the linear transformation from* (12).

Proof. Observe $u \in W_{d-t}$ by Remark 7.3. We assume $d - t \geqslant 1$; otherwise $lu = 0$ since $lW_0 = 0$. Observe $lu \in W_{d-t-1}$ by Remark 7.4. In order to show $lu = 0$ we show $lu \in W_{d-t} + \cdots + W_d$. Using Lemma 14.1(ii) and Definition 9.3 we have $c^{-1}l = \tilde{A}^* - B^{-1}$. Thus

$$c^{-1}lu = \tilde{A}^*v - B^{-1}v + c^{-1}l(u - v). \tag{51}$$

We are going to show that each of the three terms on the right in (51) is contained in $W_{d-t} + \cdots + W_d$. By the definition of t we have $W = E_0 W + \cdots + E_t W$ so $W \subseteq V_0 + \cdots + V_t$ in view of Definition 12.1(i). By this and (11) we find $W \subseteq W_{d-t} + \cdots + W_d$. By construction $v \in W$ so $\tilde{A}^*v \in W$. By these comments $\tilde{A}^*v \in W_{d-t} + \cdots + W_d$. We mentioned $v \in W$ so $v \in W_{d-t} + \cdots + W_d$. Each of $\{W_i\}_{i=d-t}^d$ is an eigenspace for B^{-1} so $B^{-1}v \in W_{d-t} + \cdots + W_d$. Since $v \in W_{d-t} + \cdots + W_d$ and since $u = G_{d-t}v$ we find $u - v \in W_{d-t+1} + \cdots + W_d$. Now $c^{-1}l(u - v) \in W_{d-t} + \cdots + W_{d-1}$ so $c^{-1}l(u - v) \in W_{d-t} + \cdots + W_d$. We have now shown that each of the three terms on the right in (51) is contained in $W_{d-t} + \cdots + W_d$. Therefore $lu \in W_{d-t} + \cdots + W_d$. Recall $lu \in W_{d-t-1}$. By this and since $\{W_i\}_{i=0}^d$ is a decomposition of V we find $lu = 0$. $\square$

Lemma 16.4. *With reference to Definitions* 7.1 *and* 12.1(*i*) *for $0 \leqslant i \leqslant d$ the action of E_i on W_{d-i} coincides with*

$$\sum_{h=0}^{i} \frac{r^h}{(q^{2i-d} - q^{2i-d-2})(q^{2i-d} - q^{2i-d-4}) \cdots (q^{2i-d} - q^{2i-d-2h})}$$

where r is the linear transformation from (12).

Proof. Pick $w \in W_{d-i}$. We find $E_i w$. By (11) and since $E_i w \in V_i$ we find $E_i w \in W_{d-i} + \cdots + W_d$. Consequently there exist $w_s \in W_s (d - i \leqslant s \leqslant d)$ such that $E_i w = \sum_{s=d-i}^d w_s$. By (10) and Remark 7.4 we have for $0 \leqslant j \leqslant d$ that r acts on W_j as $A - q^{d-2j}I$. Using this and since $(A - q^{2i-d}I)E_i = 0$ we find

$$0 = \left(A - q^{2i-d}I\right) E_i w$$

$$= \left(A - q^{2i-d}I\right) \sum_{s=d-i}^{d} w_s$$

$$= \sum_{s=d-i}^{d} \left(r + q^{d-2s} - q^{2i-d}\right) w_s.$$

Rearranging the terms above we find $0 = \sum_{s=d-i+1}^d w_s'$ where

$$w_s' = r w_{s-1} + (q^{d-2s} - q^{2i-d})w_s \quad (d - i + 1 \leqslant s \leqslant d).$$

Since $w_s' \in W_s$ for $d - i + 1 \leqslant s \leqslant d$ and since $\{W_i\}_{i=0}^d$ is a decomposition of V we find $w_s' = 0$ for $d - i + 1 \leqslant s \leqslant d$. Consequently

$$w_s = (q^{2i-d} - q^{d-2s})^{-1} r w_{s-1} \quad (d - i + 1 \leqslant s \leqslant d).$$

By Lemma 12.3 and since $w_{d-i} = G_{d-i}E_i w$ we find $w_{d-i} = w$. From these comments we obtain the desired result. $\square$

Lemma 16.5. *With reference to Assumption* 3.1 *and Definition* 9.3 *let W denote an irreducible* $(A, \widetilde{A}^*)$-*submodule of V and let t denote the endpoint of W. With reference to Definition* 12.1(*i*) *and Definition* 7.1 *pick* $v \in E_t W$ *and write* $u = G_{d-t} v$. *Then*

$$l^i r^i u \in \text{Span}(u) \quad (0 \leqslant i \leqslant t) \tag{52}$$

where r, l *are the linear transformations from* (12).

Proof. We may assume $v \neq 0$; otherwise the result is trivial. Define

$$\Delta_i = (\widetilde{A}^* - q^{2t-d}I)(\widetilde{A}^* - q^{2t-d-2}I) \cdots (\widetilde{A}^* - q^{2t-d-2i+2}I). \tag{53}$$

Since Δ_i is a polynomial in $\widetilde{A}^*$ we find $\Delta_i W \subseteq W$. In particular $\Delta_i v \in W$ so $E_t \Delta_i v \in E_t W$. The vector v spans $E_t W$ by Lemma 16.2 so there exists $m_i \in \mathbb{K}$ such that $E_t \Delta_i v = m_i v$. By this and since $E_t v = v$ we find $E_t(\Delta_i - m_i I)v = 0$. Now $(\Delta_i - m_i I)v \in E_0 W + \cdots + E_{t-1} W$ in view of Definition 16.1. Observe $E_0 W + \cdots + E_{t-1} W \subseteq V_0 + \cdots + V_{t-1}$ where the V_j are from Assumption 3.1. By these comments and (11) we find $(\Delta_i - m_i I)v \in W_{d-t+1} + \cdots + W_d$. Consequently $G_{d-t}(\Delta_i - m_i I)v = 0$. Recall $G_{d-t} v = u$ so

$$G_{d-t} \Delta_i v = m_i u. \tag{54}$$

We now evaluate $G_{d-t} \Delta_i v$. Observe $v = E_t u$ by Lemma 12.3 and since $u = G_{d-t} v$. By Lemma 16.4 there exist nonzero scalars $\gamma_h \in \mathbb{K}(0 \leqslant h \leqslant t)$ such that $v = \sum_{h=0}^t \gamma_h r^h u$. For $0 \leqslant h \leqslant t$ we compute $G_{d-t} \Delta_i r^h u$. Keep in mind $r^h u \in W_{d-t+h}$ by Remark 7.4. First assume $h < i$. Using Lemma 9.4 and (53) we find $\Delta_i r^h u$ is contained in $W_{d-t+h-i} + \cdots + W_{d-t-1}$ so $G_{d-t} \Delta_i r^h u = 0$. Next assume $h = i$. Using Lemma 14.1(ii) and Definition 9.3 we have $c^{-1} l = \widetilde{A}^* - B^{-1}$ and so $c^{-1} l|_{W_j} = (\widetilde{A}^* - q^{d-2j}I)|_{W_j}$ $(0 \leqslant j \leqslant d)$. Using this and (53) we find $(\Delta_i - c^{-i} l^i) r^i u$ is contained in $W_{d-t+1} + \cdots + W_{d-t+i}$. By this and since $c^{-i} l^i r^i u \in W_{d-t}$ we find $G_{d-t} \Delta_i r^i u = c^{-i} l^i r^i u$. Next assume $h > i$. Using Lemma 9.4 and (53) we find $\Delta_i r^h u$ is contained in $W_{d-t+h-i} + \cdots + W_{d-t+h}$. By this and since $h > i$ we find $G_{d-t} \Delta_i r^h u = 0$. By these comments we find $G_{d-t} \Delta_i v = \gamma_i c^{-i} l^i r^i u$. Combining this and (54) we obtain (52). $\square$

Lemma 16.6. *With reference to Assumption* 3.1 *and Definition* 9.3 *let W denote an irreducible* $(A, \widetilde{A}^*)$-*submodule of V and let t denote the endpoint of W. Then* $t = d$. *Moreover, the following holds. With reference to Definition* 12.1(*i*) *and Definition* 7.1 *pick a nonzero* $v \in E_d W$ *and write* $u = G_0 v$. *Let* r, l *be the linear transformations from* (12). *Then V is spanned by the vectors of the form*

$$l^{i_1} r^{i_2} l^{i_3} r^{i_4} \cdots r^{i_n} u$$

where $i_1, i_2, \ldots, i_n$ *ranges over all sequences such that n is a nonnegative even integer, and* $i_1, i_2, \ldots, i_n$ *are integers satisfying* $0 \leqslant i_1 < i_2 < \cdots < i_n \leqslant d$.

Proof. Let t is the endpoint of W. Pick a nonzero $v \in E_t W$ and write $u = G_{d-t} v$. Observe $0 \neq u \in W_{d-t}$ by Lemma 12.3. By Lemma 16.3 and Remark 7.4,

$$lu = 0, \quad r^{t+1} u = 0. \tag{55}$$

By Lemma 16.5,

$$l^i r^i u \in \text{Span}(u) \quad (0 \leqslant i \leqslant t). \tag{56}$$

Let W' denote the subspace of V spanned by all vectors of the form

$$l^{i_1} r^{i_2} l^{i_3} r^{i_4} \cdots r^{i_n} u, \tag{57}$$

where $i_1, i_2, \ldots, i_n$ ranges over all sequences such that n is a nonnegative even integer, and $i_1, i_2, \ldots, i_n$ are integers satisfying $0 \leqslant i_1 < i_2 < \cdots < i_n \leqslant t$. Observe $u \in W'$ so $W' \neq 0$. In order to show $t = d$ we show $W' = V$ and $W' \subseteq W_{d-t} + \cdots + W_d$. We now show $W' = V$. To do this we show that W' is

invariant under each of A, A^*. Recall $u \in W_{d-t}$ so u is an eigenvector for B and B^{-1}. Recall the $\mathbb{K}$-algebra $A_q(\alpha)$ from Definition 15.1. By Lemma 14.2 and Lemma 14.3 there exists an $A_q(\alpha)$-module structure on V with $\alpha = q^{-4}(q - q^{-1})^3[3]!$ where x, y, z, z^{-1} act as r, l, B, B^{-1} respectively. With respect to this $A_q(\alpha)$-module structure we have $W' = A_q(\alpha)u$ in view of Lemma 15.8, Theorem 15.9 and (55), (56). It follows that W' is invariant under each of r, l, B, B^{-1}. By Lemma 14.1 $A = r + B^{-1}$ and $A^* = l + B + cB^{-1}$. Using this we have that W' is invariant under each of A, A^*. Since A, A^* is a tridiagonal pair on V we have $W' = V$. We now show $W' \subseteq W_{d-t} + \cdots + W_d$. By Remark 7.4 the vector (57) is contained in W_{d-t+i} where $i = \sum_{h=1}^n i_h(-1)^h$. From the construction $0 \leqslant i \leqslant t$ so $W_{d-t+i} \subseteq W_{d-t} + \cdots + W_d$. Therefore the vector (57) is contained in $W_{d-t} + \cdots + W_d$ so $W' \subseteq W_{d-t} + \cdots + W_d$. We have shown $W' = V$ and $W' \subseteq W_{d-t} + \cdots + W_d$. Since $\{W_i\}_{i=0}^d$ is a decomposition of V we find $t = d$ and the result follows. □

Lemma 16.7. *With reference to Assumption* 3.1 *and Definition* 9.3 *let W denote an irreducible $(A, \widetilde{A}^*)$-submodule of V. Then $V_d \subseteq W$.*

Proof. Recall by Lemma 16.6 that the endpoint of W is d. So $E_d W \neq 0$ by Definition 16.1. We have $U_0 = W_0$ by (7),(11) and so $\dim(W_0) = 1$ by Lemma 5.1. Using this and Lemma 12.3 we find $\dim(V_d) = 1$. We have $0 \neq E_d W \subseteq V_d$ so $E_d W = V_d$. But $E_d W \subseteq W$ by (28) so $V_d \subseteq W$. □

17. $A, \widetilde{A}^*$ is a tridiagonal pair

In this section we show $A, \widetilde{A}^*$ is a q-geometric tridiagonal pair of V if and only if $P(q^{2d-2}(q - q^{-1})^{-2}) \neq 0$. The proof of this depends on the following lemma.

Lemma 17.1. *With reference to Assumption* 3.1, *Definitions* 5.3, *and* 9.3 *the following are equivalent*:

(i) V *is irreducible as an $(A, \widetilde{A}^*)$-module.*
(ii) $P(q^{2d-2}(q - q^{-1})^{-2}) \neq 0$.

Proof. (i) $\Rightarrow$ (ii) We assume $P(q^{2d-2}(q - q^{-1})^{-2}) = 0$ and derive a contradiction. Define
$$X_i = (V_i + \cdots + V_d) \cap (\widetilde{V}_{d-i+1}^* + \cdots + \widetilde{V}_d^*) \quad (1 \leqslant i \leqslant d)$$
where the V_j are from Assumption 3.1 and the $\widetilde{V}_j^*$ are from Definition 9.7. Further define $X = X_1 + \cdots + X_d$. We will show that X is an $(A, \widetilde{A}^*)$-submodule of V and $X \neq V, X \neq 0$. We first show $AX \subseteq X$. For $1 \leqslant i \leqslant d$ we have $(A - q^{2i-d}I) \sum_{j=i}^d V_j = \sum_{j=i+1}^d V_j$ by Assumption 3.1 and $(A - q^{2i-d}I) \sum_{j=d-i+1}^d \widetilde{V}_j^* \subseteq \sum_{j=d-i}^d \widetilde{V}_j^*$ by Lemma 10.2(ii). By these comments
$$(A - q^{2i-d}I)X_i \subseteq X_{i+1} \quad (1 \leqslant i \leqslant d - 1) \quad (A - q^d I)X_d = 0$$
and it follows $AX \subseteq X$. We now show $\widetilde{A}^*X \subseteq X$. For $1 \leqslant i \leqslant d$ we have $(\widetilde{A}^* - q^{2i-d-2}I) \sum_{j=i}^d V_j = \sum_{j=i-1}^d V_j$ by Lemma 10.2(i) and $(\widetilde{A}^* - q^{2i-d-2}I) \sum_{j=d-i+1}^d \widetilde{V}_j^* \subseteq \sum_{j=d-i+2}^d \widetilde{V}_j^*$ by Definition 9.7. By these comments
$$(\widetilde{A}^* - q^{2i-d-2}I)X_i \subseteq X_{i-1} \quad (2 \leqslant i \leqslant d - 1) \quad (\widetilde{A}^* - q^{-d}I)X_1 = 0$$
and it follows $\widetilde{A}^*X \subseteq X$. We have now shown that X is an $(A, \widetilde{A}^*)$-submodule of V. We now show $X \neq V$. For $1 \leqslant i \leqslant d$ we have $X_i \subseteq V_i + \cdots + V_d$ so $X_i \subseteq V_1 + \cdots + V_d$. It follows $X \subseteq V_1 + \cdots + V_d$ and so $X \neq V$. We now show $X \neq 0$. To do this we display a nonzero vector in X_d. Pick a nonzero vector $u \in U_0$. Applying Theorem 13.1 we find $\widetilde{E}_0^* E_d u = 0$. Write $v = E_d u$ and notice $v \in V_d$. By (7) and (11) we find $U_0 = W_0$ and so $v \neq 0$ by Lemma 12.3. Observe $\widetilde{E}_0^* v = 0$ so $v \in \widetilde{V}_1^* + \cdots + \widetilde{V}_d^*$ by Definition 12.1(ii). From these comments $v \in X_d$. We have displayed a nonzero vector v contained in X_d. Of course $X_d \subseteq X$ so $X \neq 0$. We have now shown that X is an $(A, \widetilde{A}^*)$-submodule of V and $X \neq V, X \neq 0$. This contradicts our assumption that V is irreducible as an $(A, \widetilde{A}^*)$-module. We conclude $P(q^{2d-2}(q - q^{-1})^{-2}) \neq 0$.

(ii) $\Rightarrow$ (i) Let W denote an irreducible $(A, \widetilde{A}^*)$-submodule of V. We show $W = V$. To do this we show W is invariant under each of A, A^*. By construction W is invariant under A. In order to show W is invariant under A^* we show W is invariant under B. We define $\overline{W} = \{ w \in W \mid Bw \in W \}$ and show $\overline{W} = W$. Using (14) we find $A\overline{W} \subseteq \overline{W}$. Using (16) we find $\widetilde{A}^*\overline{W} \subseteq \overline{W}$. We now show $\overline{W} \neq 0$. Let $0 \neq v \in V_d$. By Lemma 16.7 $v \in W$. Combining (7) and (11) we find $U_0 = W_0$. Using this and Lemma 12.3 (with $i = d$) we have that $E_d : U_0 \to V_d$ is a bijection. So there exists a nonzero $u \in U_0$ such that $E_d u = v$. Using Theorem 13.1 we find $\widetilde{E}_0^* v = q^{2d(1-d)} c^{-d} P(q^{2d-2}(q - q^{-1})^{-2}) u$. Since $P(q^{2d-2}(q - q^{-1})^{-2})$ and u are both nonzero we find $\widetilde{E}_0^* v \neq 0$. Using (29) and since $v \in W$ we have $\widetilde{E}_0^* v \in W$. Using Lemma 9.4 and Lemma 9.6 we find $\widetilde{V}_0^* = W_0$. Hence $\widetilde{E}_0^* v \in W_0$ and so $\widetilde{E}_0^* v \in \overline{W}$. By these comments we find $\overline{W} \neq 0$. We have now shown $\overline{W}$ is nonzero and invariant under each of $A, \widetilde{A}^*$. Therefore $\overline{W} = W$ since W is an irreducible $(A, \widetilde{A}^*)$-module. We have now shown W is invariant under B. By construction W is invariant under $\widetilde{A}^*$. So by Definition 9.3 W is invariant under A^*. We now know that W is nonzero and invariant under each of A, A^*. Since A, A^* is a tridiagonal pair on V we find $W = V$ and the result follows. $\square$

Lemma 17.2. *With reference to Assumption* 3.1, *Definitions* 5.3, *and* 9.3 *the following holds.* $A, \widetilde{A}^*$ *is a q-geometric tridiagonal pair on V if and only if $P(q^{2d-2}(q - q^{-1})^{-2}) \neq 0$.*

Proof. Immediate from Assumption 3.1, Lemmas 9.6, 10.2, and 17.1. $\square$

18. The proof of Theorem 5.4

In this section we give a proof of our main result by providing the required action of $\boxtimes_q$ on V.

Proof of Theorem 5.4. (i) $\Rightarrow$ (ii) By [20, Theorem 12.3] the action of x_{30} on V is diagonalizable and the set of distinct eigenvalues is $\{q^{2i-d} \mid 0 \leqslant i \leqslant d\}$. For $0 \leqslant i \leqslant d$ let X_i denote the eigenspace of x_{30} corresponding to the eigenvalue q^{2i-d}. First we show x_{30} acts on V as B from Definition 9.1. By Definition 2.1 $q x_{01} x_{30} - q^{-1} x_{30} x_{01} = (q - q^{-1})I$ and so we have $(x_{01} - q^{d-2i}I)X_i \subseteq X_{i+1}$ for $0 \leqslant i \leqslant d$ in view of Lemma 8.2. Using Definition 2.1 we find $q x_{30}(x_{30} + c x_{23}) - q^{-1}(x_{30} + c x_{23})x_{30} = (q - q^{-1})(x_{30}^2 + cI)$ and so $(x_{30} + c x_{23} - q^{2i-d}I - c q^{d-2i}I)X_i \subseteq X_{i-1}$ for $0 \leqslant i \leqslant d$ in view of Lemma 8.3. Recall by construction x_{01} acts as A and $x_{30} + c x_{23}$ acts as A^*. Using these comments and [16, Theorem 4.6] (with V_i replaced by V_{d-i}) we find $X_i = (V_0^* + \cdots + V_i^*) \cap (V_0 + \cdots + V_{d-i})$ for $0 \leqslant i \leqslant d$. Using this and Remark 7.2 we find $X_i = W_i$ for $0 \leqslant i \leqslant d$. In view of Definition 9.1 we have now shown x_{30} acts on V as B. Using this and Definition 9.3 we find x_{23} acts as $\widetilde{A}^*$. By [20, Theorem 10.3, 24, Theorem 2.7] we find $A, \widetilde{A}^*$ is a q-geometric tridiagonal pair of V. So $P(q^{2d-2}(q - q^{-1})^{-2}) \neq 0$ by Lemma 17.2.

(ii) $\Rightarrow$ (i) By Lemma 17.2 $A, \widetilde{A}^*$ is a q-geometric tridiagonal pair on V. Using this, [23, Theorem 2.7, 21, Theorem 10.4] there exists a unique irreducible $\boxtimes_q$-module structure on V such that x_{01} acts as A and x_{23} acts as $\widetilde{A}^*$. It remains to show that $x_{30} + c x_{23}$ acts on V as A^*. To do this we show x_{30} acts on V as B from Definition 9.1. By [20, Theorem 12.3] the action of x_{30} on V is diagonalizable and the set of distinct eigenvalues is $\{q^{2i-d} \mid 0 \leqslant i \leqslant d\}$. For $0 \leqslant i \leqslant d$ let X_i denote the eigenspace of x_{30} corresponding to the eigenvalue q^{2i-d}. Using [20, Theorem 16.4] we find $X_i = (\widetilde{V}_0^* + \cdots + \widetilde{V}_i^*) \cap (V_0 + \cdots + V_{d-i})$ for $0 \leqslant i \leqslant d$. Recall by Definition 9.1 that for $0 \leqslant i \leqslant d$, W_i is the eigenspace of B with eigenvalue q^{2i-d}. We show $X_i = W_i$ for $0 \leqslant i \leqslant d$. Combining Lemmas 8.4 and (16) we find $W_0 + \cdots + W_i = \widetilde{V}_0^* + \cdots + \widetilde{V}_i^*$ for $0 \leqslant i \leqslant d$. Using this and (11) we find $V_0^* + \cdots + V_i^* = \widetilde{V}_0^* + \cdots + \widetilde{V}_i^*$ for $0 \leqslant i \leqslant d$. Using this and Remark 7.2 we find $X_i = W_i$ for $0 \leqslant i \leqslant d$. We have now shown x_{30} acts on V as B. Recall by construction x_{23} acts as $\widetilde{A}^*$. Therefore by Definition 9.3 $x_{30} + c x_{23}$ acts as A^* and the result follows. $\square$

For the sake of completeness we now make a few comments regarding the $\boxtimes_q$-module structure on V given in Theorem 5.4.

Lemma 18.1. *With reference to Assumption* 3.1, *Theorem* 5.4, *and Definition* 9.7 *the following holds. For each generator x_{ij} of $\boxtimes_q$ and for $0 \leqslant n \leqslant d$ the eigenspace of x_{ij} corresponding to the eigenvalue q^{2n-d} is given as follows.*

Generator	Eigenspace corresponding to eigenvalue q^{2n-d}
x_{01}	V_n
x_{23}	$\widetilde{V}^*_{d-n}$
x_{30}	$(\widetilde{V}^*_0 + \cdots + \widetilde{V}^*_n) \cap (V_0 + \cdots + V_{d-n})$
x_{12}	$(\widetilde{V}^*_n + \cdots + \widetilde{V}^*_d) \cap (V_{d-n} + \cdots + V_d)$
x_{31}	$(\widetilde{V}^*_0 + \cdots + \widetilde{V}^*_n) \cap (V_n + \cdots + V_d)$
x_{13}	$(\widetilde{V}^*_0 + \cdots + \widetilde{V}^*_{d-n}) \cap (V_{d-n} + \cdots + V_d)$
x_{20}	$(\widetilde{V}^*_{d-n} + \cdots + \widetilde{V}^*_d) \cap (V_0 + \cdots + V_{d-n})$
x_{02}	$(\widetilde{V}^*_n + \cdots + \widetilde{V}^*_d) \cap (V_0 + \cdots + V_n)$

Proof. The first row of the table follows immediately from Theorem 5.4. In the proof of Theorem 5.4 we showed x_{23} acts on V as $\widetilde{A}^*$. Using this we immediately obtain the second row of the table. The remaining six rows of the table follow from the first two rows and [20, Theorem 16.4]. □

Lemma 18.2. *With reference to Assumption* 3.1 *and the $\boxtimes_q$-module structure on V given in Theorem* 5.4 *the following holds.*

(i) x_{30} *acts on V as B.*
(ii) x_{23} *acts on V as $\widetilde{A}^*$.*
(iii) x_{31} *acts on V as K.*
(iv) x_{13} *acts on V as K^{-1}.*

where $B, \widetilde{A}^*, K$ are from Definitions 9.1, 9.3, *and* 11.1, *respectively.*

Proof. (i) and (ii): These were shown in the proof of Theorem 5.4.

(iii) and (iv): Recall for $0 \leqslant i \leqslant d$ that U_i is the eigenspace for K corresponding to the eigenvalue q^{2i-d}. In the proof of Theorem 5.4 we showed for $0 \leqslant i \leqslant d$ that $V_0^* + \cdots + V_i^* = \widetilde{V}_0^* + \cdots + \widetilde{V}_i^*$. Using this and Lemma 4.2 we find $U_i = (\widetilde{V}_0^* + \cdots + \widetilde{V}_i^*) \cap (V_i + \cdots + V_d)$ for $0 \leqslant i \leqslant d$. The result now follows from rows five and six of the table in Lemma 18.1. □

Acknowledgments

References

[1] H. Al-Najjar, B. Curtin, A family of tridiagonal pairs, Linear Algebra Appl., 390 (2004) 369–384.
[2] H. Al-Najjar, B. Curtin, A family of tridiagonal pairs related to the quantum affine algebra $U_q(\widehat{sl}_2)$, Electron. J. Linear Algebra 13 (2005) 1–9.
[3] H. Al-Najjar, B. Curtin, A bilinear form for tridiagonal pairs of q-Serre type, Linear Algebra Appl. 428 (11–12) (2008) 2688–2698.

[4] P. Baseilhac, An integrable structure related with tridiagonal algebras, Nucl. Phys. B 705 (3) (2005) 605–619.
[5] P. Baseilhac, A family of tridiagonal pairs and related symmetric functions, J. Phys. A 39 (38) (2006) 11773–11791.
[6] G. Benkart, P. Terwilliger, Irreducible modules for the quantum affine algebra $U_q(\hat{s}l_2)$ and its Borel subalgebra, J. Algebra 282 (2004) 172–194.
[7] G. Benkart, P. Terwilliger, The universal central extension of the three-point $\mathfrak{s}l_2$ loop algebra, Proc. Amer. Math. Soc. 135 (6) (2007) 1659–1668.
[8] J. Bowman, Irreducible modules for the quantum affine algebra $U_q(\mathfrak{g})$ and its Borel subalgebra $U_q(\mathfrak{g})^{\geqslant 0}$, J. Algebra 316 (1) (2007) 231–253.
[9] B. Curtin, Modular Leonard triples, Linear Algebra Appl. 424 (2–3) (2007) 510–539.
[10] B. Curtin, Spin Leonard pairs, Ramanujan J. 13 (1–3) (2007) 319–332.
[11] A. Elduque, The S_4-action on the tetrahedron algebra, Proc. Roy. Soc. Edinburgh Sect. A 137 (6) (2007) 1227–1248.
[12] D. Funk-Neubauer, Raising/lowering maps and modules for the quantum affine algebra $U_q(\hat{s}l_2)$, Comm. Algebra 35 (7) (2007) 2140–2159.
[13] B. Hartwig, Three mutually adjacent Leonard pairs, Linear Algebra Appl. 408 (2005) 19–39.
[14] B. Hartwig, The tetrahedron algebra and its finite dimensional irreducible modules, Linear Algebra Appl. 422 (1) (2007) 219–235.
[15] B. Hartwig, P. Terwilliger, The tetrahedron algebra, the Onsager algebra, and the $\mathfrak{s}l_2$ loop algebra, J. Algebra 308 (2) (2007) 840–863.
[16] T. Ito, K. Tanabe, P. Terwilliger, Some Algebra Related to P- and Q-polynomial Association Schemes, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. (Providence, RI), American Mathematical Society, 2001., pp. 167–192.
[17] T. Ito, P. Terwilliger, Finite dimensional irreducible modules for the three-point $\mathfrak{s}l_2$ loop algebra, Comm. Algebra 36(12) (2008) 4557–4598.
[18] T. Ito, P. Terwilliger, The shape of a tridiagonal pair, J. Pure Appl. Algebra 188 (2004) 145–160.
[19] T. Ito, P. Terwilliger, q-Inverting pairs of linear transformations and the q-tetrahedron algebra, Linear Algebra Appl. 426 (2–3) (2007) 516–532.
[20] T. Ito, P. Terwilliger, The q-tetrahedron algebra and its finite dimensional irreducible modules, Comm. Algebra 35 (11) (2007) 3415–3439.
[21] T. Ito, P. Terwilliger, Tridiagonal pairs and the quantum affine algebra $U_q(\hat{s}l_2)$, Ramanujan J. 13 (1–3) (2007) 39–62.
[22] T. Ito, P. Terwilliger, Tridiagonal pairs of Krawtchouk type, Linear Algebra Appl. 427 (2–3) (2007) 218–233.
[23] T. Ito, P. Terwilliger, Two non-nilpotent linear transformations that satisfy the cubic q-Serre relations, J. Algebra Appl. 6 (3) (2007) 477–503.
[24] K. Nomura, A refinement of the split decomposition of a tridiagonal pair, Linear Algebra Appl. 403 (2005) 1–23.
[25] K. Nomura, Tridiagonal pairs and the Askey–Wilson relations, Linear Algebra Appl. 397 (2005) 99–106.
[26] K. Nomura, Tridiagonal pairs of height one, Linear Algebra Appl. 403 (2005) 118–142.
[27] K. Nomura, P. Terwilliger, The structure of a tridiagonal pair, Linear Algebra Appl. 429(7) (2008) 1647–1662.
[28] K. Nomura, P. Terwilliger, The determinant of $AA^* - A^*A$ for a Leonard pair a, a^*, Linear Algebra Appl. 416 (2–3) (2006) 880–889.
[29] K. Nomura, P. Terwilliger, Matrix units associated with the split basis of a Leonard pair, Linear Algebra Appl. 418 (2–3) (2006) 775–787.
[30] K. Nomura, P. Terwilliger, Some trace formulae involving the split sequences of a Leonard pair, Linear Algebra Appl. 413 (2006) 189–201.
[31] K. Nomura, P. Terwilliger, Affine transformations of a Leonard pair, Electron. J. Linear Algebra 16 (2007) 389–417.
[32] K. Nomura, P. Terwilliger, Balanced Leonard pairs, Linear Algebra Appl. 420 (1) (2007) 51–69.
[33] K. Nomura, P. Terwilliger, Linear transformations that are tridiagonal with respect to both eigenbases of a Leonard pair, Linear Algebra Appl. 420 (1) (2007) 198–207.
[34] K. Nomura, P. Terwilliger, The split decomposition of a tridiagonal pair, Linear Algebra Appl. 424 (2–3) (2007) 339–345.
[35] K. Nomura, P. Terwilliger, Sharp tridiagonal pairs, Linear Algebra Appl. 429 (1) (2008) 79–99.
[36] K. Nomura, P. Terwilliger, The switching element for a Leonard pair, Linear Algebra Appl. 428 (4) (2008) 1083–1108.
[37] K. Nomura, P. Terwilliger, Towards a classification of the tridiagonal pairs, Linear Algebra Appl. 429 (2–3) (2008) 503–518.
[38] P. Terwilliger, The incidence algebra of a uniform poset, Math. Appl. 20 (1990) 193–212.
[39] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other, Linear Algebra Appl. 330 (2001) 149–203.
[40] P. Terwilliger, Physics and Combinatorics, World Scientific Publishing, Nagoya, River Edge, NJ, 2001., pp. 377–398.
[41] P. Terwilliger, Leonard pairs from 24 points of view, Rocky Mountain J. Math. 32 (2) (2002) 827–888.
[42] P. Terwilliger, Introduction to Leonard pairs, J. Comput. Appl. Math. 153 (2) (2003) 463–475.
[43] P. Terwilliger, Leonard pairs and the q-Racah polynomials, Linear Algebra Appl. 387 (2004) 235–276.
[44] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the parameter array, Des. Codes Cryptogr. 34 (2005) 307–332.
[45] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the split decomposition, J. Comput. Appl. Math. 178 (2005) 437–452.
[46] P. Terwilliger, Two linear transformations each tridiagonal with respect to an eigenbasis of the other; the *TD-D* and the *LB-UB* canonical form, J. Algebra 291 (2005) 1–45.
[47] P. Terwilliger, An algebraic approach to the Askey scheme of orthogonal polynomials, Lecture Notes in Math., Springer, Berlin, 2006 ., pp. 255–330.
[48] P. Terwilliger, R. Vidunas, Leonard pairs and the Askey–Wilson relations, J. Algebra Appl. 3 (2004) 411–426.
[49] M. Vidar, Tridiagonal pairs of shape (1, 2, 1), Linear Algebra Appl. 429 (1) (2008) 403–428.
[50] R. Vidunas, Normalized Leonard pairs and Askey–Wilson relations, Linear Algebra Appl. 422 (1) (2007) 39–57.