On Dresher’s inequalities for width-integrals

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In the paper, we establish a reversed Dresher’s integral inequality, based on the Minkowski inequality and an inequality due to Radon. Further, we prove Dresher-type inequalities for width-integrals of convex bodies and mixed projection bodies, respectively.

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1. Notations and preliminaries

The setting for this paper is $n$-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve the letter $u$ for unit vectors, and the letter $B$ is reserved for the unit ball centered at the origin. The surface of $B$ is $S^{n-1}$. The volume of the unit $n$-ball is denoted by $\omega_n$. For $u \in S^{n-1}$, let $E_u$ denote the hyperplane, through the origin, that is orthogonal to $u$. We use $K^u$ to denote the image of $K$ under an orthogonal projection onto the hyperplane $E_u$. If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, we denote by $v(K_1^u, \ldots, K_{n-1}^u)$ the mixed volume of the figures $K_1^u, \ldots, K_{n-1}^u$ in the space $E_u$. If $K_1 = \cdots = K_{n-1} = K$, then we write $v(K^u)$ for $v(K_1^u, \ldots, K_{n-1}^u)$.

We use $V(K)$ for the $n$-dimensional volume of convex body $K$. Let $h(K, \cdot) : S^{n-1} \to \mathbb{R}$ denote the support function of $K \in \mathcal{K}^n$; i.e. for $u \in S^{n-1}$

$$h(K, u) = \text{Max}\{u \cdot x : x \in K\},$$

where $u \cdot x$ denotes the usual inner product $u$ and $x$ in $\mathbb{R}^n$.

Let $\delta$ denote the Hausdorff metric on $\mathcal{K}^n$, i.e. for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $| \cdot |_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$ ([1], also see [2]).

1.1. Width-integrals of convex bodies

For $u \in S^{n-1}$, $b(K, u) := \frac{1}{2}(h(K, u) + h(K, -u))$ is defined to be half the width of $K$ in the direction $u$. Two convex bodies $K$ and $L$ are said to have similar width if there exists a constant $\lambda > 0$ such that $b(K, u) = \lambda b(L, u)$ for all $u \in S^{n-1}$. Width-integrals were first considered by Blaschke (see [3]). The width-integral of index $i$ is defined by Lutwak [4]. For $K \in \mathcal{K}^n$, $i \in \mathbb{R}$

$$B_i(K) = \frac{1}{n} \int_{S^{n-1}} b(K, u)^{n-i} dS(u).$$

(1.1)

This definition implies that the width-integral of index $i$ is non-negative and continuous.
The following results (see [4]) will be used later

\[ b(K + L, u) = b(K, u) + b(L, u). \]  
\[ B_{2n} \leq V(K^n), \]  
with equality if and only if \( K \) is centered with respect to \( p \).

1.2. Polar mixed projection bodies

The projection body \( \Pi K \) of a convex body \( K \in \mathcal{K}^n \), is defined as the convex body whose support function is given by (see [5])

\[ h(\Pi K, u) = v(K^u), \quad u \in S^{n-1}. \]

If \( K_1, \ldots, K_{n-1} \in \mathcal{K}^n \), then \( \Pi(K_1, \ldots, K_{n-1}) \) denote the mixed projection body of \( K_1, \ldots, K_{n-1} \). Support function of \( \Pi(K_1, \ldots, K_{n-1}) \) is defined by

\[ h(\Pi(K_1, \ldots, K_{n-1}), u) = v(K_1^u, \ldots, K_{n-1}^u). \]

It is easy to see that \( \Pi(K_1, \ldots, K_{n-1}) \) is centered.

The following property (see [5]) will be used later. If \( K, L, K_3, \ldots, K_n \in \mathcal{K}^n \), \( \lambda, \mu > 0 \) and \( M = (K_3, \ldots, K_n) \),

\[ \Pi(\lambda K + \mu L, M) = \lambda \Pi(K, M) + \mu \Pi(L, M). \]  
(1.4)

If \( K \) is a convex body that contains the origin in its interior, the polar body of \( K, K^* \), is defined by

\[ K^* := \{ x \in \mathbb{R}^n | x \cdot y \leq 1, y \in K \}. \]

If \( K_1, \ldots, K_{n-1} \in \mathcal{K}^n \), then polar of the mixed projection body \( \Pi(K_1, \ldots, K_{n-1}) \) is usually written as \( \Pi^*(K_1, \ldots, K_{n-1}) \). If \( K_1 = K + L \) and \( M = (K_3, \ldots, K_n) \), then the polar mixed projection body \( \Pi^*(K_1, K_3, \ldots, K_n) \) is written as \( \Pi^*(K + L, M) \).

2. Reversed Dresher’s inequality

The well-known inequality due to Beckenbach can be stated as follows ([6], p. 27).

**Theorem 2.1.** If \( 1 \leq p \leq 2 \), and \( f(x), g(x) > 0 \), then

\[ \frac{\int (f(x) + g(x))^p dx}{\int (f(x) + g(x))^{p-1} dx} \leq \frac{\int f(x)^p dx}{\int f(x)^{p-1} dx} + \frac{\int g(x)^p dx}{\int g(x)^{p-1} dx}. \]  
(2.1)

An extension of the Beckenbach’s inequality was obtained by Dresher [7] by an ingenious method, using moment-space theory.

**Theorem 2.2.** If \( p \geq 1 \geq r \geq 0 \), \( f(x), g(x) \geq 0 \), and \( \phi(x) \) is a distribution function, then

\[ \left( \frac{\int (f + g)^p d\phi}{\int (f + g)^r d\phi} \right)^{1/(p-r)} \leq \left( \frac{\int f^p d\phi}{\int f^r d\phi} \right)^{1/(p-r)} + \left( \frac{\int g^p d\phi}{\int g^r d\phi} \right)^{1/(p-r)}. \]  
(2.2)

In the following, an inverse of inequality (2.2) is established.

**Theorem 2.3.** If \( p \leq 0 \leq r \leq 1 \), \( f(x), g(x) \geq 0 \), and \( \phi(x) \) is a distribution function, then

\[ \left( \frac{\int (f + g)^p d\phi}{\int (f + g)^r d\phi} \right)^{1/(p-r)} \geq \left( \frac{\int f^p d\phi}{\int f^r d\phi} \right)^{1/(p-r)} + \left( \frac{\int g^p d\phi}{\int g^r d\phi} \right)^{1/(p-r)}, \]  
(2.3)

with equality if and only if the functions \( f \) and \( g \) are proportional.

We need the following lemma to prove Theorem 2.3.

**Lemma 2.4 ([8], p. 61, Radon’s Inequality).** If \( a \) and \( b \) are positive and \( 0 < p < 1 \), then

\[ \sum \frac{a^p}{b^{p-1}} < \left( \frac{\sum a^p}{\sum b^{p-1}} \right)^{p-1}, \]  
(2.4)

unless (a) and (b) are proportional.

**Proof.** If \( \alpha_1 \geq 0, \alpha_1 \geq 0, \beta_1 > 0, \) and \( \beta_2 > 0, \) and \( -1 < \lambda < 0 \), then from Radon’s inequality, we have

\[ \frac{\alpha_1^{\lambda+1}}{\beta_1^\lambda} + \frac{\alpha_2^{\lambda+1}}{\beta_2^\lambda} \leq \frac{(\alpha_1 + \alpha_2)^{\lambda+1}}{(\beta_1 + \beta_2)^\lambda}. \]  
(2.5)

with equality if and only if \( (\alpha) \) and \( (\beta) \) are proportional. Let
\[
\alpha_1 = \left( \int f^p d\phi \right)^{1/p}, \quad \beta_1 = \left( \int f^r d\phi \right)^{1/r}, \tag{2.6}
\]
\[
\alpha_2 = \left( \int g^p d\phi \right)^{1/p}, \quad \beta_2 = \left( \int g^r d\phi \right)^{1/r}, \tag{2.7}
\]
and let
\[
\lambda = \frac{r}{p-r}. \tag{2.8}
\]
From (2.5)–(2.8), we obtain
\[
\frac{\alpha_1^{\lambda+1}}{\beta_1^\lambda} + \frac{\alpha_2^{\lambda+1}}{\beta_2^\lambda} = \left( \frac{\int f^p d\phi}{\int f^r d\phi} \right)^{\lambda/\alpha} + \left( \frac{\int g^p d\phi}{\int g^r d\phi} \right)^{\lambda/r} = \left( \frac{\int f^p d\phi}{\int f^r d\phi} \right)^{1/(p-r)} + \left( \frac{\int g^p d\phi}{\int g^r d\phi} \right)^{1/(p-r)} \leq \left[ \left( \frac{\int f^p d\phi}{\int f^r d\phi} \right)^{1/p} + \left( \frac{\int g^p d\phi}{\int g^r d\phi} \right)^{1/p} \right]^{\lambda/(p-r)}. \tag{2.9}
\]
Since \(-1 < \lambda = \frac{r}{p-r} < 0\), we may assume \(p < 0 < r\), and by Minkowski inequality for \(p < 0\) and \(0 < r \leq 1\), we have respectively
\[
\left[ \left( \int f^p d\phi \right)^{1/p} + \left( \int g^p d\phi \right)^{1/p} \right]^p \geq \int (f + g)^p d\phi, \tag{2.10}
\]
with equality if and only if \(f\) and \(g\) are proportional, and
\[
\left[ \left( \int f^r d\phi \right)^{1/r} + \left( \int g^r d\phi \right)^{1/r} \right]^r \leq \int (f + g)^r d\phi, \tag{2.11}
\]
with equality if and only if \(f\) and \(g\) are proportional.

From equality conditions for (2.5), (2.10) and (2.11), it follows that the sign of equality in (2.3) holds if and only if \(f\) and \(g\) are proportional.

From (2.9)–(2.11), inequality (2.3) follows. \(\square\)

3. Dresher’s inequalities for width-integrals

3.1. Dresher-type inequality for width-integrals of convex bodies

**Theorem 3.1.** If \(K, L \in \mathcal{K}_n\) and let \(p \leq r \leq 1\) and \(p, r \in \mathbb{R}\), then
\[
\frac{B_{n-p}(K + L)^{(p-r)}}{B_{n-r}(K + L)} \geq \frac{B_{n-r}(K)}{B_{n-r}(L)}, \tag{3.1}
\]
with equality if and only if \(K\) and \(L\) have similar width.

The inequality is reversed for \(p \geq 1 \geq r \geq 0\).

**Remark 3.2.** Let \(p = n - i\) and \(r = n - j\) in (3.1), and in view of \(p \leq 0 \leq r \leq 1 \Rightarrow i \geq n \geq j \geq n - 1\), we have
\[
\frac{B_i(K + L)^{(1/(1-i))}}{B_j(K + L)} \geq \frac{B_i(K)^{(1/(1-i))}}{B_j(K)} + \frac{B_i(L)^{(1/(1-i))}}{B_j(L)}, \tag{3.2}
\]
with equality if and only if \(K\) and \(L\) have similar width.

Taking for \(j = n\) in (3.2) and in view of \(B_n(K) = \frac{1}{n} \int_{S^{n-1}} dS(u) = n\omega_n\), a constant, we obtain
\[
B_i(K + L)^{(1/(n-i))} \geq B_i(K)^{(1/(n-i))} + B_i(L)^{(1/(n-i))}, \quad i \geq n,
\]
with equality if and only if \(K\) and \(L\) have similar width. This was previously obtained by Lutwak [4] for \(i < n - 1\).

Taking for \(j = n\) and \(i = 2n\) in (3.2), we obtain
\[
B_{2n}(K + L)^{-1/n} \geq B_{2n}(K)^{-1/n} + B_{2n}(L)^{-1/n}, \tag{3.3}
\]
with equality if and only if \(K\) and \(L\) have similar width.
If $K$, $L$ are centered at the origin so that $K + L$ is centered, then from (1.3) and (3.3), we have
\[ V((K + L)^*)^{-1/n} \geq V(K^*)^{-1/n} + V(L^*)^{-1/n}, \]
with equality if and only if $K$ and $L$ have similar width.

**Proof of Theorem 3.3.** From (1.1) and (1.2), we have
\[ B_{n-p}(K + L) = \frac{1}{n} \int_{S^{n-1}} b(K + L, u)^p dS(u) = \frac{1}{n} \int_{S^{n-1}} (b(K, u) + b(L, u))^p dS(u) \quad (3.4) \]
and
\[ B_{n-r}(K + L) = \frac{1}{n} \int_{S^{n-1}} (b(K, u) + b(L, u))^r dS(u) \quad (3.5) \]
From (3.4) and (3.5) and in view of inequality (2.3), we obtain
\[ \left( \frac{B_{n-p}(K + L)}{B_{n-r}(K + L)} \right)^{1/(p-r)} = \left( \frac{\int_{S^{n-1}} (b(K, u) + b(L, u))^p dS(u)}{\int_{S^{n-1}} (b(K, u) + b(L, u))^r dS(u)} \right)^{1/(p-r)} \geq \left( \frac{\int_{S^{n-1}} b(K, u)^p dS(u)}{\int_{S^{n-1}} b(K, u)^r dS(u)} \right)^{1/(p-r)} \]
\[ = \left( \frac{B_{n-p}(K)}{B_{n-r}(K)} \right)^{1/(p-r)} + \left( \frac{B_{n-p}(L)}{B_{n-r}(L)} \right)^{1/(p-r)}, \]
with equality if and only if $K$ and $L$ have similar width. □

3.2. Dresher-type inequality for width-integrals of mixed projection bodies

**Theorem 3.3.** Let $K$, $L$, $K_3, \ldots, K_n$ be convex bodies in $\mathbb{R}^n$ and $M = (K_3, \ldots, K_n)$. If $p \leq 0 \leq r \leq 1$ and $p$, $r \in \mathbb{R}$, then
\[ \left( \frac{B_{n-p}(\Pi(K + L, M))}{B_{n-r}(\Pi(K + L, M))} \right)^{1/(p-r)} \geq \left( \frac{B_{n-p}(\Pi(K, M))}{B_{n-r}(\Pi(K, M))} \right)^{1/(p-r)} + \left( \frac{B_{n-p}(\Pi(L, M))}{B_{n-r}(\Pi(L, M))} \right)^{1/(p-r)}, \quad (3.6) \]
with equality if and only if $\Pi(K, M)$ and $\Pi(L, M)$ have similar width.

The inequality is reversed for $p \geq 1 \geq r \geq 0$.

**Remark 3.4.** Let $p = n - i$ and $r = n - j$ in (3.6), in view of $p \leq 0 \leq r \leq 1 \Rightarrow i \geq n \geq j \geq n - 1$, we have
\[ \left( \frac{B_i(\Pi(K + L, M))}{B_j(\Pi(K + L, M))} \right)^{1/(i-j)} \geq \left( \frac{B_i(\Pi(K, M))}{B_j(\Pi(K, M))} \right)^{1/(i-j)} + \left( \frac{B_i(\Pi(L, M))}{B_j(\Pi(L, M))} \right)^{1/(i-j)}, \quad (3.7) \]
with equality if and only if $\Pi(K, M)$ and $\Pi(L, M)$ have similar width.

Taking for $j = n$ and $i = 2n$ in (3.7) we have
\[ B_{2n}(\Pi(K + L, M))^{-1/n} \geq B_{2n}(\Pi(K, M))^{-1/n} + B_{2n}(\Pi(L, M))^{-1/n}, \quad (3.8) \]
with equality if and only if $\Pi(K, M)$ and $\Pi(L, M)$ have similar width.

From (1.3) and (3.8) and in view of the projection body centered at the origin, we have
\[ V(\Pi^*(K + L, M))^{-1/n} \geq V(\Pi^*(K, M))^{-1/n} + V(\Pi^*(L, M))^{-1/n}, \]
with equality if and only if $\Pi(K, M)$ and $\Pi(L, M)$ have similar width. This was contained in [9,10].

**Proof of Theorem 3.3.** From (1.1), (1.2) and (1.4), we have
\[ B_{n-p}(\Pi(K + L, M)) = \frac{1}{n} \int_{S^{n-1}} (b(\Pi(K, M), u) + b(\Pi(L, M), u))^p dS(u) \quad (3.9) \]
and
\[ B_{n-r}(\Pi(K + L, M)) = \frac{1}{n} \int_{S^{n-1}} (b(\Pi(K, M), u) + b(\Pi(L, M), u))^r dS(u). \quad (3.10) \]
From (3.9) and (3.10) and in view of inequality (2.3), we obtain

\[
\left( \frac{B_{n-p}(\Pi(K + L, M))}{B_{n-r}(\Pi(K + L, M))} \right)^{1/(p-r)} = \left( \frac{\int_{S^{n-1}} b(\Pi(K, M), u) \, dS(u)}{\int_{S^{n-1}} b(\Pi(K, M), u) + b(\Pi(L, M), u) \, dS(u)} \right)^{1/(p-r)} \\
\geq \left( \frac{\int_{S^{n-1}} b(\Pi(K, M), u) \, dS(u)}{\int_{S^{n-1}} b(\Pi(K, M), u) \, dS(u)} \right)^{1/(p-r)} + \left( \frac{\int_{S^{n-1}} b(\Pi(L, M), u) \, dS(u)}{\int_{S^{n-1}} b(\Pi(L, M), u) \, dS(u)} \right)^{1/(p-r)} \\
= \left( \frac{B_{n-p}(\Pi(K, M))}{B_{n-r}(\Pi(K, M))} \right)^{1/(p-r)} + \left( \frac{B_{n-p}(\Pi(L, M))}{B_{n-r}(\Pi(L, M))} \right)^{1/(p-r)},
\]

with equality if and only if \( \Pi(K, M) \) and \( \Pi(L, M) \) have similar width. \( \Box \)

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