JOURNAL OF COMBINATORIAL THEORY, Series B 53, 143–148 (1991)

Note

Multicolored Forests in Bipartite Decompositions of Graphs

NOGA ALON

IBM Almaden Research Center, San Jose, California 95120, and Sackler Faculty of Exact Sciences, Tel Aviv University, Israel

AND

RICHARD A. BRUALDI* AND BRYAN L. SHADER†

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

Communicated bt the Managing Editors

Received December 5, 1989

We show that in any edge-coloring of the complete graph K_n on n vertices, such that each color class forms a complete bipartite graph, there is a spanning tree of K_n, no two of whose edges have the same color. This strengthens a theorem of Graham and Pollak and verifies a conjecture of de Caen. More generally we show that in any edge-coloring of a graph G with p positive and q negative eigenvalues, such that each color class forms a complete bipartite graph, there is a forest of at least $\max\{p, q\}$ edges, no two of which have the same color. In the case where G is bipartite there is always such a forest which is a matching.

A *bipartite decomposition* of a graph G is an edge-coloring of G such that each color class is the set of all edges of a complete bipartite subgraph of G. A well-known theorem of Graham and Pollak ([4, 5], see also [6, Problem 11.22]) asserts that the number of colors in any bipartite decomposition of K_n is at least $n-1$. Simple proofs of this theorem were given by Tverberg [10] and Peck [7]. See also [1] for an extension of the result to hypergraphs. The Graham–Pollak result is, of course, sharp, and there are many non-isomorphic bipartite decompositions of K_n using exactly

* Research partially supported by National Science Foundation Grant DMS-8901445 and National Security Agency Grant MDA904-89-H-2060.

† Current address: Department of Mathematics, University of Wyoming, Laramie, WY 82071.

0095-8956/91 $3.00

$n-1$ colors. All the proofs of this result mentioned above apply some simple ideas from linear algebra.

D. de Caen [2] conjectured that in any bipartite decomposition of K_n using $n-1$ colors there is a *multicolored tree*, i.e., a spanning tree of K_n, no two of whose edges have the same color. In this note we prove the following stronger result.

THEOREM 1. *In any bipartite decomposition of K_n there is a spanning tree of K_n, no two of whose edges have the same color.*

Graham and Pollak obtained their result as a special case of a more general theorem which asserts that the number of colors in any bipartite decomposition of an arbitrary graph G is at least the maximum of the number of positive and the number of negative eigenvalues of G. Since K_n has $n-1$ negative eigenvalues, Theorem 1 is a special case of the following more general result.

THEOREM 2. *Let G be a graph with p positive and q negative eigenvalues. Then in any bipartite decomposition of G there is a forest with* $\max\{p, q\}$ *edges, no two of which have the same color.*

Our proof combines the interlacing inequalities for symmetric matrices with the following well-known theorem of Rado, usually called the Rado–Hall Theorem.

THEOREM 3 (Rado [8, 11]). *Let $\{C_i : i \in I\}$ be a finite family of finite subsets of a vector space and let t be an integer with $0 \leqslant t \leqslant |I|$. Then there exists a subfamily of cardinality t which has a linearly independent set of distinct representatives if and only if* $\operatorname{rank}(\bigcup_{j \in J} C_j) \geqslant |J| - (|I| - t)$ *for all $J \subseteq I$, where* $\operatorname{rank}(W)$ *is the dimension of the subspace spanned by W.*

Let G be a graph with n vertices and m edges, and let B be the n by m vertex–edge incidence matrix of G. By identifying the edges of G with the columns of B, considered as vectors over $GF(2)$, we see that a set of edges is linearly independent if and only if it determines a forest. Thus the rank of a set E of edges equals $n-k$, where k is the number of connected components of the spanning subgraph of G with edge set E.

Proof of Theorem 2. Let $\{C_i : i \in I\}$ be the family of color classes of a bipartite decomposition of G. Let J be a subset of I, and let H be the spanning subgraph of G with edge set $\bigcup_{i \in J} C_i$. Suppose that k is the number of connected components of H and let $v_1, v_2, ..., v_k$ be k vertices, one from each component of H. Let G' be the subgraph of G induced on the set $\{v_1, v_2, ..., v_k\}$, and let E' be the set of edges of G'. The adjacency matrix of G' is a principal submatrix of order k of the adjacency matrix of G. By the interlacing inequalities for symmetric matrices, G' has at least $q-(n-k)$ negative eigenvalues and at least $p-(n-k)$ positive eigenvalues. By applying the general Graham–Pollak theorem to G', we conclude that every bipartite decomposition of G' requires at least

$$k-n+\max\{p, q\} \tag{1}$$

colors. Since $\{C_i \cap E' : i \in I-J\}$ is a bipartite decomposition of G',

$$|I|-|J| \geqslant k-n+\max\{p, q\}.$$

Hence

$$\operatorname{rank}\left(\bigcup_{i \in J} C_i\right)=n-k \geqslant |J|-(|I|-\max\{p, q\}).$$

By Theorem 3, there is a subfamily of $\max\{p, q\}$ color classes having an independent set of distinct representatives, that is, a forest of $\max\{p, q\}$ edges, each with a different color. ∎

Remarks. (1) Let G_i be the complete bipartite graph with edge set C_i, and let T_i be the set of edges of a spanning tree of G_i $(i \in I)$. Because $\operatorname{rank}(\bigcup_{i \in J} C_i)=\operatorname{rank}(\bigcup_{i \in J} T_i)$ there is in fact a multicolored forest with $\max\{p, q\}$ edges, each of which belongs to some T_i.

(2) If G is the complete graph K_n, the interlacing inequalities can be avoided. This is because q equals $n-1$ and (1) equals $k-1$, and the graph G' is K_k. By the Graham–Pollak theorem every bipartite decomposition of K_k requires at least $k-1$ colors.

(3) In the case of a bipartite decomposition of K_n with exactly $n-1$ colors, we showed that the number of connected components of a spanning subgraph with edge set equal to the union of t color classes is at most $n-t$. Thus a necessary condition for t edge-disjoint complete bipartite graphs to be extendable to a bipartite decomposition of K_n with exactly $n-1$ colors is that the resulting spanning subgraph of K_n have at most $n-t$ connected components.

If we have a bipartite decomposition of K_n with n colors, then we can find a special type of multicolored graph with n edges. A *near-tree* is a connected graph with the same number of edges as vertices, whose unique cycle has odd length. A *near-forest* is a graph, each of whose connected components is a near-tree.

We now consider the columns of the n by m vertex–edge incidence matrix B as vectors over the real field. Now a set F of edges of the graph G is linearly independent if and only if each connected component of the graph G_F spanned by F is a tree or a near-tree. If $|F|=n$, then F is linearly independent if and only if G_F is a near-forest. The rank of a set E of edges now equals $n-l$, where l is the number of bipartite connected components of the spanning subgraph of G with edge set E. The linearly independent sets of edges are the independent sets of the matroid $P_3(G)$ defined on the edges of G in [9].

THEOREM 4. *In any bipartite decomposition of K_n with at least n colors there is a spanning near-forest, no two of whose edges have the same color.*

Proof. Let $\{C_i : i \in I\}$ be the family of color classes of a bipartite decomposition of K_n with $|I| \geqslant n$. Let J be a subset of I. By Rado's theorem (with $t=n$) it suffices to show that the spanning subgraph H of K_n with edge set $\bigcup_{i \in J} C_i$ has at most $|I|-|J|$ bipartite connected components. Let k be the number of components of H and assume that l of them are bipartite. First suppose that $k > l$. As in the proof of Theorem 2 (see Remark (2)), $|I|-|J| \geqslant k-1$. Hence $l \leqslant |I|-|J|$. Now suppose that $l=k$ and hence all components of H are bipartite. If each component has at most one edge, then $2\,|J|+(l-|J|)=n \leqslant |I|$ and hence $|I|-|J| \geqslant l$. Hence we may assume that some component has two vertices u and v which are not adjacent in H. Let $w_1, w_2, ..., w_{l-1}$ be $l-1$ vertices, one from each of the other components of H. By applying the Graham–Pollak theorem to the complete graph K_{l+1} induced on the set $\{u, v, w_1, ..., w_{l-1}\}$ we conclude that $|I|-|J| \geqslant l$. ▮

By using the interlacing inequalities and the general Graham–Pollak theorem, the following more general result can be obtained.

THEOREM 5. *Let G be a graph with p positive and q negative eigenvalues. Then in any bipartite decomposition of G with at least* $\max\{p, q\}+1$ *colors, there is a graph with* $\max\{p, q\}+1$ *edges, no two of which have the same color, where each connected component is either a tree or a near-tree.*

The proof of Theorem 4 can be modified to obtain a similar result on clique decompositions of complete graphs. A *clique decomposition* of a graph G is an edge-coloring of G such that each color class is the set of all edges of a complete subgraph of G. The decomposition is non-trivial if it uses at least two colors. A well-known inequality of Fisher (see, e.g., [6, Problem 13.15]) asserts that the number of colors in any non-trivial clique decomposition of K_n is at least n. By combining this with the method used in the proof of Theorem 4 one can obtain the following result, whose detailed proof is left to the reader.

THEOREM 6. *In any non-trivial clique decomposition of K_n there is a spanning near-forest, no two of whose edges have the same color.*

Let G be a spanning bipartite subgraph of the complete bipartite graph $K_{n,n}$ with bipartition $U = \{u_1, ..., u_n\}$ and $V = \{v_1, ..., v_n\}$. Let A be the adjacency matrix of G of order $2n$ and let ρ be the rank of A. Theorem 2 asserts that in any bipartite decomposition of G there is a multicolored forest with $\rho/2$ edges. We can show that in fact a stronger assertion holds; there exists a multicolored matching with $\rho/2$ edges.

Without loss of generality we assume that

$$A = \begin{bmatrix} O & X \\ X^{\mathrm{T}} & O \end{bmatrix},$$

where X has order n.

THEOREM 7. *In any bipartite decomposition of the bipartite graph G there is a matching with $r = \operatorname{rank}(X)$ edges, no two of which have the same color.*

Proof. A bipartite decomposition of G with c colors corresponds to a factorization $X = YZ$ into (0, 1)-matrices of sizes n by c and c by n, respectively. Clearly X has a nonsingular r by r submatrix. By renumbering the vertices in V, if necessary, we may assume, without loss of generality, that X has a non-singular principal submatrix of order r. Hence there is a subset L of $\{1, 2, ..., n\}$ of cardinality r such that the r by c submatrix $Y[L, *]$ of Y determined by L has rank r, and the c by r submatrix $Z[*, L]$ of Z determined by L has rank r. By the Cauchy–Binet theorem there is a subset M of $\{1, 2, ..., c\}$ of cardinality r such that the matrices $Y[L, M] = [y_{ij} : i \in L, j \in M]$ and $Z[M, L] = [z_{ji} : j \in M, i \in L]$ of order r are non-singular. There exists a bijection $\sigma: L \to M$ such that $\prod_{i \in L} y_{i\sigma(i)} \neq 0$ and a bijection $\tau: M \to L$ such that $\prod_{j \in M} z_{j\tau(j)} \neq 0$. It follows that the set of edges $\{u_{\sigma^{-1}(j)}, v_{\tau(j)}\}$ with $j \in M$ is a multicolored matching of r edges. ∎

As a special case we conclude, e.g., that in every bipartite decomposition of the complete bipartite graph minus a perfect matching there is a perfect matching, no two of whose edges have the same color. Note that the assertion of the last theorem holds even if G does not have color classes with equal cardinalities; we simply restrict the decomposition to an induced subgraph with color classes of equal cardinality, the rank of whose adjacency matrix is equal to that of the adjacency matrix of G.

For several other results concerning bipartite decompositions of bipartite graphs see [3].

References

1. N. Alon, Decomposition of the complete r-graph into complete r-partite r-graphs, *Graphs Combin.* **2** (1986), 95–100.
2. D. de Caen, private communication.
3. D. de Caen and D. A. Gregory, On the decomposition of a directed graph into complete bipartite subgraphs, *Ars Combin.* **23B** (1987), 139–146.
4. R. L. Graham and H. O. Pollak, On the addressing problem for loop switching, *Bell Syst. Tech. J.* **50** (1971), 2495–2519.
5. R. L. Graham and H. O. Pollak, On embedding graphs in squashed cubes, *in* "Lecture Notes in Mathematics," Vol. 303, pp. 99–110, Springer-Verlag, New York/Berlin/Heidelberg, 1973.
6. L. Lovász, "Combinatorial Problems and Exercises," North-Holland, Amsterdam, 1979.
7. G. W. Peck, A new proof of a theorem of Graham and Pollak, *Discrete Math.* **49** (1984), 327–328.
8. R. Rado, A theorem on independence relations, *Quart. J. Math. Oxford* **13** (1942), 83–89.
9. J. M. S. Simões-Pereira, Subgraphs as circuits and bases of matroids, *Discrete Math.* **12** (1975), 79–88.
10. H. Tverberg, On the decomposition of K_n into complete bipartite graphs, *J. Graph Theory* **6** (1982), 493–494.
11. D. J. A. Welsh, "Matroid Theory," Academic Press, London, 1976.