Analysis method of planar cracks of arbitrary shape in the isotropic plane of a three-dimensional transversely isotropic magnetoelectroelastic medium

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Abstract

The hyper-singular boundary integral equation method of crack analysis in three-dimensional transversely isotropic magnetoelectroelastic media is proposed. Based on the fundamental solutions or Green’s functions of three-dimensional transversely isotropic magnetoelectroelastic media and the corresponding Somigliana identity, the boundary integral equations for a planar crack of arbitrary shape in the plane of isotropy are obtained in terms of the extended displacement discontinuities across crack faces. The extended displacement discontinuities include the displacement discontinuities, the electric potential discontinuity and the magnetic potential discontinuity, and correspondingly the extended tractions on crack face represent the conventional tractions, the electric displacement and the magnetic induction boundary values. The near crack tip fields and the intensity factors in terms of the extended displacement discontinuities are derived by boundary integral equation approach. A solution method is proposed by use of the analogy between the boundary integral equations of the magnetoelectroelastic media and the purely elastic materials. The influence of different electric and magnetic boundary conditions, i.e., electrically and magnetically impermeable and permeable conditions, electrically impermeable and magnetically permeable condition, and electrically permeable and magnetically impermeable condition, on the solutions is studied. The crack opening model is proposed to consider the real crack opening and the electric and magnetic fields in the crack cavity under combined mechanical-electric-magnetic loadings. An iteration approach is presented for the solution of the non-linear model. The exact solution is obtained for the case of uniformly applied loadings on the crack faces. Numerical results for a square crack under different electric and magnetic boundary conditions are displayed to demonstrate the proposed method.

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Keywords: Magnetoelectroelastic medium; Planar crack; Extended displacement discontinuity; Boundary integral equation; Extended stress intensity factor

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1. Introduction

Due to possessing the mechanical-electric-magnetic coupling effect, magnetoelectroelastic composite materials are finding more and more applications in microwave electronics, optoelectronics and electronic instrumentation and smart structures (Van Run et al., 1974). Because defects are often unavoidable in these materials and the defects affect the integrity and reliability of the structures, the study of defects in magnetoelectroelastic materials and structures has been receiving more and more attentions. For example, Alshits et al. (1995) gave Green’s functions for the angularly inhomogeneous piezoelectric piezomagnetic magnetoelastic anisotropic media subject to line defect. Chung and Ting (1995) obtained two-dimensional Green’s functions for a magnetoelectroelastic anisotropic medium with an elliptical cavity or rigid inclusion. Liu et al. (2001) derived the two-dimensional Green’s functions for anisotropic magnetoelectroelastic solids with an elliptical cavity or a crack taking into account the electric-magnetic fields inside the cavity. Similar to the fracture analysis of piezoelectric materials (McMeeking, 2004; Zhang et al., 2002), two common approximate approaches are used to describe the electric and magnetic boundary conditions. One approach treats the crack as an electrically and magnetically impermeable slit, while the other approach treats the crack as an electrically and magnetically permeable slit. Wang and Mai (2003) gave closed-form expressions for the energy release rate of an impermeable or permeable crack in a piezomagnetic/piezoelectric solid. Gao et al. (2003a,b, 2004), Gao and Noda (2004) and Zhou et al. (2004) studied permeable crack problems in two-dimensional magnetoelectroelastic solids, while Tian and Rajapakse (2005) studied the impermeable cracks. For three-dimensional problems, Pan (2002) derived Green’s functions in anisotropic magneto-electro-elastic full space, half space, and bimaterials. Ding et al. (2005) gave Green’s functions for two-phase transversely isotropic magnetoelectroelastic media. Hou and Leung (2004) derived the solution for a permeable penny-shaped crack. Chen et al. (2004) obtained the exact three-dimensional expressions for the full-space magneto-electro-thermo-elastic field for an impermeable penny-shaped crack subject to a uniform load on the crack surfaces. Very recently, Zhao et al. (2006a,b) studied an elliptical or ellipsoidal cavity in the two- or three-dimensional magnetoelectroelastic medium based on the exact electric and magnetic boundary conditions. When the minor axis of the cavity is reduced to zero, the exact solution for the two- or three-dimensional cavity is reduced to the exact solution for a two- or three-dimensional crack. The results show that an electrically permeable (or impermeable) crack is not certainly a magnetically permeable (or impermeable) crack. The electric permeability and the magnetic permeability of a crack in magnetoelectroelastic medium are two different properties to electric field and magnetic field. There exist another two extreme cases or boundary conditions, i.e., electrically impermeable and magnetically permeable condition, and electrically permeable and magnetically impermeable condition. This indicates that the fracture problem of magnetoelectroelastic materials is more complicated compared with that of piezoelectric materials.

In the present work planar cracks of arbitrary shape with different electric and magnetic boundary conditions in the isotropic plane in three-dimensional media are studied by boundary integral equation method. Following this introduction, the basic governing equations for three-dimensional magnetoelectroelastic medium are given in Section 2. In Section 3, the boundary integral equation method is described in details for impermeable crack. The influence of different electric and magnetic crack boundary conditions is discussed in Section 4. In Section 5, a non-linear crack model is proposed to consider the influence of the crack opening. Numerical results of the normalized extended stress intensity factors for a square crack are displayed in Section 6. And in Section 7, the present paper is concluded.

2. Basic equation

For a three-dimensional transversely isotropic magneto-electro-elastic medium with the poling direction being along the z-axis in the \( oxyz \) Cartesian coordinate system, in the absence of body force, electric charge and electric current, the governing equations are given by

\[
\begin{align*}
\sigma_{ij,j} &= 0, \\
D_{ij,i} &= 0, \quad i, j = 1, 2, 3(x, y, z), \\
B_{ij,i} &= 0.
\end{align*}
\]  

\( 1 \)
The constitutive equations can be expressed as
\[
\begin{align*}
\sigma_{ij} &= c_{ijkl}(u_{k,i} + u_{l,j})/2 + e_{ijl}\varphi_k + f_{ijl}\psi_k, \\
D_i &= e_{ijkl}(u_{k,i} + u_{l,j})/2 - e_{ikl}\varphi_k - g_{ikl}\psi_k, \\
B_i &= f_{ikl}(u_{k,i} + u_{l,j})/2 - g_{ikl}\varphi_k - \mu_{ikl}\psi_k,
\end{align*}
\]  
where \(\sigma_{ij}, D_i\) and \(B_i\) denote the stress, electric displacement and magnetic induction components, respectively. \(u_i (u_1 = u, u_2 = v\text{ and } u_3 = w)\) denote the displacement components, and \(\varphi\) and \(\psi\) denote, respectively, the electric potential and magnetic potential. \(c_{ijkl}, e_{ijkl}, f_{ijkl}, e_{ijl}, g_{ikl}\) and \(\mu_{ikl}\) are the elastic constants, piezoelectric constants, piezomagnetic constants, dielectric permittivity, electromagnetic constants and magnetic permeability, respectively. A subscript comma denotes the partial differentiation with respect to the coordinate. For a transversely isotropic medium, the independent material constants are given in Appendix A, in which the contracted subscript notations are used.

3. Boundary integral equation method for electrically and magnetically impermeable planar crack of arbitrary shape in the isotropic plane

3.1. Boundary condition

An arbitrarily shaped planar crack \(S\) lives on the \(oxy\) plane, i.e., the plane of isotropy of an infinite transversely isotropic magnetoelectroelastic medium. The upper and lower surfaces of crack \(S\) are denoted by \(S^+\) and \(S^-\), respectively, as shown in Fig. 1. The outer normal vectors of \(S^+\) and \(S^-\) are respectively given by
\[
\{n_i\}^+ = \{0, 0, -1\}, \quad \{n_i\}^- = \{0, 0, 1\}.
\]
The study in the present paper will be based on four kind boundary conditions. The electrically and magnetically impermeable condition is given by
\[
\begin{align*}
D_z(x, y, 0^+) &= D_z(x, y, 0^-) = 0, \\
B_z(x, y, 0^+) &= B_z(x, y, 0^-) = 0, \\
(x, y) &\in S,
\end{align*}
\]  
where \(D_z\) and \(B_z\) are the electric displacement and the magnetic induction in the \(z\)-direction, while the electrically and magnetically permeable condition is expressed by

![Fig. 1. An arbitrarily shaped planar crack in the oxy plane.](image-url)
where \( D'_z \) and \( B'_z \) denotes the electric displacement and the magnetic induction in the \( z \)-axis direction in the crack cavity, respectively. The other two boundary conditions, i.e., electrically impermeable and magnetically permeable condition, and electrically permeable and magnetically impermeable condition, are given by Eqs. (4c) and (4d), respectively

\[
\begin{align*}
D_z(x, y, 0^+) = D_z(x, y, 0^-) &= D'_z, & \varphi(x, y, 0^+) = \varphi(x, y, 0^-), \\
B_z(x, y, 0^+) = B_z(x, y, 0^-) &= B'_z, & \psi(x, y, 0^+) = \psi(x, y, 0^-),
\end{align*}
\]

It is known that a crack problem can be regarded as the superposition of two problems. One is the no crack problem with the given applied loadings. And the other is the perturbed problem with the loadings being only applied on crack faces. The first problem is analyzed to obtain the extended tractions on the crack faces of the perturbed problem, which are written as

\[
p_j|_S = -p_j|_{S'}, \quad \omega|_S = -\omega|_{S'}, \quad \gamma|_S = -\gamma|_{S'}, \quad (i = 1, 2, 3 \text{ or } x, y, z). \tag{5}
\]

The extended tractions are related to the general stresses by

\[
p_j = \sigma_{ij} n_i, \quad \omega = D_i n_i, \quad \gamma = B_i n_i, \tag{6}
\]

Eq. (5) demonstrates that the loadings on the upper and lower crack surfaces are equal but opposite in sign.

### 3.2. Boundary integral equations

By using of the fundamental solutions or Green’s function given in Appendix B and C and the Somigliana identity for magnetoelectroelastic media under the electrically and magnetically impermeable condition, the displacements \( u_i \), the electric potential \( \varphi \) and the magnetic potential \( \psi \) at any internal point \((x, y, z)\) can be expressed in the following integral forms

\[
\begin{align*}
u(x, y, z) &= -\int_S \left[ p^F_{ij} u_i + \omega^F_i + \Gamma^F_i \psi \right] dS - \int_S \left[ p^D_{ij} u_i + \Omega^D_i + \Gamma^D_i \psi \right] dS + \int_S \left[ p^0_{ij} u_i + \omega^0_i + \Gamma^0_i \psi \right] dS + \int_S \left[ p^1_{ij} u_i + \omega^1_i + \Gamma^1_i \psi \right] dS, \\
\varphi(x, y, z) &= -\int_S \left[ p^F_{ij} u_i + \omega^F_i + \Gamma^F_i \psi \right] dS - \int_S \left[ p^D_{ij} u_i + \Omega^D_i + \Gamma^D_i \psi \right] dS + \int_S \left[ p^0_{ij} u_i + \omega^0_i + \Gamma^0_i \psi \right] dS + \int_S \left[ p^1_{ij} u_i + \omega^1_i + \Gamma^1_i \psi \right] dS, \\
\psi(x, y, z) &= -\int_S \left[ p^F_{ij} u_i + \omega^F_i + \Gamma^F_i \psi \right] dS - \int_S \left[ p^D_{ij} u_i + \Omega^D_i + \Gamma^D_i \psi \right] dS + \int_S \left[ p^0_{ij} u_i + \omega^0_i + \Gamma^0_i \psi \right] dS + \int_S \left[ p^1_{ij} u_i + \omega^1_i + \Gamma^1_i \psi \right] dS,
\end{align*}
\]

where \( p^F_{ij}, \omega^F_i, \Gamma^F_i \) and \( U^F_{ij}, \Phi^F, \Psi_F \) denote the extended tractions and the extended displacements of the fundamental solutions corresponding to the unit point force in the \( i \)-th direction, respectively, while \( p^D_{ij}, \Omega^D_i, \Gamma^D_i \) and \( U^D_{ij}, \Phi^D, \Psi_D \) correspond to the unit point electric charge and \( p^0_{ij}, \Omega^0_i, \Gamma^0_i \) and \( U^0_{ij}, \Phi^0, \Psi^0 \) correspond to the unit point electric current

\[
\begin{align*}
P^F_{ij} &= \sigma^F_{ij} n_k, & \Omega^F_i &= D^F_i n_k, & \Gamma^F_i &= B^F_i n_k, \\
P^D_{ij} &= \sigma^D_{ij} n_k, & \Omega^D_i &= D^D_i n_k, & \Gamma^D_i &= B^D_i n_k, \\
P^0_{ij} &= \sigma^0_{ij} n_k, & \Omega^0_i &= D^0_i n_k, & \Gamma^0_i &= B^0_i n_k, \\
P^1_{ij} &= \sigma^1_{ij} n_k, & \Omega^1_i &= D^1_i n_k, & \Gamma^1_i &= B^1_i n_k,
\end{align*}
\]

where the upper index \( F, D \) and \( B \) refer to the variables corresponding to a point force, point electric charge and point electric current, respectively. From the fundamental solutions, one has the following relations on crack surfaces
\[ P_{ij}^{F} |_{S^+} = -P_{ij}^{F} |_{S^-}, U_{ij}^{F} |_{S^+} = U_{ij}^{F} |_{S^-}, \]
\[ \Omega_{ij}^{F} |_{S^+} = -\Omega_{ij}^{F} |_{S^-}, \Phi_{ij}^{F} |_{S^+} = \Phi_{ij}^{F} |_{S^-}, \]
\[ \Gamma_{ij}^{F} |_{S^+} = -\Gamma_{ij}^{F} |_{S^-}, \Psi_{ij}^{F} |_{S^+} = \Psi_{ij}^{F} |_{S^-}, \]
\[ P_{ij}^{D} |_{S^+} = -P_{ij}^{D} |_{S^-}, U_{ij}^{D} |_{S^+} = U_{ij}^{D} |_{S^-}, \]
\[ \Omega_{ij}^{D} |_{S^+} = -\Omega_{ij}^{D} |_{S^-}, \Phi_{ij}^{D} |_{S^+} = \Phi_{ij}^{D} |_{S^-}, \]
\[ \Gamma_{ij}^{D} |_{S^+} = -\Gamma_{ij}^{D} |_{S^-}, \Psi_{ij}^{D} |_{S^+} = \Psi_{ij}^{D} |_{S^-}. \]

Substituting Eqs. (5) and (9) into (7) yields
\[ u_{i}(x, y, z) = -\int_{S^+} [P_{ij}^{F} ||u_{j}|| + \Omega_{ij}^{F} ||\phi|| + \Gamma_{ij}^{F} ||\psi||]dS, \]
\[ -\varphi(x, y, z) = -\int_{S^+} [P_{ij}^{D} ||u_{j}|| + \Omega_{ij}^{D} ||\phi|| + \Gamma_{ij}^{D} ||\psi||]dS, \]
\[ -\psi(x, y, z) = -\int_{S^+} [P_{ij}^{B} ||u_{j}|| + \Omega_{ij}^{B} ||\phi|| + \Gamma_{ij}^{B} ||\psi||]dS, \]

where \(||u||, ||\phi||\) and \(||\psi||\) are the displacement discontinuities, the electric potential discontinuity and the magnetic potential discontinuity across the crack faces, respectively, namely
\[ ||u_{i}(\xi, \eta)|| = u_{i}(\xi, \eta, 0^+) - u_{i}(\xi, \eta, 0^-), \]
\[ ||\phi(\xi, \eta)|| = \phi(\xi, \eta, 0^+) - \phi(\xi, \eta, 0^-), \quad (\xi, \eta) \in S, \]
\[ ||\psi(\xi, \eta)|| = \psi(\xi, \eta, 0^+) - \psi(\xi, \eta, 0^-). \]

Inserting the fundamental solutions into Eq. (10) yields the following concrete expressions
\[ u = \int_{S^+} \left\{ ||u|| \left[ -\alpha_{ij} \frac{1}{R(3z + R)} - \frac{(\eta - y)^2}{R^2(R + z)} - \frac{(\eta - y)^2}{R^2(3R + z)} \right] + \sum_{i=1}^{4} \alpha_{ij} \frac{1}{R(8R + z)} \left[ -\frac{(\xi - x)^2}{R^2(R + z)} - \frac{(\xi - x)^2}{R^2(3R + z)} \right] \right\} dS(\xi, \eta), \]
\[ v = \int_{S^+} \left\{ ||v|| \left[ -\alpha_{ij} \frac{1}{R(3z + R)} \left[ \frac{1}{R^2(R + z)} + \frac{1}{R^2(3R + z)} \right] - (\xi - x)(\eta - y) \sum_{i=1}^{4} \alpha_{ij} \frac{1}{R(8R + z)} \left[ \frac{1}{R^2(R + z)} + \frac{1}{R^2(3R + z)} \right] \right] \right\} dS(\xi, \eta), \]
\[ w = \int_{S^+} \left\{ ||w|| \left[ \frac{4}{R(3z + R)} \sum_{i=1}^{4} \frac{\alpha_{ij} \frac{1}{R^4}}{R^4} + ||\varphi|| \frac{4}{R(3z + R)} \sum_{i=1}^{4} \frac{\alpha_{ij} \frac{1}{R^4}}{R^4} - ||\psi|| \frac{4}{R(3z + R)} \sum_{i=1}^{4} \frac{\alpha_{ij} \frac{1}{R^4}}{R^4} \right] \right\} dS(\xi, \eta), \]
and \( s_j \) are the roots of the material characteristic equation, \( \omega_{ij}, \theta_{ij}, A_i, B_i, C_i \) and \( D_j \) are all material related constants given in Appendixes A, B, C.

Inserting Eqs. (12) into the constitutive Eq. (2), one can obtain the extended stress fields expressed in terms of the extended displacement discontinuities across the crack faces. Substituting the obtained results into the boundary condition in Eq. (5) yields the boundary integral equations for an arbitrarily shaped planar crack

\[
\int_{S^+} \left\{ [L_{11}(1 - 3 \cos^2 \theta) + L_{12}(1 - 3 \sin^2 \theta)] ||u|| + L_{13} \cos \theta \sin \theta ||v|| \right\} \frac{1}{r^3} dS = -p_x(x, y),
\]

\[
\int_{S^+} \left\{ L_{13} \cos \theta \sin \theta ||u|| + [L_{12}(1 - 3 \cos^2 \theta) + L_{11}(1 - 3 \sin^2 \theta)] ||v|| \right\} \frac{1}{r^3} dS = -p_y(x, y),
\]

\[
\int_{S^+} [L_{31} ||w|| + L_{32} ||\varphi|| + L_{33} ||\psi||] \frac{1}{r^3} dS = -\omega(x, y),
\]

\[
\int_{S^+} [L_{41} ||w|| + L_{42} ||\varphi|| + L_{43} ||\psi||] \frac{1}{r^3} dS = -\gamma(x, y),
\]

where

\[
\begin{align*}
    r^2 &= (\xi - x)^2 + (\eta - y)^2, \quad \cos \theta = (\xi - x)/r, \quad \sin \theta = (\eta - y)/r
\end{align*}
\]

and the material related constants \( L_{ij} \) are given by

\[
L_{11} = \sum_{i=1}^{4} \omega_{i1} (-c_{44} D_i s_i + c_{44} A_i - e_{12} B_i - f_{13} C_i),
\]

\[
L_{12} = c_{44} D_3 s_3,
\]

\[
L_{13} = \sum_{i=1}^{4} \omega_{i1} (c_{44} D_i s_i + c_{44} A_i - e_{13} B_i - f_{15} C_i) + c_{44} D_5 s_5;
\]

\[
L_{31} = -\sum_{i=1}^{4} \theta_{i1} [-c_{13} D_i + (c_{33} A_i - e_{33} B_i - f_{33} C_i)] s_i,
\]

\[
L_{32} = -\sum_{i=1}^{4} \theta_{i1} [-e_{31} D_i + (e_{33} A_i + e_{35} B_i + g_{33} C_i)] s_i,
\]

\[
L_{33} = -\sum_{i=1}^{4} \theta_{i1} [-f_{31} D_i + (f_{33} A_i + g_{33} B_i + \mu_{33} C_i)] s_i;
\]

\[
L_{41} = -\sum_{i=1}^{4} \theta_{i2} [-c_{13} D_i + (c_{33} A_i - e_{33} B_i - f_{33} C_i)] s_i,
\]

\[
L_{42} = -\sum_{i=1}^{4} \theta_{i2} [-e_{31} D_i + (e_{33} A_i + e_{35} B_i + g_{33} C_i)] s_i,
\]

\[
L_{43} = -\sum_{i=1}^{4} \theta_{i2} [-f_{31} D_i + (f_{33} A_i + g_{33} B_i + \mu_{33} C_i)] s_i;
\]

\[
L_{51} = -\sum_{i=1}^{4} \theta_{i3} [-c_{13} D_i + (c_{33} A_i - e_{33} B_i - f_{33} C_i)] s_i,
\]

\[
L_{52} = -\sum_{i=1}^{4} \theta_{i3} [-e_{31} D_i + (e_{33} A_i + e_{35} B_i + g_{33} C_i)] s_i,
\]

\[
L_{53} = -\sum_{i=1}^{4} \theta_{i3} [-f_{31} D_i + (f_{33} A_i + g_{33} B_i + \mu_{33} C_i)] s_i.
\]
In Eqs. (14)–(18), the kernel functions have the singularity of $O(1/r^3)$, and hence the integral equations are hyper-singular. The displacement discontinuities $\|u\|$ and $\|v\|$ on the crack faces are coupled by Eqs. (14) and (15), while the displacement discontinuity $\|w\|$, the electric potential discontinuity $\|\varphi\|$ and the magnetic potential discontinuity $\|\psi\|$ are coupled by Eqs. (16)–(18).

3.3. Singular behavior and the intensity factors

Choose an arbitrary point $o$ on the crack border $\Gamma$ to analyze the singular behavior. The border $\Gamma$ of the crack is smooth at point $o$. Without loss in generality, the Cartesian coordinate system $oxyz$ is placed such that the $y$-axis and $x$-axis are tangent and normal to $\Gamma$, respectively, while the $z$-axis is normal to the crack plane $S$ as shown in Fig. 2.

Let the infinitesimal $\varepsilon$ denote the radius of a circle $\Sigma$ centered at point $o$ contained in $S$. The integrals in Eqs. (14)–(18) should be finite in $\Sigma$ and can be written as

$$\begin{align*}
\int_{\Sigma} \left\{ [L_{11}(1 - 3 \cos^2 \theta) + L_{12}(1 - 3 \sin^2 \theta)] \|u\| + L_{13} \cos \theta \|v\| \right\} \frac{1}{r^3} dS &= R_x(x, y), \\
\int_{\Sigma} \left\{ L_{13} \cos \theta \|u\| + [L_{12}(1 - 3 \cos^2 \theta) + L_{11}(1 - 3 \sin^2 \theta)] \|v\| \right\} \frac{1}{r^3} dS &= R_y(x, y), \\
\int_{\Sigma} \left\{ L_{31} \|w\| + L_{32} \|\varphi\| + L_{33} \|\psi\| \right\} \frac{1}{r^3} dS &= R_z(x, y), \\
\int_{\Sigma} \left\{ L_{41} \|w\| + L_{42} \|\varphi\| + L_{43} \|\psi\| \right\} \frac{1}{r^3} dS &= R_\varphi(x, y), \\
\int_{\Sigma} \left\{ L_{51} \|w\| + L_{52} \|\varphi\| + L_{53} \|\psi\| \right\} \frac{1}{r^3} dS &= R_\psi(x, y),
\end{align*}$$

(20)

where $R_x(x, y)$, $R_y(x, y)$, $R_z(x, y)$, $R_\varphi(x, y)$ and $R_\psi(x, y)$ are all finite functions for $(x, y) \in \Sigma$.

Based on the elastic theory, the displacement near point $o$ can be obtained by superposing the plane strain and the anti-plane displacements in the $oxz$ plane. So the extended displacement discontinuities at the neighborhood of point $o$ are given by

$$\begin{align*}
\|u\| &= A_x(o) x^{\varepsilon_x}, \quad \|v\| = A_y(o) x^{\varepsilon_y}, \quad \|w\| = A_z(o) x^{\varepsilon_z}, \\
\|\varphi\| &= A_\varphi(o) x^{\varepsilon_\varphi}, \quad \|\psi\| = A_\psi(o) x^{\varepsilon_\psi},
\end{align*}$$

(21)

where the coefficients $A_x$, $A_y$, $A_z$, $A_\varphi$ and $A_\psi$ depend on the location of point $o$, and $\varepsilon_x$, $\varepsilon_y$, $\varepsilon_z$, $\varepsilon_\varphi$ and $\varepsilon_\psi$ are the singular indices of the extended displacements and their values are between (0,1).

Inserting Eq. (21) into Eq. (20), and using the following integrals
The existence of non-trivial solution of Eq. (23) requires
\[
\begin{align*}
&\int_{\Sigma} \frac{||u||}{r^3} \, d\zeta d\eta = -2x^{a-1}A_x(o)\pi \alpha_x \cot \pi \alpha_x, \\
&\int_{\Sigma} \frac{||u||}{r^3} \cos \theta d\zeta d\eta = -\frac{4}{3}x^{a-1}A_x(o)\pi \alpha_x \cot \pi \alpha_x, \\
&\int_{\Sigma} \frac{||u||}{r^3} \sin \theta \cos \theta d\zeta d\eta = 0, \\
&\int_{\Sigma} \frac{||u||}{r^3} \sin^2 \theta d\zeta d\eta = -\frac{2}{3}x^{a-1}A_x(o)\pi \alpha_x \cot \pi \alpha_x, \\
&\int_{\Sigma} \frac{||u||}{r^3} d\zeta d\eta = -2x^{a-1}A_x(o)\pi \alpha_x \cot \pi \alpha_x, \\
\end{align*}
\]
and letting \(r \to 0\), we obtain
\[
\begin{align*}
L_{11}A_x(o) \cot \pi \alpha_x &= 0, \\
L_{11}A_y(o) \cot \pi \alpha_x &= 0, \\
L_{31}A_x(o) \cot \pi \alpha_x + L_{32}A_{\phi}(o) \cot \pi \alpha_{\phi} + L_{33}A_{\phi}(o) \cot \pi \alpha_{\phi} &= 0, \\
L_{41}A_x(o) \cot \pi \alpha_x + L_{42}A_{\phi}(o) \cot \pi \alpha_{\phi} + L_{43}A_{\phi}(o) \cot \pi \alpha_{\phi} &= 0, \\
L_{51}A_x(o) \cot \pi \alpha_x + L_{52}A_{\phi}(o) \cot \pi \alpha_{\phi} + L_{53}A_{\phi}(o) \cot \pi \alpha_{\phi} &= 0. \\
\end{align*}
\]
The existence of non-trivial solution of Eq. (23) requires
\[
\cot \pi \alpha_x = \cot \pi \alpha_y = \cot \pi \alpha_z = \cot \pi \alpha_{\phi} = 0. \\
\]
Finally, one obtains the singular indexes
\[
\alpha_x = \alpha_y = \alpha_z = \alpha_{\phi} = \frac{1}{2}. \\
\]
This result reveals that the extended displacements near the crack tip have the classical singularity \(r^{\frac{1}{2}}\) as in the fracture mechanics of conventional elastic materials.

Making use of Eq. (10) and the constitutive Eq. (2), the extended stresses at points \((-\rho, y, 0)\) \((\rho > 0)\) near the point \(o\) are expressed
\[
\begin{align*}
\sigma_{xz} &= \int_{S^+} \left\{ [L_{11}(1 - 3 \cos^2 \theta) + L_{12}(1 - 3 \sin^2 \theta)] ||u|| + L_{13} \cos \theta \sin \theta ||v|| \right\} \frac{1}{r^3} \, dS, \\
\sigma_{zy} &= \int_{S^+} \left\{ L_{13} \cos \theta \sin \theta ||u|| + [L_{12}(1 - 3 \cos^2 \theta) + L_{11}(1 - 3 \sin^2 \theta)] ||v|| \right\} \frac{1}{r^3} \, dS, \\
\sigma_{zz} &= \int_{S^+} \left\{ L_{31} ||w|| + L_{32} ||\phi|| + L_{33} ||\psi|| \right\} \frac{1}{r^3} \, dS, \\
D_z &= \int_{S^+} \left\{ L_{41} ||w|| + L_{42} ||\phi|| + L_{43} ||\psi|| \right\} \frac{1}{r^3} \, dS, \\
B_z &= \int_{S^+} \left\{ L_{51} ||w|| + L_{52} ||\phi|| + L_{53} ||\psi|| \right\} \frac{1}{r^3} \, dS. \\
\end{align*}
\]
Substituting Eq. (21) into (26) gives
\[
\begin{align*}
\sigma_{xz} &= -L_{11}A_x(o)\pi/\sqrt{\rho}, \\
\sigma_{zy} &= -L_{12}A_y(o)\pi/\sqrt{\rho}, \\
\sigma_{zz} &= [L_{31}A_x(o) + L_{32}A_{\phi}(o) + L_{33}A_{\phi}(o)]\pi/\sqrt{\rho}, \\
D_z &= [L_{41}A_x(o) + L_{42}A_{\phi}(o) + L_{43}A_{\phi}(o)]\pi/\sqrt{\rho}, \\
B_z &= [L_{51}A_x(o) + L_{52}A_{\phi}(o) + L_{53}A_{\phi}(o)]\pi/\sqrt{\rho}. \\
\end{align*}
\]
Defining the intensity factors
\[ K_F^I = \lim_{\rho \to 0} \sqrt{2\pi \rho} \sigma_{zz}(-\rho, y, 0), \]
\[ K_F^D = \lim_{\rho \to 0} \sqrt{2\pi \rho} D_z(-\rho, y, 0), \]
\[ K_F^I = \lim_{\rho \to 0} \sqrt{2\pi \rho} B_z(-\rho, y, 0), \]
\[ K_F^I = \lim_{\rho \to 0} \sqrt{2\pi \rho} \sigma_{zz}(-\rho, y, 0), \]
\[ K_F^I = \lim_{\rho \to 0} \sqrt{2\pi \rho} \sigma_{zz}(-\rho, y, 0), \]
\[ K_F^I = \lim_{\rho \to 0} \sqrt{2\pi \rho} \sigma_{zz}(-\rho, y, 0), \]
\[ K_F^I = \lim_{\rho \to 0} \sqrt{2\pi \rho} \sigma_{zz}(-\rho, y, 0), \]
\[ K_F^I = \lim_{\rho \to 0} \sqrt{2\pi \rho} \sigma_{zz}(-\rho, y, 0), \]
\[ \text{inserting Eq. (27) into Eq. (28) yields} \]
\[ K_F^I = \sqrt{2\pi} [L_{31} A_x(o) + L_{32} A_\phi(o) + L_{33} A_\psi(o)], \]
\[ K_F^D = \sqrt{2\pi} [L_{41} A_x(o) + L_{42} A_\phi(o) + L_{43} A_\psi(o)], \]
\[ K_F^I = \sqrt{2\pi} [L_{51} A_x(o) + L_{52} A_\phi(o) + L_{53} A_\psi(o)], \]
\[ K_F^I = -\sqrt{2\pi} L_{11} A_x(o), \]
\[ K_F^I = -\sqrt{2\pi} L_{12} A_x(o). \]

Considering Eq. (21), the intensity factors in Eq. (29) can be expressed in terms of the extended displacement discontinuities
\[ K_F^I = \sqrt{2\pi} \lim_{x \to 0} [L_{31} ||w|| + L_{32} ||\phi|| + L_{33} ||\psi||]/\sqrt{x}, \]
\[ K_F^D = \sqrt{2\pi} \lim_{x \to 0} [L_{41} ||w|| + L_{42} ||\phi|| + L_{43} ||\psi||]/\sqrt{x}, \]
\[ K_F^I = \sqrt{2\pi} \lim_{x \to 0} [L_{51} ||w|| + L_{52} ||\phi|| + L_{53} ||\psi||]/\sqrt{x}, \]
\[ K_F^I = -\sqrt{2\pi} \lim_{x \to 0} L_{11} ||u||/\sqrt{x}, \]
\[ K_F^I = -\sqrt{2\pi} \lim_{x \to 0} L_{12} ||v||/\sqrt{x}. \]

3.4. Solution method

In this subsection, a solution method will be presented by use of the analogy between the extended displacement discontinuity integral equations of magnetoelastic medium and those of conventional elastic solids.

Consider the same crack \( S \) in an infinite conventional elastic medium subjected to the tractions \( t_x, t_y, \) and \( t_z \) respectively along the \( x-, y- \) and \( z- \) axis, the displacement discontinuity boundary integral equations take the form (Ioakimidis, 1987; Lin’kov and Mogilevshaya, 1986; Zhao et al., 1994)
\[ \frac{E}{8\pi(1 - \nu^2)} \int_{S'} \left\{ \left[(1 - 2\nu) + 3v \sin^2 \theta \right] ||U|| + 3v \sin \theta \cos \theta ||V|| \right\} \frac{1}{r^3} dS = -t_x(x, y), \]
\[ \frac{E}{8\pi(1 - \nu^2)} \int_{S'} \left\{ 3v \sin \theta \cos \theta ||U|| + \left[(1 - 2\nu) + 3v \cos^2 \theta \right] ||V|| \right\} \frac{1}{r^3} dS = -t_y(x, y), \]
\[ \frac{E}{8\pi(1 - \nu^2)} \int_{S'} ||W|| \frac{1}{r^3} dS = -t_z(x, y), \]
where \( E \) and \( \nu \) are respectively the Young’s modulus and Poisson ratio, and \( ||U||, ||V|| \) and \( ||W|| \) are the displacement discontinuities. It can be seen that Eqs. (14) and (15) are identical to Eqs. (31) and (32), which are the governing equations of asymmetric problem. The solution of the asymmetric problem can be obtained directly from those of the corresponding elastic material, which has been studied intensively and extensively. We only consider the symmetric problem in the present paper.
Let \( W_{p_2} \), \( W_{o_2} \) and \( W_{f_2} \) be the solutions of Eq. (33) corresponding to \( t_z = p_2, t_z = o_2 \) and \( t_z = g \), respectively. The mode I intensity factors along the crack border are given by (Zhao et al., 1994)

\[
\begin{pmatrix}
K^p_1 \\
K^o_1 \\
K^f_1 \\
\end{pmatrix} = \frac{E}{8(1-v^2)} \ln_{\rho\to 0} \left( \frac{2\pi}{\rho} \begin{pmatrix}
W_{p_2} & W_{o_2} & W_{f_2}
\end{pmatrix} \right)
\]

(34)

Therefore, the solution to Eqs. (16)–(18) can be expressed by

\[
\begin{align*}
L_{31} \|w\| + L_{32} \|\varphi\| + L_{33} \|\psi\| &= \frac{E}{8\pi(1-v^2)} W_{p_2}, \\
L_{41} \|w\| + L_{42} \|\varphi\| + L_{43} \|\psi\| &= \frac{E}{8\pi(1-v^2)} W_{o_2}, \\
L_{51} \|w\| + L_{52} \|\varphi\| + L_{53} \|\psi\| &= \frac{E}{8\pi(1-v^2)} W_{f_2}.
\end{align*}
\]

(35)

Solving Eq. (35) obtains \( \|w\|, \|\varphi\| \) and \( \|\psi\| \), and substituting them into Eq. (30) gives

\[
\begin{align*}
K^p_1 &= G^{p_2} K^p_1 + G^{o_2} K^o_1 + G^{f_2} K^f_1, \\
K^o_1 &= G^{p_2} K^p_1 + G^{o_2} K^o_1 + G^{f_2} K^f_1, \\
K^f_1 &= G^{p_2} K^p_1 + G^{o_2} K^o_1 + G^{f_2} K^f_1,
\end{align*}
\]

(36)

where the coefficients \( G \) with different superscripts are related to the material constants \( L_{ij} \) in Eqs. (19). After lengthy manipulations, one obtains

\[
\begin{align*}
G^{p_2} = G^{o_2} = G^{f_2} &= 1, \\
G^{p_2} = G^{o_2} = G^{f_2} &= G^{p_2} = G^{o_2} = 0.
\end{align*}
\]

(37)

Thus, Eq. (36) can be rewritten as

\[
\begin{align*}
K^p_1 &= K^p_1, \\
K^o_1 &= K^o_1, \\
K^f_1 &= K^f_1.
\end{align*}
\]

(38)

The results show that the Mode I stress intensity factor, the electric displacement and the magnetic induction intensity factors for impermeable crack depend on the mechanical, electrical and magnetic loadings, respectively, regardless of the material properties, the loadings and the geometry of the planar crack. They can be calculated by Eq. (38) directly using the corresponding solutions of purely elastic material that has been studied intensively and extensively. The similar conclusion was also reached for planar cracks in transversely isotropic piezoelectric materials (Zhao et al., 1997).

### 4. Effects of different electric and magnetic boundary conditions on solution

In the impermeable crack model, the electric field and the magnetic field in the crack cavity are not considered as given in Eq. (4a). The other three kinds of boundary conditions, i.e., electrically and magnetically permeable condition, electrically impermeable and magnetically permeable condition, electrically permeable and magnetically impermeable condition, are discussed in the following three subsections. Considering the electric and magnetic fields in crack cavity and using the same procedure in last section and the Gauss theory, the extended displacement discontinuity boundary integral equations are derived.
\[ \int_{\Gamma^3} [L_{31} \|w\| + L_{32} \|\varphi\| + L_{33} \|\psi\|] \frac{1}{R^3} dS = -p_c(x, y), \]
\[ \int_{\Gamma^3} [L_{41} \|w\| + L_{42} \|\varphi\| + L_{43} \|\psi\|] \frac{1}{R^3} dS = -\omega(x, y) + D_c^z, \]
\[ \int_{\Gamma^3} [L_{51} \|w\| + L_{52} \|\varphi\| + L_{53} \|\psi\|] \frac{1}{R^3} dS = -\gamma(x, y) + B_c^z. \]  
\(\text{(39)}\)

Comparing Eq. (39) with Eqs. (16)–(18), it can be seen that the effect of electric displacement and the magnetic induction in the crack cavity is equivalent to changing the applied electric and magnetic loadings on crack faces.

4.1. Electrically and magnetically permeable crack

Eq. (4b) can be rewritten as
\[ \|\varphi\| = \varphi(x, y, 0^+) - \varphi(x, y, 0^-) = 0, \]
\[ \|\psi\| = \psi(x, y, 0^+) - \psi(x, y, 0^-) = 0. \]  
\(\text{(40)}\)

Substituting Eq. (40) into (39) gives
\[ -D_c^z = -\omega(x, y) + \frac{L_{41}}{L_{31}} p_c(x, y), \]
\[ -B_c^z = -\gamma(x, y) + \frac{L_{51}}{L_{31}} p_c(x, y). \]  
\(\text{(41)}\)

Therefore the electric displacement and magnetic induction intensity factors are obtained
\[ K^D_1 = \frac{L_{41}}{L_{31}} K^\varphi_1 = \frac{L_{41}}{L_{31}} K^p_1, \]
\[ K^B_1 = \frac{L_{51}}{L_{31}} K^\psi_1 = \frac{L_{51}}{L_{31}} K^\omega_1. \]  
\(\text{(42)}\)

Eq. (42) shows that the intensity factors of the electrically and magnetically permeable crack depend only on the mechanical loading. The electric displacement intensity factor and the magnetic induction intensity factor are induced by the mechanical loading through piezoelectric effect and piezomagnetic effect, respectively.

4.2. Electrically impermeable and magnetically permeable crack

The boundary condition in Eq. (4c) gives
\[ D_c^z = 0, \quad \|\psi\| = 0. \]  
\(\text{(43)}\)

Substituting Eq. (43) into (39) yields the magnetic induction in the crack cavity
\[ -B_c^z = -\gamma(x, y) + \frac{L_{51} L_{42} - L_{52} L_{41}}{L_{31} L_{42} - L_{32} L_{41}} p_c(x, y) + \frac{L_{52} L_{31} - L_{51} L_{32}}{L_{31} L_{42} - L_{32} L_{41}} \omega(x, y). \]  
\(\text{(44)}\)

Furthermore the intensity factors are obtained
\[ K^\psi_1 = K^\omega_1, \]
\[ K^D_1 = K^\varphi_1, \]
\[ K^B_1 = \frac{L_{51} L_{42} - L_{52} L_{41}}{L_{31} L_{42} - L_{32} L_{41}} K^p_1 + \frac{L_{52} L_{31} - L_{51} L_{32}}{L_{31} L_{42} - L_{32} L_{41}} K^\omega_1. \]  
\(\text{(45)}\)

The result shows that the intensity factors are independent of the magnetic loading.
4.3. Electrically permeable and magnetic impermeable crack

From Eq. (4d), one has

$$\|\varphi\| = 0, \quad B^e_\zeta = 0.$$  \hspace{1cm} (46)

Substituting Eq. (46) into Eq. (39), the electric displacement in the crack cavity is derived

$$-D^e_\zeta = -\omega(x,y) + \frac{L_{41}L_{53} - L_{43}L_{51}}{L_{31}L_{53} - L_{33}L_{51}} p_e(x,y) + \frac{L_{43}L_{34} - L_{41}L_{33}}{L_{31}L_{53} - L_{33}L_{51}} \gamma(x,y).$$  \hspace{1cm} (47)

Finally, the intensity factors are obtained

$$K^p_1 = K^p_1,$$

$$K^d_1 = \frac{L_{41}L_{53} - L_{43}L_{51}}{L_{31}L_{53} - L_{33}L_{51}} K^p_1 + \frac{L_{31}L_{43} - L_{33}L_{41}}{L_{31}L_{53} - L_{33}L_{51}} K^1_1,$$

$$K^p_1 = K^1_1.$$  \hspace{1cm} (49)

They are independent of the electrically loading on the crack faces.

The four crack face conditions discussed in the last two sections are four approximate boundary conditions.

The recent study (Zhao et al., 2006a,b) shows that the permeability of the material in the crack cavity and the opening of the crack have great effect on the solution, while the crack opens under applied mechanical-electric loading. So it is a typical geometric non-linear problem. A crack opening model will be adopted in the next section to deal with this problem.

5. The crack opening model for planar cracks of arbitrary shape in the isotropic plane

The extended displacement discontinuity boundary integral equations are still given by Eq. (39), but the electric displacement $D^e_\zeta$ and the magnetic induction $B^e_\zeta$ in the crack cavity are related to the extended displacement discontinuity by

$$D^e_\zeta = -\varepsilon^e\|\varphi\|/\|w\|, \quad B^e_\zeta = -\mu^e\|\psi\|/\|w\|. $$  \hspace{1cm} (50)

Eqs. (39) and (50) determine the extended displacement discontinuities on the crack faces, and the electric displacement $D^e_\zeta$ and the magnetic induction $B^e_\zeta$ in the crack cavity. This model is developed from Hao and Shen (1994) for piezoelectric materials. It can be seen that these equations are non-linear and the analytical solution is hard to be obtained. An iteration approach is proposed to solve the non-linear problem.

At first, the crack is treated as impermeable one by assuming

$$D^e_\zeta(x,y) = 0,$$

$$B^e_\zeta(x,y) = 0.$$  \hspace{1cm} (51)

Inserting Eq. (51) into Eq. (39) yields the extended displacement discontinuities. Then, the new values of the electric displacements and the magnetic induction in the crack cavity can be calculated from the obtained solutions by Eq. (50). Continuing the iteration can give the solution with the preset accuracy being satisfied.

For the case of uniformly applied loadings on crack faces, the exact solution can be derived. The constant extended tractions on the upper and lower crack faces are given by

$$p_e(x,y,0^+) = -p_e(x,y,0^-) = p_0,$$

$$\omega(x,y,0^+) = -\omega(x,y,0^-) = \omega_0,$$

$$\gamma(x,y,0^+) = -\gamma(x,y,0^-) = \gamma_0.$$  \hspace{1cm} (52)

From the above method, the first iteration gives the electric displacement and the magnetic induction in the crack cavity.
\[ D^c_z = -\varepsilon^c \left( \frac{L_{43}L_{51} - L_{41}L_{53}}{\varepsilon^c} \right) e_0 + \left( \frac{L_{31}L_{53} - L_{33}L_{51}}{\varepsilon^c} \right) \varepsilon^c + \left( \frac{L_{33}L_{41} - L_{31}L_{43}}{\varepsilon^c} \right) \sigma_{0z}, \]
\[ B^c_z = -\mu^c \left( \frac{L_{41}L_{52} - L_{42}L_{51}}{\mu^c} \right) p_0 + \left( \frac{L_{32}L_{51} - L_{31}L_{52}}{\mu^c} \right) \sigma_{0z} + \left( \frac{L_{31}L_{42} - L_{32}L_{41}}{\mu^c} \right) \sigma_{0z}. \]

They are also constant. Therefore it can be concluded that the final electric displacement and the magnetic induction in the crack cavity are uniform fields. From Eqs. (39) and (50), a set of algebraic equations is deduced
\[ \left( L_{31} - L_{32} \frac{D^c_z}{\varepsilon^c} - L_{33} \frac{B^c_z}{\mu^c} \right) (\varepsilon_{0z} - D^c_z) = \left( L_{41} - L_{42} \frac{D^c_\varepsilon}{\varepsilon^c} - L_{43} \frac{B^c_\varepsilon}{\mu^c} \right) p_0, \]
\[ \left( L_{31} - L_{32} \frac{D^c_z}{\varepsilon^c} - L_{33} \frac{B^c_z}{\mu^c} \right) (\sigma_{0z} - B^c_z) = \left( L_{51} - L_{52} \frac{D^c_\varepsilon}{\varepsilon^c} - L_{53} \frac{B^c_\varepsilon}{\mu^c} \right) p_0. \]

Solving Eq. (54) gives the electric displacement \( D^c_z \) and the magnetic induction \( B^c_z \) in the crack cavity. Only one solution is the real solution, which satisfies the crack opening condition
\[ ||w|| \geq 0. \]

6. Numerical example

Consider a square crack of side length \( 2a \) in a composite made of BaTiO\(_3\) as the inclusion and CoFe\(_2\)O\(_4\) as the matrix. The crack is perpendicular to the poling direction. The volume fraction of the inclusions is denoted by \( V_i \). The material constants are given as follows (Huang et al., 1998)

BaTiO\(_3\):
\[ c_{11} = 166 \text{ GPa}, \quad c_{33} = 162 \text{ GPa}, \quad c_{44} = 43 \text{ GPa}, \quad c_{12} = 77 \text{ GPa}, \quad c_{13} = 78 \text{ GPa}, \]
\[ e_{31} = -4.4 \text{ C/m}^2, \quad e_{33} = 18.6 \text{ C/m}^2, \quad e_{15} = 11.6 \text{ C/m}^2, \]
\[ e_{11} = 11.2 \times 10^{-9} \text{ C}^2/(\text{Nm}^2), \quad e_{33} = 12.6 \times 10^{-9} \text{ C}^2/(\text{Nm}^2), \]
\[ \mu_{11} = 5.0 \times 10^{-6} \text{ Ns}^2/\text{C}^2, \quad \mu_{33} = 10.0 \times 10^{-6} \text{ Ns}^2/\text{C}^2, \]

CoFe\(_2\)O\(_4\):
\[ c_{11} = 286 \text{ GPa}, \quad c_{33} = 269.5 \text{ GPa}, \quad c_{44} = 45.3 \text{ GPa}, \quad c_{12} = 173.0 \text{ GPa}, \]
\[ c_{13} = 170.5 \text{ GPa}, \]
\[ f_{31} = 580.3 \text{ N/(Am)}, \quad f_{33} = 699.7 \text{ N/(Am)}, \quad f_{15} = 550. \text{ N/(Am)}, \]
\[ e_{11} = 0.08 \times 10^{-9} \text{ C}^2/(\text{Nm}^2), \quad e_{33} = 0.093 \times 10^{-9} \text{ C}^2/(\text{Nm}^2), \]
\[ \mu_{11} = 590 \times 10^{-6} \text{ Ns}^2/\text{C}^2, \quad \mu_{33} = 157 \times 10^{-6} \text{ Ns}^2/\text{C}^2, \]

Vacuum:
\[ \varepsilon^e = 8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2, \quad \mu^e = 1.257 \times 10^{-6} \text{ Ns}^2/\text{C}^2. \]

The following mixture rule is used to calculate the composite material constants correspondingly from those of the inclusion and matrix
\[ A^c = A^i V_i + A^m (1 - V_i), \]
where the superscripts c, i and m represent the composite, inclusion and matrix respectively.

For \( V_i = 0.5 \) the material related constants are calculated to be
\[ L_{31} = 5.82684 \times 10^9, \quad L_{32} = 0.439827, \quad L_{33} = 12.101, \]
\[ L_{41} = 0.439827, \quad L_{42} = -0.503804 \times 10^{-9}, \quad L_{43} = -0.558533 \times 10^{-9}, \]
\[ L_{51} = 12.101, \quad L_{52} = -0.558533 \times 10^{-9}, \quad L_{53} = -12588.2 \times 10^{-9}. \]
The extended tractions on the crack faces are
\[ p_z = 100 \text{ MPa}, \quad \omega = 1.0 \text{ C/m}^2, \quad \gamma = 10 \text{ N/Am}, \] \hspace{1cm} (61)

The square crack problem in pure elasticity was well studied, and the stress intensity factor along the side was available (Murakami et al., 1992). By use of the solution and the method proposed in the present paper, the extended intensity factors can be obtained. They are plotted in Figs. 3–6 respectively for the four kinds of boundary conditions, where the extended intensity factors are normalized by
\[ F = \frac{K_I^P}{(\sqrt{\pi}a \omega)}, \quad F_D = \frac{K_D^P}{(\sqrt{\pi}a \omega)}, \quad F_B = \frac{K_B^P}{(\sqrt{\pi}a \gamma)}. \] \hspace{1cm} (62)

The three dimensionless intensity factors for impermeable condition are equal as shown in Fig. 3, with the maximum value being 0.756 at the middle point of the side. Fig. 4 displays the intensity factors for permeable condition. The electric displacement and the magnetic induction intensity factors are both smaller than that under impermeable condition. Figs. 5 and 6 depict the intensity factors under the electrically impermeable and magnetically permeable condition and electrically permeable and magnetically impermeable condition, respectively. The electric displacement intensity factor under the electrically impermeable and magnetically permeable condition is the same as that under impermeable condition. Similarly, the magnetic induction intensity factor under the electrically permeable and magnetically impermeable condition is the same as that under impermeable condition.

Fig. 3. Normalized extended intensity factors under electrically and magnetically impermeable boundary condition.

Fig. 4. Normalized extended intensity factors under electrically and magnetically permeable boundary condition.
Solving Eq. (54) yields the analytical electric displacement and the magnetic induction in the crack cavity 

\[ D_c = \frac{1.3884}{10^{-2}} \text{C/m}^2, \quad B_c = 9.15062 \text{N/Am}. \] (63)

The iteration results for the opening model are shown in Fig. 7. It can be seen that the numerical results converge to the exact solution very fast.
Fig. 8 shows the dimensionless intensity factors by the crack opening model. The electric displacement and the magnetic induction intensity factors are both larger than that under impermeable condition. The electric displacement intensity factor increases about 10%, and the magnetic induction intensity factor about 50%.

7. Concluding remarks

The extended displacement discontinuity boundary integral equation approach is described systematically for a planar crack with different electric and magnetic boundary conditions in the isotropic plane of an infinite transversely isotropic magnetoelectroelastic medium. The planar crack can be of arbitrary shape, so the method and the conclusions are applicable to coplanar cracks. The revealed analogy between the boundary integral equations of the magnetoelectroelastic materials and the purely elastic materials shows that the solutions can be obtained directly from those of the corresponding purely elastic problems because the fracture problem of purely elastic material under mechanical loading has been studied intensively and extensively.

The proposed method is applicable to crack problems in piezoelectric, piezomagnetic and purely elastic materials. The displacement discontinuity method or the hyper-singular boundary integral equation method for purely elastic problem has been studied intensively, see for example Lin’kov and Mogilevshaya (1986), Ioakimidis (1987), Zhao et al. (1994) and Wen (1996), while the method for piezoelectric material can be referred to Zhao et al. (1997) and Zhang et al. (2002).

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Appendix A. The material characteristic equation

The material characteristic equation for a transversely isotropic magnetoelectroelastic medium is derived by using the method of Wang and Shen (2002) as

\[ \mathbf{w}^T (\mathbf{A} - \mu \mathbf{B})^{-1} \mathbf{w} = c_{11} - c_{44} \mu^{-1}, \]  

where

\[
\mathbf{w} = \begin{bmatrix} c_{13} + c_{44} \\ e_{15} + e_{31} \\ f_{15} + f_{31} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} c_{33} & e_{33} & f_{33} \\ e_{33} & -\varepsilon_{33} & -g_{33} \\ f_{33} & -g_{33} & -\mu_{33} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} c_{44} & e_{15} & f_{15} \\ e_{15} & -\varepsilon_{11} & -g_{11} \\ f_{15} & -g_{11} & -\mu_{11} \end{bmatrix}. \]
where the contracted notations of subscripts are used.

Both $\mathbf{A}$ and $\mathbf{B}$ are real and symmetric matrices. Considering the following eigenvalue problem

$$\mathbf{A} \zeta = \delta \mathbf{B} \zeta,$$

we obtain the eigen-matrix

$$\mathbf{Y} = [\zeta_1, \zeta_2, \zeta_3].$$

Thus, Eq. (A.1) can be reduced to

$$c_{11} - c_{44} \lambda^{-1} = \mathbf{w}^T(\mathbf{Y}^{-1} \mathbf{B}^{-1} \mathbf{A} \mathbf{Y} - \lambda \mathbf{I})^{-1} \mathbf{w},$$

where

$$\mathbf{w} = \mathbf{Y}^T \mathbf{w}, \quad \mathbf{Y} = \mathbf{Y}^{-1} \mathbf{B}^{-1} \mathbf{w}.$$

Finally, Eq. (A.5) becomes

$$\hat{w}_1 \hat{w}_1 + \hat{w}_2 \hat{w}_2 + \hat{w}_3 \hat{w}_3 + c_{44} \hat{w}_1 = c_{11}.$$

Solving Eq. (A.7) obtains the four characteristic roots $\lambda_i \equiv 1/\xi_i^2 (i = 1, 2, 3, 4)$, therefore, determines four sets $k_{ji} (j = 1, 2, 3)$ by

$$c_{44} + k_{11}(c_{13} + c_{44}) + k_{21}(e_{15} + e_{31}) + k_{31}(f_{15} + f_{31}) = k_{11}e_{33} + k_{11}e_{44} + k_{21}e_{15} + k_{31}f_{15} = k_{11}e_{33} - k_{21}e_{33} - k_{31}g_{33} = k_{11}e_{15} + k_{11}e_{31} - k_{21}e_{15} - k_{31}g_{11}.$$

Finally, the fifth characteristic root is denoted by

$$\lambda_5 = \frac{c_{44}}{c_{66}}, \quad c_{66} = \frac{c_{11} - c_{12}}{2}.$$

**Appendix B. Fundamental solutions corresponding to unit point force $P_3$ in the $z$-direction**

Using the derivation procedure of Ding *et al.* (2005), the fundamental solutions are obtained

$$\tau_{sm} = P_3 \sum_{i=1}^{4} \frac{\omega_{im} A_i x}{R^3_i},$$

$$\tau_{ym} = P_3 \sum_{j=1}^{4} \frac{\omega_{jm} A_j y}{R^3_i},$$

$$\sigma_m = P_3 \sum_{i=1}^{4} \frac{\varphi_{im} A_i z_i}{R^3_i},$$

where $z_i = s_i z, \quad \tau_{x1} = \tau_{x2} = D_x, \quad \tau_{x3} = B_x, \quad \tau_{y1} = \tau_{y2} = D_y, \quad \tau_{y3} = B_y, \quad \sigma_1 = \sigma_{22}, \quad \sigma_2 = D_z$ and $\sigma_3 = B_z$, and the material related constants are given by

$$\omega_{im}, \varphi_{im}, \omega_{jm}.$$
\[ a_{im} = s_i k_{mi}, \quad (B.2) \]
\[ \xi_i = (c_{ij} x_i + e_{ij} x_j + f_{ij} s_i) s_i - c_{ij}, \]
\[ \omega_{51} = e_{45} s_5, \quad \omega_{52} = e_{15} s_5, \quad \omega_{53} = f_{15} s_5, \]
\[ \omega_{1i} = e_{45}(s_i + x_i) + e_{15} x_i + f_{15} x_i, \]
\[ \omega_{2i} = e_{15}(s_i + x_i) - e_{11} x_i - g_{11} x_i, \]
\[ \omega_{3i} = f_{15}(s_i + x_i) - g_{11} x_i - \mu_{11} x_i, \]
\[ \vartheta_{i1} = (c_{ij} x_i + e_{ij} x_j + f_{ij} x_i s_i) s_i - c_{ij}, \]
\[ \vartheta_{i2} = (e_{ij} x_i - \omega_{ij} x_i - \mu_{ij} x_i) s_i - e_{ij}, \]
\[ \vartheta_{i3} = (f_{ij} x_i - \omega_{ij} x_i - \mu_{ij} x_i) s_i - f_{ij}, \]

and
\[ 4 \pi \sum_{i=1}^{4} A_i = 0, \quad 4 \pi \sum_{i=1}^{4} \vartheta_{i1} A_i = -1, \quad (B.4) \]
\[ 4 \pi \sum_{i=1}^{4} \vartheta_{i2} A_i = 0, \quad 4 \pi \sum_{i=1}^{4} \vartheta_{i3} A_i = 0. \]

Solutions corresponding to unit point electric charge \( P_4 \) and point electric current \( P_5 \) are in the same form as Eq. (31), but respectively by using of \( P_4 \) and \( B_i \) and \( P_5 \) and \( C_i \) instead of \( P_3 \) and \( A_i \). The coefficients \( B_i \) and \( C_i \) are determined respectively by
\[ \sum_{i=1}^{4} B_i = 0, \quad 4 \pi \sum_{i=1}^{4} \vartheta_{i1} B_i = 0, \quad (B.5) \]
\[ 4 \pi \sum_{i=1}^{4} \vartheta_{i2} B_i = 1, \quad 4 \pi \sum_{i=1}^{4} \vartheta_{i3} B_i = 0, \]
\[ \sum_{i=1}^{4} C_i = 0, \quad 4 \pi \sum_{i=1}^{4} \vartheta_{i1} C_i = 0, \quad (B.6) \]
\[ 4 \pi \sum_{i=1}^{4} \vartheta_{i2} C_i = 0, \quad 4 \pi \sum_{i=1}^{4} \vartheta_{i3} C_i = 1. \]

**Appendix C. Fundamental solutions corresponding to unit point force \( P_1 \) in the \( x \)-direction**

This fundamental solutions take the forms
\[ \tau_{xim} = -P_1 \omega_{3m} D_3 \left[ \frac{1}{R_3(R_5 - z_5)} - \frac{y^2}{R_3^3(R_5 - z_5)} - \frac{y^2}{R_3^2(R_5 - z_5)^2} \right], \quad (C.1) \]
\[ \tau_{3im} = -P_1 \omega_{3m} D_3 x y \left[ \frac{1}{R_3(R_5 - z_5)} + \frac{1}{R_3^2(R_5 - z_5)^2} \right], \quad (C.2) \]
\[ \sigma_m = P_1 \sum_{i=1}^{4} \vartheta_{im} \frac{D_i x}{R_i} \left[ \frac{1}{R_i(R_i - z_i)} + \frac{1}{R_i^2(R_i - z_i)^2} \right], \quad (C.3) \]

where the coefficients \( D_i \) are given by
\[
\sum_{i=1}^{4} \alpha_{im} D_i = 0,
\]
\[
s_5 D_5 + \sum_{i=1}^{4} s_i D_i = 0.
\]
\[2\pi c_{45s_5} D_5 - 2\pi \sum_{i=1}^{4} \omega_{ii} D_i = -1. \tag{C.4}
\]

References


