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The Zeros of Basic Bessel Functions, the Functions $J_{\nu+ax}(x)$, and Associated Orthogonal Polynomials*

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The modified Lommel polynomials are generalized and their orthogonality relation is obtained. As a byproduct we prove that the non-zero roots of $J_{\nu+ax}(x)$ are real, simple and interlace with those of $J_{\nu+ax-1}(x)$. The q-Lommel polynomials are shown to play in the theory of basic Bessel function the role played by the Lommel polynomials in the Bessel function theory. The q-Lommel polynomials are proved to be orthogonal with respect to a purely discrete measure with bounded support. This is then used to prove that the prositive zeros of $x^{-\nu}J_\nu^{(2)}(x; q)$ are real simple and interlace with the zeros of $x^{-\nu-1}J_{\nu+1}^{(2)}(x; q)$, when $\nu > -1$. We also establish the complete monotonicity of $-1x^{-1/2}J_{\nu+1}^{(2)}(i\sqrt{x}; q)/J_\nu^{(2)}(i\sqrt{x}; q)$.

1. INTRODUCTION

The Bessel Function

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)} \tag{1.1}$$

satisfies

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x). \tag{1.2}$$

By iterating the above recursion one expects that $J_{\nu+m}(x)$ may be expressed as $f_m(x) J_\nu(x) + g_m(x) J_{\nu-1}(x)$, where $f_m(1/x)$ and $g_m(1/x)$ are polynomials of degrees m and $m-1$, respectively. These two polynomials can be expressed in terms of a single set of polynomials called the Lommel polynomials. Indeed

$$J_{\nu+m}(x) = R_{m,\nu}(x) J_\nu(x) - R_{m-1,\nu+1}(x) J_{\nu-1}(x) \tag{1.3}$$

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(Watson [26, pp. 294, 295]). These polynomials satisfy the recurrence relation [26, p. 299]

$$R_{m-1,\nu}(x) + R_{m+1,\nu}(x) = \frac{2(\nu+m)}{x} R_{m,\nu}(x), \qquad m = 1, 2, \ldots, \tag{1.4}$$

with $R_{0,\nu}(x) = 1$, $R_{1,\nu}(x) = 2\nu/x$. Following Dickinson [7] we use the notation $\{h_{n,\nu}(x)\}$ for what he called the "modified Lommel polynomials,"

$$h_{n,\nu}(x) = R_{n,\nu}(1/x). \tag{1.5}$$

Clearly $h_{n,\nu}(x)$ is polynomial in x of degree n, when $\nu > -1$. It is obvious that

$$h_{n-1,\nu}(x) + h_{n+1,\nu}(x) = 2x(\nu+n)\, h_{n,\nu}(x), \qquad n = 1, 2, \ldots, \tag{1.6}$$

and

$$h_{0,\nu}(x) = 1, \qquad h_{1,\nu}(x) = 2\nu x. \tag{1.7}$$

By a theorem of Favard [9], the recurrence relation (1.6) implies the existence of a positive Borel measure $d\mu$ on $(-\infty, \infty)$ so that

$$\int_{-\infty}^{\infty} h_{n,\nu}(x)\, h_{m,\nu}(x)\, d\mu(x) = [2(\nu+n)]^{-1}\, \delta_{m,n}, \tag{1.8}$$

with

$$(\sigma)_0 = 1, \qquad (\sigma)_n = \sigma(\sigma+1) \cdots (\sigma+n-1), \qquad n > 0. \tag{1.9}$$

Schwartz [22] computed this measure explicitly. He proved that $\mu(x)$ is a nondecreasing step function having an increase $1/j^2_{\nu-1,n}$ at the points $\pm 1/j_{\nu-1,k}$, where $\{j_{\nu,n}\}$ are the positive zeros of $J_\nu(x)$. It is known that the zeros of $x^{-\nu}J_\nu(x)$ are real and simple. Dickinson [7] rediscovered Schwartz' results.

A close look at the recursion (1.6) suggests a generalization of the modified Lommel polynomials. Let

$$\tau_n(x) = \tau_n(x; \nu, a) \tag{1.10}$$

be a sequence of polynomials defined recursively by

$$\begin{gathered} \tau_0(x) = 1, \qquad \tau_1(x) = 2\nu x + 2a. \\ \tau_{n+1}(x) = 2[x(\nu+n) + a]\, \tau_n(x) - \tau_{n-1}(x). \end{gathered} \tag{1.11}$$

Clearly

$$\tau_n(x; \nu, 0) = h_{n,\nu}(x). \tag{1.12}$$

One might expect that the polynomials $\{\tau_n(x)\}$ will still be connected with Bessel functions. We shall prove that $\{\tau_n(x)\}$ are orthogonal with respect to a purely discrete measure. This measure is not symmetric any more because the polynomials $\{\tau_n(x)\}$ are not symmetric. As a byproduct we prove that $J_{\nu+ax}(x)$ has infinitely many zeros for $\nu > -1$. The nonzero roots of this function are all real and simple. Furthermore the roots of $J_{\nu+ax}(x)$ and $J_{\nu+ax+1}(x)$ interlace. It is surprising that these results which do not seem to be directly related to orthogonal polynomials follow from analyzing certain orthogonal polynomials. A direct proof of these results will be of interest.

Jackson [17] defines a basic or q-analogue of the Bessel function by

$$J_\nu^{(1)}(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{(q; q)_n (q^{\nu+1}; q)_n}, \qquad 0 < q < 1, \tag{1.13}$$

where

$$(a; q)_0 = 1,\ (a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j),\ (a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j).$$

The q-gamma function is, see Askey [3], and Jackson [17]

$$\Gamma_q(x) = (q; q)_\infty (1-q)^{1-x}/(q^x; q)_\infty, \qquad 0 < q < 1, \tag{1.15}$$

while

$$\Gamma_q(x) = (q^{-1}; q^{-1})_\infty (-1+q)^{1-x} q^{x(x-1)/2}/(q^{-x}; q^{-1})_\infty, \qquad q > 1. \tag{1.16}$$

It is known that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1$. This implies that $J_\nu^{(1)}(x(1-q); q) \to J_\nu(x)$ as $q \to 1^-$, since the function $(q^{\nu+1}; q)_\infty/(q; q)_\infty$ appearing in (1.13) is $(1-q)^{-\nu}/\Gamma_q(\nu+1)$. There is another q-Bessel function that results by replacing q by q^{-1} in the series defining $J_\nu^{(1)}(x; q)$ and modifying the multiplicative factor. This function is essentially

$$J_\nu^{(2)}(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{(q; q)_n (q^{\nu+1}; q)_n} q^{n(\nu+n)}, \qquad 0 < q < 1. \tag{1.17}$$

Jackson's original notation is different. He also restricted himself to the cases $\nu = 0, \pm 1, \pm 2, \ldots$, see [13–16]. Hahn [11] also investigated q-Bessel functions.

Now that we have basic analogues of the Bessel functions we might again express $J^{(k)}_{m+\nu}(x;q)$ in terms of $J^{(k)}_{\nu}(x;q)$ and $J^{(k)}_{\nu-1}(x;q)$, $k=1,2$, thus giving rise to a basic analogue of the Lommel polynomials. We shall prove that both $J^{(1)}_{\nu}(x;q)$ and $J^{(2)}_{\nu}(x;q)$ satisfy the same recurrence relation, namely,

$$q^{\nu}J^{(k)}_{\nu+1}(x;q)=\frac{2(1-q^{\nu})}{x}J^{(k)}_{\nu}(x;q)-J^{(k)}_{\nu-1}(x;q), \qquad k=1,2, \tag{1.18}$$

then by iteration we shall prove that

$$q^{m\nu+m(m-1)/2}J^{(k)}_{\nu+m}(x;q)=R_{m,\nu}(x;q)\,J^{(k)}_{\nu}(x;q) - R_{m-1,\nu+1}(x;q)\,J^{(k)}_{\nu-1}(x;q), \qquad k=1,2. \tag{1.19}$$

The polynomials $\{R_{n,\nu}(x;q)\}$ will turn out to satisfy the recursion

$$R_{n+1,\nu}(x;q)=\frac{2(1-q^{\nu+n})}{x}R_{n,\nu}(x;q)-q^{n+\nu-1}R_{n-1,\nu}(x;q), \qquad n=1,2,..., \tag{1.20}$$

with $R_{0,\nu}(x,q)=1$ and $R_{1,\nu}(x;q)=2(1-q^{\nu})/x$. Letting

$$h_{n,\nu}(x;q)=R_{n,\nu}(1/x;q), \tag{1.21}$$

we see that

$$h_{n+1,\nu}(x;q)=2x(1-q^{n+\nu})\,h_{n,\nu}(x;q)-q^{n+\nu-1}h_{n-1,\nu}(x;q), \qquad n=1,2,..., \\ h_{0,\nu}(x;q)=1, \qquad h_{1,\nu}(x;q)=2x(1-q^{\nu}). \tag{1.22}$$

The above results will be proved in Section 2 together with several miscellaneous relationships involving the polynomials $R_{n,\nu}(x;q)$. In Section 3 we derive generating functions for the modified Lommel polynomials $\{h_{n,\nu}(x)\}$, their basic analogue $\{h_{n,\nu}(x;q)\}$ and the generalized polynomials $\{\tau_n(x;\nu,a\}$. We also determine the asymptotic behaviour of $h_{n,\nu}(x;q)$ for large n and fixed x. This formula corresponds to Hurwitz' theorem (Watson [26, p. 302]),

$$\lim_{n\to\infty}(2x)^{1-\nu-n}\,h_{n,\nu}(x)/\Gamma(n+\nu)=J_{\nu-1}(1/x). \tag{1.23}$$

Explicit formulas for $\{h_{n,\nu}(x;q)\}$ and $\{\tau_n(x;\nu,a)\}$ are also found. Section 4 is devoted to obtaining the measures with respect to which the polynomials $\{h_{n,\nu}(x;q)\}$ and $\{\tau_n(x;\nu,a)\}$ are orthogonal. We shall prove that the first measure is purely discrete with atoms at the points $\{\pm 1/j_{\nu-1,n}(q)\}$, where $0<j_{\nu,1}(q)<j_{\nu,2}(q)<\cdots<j_{\nu,n}(q)<\cdots$ are the positive roots of $J^{(2)}_{\nu}(x;q)$. Hahn [11] mentions that $J^{(2)}_{\nu}(x;q)$ has infinitely many real zeros. We were unable to find any more information about these zeros in the literature. We prove that for $\nu > -1$ *all* the non-zero roots of $J_\nu^{(2)}(x; q)$ are real and simple and that the zeros of $J_\nu^{(2)}(x; q)$ and $J_{\nu+1}^{(2)}(x; q)$ interlace. We also prove that the zeros of $J_{\nu+ax}(x)$ have similar properties. These results are known to hold for the Bessel functions $J_\nu(x)$ and are usually proved by Sturmian arguments; see Watson [26] and Szegö [25]. These techniques, however, are not available for q-difference equations. The function $J_{\nu+ax}(x)$ does not seem to satisfy any differential equation for $a \neq 0$. The methods used in the present paper originated in Askey and Ismail's work [5] and some of these ideas go back to Pollaczek [21].

We do not expect the zeros of $J_\nu^{(1)}(x; q)$ to have similar properties because the series (1.13) $J_\nu^{(1)}(x; q)$ converges only for $|x| < 2$. It is also known that [11]

$$(-x^2/4; q)_\infty J_\nu^{(1)}(x; q) = J_\nu^{(2)}(x; q), \qquad |x| < 2,$$

which shows that $J_\nu^{(1)}(x; q)$ has only finitely many non-zero roots in $|x| < 2$ because each such root is also a zero of the entire function $x^{-\nu} J_\nu^{(2)}(x; q)$.

In Section 4 we also show that $-ix^{-1/2} J_{\nu+1}^{(2)}(i\sqrt{x}; q)/J_\nu^{(2)}(i\sqrt{x}; q)$ is a completely monotonic function of x for $\nu > -1$. This is a q-analogue of the complete monotonicity of $x^{-1/2} I_{\nu+1}(\sqrt{x})/I_\nu(\sqrt{x})$. Recently infinite divisibility problems in probability theory lead naturally to questions of complete monotonicity of quotients of special functions, see Grosswald [10], Ismail and Kelker [12], Kent [18] and Wendel [27].

Maki [19] observed that the polynomials $R_{n,\nu}(x)$ are polynomials in ν if x is considered as a parameter. He proved that they are orthogonal with respect to a discrete measure with unbounded support. The polynomials $R_{n,\nu}(x; q)$ do not form an orthogonal polynomial set in the variable q^ν if x is considered a fixed parameter. On the other hand it easily follows from (1.20) that the polynomials

$$\theta_n(q^\nu) = q^{-\nu n/2} R_{n,\nu}(xq^{-\nu/2}; q) \tag{1.24}$$

are polynomials in q^ν with $\theta_n(q^\nu)$ of degree n. They satisfy the recurrence relation

$$\theta_{n+1}(\nu) = \frac{2}{x}(1 - \nu q^n)\, \theta_n(\nu) - q^{n-1} \theta_{n-1}(\nu), \qquad n = 1, 2, \ldots, \tag{1.25}$$

with

$$\theta_0(\nu) = 1, \qquad \theta_1(\nu) = 2(1-\nu)/x. \tag{1.26}$$

Those polynomials are orthogonal with respect to a measure with unbounded support. It will be of interest to determine this measure(s). The methods used in the present paper will not be applicable to this problem.

There are several interesting relationships involving Bessel functions and Legendre polynomials, or in general the ultraspherical (Gegenbauer) polynomials, see [25, 26]. It might be of interest to draw the reader's attention to Jackson's works [13–16], where the corresponding connections between the q-Bessel functions and the q-ultraspherical polynomials are derived. The q-ultraspherical polynomials seem to have appeared first in L. J. Rogers memoirs on expansions of infinite products where he proved the Rogers-Ramanujan's identities, see Andrews [2] and Askey and Ismail [4]. Askey and Ismail [4] found, among other things, the weight function for these polynomials.

2. The q Bessel Functions and q-Lommel Polynomials

We first prove relationship (1.18).

Proof of (1.18). Clearly

$$\begin{aligned}
J_\nu^{(1)}(x;q) &= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-)^n (x/2)^{\nu+2n} (1-q^{n+\nu+1})}{(q;q)_n (q^{\nu+1};q)_{n+1}} \\
&= \frac{(q^{\nu+2};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n} (1-q^{\nu+1}+q^{\nu+1}-q^{n+\nu+1})}{(q;q)_n (q^{\nu+2};q)_n} \\
&= \frac{2}{x}(1-q^{\nu+1}) J_{\nu+1}^{(1)}(x;q) + q^{\nu+1} \frac{(q^{\nu+2};q)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{(q;q)_{n-1}(q^{\nu+2};q)_n} \\
&= \frac{2}{x}(1-q^{\nu+1}) J_{\nu+1}^{(1)}(x;q) - q^{\nu+1} \frac{(q^{\nu+3};q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2+2n}}{(q;q)_n (q^{\nu+3};q)_n},
\end{aligned}$$

which implies (1.18) for $k=1$. The proof of (1.18) when $k=2$ is similar but instead of writing $1-q^{n+\nu+1}$ as $1-q^{\nu+1}+q^{\nu+1}-q^{n+\nu+1}$ we write it as $1-q^n+q^n-q^{n+\nu+1}$. Thus

$$\begin{aligned}
J_\nu^{(2)}(x;q) &= \frac{(q^{\nu+2};q)_\infty}{(q;q)_\infty} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{(q;q)_{n-1} (q^{\nu+2};q)_n} q^{n(\nu+n)} \right. \\
&\quad \left. + \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n} (1-q^{\nu+1})}{(q;q)_n (q^{\nu+2};q)_n} q^{n(\nu+n+1)} \right\} \\
&= \frac{2}{x}(1-q^{\nu+1}) J_{\nu+1}^{(2)}(x;q) - q^{\nu+1} J_{\nu+2}^{(2)}(x;q),
\end{aligned}$$

which is essentially (1.18) with $k=2$.

Proof of (1.19) *and* (1.20). Use straightforward induction.

The fact that $J_\nu^{(2)}(x;q)$ satisfies the q-difference equation

$$(1+qx^2/4)\,J_\nu^{(2)}(qx;q)-(q^{\nu/2}+q^{-\nu/2})\,J_\nu^{(2)}(\sqrt{q}x;q)+J_\nu^{(2)}(x;q)=0, \tag{2.1}$$

follow directly from (1.17). Similarly one can establish

$$J_\nu^{(1)}(qx;q)-(q^{\nu/2}+q^{-\nu/2})\,J_\nu^{(1)}(\sqrt{q}x;q)+(1+x^2/4)\,J_\nu^{(1)}(x;q)=0. \tag{2.2}$$

It is easy to see, via (1.13), that

$$J_\nu^{(1)}(\sqrt{q}x;q)=q^{\nu/2}\left(J_\nu^{(1)}(x;q)+\frac{x}{2}J_{\nu+1}^{(1)}(x;q)\right) \tag{2.3}$$

and that

$$J_\nu^{(1)}(\sqrt{q}x;q)=q^{-\nu/2}\left(J_\nu^{(1)}(x;q)-\frac{x}{2}J_{\nu-1}^{(1)}(x;q)\right). \tag{2.4}$$

It is clear from (2.2) that both $J_\nu^{(1)}(x;q)$ and $J_{-\nu}^{(1)}(x;q)$ satisfy the same q-difference equation (2.2). Similarly $J_{-\nu}^{(2)}(x;q)$ also satisfies (2.1). Next we derive a Wronskian type formula for $J_\nu^{(1)}(x;q)$. This Wronskian type formula will be used to obtain a power series representation for the q-Lommel polynomials $R_{n,\nu}(x)$. Let

$$f_\nu(x)=J_\nu^{(1)}(x;q)\,J_{-\nu}^{(1)}(\sqrt{q}x;q)-J_{-\nu}^{(1)}(x;q)\,J_\nu^{(1)}(\sqrt{q}x;q),$$

and assume ν is not an integer. Since

$$\begin{aligned}f_\nu(\sqrt{q}x)=J_\nu^{(1)}(\sqrt{q}x;q)\{(q^{\nu/2}+q^{-\nu/2})\,J_{-\nu}^{(1)}(\sqrt{q}x;q)-(1+x^2/4)\,J_{-\nu}^{(1)}(x;q)\}\\ -J_{-\nu}^{(1)}(\sqrt{q}x;q)\{(q^{\nu/2}+q^{-\nu/2})\,J_\nu^{(1)}(\sqrt{q}x;q)-(1+x^2/4)\,J_\nu^{(1)}(x;q)\},\end{aligned}$$

by (2.2), hence

$$f_\nu(\sqrt{q}x)=(1+x^2/4)\,f_\nu(x),$$

which by iteration yields

$$f_\nu(x)=f_\nu(0)\Big/\left(-\frac{x^2}{4};q\right)_\infty.$$

From (1.13), $f_\nu(0)$ is nothing but $(q^{\nu+1};q)_\infty\,({}^{1-\nu};q)_\infty\,(q^{-\nu/2}-q^{\nu/2})/(q;q)_\infty^2$.

Therefore for non-integral ν, we have

$$J_\nu^{(1)}(x;q)\,J_{-\nu}^{(1)}(\sqrt{q}x;q)-J_{-\nu}^{(1)}(x;q)\,J_\nu^{(1)}(\sqrt{q}x;q)$$
$$=q^{-\nu/2}(q^\nu;q)_\infty\,(q^{1-\nu};q)_\infty\Big/\left\{(q;q)_\infty^2\left(-\frac{x^2}{4};q\right)_\infty\right\}, \qquad (2.5)$$

which is a Wronskian formula and as $q \to 1$, it tends to (59), p. 12 of [8].

Using the q-difference-finite difference relationships (2.3) and (2.4) we get

$$J_\nu^{(1)}(x;q)\,J_{1-\nu}^{(1)}(x;q)+J_{-\nu}^{(1)}(x;q)\,J_{\nu-1}^{(1)}(x;q)$$
$$=\frac{2(q^\nu;q)_\infty\,(q^{1-\nu};q)_\infty}{x(q;q)_\infty^2\,(-x^2/4;q)_\infty}, \qquad (2.6)$$

an analogue of (60), p. 12 in [8].

Now we go to back to (1.18) replace ν by $-\nu$ to obtain

$$q^\nu e^{i(\nu+1)\pi}J_{-\nu-1}^{(1)}(x;q)=\frac{2(1-q^\nu)}{x}e^{i\nu\pi}J_{-\nu}^{(1)}(x;q)-e^{i(\nu-1)\pi}J_{-\nu+1}^{(1)}(x;q).$$

Thus $J_\nu^{(1)}(x;q)$ and $e^{i\nu\pi}J_{-\nu}^{(1)}(x;q)$ satisfy the same recursion (1.18), hence

$$q^{m\nu+m(m-1)/2}(-1)^m J_{-\nu-m}^{(1)}(x;q)=R_{m,\nu}(x;q)\,J_{-\nu}^{(1)}(x;q)$$
$$+R_{m-1,\nu+1}(x;q)\,J_{-\nu+1}^{(1)}(x;q).$$

Eliminating $R_{m-1,\nu+1}(x;q)$ from the above recurison relation and (1.19) we arrive at

$$q^{m\nu+m(m-1)/2}\{J_{\nu+m}^{(1)}(x;q)\,J_{1-\nu}^{(1)}(x;q)+(-1)^m J_{\nu-1}^{(1)}(x;q)\,J_{-\nu-m}^{(1)}(x;q)\}$$
$$=R_{m,\nu}(x;q)\{J_\nu^{(1)}(x;q)\,J_{1-\nu}^{(1)}(x;q)+J_{-\nu}^{(1)}(x;q)\,J_{\nu-1}^{(1)}(x;q)\},$$

which when combined with (2.6) yields

$$R_{m,\nu}(x;q)=\frac{x(q;q)_\infty^2\left(-\frac{x^2}{4};q\right)_\infty}{2(q^\nu;q)_\infty\,(q^{1-\nu};q)_\infty}\,q^{m\nu+m(m-1)/2}$$
$$\times\{J_{\nu+m}^{(1)}(x;q)\,J_{1-\nu}^{(1)}(x;q)+(-1)^m J_{\nu-1}^{(1)}(x;q)\,J_{-\nu-m}^{(1)}(x;q)\}. \qquad (2.7)$$

The above identity was established for non-integral ν. It is easy to see, however, that as ν tends to an integer the right hand side of (2.7) does tend to a limit. The relationship (2.7) provides a power series representation for the functions $R_{n,\nu}(x)$. On the other hand we know that $R_{n,\nu}(x)$ is a polynomial in $2/x$ of precise degree n. The function $x(-x^2/4; q)_\infty J^{(1)}_{\nu+m}(x; q) \cdot J^{(1)}_{1-\nu}(x; q)$ contributes only positive powers of x. Note that

$$(z; q)_\infty = \sum_{n=0}^{\infty} (-1)^n q^{n(n-1)/2} z^n/(q; q)_n. \tag{2.8}$$

Now (2.7) and (2.8) lead to the explicit formula

$$\begin{aligned} R_{m,\nu}(x; q) = (q^\nu; q)_m \sum_{p=0}^{[m/2]} \left(\frac{x}{2}\right)^{-m+2p} (-1)^p \\ \times \sum_{j+k+l=p} \frac{(-1)^j q^{j(j-1)/2}}{(q; q)_j (q; q)_k (q; q)_l (q^\nu; q)_k (q^{-\nu-m+1}, q)_l}. \end{aligned} \tag{2.9}$$

It does not seem to be possible to simplify the above expression. In the next section we shall establish the explicit formula

$$R_{m,\nu}(x; q) = \sum_{j=0}^{[m/2]} \frac{(x/2)^{2j-m} (-1)^j (q^\nu; q)_{m-j} (q; q)_{m-j}}{(q; q)_j (q^\nu; q)_j (q; q)_{m-2j}} q^{j(j+\nu-1)}, \tag{2.10}$$

see (3.6).

3. Generating Functions, Explicit Formulas and Asymptotics

We start the present section by deriving a new generating function for the modified Lommel polynomials, see (1.6) and (1.7). Let

$$H = H(x, t, \nu) = \sum_{0}^{\infty} h_{n,\nu}(x)\, t^n/n!.$$

Multiplying (1.6) by $t^n/(n-1)!$ and adding the resulting identities we derive the differential equation

$$(1 - 2xt)\frac{\partial^2 H}{\partial t^2} - 2x(\nu + 1)\frac{\partial H}{\partial t} + H = 0.$$

Making the substitution,

$$H(x, t, \nu) = (1 - 2xt)^{-\nu/2} F(x, t, \nu),$$

in the above differential equation we prove that

$$(1-2xt)^2\frac{\partial^2 F}{\partial t^2}-2x(1-2xt)\frac{\partial F}{\partial t}+\{(1-2xt)-\nu^2x^2\}F=0.$$

Now change the independent variable t to u, where $u^2=1-2xt$. Hence F satisfies the Bessel differential equation

$$u^2\frac{\partial^2 F}{\partial u^2}+u\frac{\partial F}{\partial u}+\left(\frac{u^2}{x^2}-\nu^2\right)F=0.$$

Consequently

$$H(x,t,\nu)=(1-2xt)^{-\nu/2}\left\{AJ_\nu\left(\frac{\sqrt{1-2xt}}{x}\right)+BY_\nu\left(\frac{\sqrt{1-2xt}}{x}\right)\right\},$$

where A and B may depend on x and ν but are independent of t. The conditions $H(x,0,\nu)=1$, $(\partial H/\partial t)(x,0,\nu)=h_{1,\nu}(x)=2\nu x$, and the Wronskian formula (35), p. 80 in [8] establish the new generating function for the modified Lommel polynomials

$$\sum_{n=0}^{\infty} h_{n,\nu}(x)\,t^n/n!=\frac{\pi}{2x}(1-2xt)^{-\nu/2} \tag{3.1}$$
$$\times\left\{J_\nu\left(\frac{\sqrt{1-2xt}}{x}\right)Y_{\nu-1}\left(\frac{1}{x}\right)-Y_\nu\left(\frac{\sqrt{1-2xt}}{x}\right)J_{\nu-1}\left(\frac{1}{x}\right)\right\}.$$

A generating function for $\{\tau_n(x)\}$ can be established in a similar way. Note that we can view the polynomials $\{\tau_n(x;\nu,a)\}$ as $\{h_{n,\nu+a/x}(x)\}$, see (1.6), (1.7) and (1.11). The above derivation of (3.1) treated both x and ν as constants. Hence (3.1) yields

$$\sum_{n=0}^{\infty}\tau_n(x;\nu,a)\,t^n/n!=\frac{\pi}{2x}(1-2xt)^{-(\nu+a/x)/2} \tag{3.2}$$
$$\times\left\{J_{\nu+a/x}\left(\frac{\sqrt{1-2xt}}{x}\right)Y_{\nu-1+a/x}(1/x)-Y_{\nu+a/x}\left(\frac{\sqrt{1-2xt}}{x}\right)J_{\nu-1+a/x}(1/x)\right\}.$$

We now derive a generating function for the polynomials $h_{n,\nu}(x;q)$ defined in (1.20). Let

$$G(x,t)=\sum_{n=0}^{\infty}h_{n,\nu}(x;q)\,t^n.$$

If we multiply the three term recurrence relation (1.22) by t^{n+1} and add the resulting formulas for $n = 1, 2, \ldots$ we will see that the generating function $G(x, t)$ will satisfy the first order q difference equation $(1 - 2xt)\, G(x, t) = 1 - 2xtq^{\nu}(1 + \frac{1}{2}t/x)\, G(x, qt)$, which can be iterated to give, since $G(x, 0) = 1$,

$$\sum_{n=0}^{\infty} h_{n,\nu}(x; q)\, t^n = \sum_{l=0}^{\infty} \frac{(-2xtq^{\nu})^l\, (-\frac{1}{2}t/x; q)_l}{(2xt; q)_{l+1}}\, q^{l(l-1)/2}. \tag{3.3}$$

The generating function (3.3) has two important consequences. The first is that it enables us to obtain an explicit formula for $h_{n,\nu}(x; q)$. The second is that we can apply Darboux' method to it and determine the asymptotic behaviour of $h_{n,\nu}(x; q)$ for large n and fixed x in the complex plane. Eventually we shall use these asymptotic estimates to obtain the measure which these polynomials are orthogonal with respect to. We now derive an explicit formula for $h_{n,\nu}(x; q)$. Using the Gauss binomial theorem (Szegö [25, p. 33])

$$(-u; q)_l = \sum_{j=0}^{l} \frac{(q; q)_l\, u^j}{(q; q)_j\, (q; q)_{l-j}}\, q^{j(j-1)/2}, \tag{3.4}$$

and Heine binomial theorem (Slater [23, p. 248])

$$\frac{1}{(u; q)_{l+1}} = \frac{(uq^{l+1}; q)_{\infty}}{(u; q)_{\infty}} = \sum_{k=0}^{\infty} \frac{(q^{l+1}; q)_k}{(q; q)_k}\, u^k, \tag{3.5}$$

we obtain upon equating like powers of t in (3.3)

$$\begin{aligned} h_{n,\nu}(x; q) &= \sum_{j+k+l=n} \frac{(-q^{\nu})^l\, q^{l(l-1)/2 + j(j-1)/2} (q; q)_{l+k}}{(q; q)_j\, (q; q)_{l-j}\, (q; q)_k}\, (2x)^{l+k-j} \\ &= \sum_{j=0}^{[n/2]} \frac{(2x)^{n-2j}}{(q; q)_j}\, q^{j(j-1)/2} (q; q)_{n-j} \\ &\quad \times \sum_{l=j}^{n-j} \frac{(q^{\nu})^l\, (q^{j-n}; q)_l}{(q; q)_{l-j}\, (q; q)_{n-j}}\, q^{l(n-j)} \end{aligned}$$

in view of [23, p. 241],

$$(a; q)_{s-l} = \frac{(a; q)_s\, q^{l(l+1)/2}\, q^{-sl}}{(q^{1-s}/a; q)_l\, (-a)^l}.$$

With help of (3.5) and some straightforward manipulations we are lead to

$$h_{n,\nu}(x; q) = \sum_{j=0}^{[n/2]} \frac{(2x)^{n-2j}\, (q^{j-n}; q)_j\, (q^{\nu+j}; q)_{\infty}}{(q; q)_j\, (q^{n-j+\nu}; q)_{\infty}}\, q^{j(2\nu+2n-j-1)/2}.$$

Observe that

$$(q^{j-n}; q)_j = (-1)^j q^{j(3j-2n-1)/2}(q; q)_{n-j}/(q; q)_{n-2j}.$$

Thus

$$h_{n,\nu}(x; q) = \sum_{j=0}^{[n/2]} \frac{(2x)^{n-2j}(-1)^j (q^\nu; q)_{n-j} (q; q)_{n-j}}{(q; q)_j (q^\nu; q)_j (q; q)_{n-2j}} q^{j(j+\nu-1)}. \tag{3.6}$$

The explicit formula (3.6) is the exact analogue of (3), p. 296 in Watson [26]. As a consequence of (3.6) one can guess that for fixed x,

$$(2x)^{-n} h_{n,\nu}(x; q) \sim (q^\nu; q)_\infty \sum_{j=0}^{\infty} \frac{(1/2x)^{2j}(-1)^j q^{j(j+\nu-1)}}{(q; q)_j (q^\nu; q)_j}, \qquad \text{as} \quad n \to \infty. \tag{3.7}$$

The derivation of above asymptotic formula from (3.6) can be justified in a way similar to the argument Watson [26, p. 302] used to derive the Hurwitz formula (1.23), that is,

$$(\tfrac{1}{2}z)^{\nu+n} R_{n,\nu+1}(z)/\Gamma(\nu+n+1) \sim J_\nu(z). \tag{3.8}$$

Another way of deriving (3.7) is to apply Darboux' method, see Olver [20, pp. 309, 310] to the generating function (3.3) and the "comparison" function

$$(1-2xt)^{-1} \sum_{l=0}^{\infty} \frac{(-q^\nu)^l (-1/4x^2; q)_l}{(q; q)_l} q^{l(l-1)/2}.$$

Incidently, Hurwitz' formula (3.8) follows from (3.1) also by Darboux' method. Let us rewrite (3.7) in terms of the q-Bessel function $J_\nu^{(2)}(x; q)$. Clearly

$$\frac{(2x)^{-n-\nu+1}}{(q; q)_\infty} h_{n,\nu}(x; q) \sim J_{\nu-1}^{(2)}(1/x; q), \qquad \text{as} \quad n \to \infty, \tag{3.9}$$

and its equivalent

$$\frac{(x/2)^{n+\nu-1}}{(q; q)_\infty} R_{n,\nu}(x; q) \sim J_{\nu-1}^{(2)}(x; q), \qquad \text{as} \quad n \to \infty, \tag{3.10}$$

hold.

We now consider the polynomials $\{\tau_n(x)\}$. Recall that

$$\tau_n(x; \nu, a) = R_{n,\nu+a/x}(1/x). \tag{3.11}$$

The explicit formula for $R_{n,\nu}(x)$ (Watson [26, p. 296]) yields

$$\tau_n(x; \nu, a) = \sum_{j=0}^{[n/2]} \frac{(-1)^j (2x)^{n-2j} (\nu + a/x)_{n-j} (n-j)!}{j! (\nu + a/x)_j (n-2j)!}. \tag{3.12}$$

The quotient $(\nu + a/x)_{n-j}/(\nu + a/x)_j$ is $(\nu + j + a/x)_{n-2j}$, hence the jth term in the right hand side of (3.12) is a polynomial of degree $n - 2j$. The following asymptotic behaviour of $\tau_n(x; \nu, a)$ follows from Hurwitz' formula. Indeed

$$(2x)^{1-n-\nu-a/x} \tau_n(x; \nu, a)/\Gamma(n + \nu + a/x) \sim J_{\nu-1+a/x}(1/x). \tag{3.13}$$

4. Orthogonality Relations and the Zeros of $J_\nu^{(2)}(x; q)$ and $J_{\nu+ax}(x)$

Favard [9] proved that a sequence of polynomials $\{P_n(x)\}$ is orthogonal if and only if it satisfies a three term recursion

$$\begin{gathered} P_{n+1}(x) = (A_n x + B_n) P_n(x) - C_n P_{n-1}(x), \qquad n = 1, 2, \ldots, \\ P_0(x) = 1, \qquad P_1(x) = A_0 x + B_0, \end{gathered} \tag{4.1}$$

with

$$A_n A_{n-1} C_n > 0, \qquad n = 1, 2, \ldots. \tag{4.2}$$

A theorem of Krein, see [1, p. 236], asserts that if $\{P_n(x)\}$ is orthogonal and

$$\int_{-\infty}^{\infty} P_n(x) P_m(x)\, d\mu(x) = \lambda_n \delta_{m,n}, \tag{4.3}$$

$$a_n = \lambda_n^{-1} \int_{-\infty}^{\infty} x P_m^2(x)\, d\mu(x), \qquad b_n = (\lambda_n \lambda_{n+1})^{-1/2} \int_{-\infty}^{\infty} x P_n(x) P_{n+1}(x)\, d\mu(x), \tag{4.4}$$

then the support of $d\mu$ will be bounded if $a_n \to 0$ and $b_n \to 0$. Krein's theorem further asserts that the measure will be purely discrete and the only cluster point of its support will be at $x = 0$. It is not difficult to see that

$$\lambda_n = (A_0 \lambda_0 / A_n)\, C_1 C_2 \cdots C_n. \tag{4.5}$$

$$a_n = -B_n/A_n, \qquad b_n = (\lambda_{n+1}/\lambda_n)^{1/2}/A_n = \{C_{n+1}/(A_n A_{n+1})\}^{1/2}. \tag{4.6}$$

Krein's theorem gives a sufficient condition for the boundedness of the support of $d\mu$. A. Markoff proved that if $d\mu$ has bounded support $\int_{-\infty}^{\infty} d\mu = 1$, and $P_n(x)$ satisfies (4.1), then

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{z-t} = \lim_{n\to\infty} \{P_n^*(z)/P_n(z)\}, \qquad z \notin \sup\{d\mu\},$$

where $P_n^*(z)$ satisfies the recurrence relation (4.1) and the initial conditions

$$P_0^*(x) = 0, \qquad P_1^*(x) = A_0. \tag{4.7}$$

A proof of this theorem is in Szegö [25] and in Chihara [6]. Pollaczek's work [21] also contains another proof. A measure $d\mu$ can be recoverd from it's Stieltjes transform by applying the Perron–Stieltjes inversion formula, see Stone [24],

$$\begin{aligned} F(z) &= \int_{-\infty}^{\infty} \frac{d\mu(t)}{z-t} \text{ implies} \\ \mu(t_2) - \mu(t_1) &= \lim_{\epsilon\to 0^+} (2\pi i)^{-1} \int_{t_1}^{t_2} \{F(t-i\varepsilon) - F(t+i\varepsilon)\}\, dt. \end{aligned} \tag{4.8}$$

Now consider the polynomials $\{h_{n,\nu}(x;q)\}$. In this case A_n is $2(1-q^{\nu+n})$, $B_n = 0$ and $C_n = q^{\nu+n-1}$. Substituting these values in (4.6) we see that the assumptions in Krein's theorem are satisfied, hence Markoff's theorem is applicable. The polynomials of the second kind $h_{n,\nu}^*(x;q)$ are given by

$$h_{n,\nu}^*(x;q) = 2(1-q^\nu)\, h_{n-1,\nu+1}(x;q), \qquad n = 1, 2,\ldots \text{ and } h_{0,2}^*(x;q) = 0. \tag{4.9}$$

Let the corresponding measure be $d\alpha_\nu(x;q)$. Markoff's theorem and the asymptotic formula (3.9) yield

$$\int_{-\infty}^{\infty} \frac{d\alpha_\nu(t;q)}{z-t} = 2(1-q^\nu)\, J_\nu^{(2)}(1/z;q)/J_{\nu-1}^{(2)}(1/z;q), \qquad z \notin \operatorname{supp}\{d\mu\}. \tag{4.10}$$

Formula (4.10) and the inversion formula (4.8) reasure the purely discrete nature of the measure $d\alpha_\nu(x;q)$ because the right member of (4.10) is a single valued function of z in the complex plane. This also shows that $J_{\nu-1}^{(2)}(x;q)$ has infinitely many real zeros because the step function $\alpha_\nu(x;q)$ must have infinitely many points of increase.

THEOREM 4.1. *The basic Bessel functions $J_\nu^{(2)}(z;q)$ and $J_{\nu+1}^{(2)}(z;q)$ have no common zeros, except possibly $z = 0$, when ν is real.*

Proof. We first derive an analogue of (2.3) for $J_\nu^{(2)}(x;q)$. The definition of $J_\nu^{(2)}(x;q)$ implies

$$J_\nu^{(2)}(\sqrt{q}x;q) - q^{\nu/2}J_\nu^{(2)}(x;q)$$

$$= \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} q^{\nu/2} \sum_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}(-1)^n q^{n(\nu+n)}}{(q;q)_n (q^{\nu+1};q)_n}(q^n-1)$$

$$= \frac{(q^{\nu+2};q)_\infty}{(q;q)_\infty} q^{1+3\nu/2} \sum_{n=0}^{\infty} \frac{(x/2)^{\nu+2n+2}(-1)^n q^{n(n+\nu+1)+n}}{(q;q)_n (q^{\nu+2};q)_n},$$

which is nothing but

$$J_\nu^{(2)}(\sqrt{q}x;q) = q^{\nu/2}J_\nu^{(2)}(x;q) + q^{\nu+1/2}(x/2)\,J_{\nu+1}^{(2)}(\sqrt{q}x;q). \tag{4.11}$$

If $z \neq 0$ is a common zero of $J_\nu^{(2)}(x;q)$ and $J_{\nu+1}^{(2)}(x;q)$, then $J_\nu^{(2)}(z/\sqrt{q};q)$ also vanishes, by (4.11), and the q-difference equation (2.1) will then show that $\sqrt{q}z$ is also a zero of $J_\nu^{(2)}(x;q)$. Repeated application of the above argument show that $q^{n/2}z$, $n=0,1,\ldots,$ are zeros of $J_\nu^{(2)}(x;q)$. This contradicts the identity theorem for analytic functions, since $x^{-\nu}J_\nu^{(2)}(x;q)$ is obviously an entire function. This completes the proof of the theorem.

THEOREM 4.2. *All the zeros of $x^{-\nu}J_\nu^{(2)}(x;q)$ are real and simple for $\nu > -1$. There are infinitely many of them and their only cluster point is the point at infinity.*

Proof. Let the step function $\alpha_\nu(t;q)$ have $\{t_k\}_1^\infty$ as the set of points of increase with a jump B_k at $t=t_k$, $k=0,1,\ldots.$ Relationship (4.10) says

$$\sum_{k=0}^{\infty} B_k/(z-t_k) = 2(1-q^\nu)\,J_\nu^{(2)}(1/z;q)/J_{\nu+1}^{(2)}(1/z;q), \qquad z \notin \{t_k\}. \tag{4.12}$$

The zeros of $z^\nu J_{\nu+1}^{(2)}(1/z;q)$ are precisely the poles of $\sum_{k=0}^{\infty} B_k/(z-t_k)$, by Theorem 4.1. Hence these zeros are real and simple because they coincide with $\{t_k\}_1^\infty$. This completes the proof, because the only cluster point of $\{t_k\}_1^\infty$ is the origin, by Krein's theorem.

THEOREM 4.3. *Between any two consecutive positive roots of $x^{-\nu}J_\nu^{(2)}(x;q)$, the function $x^{-\nu-1}J_{\nu+1}^{(2)}(x;q)$ has precisely one zero when $\nu \geqslant -1$.*

Proof. The jump $B_k > 0$, is the residue of the left hand side of (4.10) at $z = t_k$. Therefore

$$0 < B_k = -2t_k^2(1-q^\nu)\,J_\nu^{(2)}(1/t_k;q)/J_{\nu-1}^{(2)\prime}(1/t_k;q),$$

that is $J_\nu^{(2)}(x;q)$ and $J_{\nu-1}^{(2)\prime}(x;q)$ have opposite signs when $x \neq 0$, is a root of $J_{\nu+1}^{(2)}(x;q)$. Clearly $J_{\nu-1}^{(2)\prime}(x;q)\, J_{\nu-1}^{(2)\prime}(y;q) < 0$ if x and y $(x < y)$ are two consecutive non-zero roots of $J_{\nu+1}^{(2)}(x;q)$. Thus $J_\nu^{(2)}(x;q)\, J_\nu^{(2)}(y;q) < 0$, which implies that $J_\nu^{(2)}(z;q)$ has a root in (x, y). There is only one such root, however, because both sides in (4.12) are differentiable between the points $\{t_k\}$ and their derivative is $-\sum_{k=0}^{\infty} B_k/(z-t_k)^2$, so is never zero. This completes the proof.

The function $x^{-\nu}J_\nu^{(2)}(x;q)$ is an even function. Let

$$0 < j_{\nu,1}(q) < j_{\nu,2}(q) < \cdots < j_{\nu,n}(q) < \cdots \tag{4.13}$$

be the infinite sequence of its positive zeros. Assume that the jumps of the step function $\alpha_\nu(x;q)$ satisfy

$$d\alpha_\nu(x;q) = (1-q^\nu)\, A_k(\nu)/j_{\nu-1,k}^2(q) \qquad \text{at} \quad x = 1/j_{\nu-1,k}(q). \tag{4.14}$$

Clearly $\alpha_\nu(x;q)$ must also have an equal jump at $-1/j_{\nu-1,k}(q)$. Observe that (4.12) will then become

$$\sum_{k=0}^{\infty} A_k(\nu+1)\{z/(j_{\nu,k}^2(s) - z^2)\} = J_{\nu+1}^{(2)}(z;q)/J_\nu^{(2)}(z;q). \tag{4.15}$$

Formula (4.15) is an analogue of the Mittag-Leffler expansion for Bessel functions, see, e.g., (3), p. 61 in Erdelyi *et al.* [8]. Clearly (4.15) implies the complete monotonicity of the function $-ix^{-1/2}J_{\nu+1}^{(2)}(ix^{1/2};q)/J_\nu^{(2)}(ix^{1/2};q)$. Since the polynomials $\{h_{n,\nu}(x;q)\}$ are symmetric, that is, $h_{n,\nu}(-x) = (-1)^n h_{n,\nu}(x)$, then

$$\begin{aligned}(1+(-1)^{m+n}) \sum_{k=0}^{\infty} A_k(\nu)\, j_{\nu-1,k}^{-2}(q)\, h_{n,\nu}(j_{\nu-1,k}^{-1}(q);q)\, h_{m,\nu}(j_{\nu-1,k}^{-1}(q);q)\\ = q^{\nu n + n(n-1)/2}\delta_{m,n}/(1-q^{\nu+n}).\end{aligned} \tag{4.16}$$

As $q \to 1$, $h_{n,\nu}(x/(1-q);q) \to h_{n,\nu}(x)$, as can be seen from (1.6), (1.7) and (1.22). Letting $q \to 1$ in (4.16) we recover the orthogonality relation for the modified Lommel polynomials

$$(1+(-1)^{m+n}) \sum_{k=0}^{\infty} h_{n,\nu}(j_{\nu-1,k}^{-1})\, h_{m,\nu}(j_{\nu-1,k}^{-1} = \delta_{m,n}/\{2(\nu+n)\}. \tag{4.17}$$

Equality (4.17) can also be proved via Krein's theorem, Markoff's theorem by using Hurwitz' formula (1.23). The orthogonality relation (4.17) appears in Dickinson [7] with a slight error in it.

We now conisder the polynomials $\tau_n(x; \nu, a)$. Here A_n, B_n and C_n are $2(n+\nu)$, $2a$ and 1, respectively. From (4.6) we see that the assumptions of Markoff's theorem are satisfied. We now repeat the argument used in the case of $\{h_{n,\nu}x; q)\}$ and prove the following theorem.

THEOREM 4.4. *All the zeros of $x^{-\nu-ax}J_{\nu+ax}(x)$ are real and simple when $\nu > -1$ and a is real. There are infinitely many of them and their only cluster point is the point at infinity. Between any two consecutive zeros of $x^{-\nu-ax}J_{\nu+ax}(x)$, the function $x^{-\nu-ax-1}J_{\nu+ax-1}(x)$ has only one zero.*

Let $\{\eta_j\}$ be the sequence of zeros of $x^{-\nu-ax-1}J_{\nu+ax-1}(x)$.

$$\gamma_j = 2\nu \frac{d}{dx} J_{\nu+a/x}(1/x)/J_{\nu-1+ax}(1/x)\,|_{x=1/\eta_j}. \tag{4.18}$$

The orthogonality relation for $\{\tau_n(x; \nu, a)\}$ is

$$\sum_{j=0}^{\infty} \gamma_j \tau_m(\eta_j^{-1}; \nu, a)\, \tau_m(\eta_j^{-1}; \nu, a) = \delta_{m,n}/\{2(\nu+n)\}. \tag{4.19}$$

We conclude by mentioning a basic analogue of $\tau_n(x; \nu, a)$ and of the functions $J_{\nu+ax}(x)$. Define the polynomials $\{\tau_{n,q}(x; \nu, a)\}$ or simply $\{\tau_{n,q}(x)\}$ by

$$\tau_{0,q}(x; \nu, a) = 1, \qquad \tau_{1,q}(x; \nu, a) = 2x(1-q^{\nu}) + 2aq^{\nu} \tag{4.20}$$
$$\tau_{n+1,q}(x; \nu, a) = 2[x(1-q^{n+\nu}) + aq^{n+\nu}]\, \tau_{n,q}(x; \nu, a) - q^{n+\nu-1}\tau_{n-1,q}(x; \nu, a).$$

A generating function is

$$\sum_{n=0}^{\infty} \tau_{n,q}(x; \nu, a)\, t^n = \sum_{k=0}^{\infty} \frac{[2tq^{\nu}(a-x)]^k\, q^{k(k-1)/2}}{(2xt; q)_{k+1}} \left(\frac{t}{2(a-x)}; q\right)_k. \tag{4.21}$$

Darboux' method is applicable to (4.21) and leads to

$$(2x)^{-n}\, \tau_{n,q}(x; \nu, a) \sim \sum_{k=0}^{\infty} \frac{[(ax)\, q^{\nu}/x]^k}{(q; q)_k}\, q^{k(k-1)/2} \left(\frac{1}{4x(a-x)}; q\right)_k,$$

which, in view of Gauss' binomial theorem (3.4) can be expressed as

$$(2x)^{-n}\, \tau_{n,q}(x; \nu, a) \sim (q^{\nu}(1-a(x); q)_{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j\, (x/2)^{2j}\, q^{j(\nu-1+j)}}{(q; q)_j\, (q^{\nu}(1-a/x); q)_j}. \tag{4.22}$$

Let us introduce the notation

$$\mathscr{F}(x, \nu, a/q) = \frac{(q^{\nu+1}(1-ax); q)_{\infty}}{(q; q)_{\infty}} \sum_{j=0}^{\infty} \frac{(-1)^j\, (x/2)^{\nu+2j}\, q^{j(\nu+j)}}{(q; q)_j\, (q^{\nu+1}(1-ax); q)_j}. \tag{4.23}$$

The polynomials of the second kind are

$$\tau^*_{n,q}(x; \nu, a) = 2(1 - q^\nu)\, \tau_{n-1,q}(x; \nu + 1, a). \tag{4.24}$$

The theorems of Favard, Krein and Markoff are applicable and if we denote the corresponding measure by $dA(t)$, normalize by $\int_{-\infty}^{\infty} dA(t) = 1$, then from (4.22), (4.23) and (4.24) we get

$$\int_{-\infty}^{\infty} \frac{dA(t)}{z - t} = 2(1 - q^\nu)\mathscr{F}(1/x, \nu, a/q)/\mathscr{F}(1/x, \nu - 1, a/q). \tag{4.25}$$

This proves that all the non-zero roots of $\mathscr{F}(x; \nu, a/q)$ are real and simple and also establishes the interlacing property of these roots for different functions. The orthogonality relation can be similarly found. An explicit formula for $\tau_{n,q}$ can be derived from the generating function (4.21) in the same way the explicit formula (3.6) for $h_{n,\nu}(x; q)$ was derived from the generating function (3.3). The result is

$$\tau_{n,q}(x; \nu, a) = \sum_{j=0}^{[n/2]} \frac{(2x)^{n-2j}\,(-1)_j\,(q^\nu(1 - a/x); q)_{n-j}\,(q; q)_{n-j}}{(q; q)_j\,(q^\nu(1 - a/x); q)_j\,(q; q)_{n-2j}}\, q^{j(j+\nu-1)}. \tag{4.26}$$

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Note Added in Proof

In order to show that $x = 0$ is not a mass point for $h_{n,\nu}(x; q)$ one needs to show that $\sum_0^\infty (\tilde{h}_{n,\nu}(0; q))^2$ diverges, $\tilde{h}_{n,\nu}$ being the orthonormal polynomials. This is easy to do since $\tilde{h}_{n,\nu}(x; q) = h_{n,\nu}(x; q)(1 - q^{\nu+n})^{1/2}\, q^{-n\nu - n(n-1)/2}$ and $h_{n,\nu}(0; q)$ can be computed from the explicit formula (3.6).

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