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Journal of Computational and Applied Mathematics 79 (1997) 277–287

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Comparison of two methods for the computation of singular solutions in elliptic problems

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Received 7 January 1996

Abstract

We compare two numerical methods for the solution of elliptic problems with boundary singularities. The first is the integrated singular basis function method (ISBFM), a finite-element method in which the solution is approximated by standard polynomial basis functions supplemented by the leading terms of the local (singular) solution expansion. A double application of Green's theorem reduces all Galerkin integrals containing singular terms to boundary integrals with nonsingular integrands. The originally essential boundary conditions are weakly enforced by means of Lagrange multipliers. The second method is a singular function boundary integral method which can be viewed as a modification of the ISBFM. The solution is approximated only by the leading terms of the local solution expansion. The discretized equations are boundary integrals and the dimension of the problem is reduced by one. The two methods are applied to the cracked-beam problem giving very accurate estimates of the leading singular coefficients. Comparisons are made and their limitations are discussed.

Keywords: Elliptic problems; Singularities; Convergence

AMS classification: 65N12; 65N30

1. Introduction

We investigate the performance of two special numerical methods in the solution of elliptic problems with singularities. We limit ourselves to problems with a boundary singularity caused by a sudden change of the boundary conditions. Generally, the form of the singularity is easily obtained

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PII S0377-0427(96)00173-2

using separation of variables. For the two-dimensional Laplace equation, the asymptotic solution in polar coordinates (r,θ), centered at the singular point, is given by

$$u(r,\theta)=\sum_{j=1}^{\infty}\alpha_j W^j(r,\theta)=\sum_{j=1}^{\infty}\alpha_j r^{\mu_j} f_j(\theta), \quad (r,\theta)\in V, \tag{1}$$

where V is a simply connected domain, u is the dependent variable, α_j are the unknown singular coefficients, μ_j are the singularity powers arranged in ascending order, and the functions $f_j(\theta)$ represent the θ dependence of the eigensolution. The values of μ_j and the form of $f_j(\theta)$ are determined by the boundary conditions along the parts of the boundary that cause the singularity. The functions W^j, referred to here as singular functions, satisfy the governing equation in the domain and the boundary conditions along the parts of the boundary that cause the singularity. The singular coefficients α_j depend on the global problem and are often desirable in many applications. As an example, in fracture mechanics, the first coefficient is the stress intensity factor, a measure of the stress at which fracture occurs [2].

It is well known that standard numerical methods like finite-element, boundary-element, finite-difference, and spectral methods perform poorly in the neighborhood of singular points. Grid refinement is a common practice aiming to improve the accuracy and the convergence rate. Special adaptive grid refinement schemes for the finite element method are reviewed in [7, 1]. Despite their computational cost, their efficiency is not always satisfactory.

The incorporation of the form of the singularity in the numerical scheme is generally more effective than grid refinement and leads to faster convergence and more accurate solutions. Special numerical methods for the solution of singular Laplacian problems include finite-difference, global-element, boundary-element, finite-element and other methods [8, 4].

In two previous papers [8, 4], we developed two singular methods for the solution of Laplacian problems with boundary singularities which provide accurate estimates of the leading singular coefficients. The first method is the integrated singular basis function method (ISBFM), a special finite-element method in which the solution is approximated by two sets of trial functions [8, 3]. The first set consists of standard biquadratic functions Φ^j approximating the 'regular' part u^{r} of the solution and the second one consists of the leading N_α singular functions W^j approximating the 'singular' part u^{s} of the solution:

$$\bar{u}=\bar{u}^{\mathrm{r}}+\bar{u}^{\mathrm{s}}=\sum_{j=1}^{N_u} u_j\Phi^j+\sum_{j=1}^{N_\alpha}\bar{\alpha}_j W^j, \tag{2}$$

where N_u is the number of nodes, u_j are the nodal values of the regular part of the solution, N_α is the number of singular functions, and $\bar{\alpha}_j$ are the approximations of the leading singular coefficients. Applying Galerkin's method and Green's theorem leads to two sets of discretized equations. A second application of Green's theorem reduces all the integrals of the second equation set to boundary integrals with nonsingular integrands. ISBFM thus avoids the need for high-order integration in the neighborhood of the singularity and improves the overall accuracy. The originally essential boundary conditions are weakly enforced by means of Lagrange multipliers. ISBFM accelerates the convergence with regular mesh refinement and converges rapidly with the number of singular functions N_α [8].

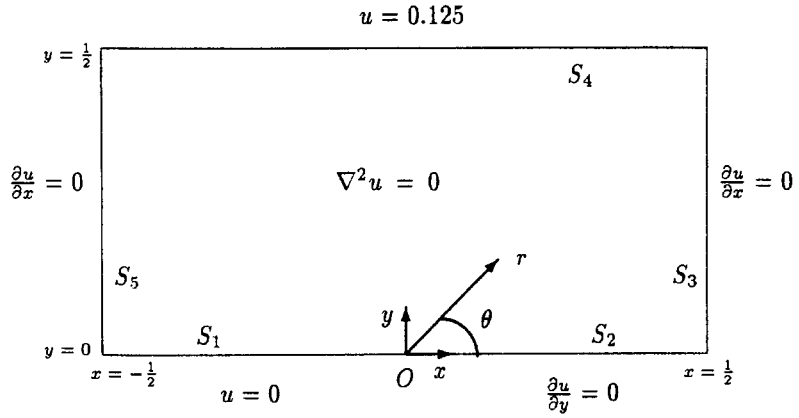

Fig. 1. The cracked-beam problem.

The second method is the singular function boundary integral method which can be viewed as a modification of the ISBFM. The solution is approximated only by the leading terms of the singularity expansion

$$\bar{u}=\sum_{j=1}^{N_a}\bar{\alpha}_j W^j. \tag{3}$$

With the double application of Green's theorem, all the discretized equations are reduced to boundary integrals. As with the ISBFM, Lagrange multipliers are used to apply the essential boundary conditions. With the singular function boundary integral method, the dimension of the problem is reduced by one and, consequently, the computational cost is considerably lower. Moreover, the convergence of the solution with the number of singular functions is exponential [5, 4].

In [4], we solved a benchmark Laplacian problem known as the Motz problem. The numerical calculations show that the singular function boundary integral method performs better than the ISBFM for this particular problem. This, however, may not be the case in all applications. As pointed out in [3], the stiffness matrix is ill-conditioned when N_α is less than the number of the Lagrange multipliers. On the other hand, stability deteriorates when N_α gets high. Therefore, the method may not perform well when a high number of Lagrange multipliers is necessary, i.e. when we have essential boundary conditions in large parts of the boundary. The objective of the present paper is to compare the performances of the two methods when solving such a problem, the cracked-beam problem.

The geometry, the governing equations and the boundary conditions for the cracked-beam problem are shown in Fig. 1. In the original problem, $\nabla^2 v=-1$ and $v=0$ on $y=\frac{1}{2}$. The transformation $u=v+\frac{1}{2}y^2$ leads to the problem considered here. A singularity arises at $x=y=0$, where the boundary condition suddenly changes from $u=0$ to $\partial u/\partial y=0$. The local solution is given by

$$u=\sum_{j=1}^{\infty}\alpha_j r^{(2j-1)/2}\cos\left[\left(\frac{2j-1}{2}\right)\theta\right]. \tag{4}$$

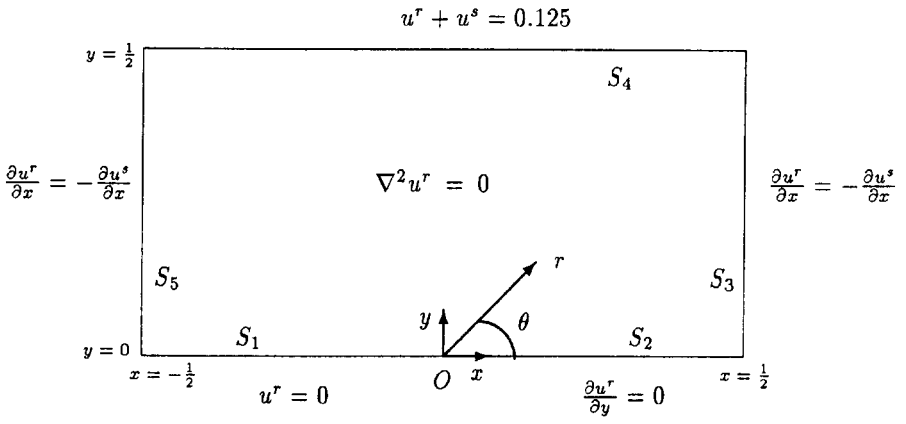

Fig. 2. The modified cracked-beam problem.

The radius of convergence of the above expansion is at least as large as 1 [9, 6], and therefore it is valid in the entire solution domain.

Estimates of the leading singular coefficients have been obtained by Wigley [10], who used an inherently iterative finite difference method and by Olson et al. [8], who used the ISBFM. The results for the leading singular coefficients in these two papers include only the first six decimal digits. Here, we would like to more systematically study the accuracy achieved by the ISBFM and the singular function boundary integral method, as far as the values of the singular coefficients are concerned. In Section 2 we give brief formulations of the two methods and in Section 3 we present the results. The conclusions are summarized in Section 4.

2. The numerical methods

2.1. The integrated singular basis function method (ISBFM)

Since the singular part u^s of the solution satisfies the governing equation, the original problem of Fig. 1 is transformed to that shown in Fig. 2. Using the Galerkin method and Green's theorem, we obtain the following discretized equations:

$$\int_S \frac{\partial \bar{u}^r}{\partial n} \Phi^i \, dS - \int_V \nabla \bar{u}^r \nabla \Phi^i \, dV = 0, \quad i = 1, 2, \ldots, N_u, \tag{5}$$

$$\int_S \frac{\partial \bar{u}^r}{\partial n} W^i \, dS - \int_V \nabla \bar{u}^r \nabla W^i \, dV = 0, \quad i = 1, 2, \ldots, N_a. \tag{6}$$

The singular volume integrals of Eq. (6) are reduced to boundary integrals after applying once more Green's theorem:

$$\int_S \left(\frac{\partial \bar{u}^r}{\partial n} W^i - \bar{u}^r \frac{\partial W^i}{\partial n} \right) dS = 0, \quad i = 1, 2, \ldots, N_a. \tag{7}$$

To impose the originally essential boundary condition on S_4, we employ Lagrange multipliers λ^j expanded in terms of quadratic basis functions M^j:

$$\lambda = \frac{\partial \bar{u}^r}{\partial y} = \sum_{j=1}^{N_\lambda} \lambda^j M^j, \tag{8}$$

where N_λ is the number of quadratic nodes on boundary S_4.

Notice that all the boundary integrals on S_1 and S_2 are ignored because for u^r we have essential or natural boundary conditions whereas u^s satisfies the boundary conditions identically. We thus obtain the following system of $N_u+N_\alpha+N_\lambda$ linear equations:

$$-\int_{S_3} \frac{\partial \bar{u}^s}{\partial x} \Phi^i \, dy + \int_{S_4} \lambda \Phi^i \, dx + \int_{S_5} \frac{\partial \bar{u}^s}{\partial x} \Phi^i \, dy + \int_V \nabla \bar{u}^r \nabla \Phi^i \, dV = 0, \quad i = 1, 2, \ldots, N_u, \tag{9}$$

$$-\int_{S_3} \left(\frac{\partial \bar{u}^s}{\partial x} W^i + \bar{u}^r \frac{\partial W^i}{\partial x} \right) dy + \int_{S_4} \left(\lambda W^i + \bar{u}^s \frac{\partial W^i}{\partial y} \right) dx$$

$$+\int_{S_5} \left(\frac{\partial \bar{u}^s}{\partial x} W^i + \bar{u}^r \frac{\partial W^i}{\partial x} \right) dy = 0.125 \int_{S_4} \frac{\partial W^i}{\partial y} \, dx, \quad i = 1, 2, \ldots, N_\alpha, \tag{10}$$

$$\int_{S_4} (\bar{u}^r + \bar{u}^s) M^i \, dx = 0.125 \int_{S_4} M^i \, dx, \quad i = 1, 2, \ldots, N_\lambda. \tag{11}$$

If we denote Eqs. (9)–(11) by $\boldsymbol{X}_1$–$\boldsymbol{X}_3$ and use the symbols $\boldsymbol{U}$, $\boldsymbol{A}$ and $\boldsymbol{\Lambda}$ for the vectors of the three sets of unknowns, we can write the above system of equations in the following block form:

$$\begin{bmatrix} \dfrac{\partial \boldsymbol{X}_1}{\partial \boldsymbol{U}} & \dfrac{\partial \boldsymbol{X}_1}{\partial \boldsymbol{A}} & \dfrac{\partial \boldsymbol{X}_1}{\partial \boldsymbol{\Lambda}} \\ \dfrac{\partial \boldsymbol{X}_2}{\partial \boldsymbol{U}} & \dfrac{\partial \boldsymbol{X}_2}{\partial \boldsymbol{A}} & \dfrac{\partial \boldsymbol{X}_2}{\partial \boldsymbol{\Lambda}} \\ \dfrac{\partial \boldsymbol{X}_3}{\partial \boldsymbol{U}} & \dfrac{\partial \boldsymbol{X}_3}{\partial \boldsymbol{A}} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{A} \\ \boldsymbol{\Lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{B}_2 \\ \boldsymbol{B}_3 \end{bmatrix}. \tag{12}$$

The stiffness matrix is symmetric. We observe that we should have $N_\alpha \geqslant N_\lambda$, if the stiffness matrix is to be nonsingular [Eqs. (11) should not be more than those in (10)]. For the numerical integration, the elements are subdivided into 10 subelements over which a 15-point Gauss–Legendre quadrature is employed. Different tests with lower-order quadratures and/or more element subdivisions showed that the quadrature used is satisfactory for the values of N_α we use here.

2.2. The singular function boundary integral method

The solution is approximated exclusively with singular functions, i.e. $N_u=0$. The final system of discretized equations consists of two equation sets as follows:

$$-\int_{S_3} \bar{u}\frac{\partial W^i}{\partial x}\,dy+\int_{S_4}\left(\lambda W^i-\bar{u}\frac{\partial W^i}{\partial y}\right)dx+\int_{S_5}\bar{u}\frac{\partial W^i}{\partial x}\,dy=0,\quad i=1,2,\ldots,N_\alpha, \tag{13}$$

$$\int_{S_4}\bar{u}M^i\,dx=0.125\int_{S_4}M^i\,dx,\quad i=1,2,\ldots,N_\lambda. \tag{14}$$

Only boundary integrals are reduced with this method. As with the ISBFM, the stiffness matrix is symmetric and the stiffness matrix is singular if $N_a<N_\lambda$.

3. Numerical results

As mentioned above, if $N_a<N_\lambda$ the stiffness matrix is singular, and as N_α increases a stronger coupling of Eqs. (13) and (14) is achieved resulting in a better conditioning of the stiffness matrix. On the other hand, the higher the order of a singular function is, the lower its values are along the boundary ($r<1$). Above a critical value of N_α the contributions of the singular functions are negligible and the conditioning of the stiffness matrix worsens again. The smoothness of computed values of the Lagrange multipliers provide a good measure of the quality of the solution in the case of the singular function boundary integral method. The presence of the finite elements in the ISBFM has a stabilizing effect and no oscillations of λ are observed.

We performed two series of runs to find the optimal values of N_α and N_λ. First we kept N_λ constant and varied N_α from 1 up to 100. In the singular function boundary integral method, λ is characterized by oscillations at all values of N_α when $N_\lambda\geqslant 29$. For $N_\lambda=25$, the calculated λ is smooth for $40\leqslant N_\alpha\leqslant 65$. For smaller values of N_λ, λ is still smooth but its approximation is, of course, less satisfactory because the boundary is less refined. Thus, the optimal value of N_λ is 25. In Fig. 3, we plot the values of λ obtained with $N_\alpha=65$ and 70. For the latter value we observe oscillations which increase in size when we further increase N_α. Similar oscillations are observed when N_α is less than 40.

In Tables 1 and 2, we show the effect of N_λ on the calculated values of various coefficients obtained using the two methods with $N_\alpha=50$. One notices that the values of the singular coefficients converge rapidly with N_λ and that highly accurate estimates are obtained at least for the leading coefficients. This is shown in Fig. 4 where we plot the absolute errors for α_1, α_3 and α_6 (calculated with the singular function boundary integral method) as functions of N_λ. Note that the convergence of the solution is exponential only at low values of N_λ. As N_λ gets higher approaching N_α, we observe a plateau and then when $N_\lambda>N_\alpha$ the error increases as expected (the conditioning of the stiffness matrix worsens). The results with the ISBFM are similar but we can make two remarks: (a) for low values of $N_\lambda(\leqslant 13)$, the error is greater than that of the singular function boundary integral method because the finite element mesh is coarse, and (b) for higher values of N_λ the error is smaller, with the exception of the second and third coefficients (compare, for example, the values of α_{12} in Tables 1 and 2).

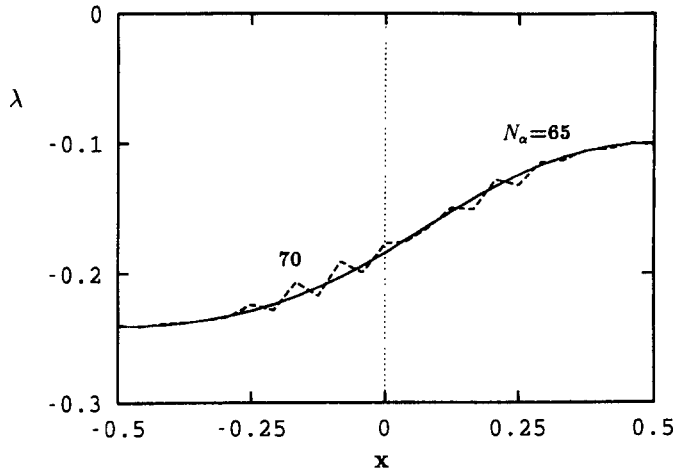

Fig. 3. Calculated Lagrange multipliers for N_α=65 and 70; N_λ=25, singular function boundary integral method.

Table 1
Convergence of the solution with N_λ; N_α=50, singular function boundary integral method

N_λ	α_1	α_3	α_6	α_9	α_{12}
5	0.191117491954	0.000024706282	−0.019626056463	−0.0015038821	0.033398790
9	0.191118628314	0.000000173464	−0.019037795763	−0.0009575540	0.004468826
13	0.191118631766	0.000000001149	−0.019033255541	−0.0006639323	0.000153706
17	0.191118631935	0.000000000115	−0.019033394588	−0.0006542267	0.000002858
21	0.191118631964	0.000000000098	−0.019033405109	−0.0006541222	−0.000000038
25	0.191118631972	0.000000000000	−0.019033403707	−0.0006541249	−0.000000001
33	0.191118631972	0.000000000000	−0.019033403708	−0.0006541248	−0.000000000
41	0.191118631972	0.000000000000	−0.019033403709	−0.0006541251	−0.000000021
49	0.191118631972	0.000000000000	−0.019033403708	−0.0006541248	−0.000000002

Table 2
Convergence of the solution with N_λ; N_α=50, ISBFM

N_λ	α_1	α_3	α_6	α_9	α_{12}
5	0.191129255940	0.000933423906	−0.034142473472	−0.195601823031	−0.2952783399
9	0.191118621246	0.000000978167	−0.019089612812	−0.003160009962	0.0371428429
13	0.191118630341	−0.000000024677	−0.019033351630	−0.000667732647	−0.0000739627
17	0.191118631965	0.000000000553	−0.019033406594	−0.000654114280	0.0000008072
21	0.191118631972	−0.000000000007	−0.019033403707	−0.000654124854	−0.0000000000
25	0.191118631972	−0.000000000024	−0.019033403708	−0.000654124845	0.0000000000
33	0.191118631972	0.000000000033	−0.019033403706	−0.000654124843	−0.0000000000
41	0.191118631972	0.000000000187	−0.019033403696	−0.000654124853	0.0000000000
49	0.191118631972	0.000000000142	−0.019033403719	−0.000654124847	−0.0000000000

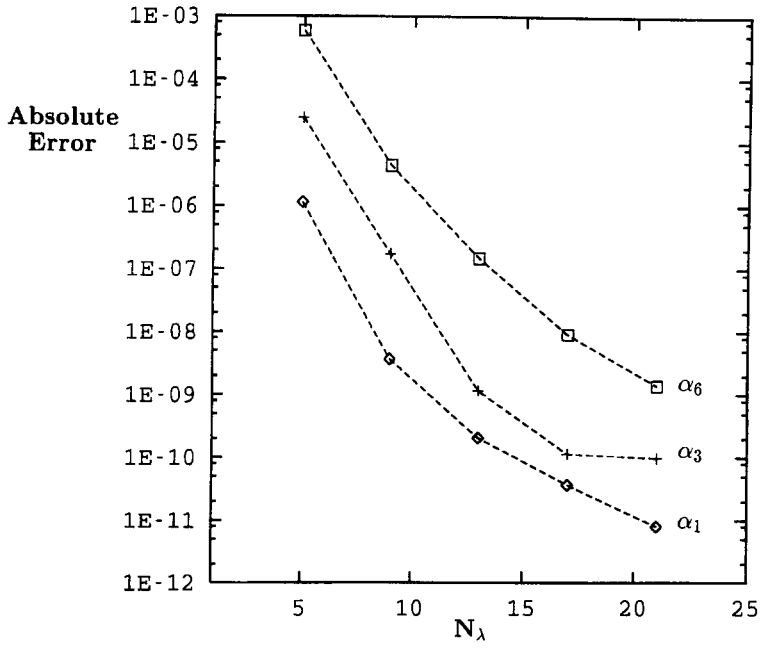

Fig. 4. Absolute errors as functions of N_λ; N_α=50, singular function boundary integral method.

Table 3
Convergence of the solution with N_α; N_λ=25, singular function boundary integral method

N_α	α_1	α_3	α_6	α_9	α_{12}
20	0.190822980843	−0.001872203606	0.011344344289	−0.1291739426	0.272657777
25	0.191118635827	−0.000006447420	−0.018538495371	−0.0086140952	0.044102680
30	0.191118629150	0.000000359486	−0.019078170017	0.0012651979	−0.023998924
35	0.191118631972	−0.000000000011	−0.019033409558	−0.0006543995	−0.000012267
40	0.191118631972	0.000000000000	−0.019033403708	−0.0006541249	0.000000001
45	0.191118631972	0.000000000000	−0.019033403708	−0.0006541248	−0.000000000
50	0.191118631972	0.000000000000	−0.019033403707	−0.0006541249	−0.000000001
55	0.191118631972	0.000000000003	−0.019033403667	−0.0006541253	0.000000022
60	0.191118631962	0.000000000136	−0.019033405761	−0.0006541177	−0.000000110

In Table 3, we show the effect of N_α on the values of some singular coefficients calculated with the singular function boundary integral method with N_λ=25. For $N_a > 60$ the stability of the solution appears to start deteriorating (the high accuracy of the leading coefficients is conserved but some oscillations appear in the last digits of the high order coefficients). A similar loss of stability is observed with the boundary method of Li et al. [5]. The optimal value of N_α is 50. Moreover, for the higher-order coefficients we observe oscillations which allow the exact determination of only few significant digits. The exponential convergence of the method is illustrated in Fig. 5 where we plot the absolute errors for α_1, α_3 and α_6 as functions of N_α ($N_\lambda = 25$). The results with the ISBFM are similar.

To compare the accuracy achieved with the two methods, it is more illustrative to take a look on Table 4 where we list the values of the singular coefficients calculated for the optimal choices of N_α

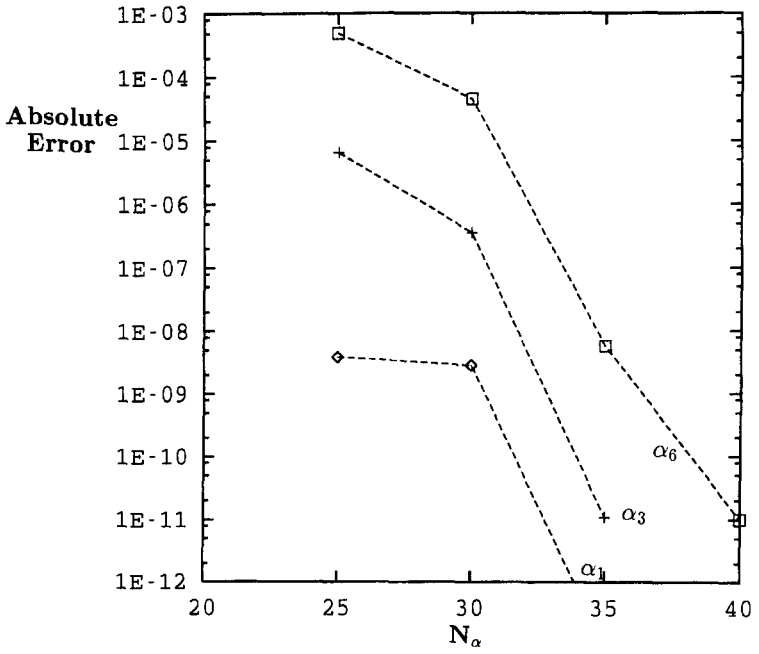

Fig. 5. Absolute errors as functions of N_α; N_λ=25, singular function boundary integral method.

and N_λ. For the cracked-beam problem, the ISBFM yields more accurate values with the exception of the second and the third singular coefficient. This is due to the fact that the corresponding contributions of the local solution expansion are contaminated by the biquadratic basis functions.

4. Conclusions

We have used the integrated singular basis function method (ISBFM) and the singular function boundary integral method for solving the cracked-beam problem. In the former method the solution is approximated by standard polynomial functions supplemented by the leading terms of the local solution expansion whereas the latter keeps only the leading singular expansion terms. Obviously, the two methods can be used only if the solution expansion is valid everywhere in the problem domain. Otherwise, the two methods can still be applied by subdividing the domain into several subdomains and using different expansions (or methods) in each of them.

The originally essential boundary conditions are applied by means of Lagrange multipliers. The number of the singular functions (N_α) should be greater than the number of the Lagrange multipliers (N_λ) because otherwise the stiffness matrix is ill-conditioned. On the other hand, large values of N_α should be avoided because the contributions of the high-order singular functions become either negligible (for $r<1$) or large (if $r>1$) beyond the limits double precision can handle. Therefore, the two methods are not suitable for problems requiring a large number of Lagrange multipliers, i.e. for problems with essential boundary conditions in large parts of the boundary (except in those causing the boundary singularity).

Table 4
Comparison of the calculated coefficients; $N_\alpha = 50$ and $N_\lambda = 25$

i	ISBFM	Integral method
1	0.191118631972	0.191118631972
2	−0.1181160720	−0.118116071967
3	0.0000000000	0.000000000000
4	0.00000000000	0.00000000000
5	−0.01254698598	−0.01254698598
6	−0.01903340371	−0.01903340371
7	0.0000000000	0.0000000000
8	0.0000000000	0.0000000000
9	−0.0006541248	−0.000654125
10	−0.0075959348	−0.007595935
11	0.0000000000	0.00000000
12	0.0000000000	0.0000000
13	−0.000505411	−0.0005054
14	−0.004477115	−0.0044771
15	0.000000000	0.000000
16	0.000000000	0.000000
17	−0.000190964	−0.00019
18	−0.00300990	−0.00301
19	0.00000000	0.0000
20	0.00000000	0.0000
21	−0.0001179	
22	−0.0022019	
23	0.0000000	
24	0.0000000	
25	−0.000072	

In the singular function boundary integral method the discretized equations are reduced to boundary integrals and therefore this method is computationally less costly. The ISBFM, however, is more stable. Both methods exhibit exponential convergence with the number of Lagrange multipliers N_λ, provided that N_λ is less than N_α, and with N_α, provided that it is greater than N_λ and not very large. Both methods give accurate estimates for the leading singular coefficients. Accuracy is lost as the order increases. The results of the ISBFM are more accurate except for the second and third coefficient because the corresponding singular function contributions are contaminated by the biquadratic basis functions.

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