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Tensor products of operator systems ☆

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Abstract

The purpose of the present paper is to lay the foundations for a systematic study of tensor products of operator systems. After giving an axiomatic definition of tensor products in this category, we examine in detail several particular examples of tensor products, including a minimal, maximal, maximal commuting, maximal injective and some asymmetric tensor products. We characterize these tensor products in terms of their universal properties and give descriptions of their positive cones. We also characterize the corresponding tensor products of operator spaces induced by a certain canonical inclusion of an operator space into an operator system. We examine notions of nuclearity for our tensor products which, on the category of C*-algebras, reduce to the classical notion. We exhibit an operator system $\mathcal{S}$ which is not completely order isomorphic to a C*-algebra yet has the property that for every C*-algebra A, the minimal and maximal tensor product of $\mathcal{S}$ and A are equal.

Keywords: Operator system; Tensor product

1. Introduction

For the last 25 years there has been a great deal of development of the theory of tensor products of operator spaces and there has been a great influx of ideas and techniques from Banach space

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theory. During the same period there has been very little development of the tensor theory of operator systems. Since the methods of [18] show that many of the basic results about operator spaces and completely bounded maps can be derived from results about operator systems and completely positive maps, we believe that further development of the tensor theory of operator systems could play an important role in operator space tensor theory, as well as having its own intrinsic merit.

In this paper we initiate the systematic study of tensor products in the category whose objects are operator systems and whose morphisms are unital completely positive maps. After setting the axiomatic foundations in Section 3, we introduce and study several particular tensor products. We thus dedicate Section 4 to the "minimal" tensor product of operator systems, which corresponds to the formation of composite quantum systems in Quantum Information Theory.

In Sections 5 and 6 we study the "maximal" and the "commuting" tensor products. The maximal tensor product is also important in Quantum Information Theory, since the states on the minimal tensor product of two finite-dimensional operator systems can be identified with the maximal tensor product of their dual spaces [10, Proposition 1.9]. We characterize the maximal tensor product in terms of a universal linearization property for jointly completely positive maps, and the commuting tensor product in terms of the maximal C*-algebraic tensor product of certain universal C*-algebras associated with the corresponding operator systems. It follows from an earlier work of Lance [15] that, given two C*-algebras, their maximal and commuting tensor products as operator systems both agree with their C*-maximal tensor product. However, we show that for general operator systems these tensor products are distinct. Thus the maximal tensor product and the commuting tensor product give two different ways to extend the C*-maximal tensor product from the category of C*-algebras to operator systems. This implies that C*-algebraic notions that can be defined in terms of the minimal and maximal C*-tensor products, such as nuclearity, weak expectation property (WEP), and exactness, can bifurcate into multiple concepts in this larger category.

In particular, we exhibit an operator system $\mathcal{S}$ which is not completely order isomorphic to a C*-algebra and which does not "factor through matrix algebras"; i.e., is not nuclear in this classical sense, but which has the property that for every C*-algebra A, the minimal and the maximal operator system tensor product structures on $\mathcal{S} \otimes A$ coincide. Similarly, we exhibit operator systems that are not nuclear in the classical sense, but which have the property that their minimal and commuting tensor products with every operator system are equal. This is achieved through a careful examination of operator subsystems of the space of all n by n matrices associated with graphs.

Since every operator space embeds in a canonical operator system, tensor products in the operator system category can be pulled back to tensor products in the operator space category. We describe the pullbacks of the operator system tensor products that we construct. In particular, we show that the tensor product induced by the maximal (respectively, minimal) operator system tensor product coincides with the operator projective (respectively, injective) tensor product. The family of tensor products on the operator space category that one can obtain as pullbacks is potentially more suited for carrying out Grothendieck's program.

In Section 7, we examine the lattice structure of operator system tensor products. This allows us to introduce maximal, one-sided and two-sided, injective tensor products, the one-sided ones being asymmetric. We also formulate a characterization of nuclearity and WEP for C*-algebras in terms of these asymmetric tensor products.

2. Preliminaries

In this section we establish the terminology and state the definitions that shall be used throughout the paper.

A **$*$-vector space** is a complex vector space V together with a map $^* : V \to V$ that is involutive (i.e., $(v^*)^* = v$ for all $v \in V$) and conjugate linear (i.e., $(\lambda v + w)^* = \bar{\lambda} v^* + w^*$ for all $\lambda \in \mathbb{C}$ and $v, w \in V$). If V is a $*$-vector space, then we let $V_h = \{x \in V\colon\ x^* = x\}$ and we call the elements of V_h the **hermitian** elements of V. Note that V_h is a real vector space.

An **ordered $*$-vector space** is a pair (V, V^+) consisting of a $*$-vector space V and a subset $V^+ \subseteq V_h$ satisfying the following two properties:

(a) V^+ is a cone in V_h;
(b) $V^+ \cap (-V^+) = \{0\}$.

In any ordered $*$-vector space we may define a partial order $\geqslant$ on V_h by defining $v \geqslant w$ (or, equivalently, $w \leqslant v$) if and only if $v - w \in V^+$. Note that $v \in V^+$ if and only if $v \geqslant 0$. For this reason V^+ is called the cone of **positive** elements of V.

If (V, V^+) is an ordered $*$-vector space, an element $e \in V_h$ is called an **order unit** for V if for all $v \in V_h$ there exists a real number $r > 0$ such that $re \geqslant v$. If (V, V^+) is an ordered $*$-vector space with an order unit e, then we say that e is an **Archimedean order unit** if whenever $v \in V$ and $re + v \geqslant 0$ for all real $r > 0$, we have that $v \in V^+$. In this case, we call the triple (V, V^+, e) an **Archimedean ordered $*$-vector space** or an **AOU space**, for short. The **state space** of V is the set $S(V)$ of all linear maps $f : V \to \mathbb{C}$ such that $f(V^+) \subseteq [0, \infty)$ and $f(e) = 1$.

If V is a $*$-vector space, we let $M_{m,n}(V)$ denote the set of all $m \times n$ matrices with entries in V and set $M_n(V) = M_{n,n}(V)$. The natural addition and scalar multiplication turn $M_{m,n}(V)$ into a complex vector space. We set $M_{m,n} := M_{m,n}(\mathbb{C})$, and let $\{E_{i,j}\colon\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m\}$ denote its canonical matrix unit system. If $X = (x_{i,j})_{i,j} \in M_{l,m}$ is a scalar matrix, then for any $A = (a_{i,j})_{i,j} \in M_{m,n}(V)$ we let XA be the element of $M_{l,n}(V)$ whose i, j-entry $(XA)_{i,j}$ equals $\sum_{k=1}^{m} x_{i,k} a_{k,j}$. We define multiplication by scalar matrices on the left in a similar way. Furthermore, when $m = n$, we define a $*$-operation on $M_n(V)$ by letting $(a_{i,j})_{i,j}^* := (a_{j,i}^*)_{i,j}$. With respect to this operation, $M_n(V)$ is a $*$-vector space. We let $M_n(V)_h$ be the set of all hermitian elements of $M_n(V)$.

Definition 2.1. Let V be a $*$-vector space. We say that $\{C_n\}_{n=1}^{\infty}$ is a *matrix ordering* on V if

(1) C_n is a cone in $M_n(V)_h$ for each $n \in \mathbb{N}$,
(2) $C_n \cap (-C_n) = \{0\}$ for each $n \in \mathbb{N}$, and
(3) for each $n, m \in \mathbb{N}$ and $X \in M_{n,m}$ we have that $X^* C_n X \subseteq C_m$.

In this case we call $(V, \{C_n\}_{n=1}^{\infty})$ a *matrix ordered $*$-vector space*. We refer to condition (3) as the *compatibility of the family* $\{C_n\}_{n=1}^{\infty}$.

Note that properties (1) and (2) show that $(M_n(V), C_n)$ is an ordered $*$-vector space for each $n \in \mathbb{N}$. As usual, when $A, B \in M_n(V)_h$, we write $A \leqslant B$ if $B - A \in C_n$.

Definition 2.2. Let $(V, \{C_n\}_{n=1}^{\infty})$ be a matrix ordered $*$-vector space. For $e \in V_h$ let

$$e_n := \begin{pmatrix} e & & \\ & \ddots & \\ & & e \end{pmatrix}$$

be the corresponding diagonal matrix in $M_n(V)$. We say that e is a *matrix order unit* for V if e_n is an order unit for $(M_n(V), C_n)$ for each n. We say that e is an *Archimedean matrix order unit* if e_n is an Archimedean order unit for $(M_n(V), C_n)$ for each n. An (*abstract*) *operator system* is a triple $(V, \{C_n\}_{n=1}^{\infty}, e)$, where V is a complex $*$-vector space, $\{C_n\}_{n=1}^{\infty}$ is a matrix ordering on V, and $e \in V_h$ is an Archimedean matrix order unit.

We note that the above definition of an operator system was first introduced by Choi and Effros in [4]. If V and V' are vector spaces and $\phi : V \to V'$ is a linear map, then for each $n \in \mathbb{N}$ the map ϕ induces a linear map $\phi^{(n)} : M_n(V) \to M_n(V')$ given by $\phi^{(n)}((v_{i,j})_{i,j}) := (\phi(v_{i,j}))_{i,j}$. If $(V, \{C_n\}_{n=1}^{\infty})$ and $(V', \{C'_n\}_{n=1}^{\infty})$ are matrix ordered $*$-vector spaces, a map $\phi : V \to V'$ is called **completely positive** (for short, **c.p.**) if $\phi^{(n)}(C_n) \subseteq C'_n$ for each $n \in \mathbb{N}$. Similarly, we call a linear map $\phi : V \to V'$ a **complete order isomorphism** if ϕ is invertible and both ϕ and ϕ^{-1} are completely positive.

We denote by $\mathcal{B}(H)$ the space of all bounded linear operators acting on a Hilbert space H. The direct sum of n copies of H is denoted by H^n and its elements are written as column vectors. A **concrete operator system** $\mathcal{S}$ is a subspace of $\mathcal{B}(H)$ such that $\mathcal{S} = \mathcal{S}^*$ and $I \in \mathcal{S}$. (Here, and in the sequel, we let I denote the identity operator.) As is the case for many classes of subspaces (and subalgebras) of $\mathcal{B}(H)$, there is an abstract characterization of concrete operator systems. If $\mathcal{S} \subseteq \mathcal{B}(H)$ is a concrete operator system, then we observe that $\mathcal{S}$ is a $*$-vector space with respect to the adjoint operation, $\mathcal{S}$ inherits an order structure from $\mathcal{B}(H)$, and has I as an Archimedean order unit. Moreover, since $\mathcal{S} \subseteq \mathcal{B}(H)$, we have that $M_n(\mathcal{S}) \subseteq M_n(\mathcal{B}(H)) \equiv \mathcal{B}(H^n)$ and hence $M_n(\mathcal{S})$ inherits an involution and an order structure from $\mathcal{B}(\mathcal{H}^n)$ and has the $n \times n$ diagonal matrix

$$\begin{pmatrix} I & & \\ & \ddots & \\ & & I \end{pmatrix}$$

as an Archimedean order unit. In other words, $\mathcal{S}$ is an abstract operator system in the sense of Definition 2.2. The following result of Choi and Effros [4, Theorem 4.4] shows that the converse is also true. For an alternative proof of the result, we refer the reader to [18, Theorem 13.1].

Theorem 2.3 *(Choi–Effros). Every concrete operator system $\mathcal{S}$ is an abstract operator system. Conversely, if $(V, \{C_n\}_{n=1}^{\infty}, e)$ is an abstract operator system, then there exists a Hilbert space $\mathcal{H}$, a concrete operator system $\mathcal{S} \subseteq \mathcal{B}(H)$, and a complete order isomorphism $\phi : V \to \mathcal{S}$ with $\phi(e) = I$.*

Thanks to the above theorem, we can identify abstract and concrete operator systems and refer to them simply as operator systems. To avoid excessive notation, we will generally refer to an operator system as simply a set $\mathcal{S}$ with the understanding that e is the order unit and $M_n(\mathcal{S})^+$ is the cone of positive elements in $M_n(\mathcal{S})$. We note that any unital C*-algebra (and all C*-algebras in the present paper will be assumed to be unital) is also an operator system in a canonical way.

There is a similar theory for arbitrary subspaces $X \subseteq \mathcal{B}(H)$, called also **concrete operator spaces**. The identification $M_n(\mathcal{B}(H)) \equiv \mathcal{B}(H^n)$ endows each $M_n(X) \subseteq M_n(\mathcal{B}(H))$ with a norm; the family of norms obtained in this way satisfies certain compatibility axioms called *Ruan's axioms*. Ruan's theorem identifies the vector spaces satisfying Ruan's axioms with the concrete operator spaces. Sources for the details include [9] and [18].

What is important for our setting is that the dual of every operator space is again an operator space [1,9] and that the dual of an operator system is a matrix-ordered space [4]. Thus the dual of an operator system carries two structures and we will need to understand the relationship between these structures.

To this end, let $\mathcal{S}$ be an operator system and let $\mathcal{S}^d$ denote its Banach space dual. For $f \in \mathcal{S}^d$, we define $f^* \in \mathcal{S}^d$ by $f^*(s) = \overline{f(s^*)}$. This operation turns $\mathcal{S}^d$ into a $*$-vector space and it is easy to check that the cone of positive linear functionals defines an order on $\mathcal{S}^d$. One can define a matrix order by declaring an element $(f_{i,j}) \in M_n(\mathcal{S}^d)$ to be **positive** if and only if the map $F : \mathcal{S} \to M_n$ given by $F(s) = (f_{i,j}(s))$ is completely positive. It follows from [4, Lemma 4.2, Lemma 4.3] that this family of sets is a matrix ordering on $\mathcal{S}^d$.

On the other hand, one defines a norm on $M_n(\mathcal{S}^d)$ by setting $\|(f_{i,j})\| = \|F\|_{cb}$, where $\|F\|_{cb}$ denotes the completely bounded norm of the mapping F. This family of norms satisfies Ruan's axioms and thus gives $\mathcal{S}^d$ the structure of an abstract operator space.

The following result compares these two structures.

Theorem 2.4. *Let $\mathcal{S}$ be an operator system. Then there exists a Hilbert space H and a weak* continuous completely positive map $\Phi : \mathcal{S}^d \to \mathcal{B}(H)$ that is a complete order isomorphism onto its range and satisfies*

$$\big\|\big(\Phi(f_{i,j})\big)\big\| \leqslant \big\|(f_{i,j})\big\| \leqslant 2\big\|\big(\Phi(f_{i,j})\big)\big\|$$

for all $(f_{i,j}) \in M_n(\mathcal{S}^d)$ and all $n \in \mathbb{N}$.

Proof. Let $\mathcal{I}_n = \{P \in M_n(\mathcal{S})^+ \colon \ \|P\| \leqslant 1\}$, so that $0 \leqslant P \leqslant e_n$ for each $P \in \mathcal{I}_n$. For each $P = (p_{i,j}) \in \mathcal{I}_n$ define $e_P : \mathcal{S}^d \to M_n$ by setting $e_P(f) = (f(p_{i,j}))$. The map e_P is completely positive by [4, Lemma 4.3] and since $\|P\| \leqslant 1$, we have that $\|e_P\|_{cb} \leqslant 1$. Note that the space $A_n = \ell^\infty(\mathcal{I}_n, M_n)$ of all bounded M_n-valued functions defined on the set $\mathcal{I}_n$ is a unital C*-algebra and that $M_k(A_n) \equiv \ell^\infty(\mathcal{I}_n, M_{kn})$ in a canonical way. Let $\phi_n : \mathcal{S}^d \to A_n$ be defined by $\phi_n(f)(P) = e_P(f)$. It follows that ϕ_n is completely positive and $\|\phi_n\|_{cb} \leqslant 1$.

Now define $\Phi : \mathcal{S}^d \to \sum_{n=1}^\infty \oplus A_n$ by letting $\Phi(f) = \sum_{n=1}^\infty \oplus \phi_n(f)$; we have that Φ is completely positive and $\|\Phi\|_{cb} \leqslant 1$. Since $(f_{i,j}) \in M_n(\mathcal{S}^d)^+$ if and only if $(e_P(f_{i,j})) \geqslant 0$ for every $P \in \mathcal{I}_m$ and every m, we have that Φ is a complete order isomorphism onto its range. It is also clear that Φ is weak* continuous.

Let $(f_{i,j}) \in M_n(\mathcal{S}^d)$ and $F : \mathcal{S} \to M_n$ be the map given by $F(s) = (f_{i,j}(s))$. Given any $x \in M_n(\mathcal{S})$ with $\|x\| \leqslant 1$ we have that

$$P = \frac{1}{2}\begin{pmatrix} e_n & x \\ x^* & e_n \end{pmatrix} \in \mathcal{I}_{2n},$$

and hence $\frac{1}{2}\|F(x)\| \leqslant \|(e_P(f_{i,j}))\| \leqslant \|(\Phi(f_{i,j}))\|$. Thus, $\|(f_{i,j})\| = \|F\|_{cb} \leqslant 2\|(\Phi(f_{i,j}))\|$, and the result follows. □

Given two operator systems, $\mathcal{S}$ and $\mathcal{T}$, we write $\mathrm{CP}(\mathcal{S},\mathcal{T})$ for the cone of all completely positive maps from $\mathcal{S}$ into $\mathcal{T}$, and we write $\mathrm{UCP}(\mathcal{S},\mathcal{T})$ for the set of all unital completely positive (abbreviated u.c.p.) maps from $\mathcal{S}$ into $\mathcal{T}$. We denote by $\mathcal{O}$ the category whose objects are operator systems and whose morphisms are unital completely positive maps. The **matricial state space** of an operator system $\mathcal{S}$ is the set $S_\infty(\mathcal{S}) = \bigcup_{n=1}^{\infty} S_n(\mathcal{S})$, where

$$S_n(\mathcal{S}) = \{\phi : \mathcal{S} \to M_n\colon\ \phi \text{ a unital completely positive map}\}.$$

The algebraic tensor product of two vector spaces V and W is denoted by $V \otimes W$. If $V^+ \subseteq V$ and $W^+ \subseteq W$ are proper cones, we let $V^+ \otimes W^+ = \{v \otimes w\colon\ v \in V^+,\ w \in W^+\}$. For $n, m \in \mathbb{N}$, we shall use the usual Kronecker identification of $M_n \otimes M_m$ with M_{mn}; thus, if $(x_{i,j}) \in M_n$ and $(y_{k,l}) \in M_m$, we identify $(x_{i,j}) \otimes (y_{k,l})$ with the matrix $(x_{i,j}y_{k,l})_{(i,k),(j,l)} \in M_{mn}$. At the level of matrix units we have $E_{i,j} \otimes E_{k,l} = E_{(i,k),(j,l)}$.

If V_1, V_2, and W are vector spaces and if $\psi : V_1 \times V_2 \to W$ is a bilinear map, then for $n, m \in \mathbb{N}$ we let $\psi^{(n,m)} : M_n(V_1) \times M_m(V_2) \to M_{nm}(W)$ be the bilinear map given by $\psi^{(n,m)}((x_{i,j})_{i,j}, (y_{k,l})_{k,l}) = (\psi(x_{i,j}, y_{k,l}))_{(i,k),(j,l)}$.

Another construction that will play a role throughout this paper is the **Archimedeanization** of an ordered (respectively, matrix ordered) $*$-vector space with an order unit e. This was first introduced in [22] for ordered spaces and extended to matrix ordered spaces in [21]. Briefly, if $(V, \{D_n\}_{n=1}^{\infty}, e)$ is a matrix ordered $*$-vector space with matrix order unit e and with the property that (V, D_1, e) is an AOU space, then the Archimedeanization is obtained by forming the smallest set of cones $C_n \subseteq M_n(V)$, such that $D_n \subseteq C_n$ and $(V, \{C_n\}_{n=1}^{\infty}, e)$ is an operator system. In [21], an explicit description of the elements of C_n is given; namely, we have that $C_n = \{p \in M_n(V)\colon\ p + re_n \in D_n,\ \text{for all } r > 0\}$. We record one fact about this process that we shall need later.

Lemma 2.5. *Let $(V, \{D_n\}_{n=1}^{\infty}, e)$ be a matrix ordered $*$-vector space with matrix order unit e and with the property that (V, D_1, e) is an AOU space. Let $(C_n)_{n=1}^{\infty}$ be the cones obtained through the Archimedeanization process. Suppose that $\mathcal{T}$ is an operator system and $\phi : V \to \mathcal{T}$ is a linear map. We have that $\phi^{(n)}(D_n) \in M_n(\mathcal{T})^+$ if and only if $\phi^{(n)}(C_n) \in M_n(\mathcal{T})^+$, for each $n \in \mathbb{N}$.*

Proof. This follows from the characterization of the Archimedeanization as the smallest set of cones turning V into an operator system. □

We shall also frequently need the following fact.

Lemma 2.6. *Let V be a vector space and $\mathcal{S}$ and $\mathcal{T}$ be operator systems with underlying vector space V. Suppose that* $\mathrm{UCP}(\mathcal{S}, \mathcal{B}(H)) = \mathrm{UCP}(\mathcal{T}, \mathcal{B}(H))$ *for every Hilbert space H. Then $\mathcal{S}$ is completely order isomorphic to $\mathcal{T}$.*

Proof. Assume, without loss of generality, that $\mathcal{S} \subseteq \mathcal{B}(H)$ is a concrete operator system. Then the identity map $\mathrm{id} : \mathcal{S} \to \mathcal{B}(H)$ is unital and completely positive. It follows that id is completely positive on $\mathcal{T}$ and hence $M_n(\mathcal{T})^+ \subseteq M_n(\mathcal{S})^+$. Reversing the argument implies that the identity map on V is a unital complete order isomorphism. □

3. Tensor products of operator systems

We start this section with the definitions of the main concepts studied in this paper. Given a pair of operator systems $(\mathcal{S}, \{P_n\}_{n=1}^{\infty}, e_1)$ and $(\mathcal{T}, \{Q_n\}_{n=1}^{\infty}, e_2)$ by an **operator system structure on $\mathcal{S} \otimes \mathcal{T}$,** we mean a family $\tau = \{C_n\}_{n=1}^{\infty}$ of cones, where $C_n \subseteq M_n(\mathcal{S} \otimes \mathcal{T})$, satisfying:

(T1) $(\mathcal{S} \otimes \mathcal{T}, \{C_n\}_{n=1}^{\infty}, e_1 \otimes e_2)$ is an operator system denoted $\mathcal{S} \otimes_\tau \mathcal{T}$,
(T2) $P_n \otimes Q_m \subseteq C_{nm}$, for all $n, m \in \mathbb{N}$, and
(T3) If $\phi : \mathcal{S} \to M_n$ and $\psi : \mathcal{T} \to M_m$ are unital completely positive maps, then $\phi \otimes \psi$: $\mathcal{S} \otimes_\tau \mathcal{T} \to M_{mn}$ is a unital completely positive map.

To simplify notation we shall generally write $C_n = M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+$. Conditions (T2) and (T3) are reminiscents of Grothendieck's axioms for tensor products of Banach spaces. Condition (T2) may be viewed as the order analogue of the cross-norm condition, while (T3) as the analogue of the property of a cross-norm of being "reasonable".

Given two operator system structures τ_1 and τ_2 on $\mathcal{S} \otimes \mathcal{T}$, we say that τ_1 **is greater than** τ_2 provided that the identity map on $\mathcal{S} \otimes \mathcal{T}$ is completely positive from $\mathcal{S} \otimes_{\tau_1} \mathcal{T}$ to $\mathcal{S} \otimes_{\tau_2} \mathcal{T}$, which is equivalent to requiring that $M_n(\mathcal{S} \otimes_{\tau_1} \mathcal{T})^+ \subseteq M_n(\mathcal{S} \otimes_{\tau_2} \mathcal{T})^+$ for every $n \in \mathbb{N}$.

By an **operator system tensor product**, we mean a mapping $\tau : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$, such that for every pair of operator systems $\mathcal{S}$ and $\mathcal{T}$, $\tau(\mathcal{S}, \mathcal{T})$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$, denoted $\mathcal{S} \otimes_\tau \mathcal{T}$.

We call an operator system tensor product τ **functorial**, if the following property is satisfied:

(T4) For any four operator systems $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$, and $\mathcal{T}_2$, we have that if $\phi \in \mathrm{UCP}(\mathcal{S}_1, \mathcal{S}_2)$ and $\psi \in \mathrm{UCP}(\mathcal{T}_1, \mathcal{T}_2)$, then the linear map $\phi \otimes \psi : \mathcal{S}_1 \otimes \mathcal{T}_1 \to \mathcal{S}_2 \otimes \mathcal{T}_2$ belongs to $\mathrm{UCP}(\mathcal{S}_1 \otimes_\tau \mathcal{T}_1, \mathcal{S}_2 \otimes_\tau \mathcal{T}_2)$.

If for all operator systems $\mathcal{S}$ and $\mathcal{T}$ the map $\theta : x \otimes y \to y \otimes x$ extends to a unital complete order isomorphism from $\mathcal{S} \otimes_\tau \mathcal{T}$ onto $\mathcal{T} \otimes_\tau \mathcal{S}$ then τ is called **symmetric**.

Given three vector spaces $\mathcal{R}, \mathcal{S}$, and $\mathcal{T}$, there is a natural isomorphism from $(\mathcal{R} \otimes \mathcal{S}) \otimes \mathcal{T}$ onto $\mathcal{R} \otimes (\mathcal{S} \otimes \mathcal{T})$. We say that an operator system tensor product τ is **associative** if for any three operator systems $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$, this natural isomorphism yields a complete order isomorphism from $(\mathcal{R} \otimes_\tau \mathcal{S}) \otimes_\tau \mathcal{T}$ onto $\mathcal{R} \otimes_\tau (\mathcal{S} \otimes_\tau \mathcal{T})$.

We say that a functorial operator system tensor product is **injective** if for all operator systems $\mathcal{S}_1 \subseteq \mathcal{S}_2$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$, the inclusion $\mathcal{S}_1 \otimes_\tau \mathcal{T}_1 \subseteq \mathcal{S}_2 \otimes_\tau \mathcal{T}_2$ is a complete order isomorphism onto its range, that is, $M_n(\mathcal{S}_1 \otimes \mathcal{T}_1) \cap M_n(\mathcal{S}_2 \otimes_\tau \mathcal{T}_2)^+ = M_n(\mathcal{S}_1 \otimes_\tau \mathcal{T}_1)^+$ for every $n \in \mathbb{N}$.

One important concept from the theory of C*-algebras that we shall be interested in generalizing is *nuclearity*.

Definition 3.1. Let α and β be operator system tensor products. An operator system $\mathcal{S}$ will be called (α, β)-nuclear if the identity map between $\mathcal{S} \otimes_\alpha \mathcal{T}$ and $\mathcal{S} \otimes_\beta \mathcal{T}$ is a complete order isomorphism for every operator system $\mathcal{T}$.

One shortcoming of the theory of operator space tensor products is that the minimal and maximal operator space tensor products of matrix algebras do not coincide. For this reason there are essentially no nuclear spaces in the operator space category. We will see that, unlike the operator space case, there is a rich theory of nuclear operator systems for the various tensor products we will introduce subsequently.

Recall that every operator system is also an operator space whose matrix norms are determined by the matrix order. In fact, if $\mathcal{S}$ is an operator system with order unit e, then $s = (s_{i,j}) \in M_n(\mathcal{S})$ satisfies $\|(s_{i,j})\| \leqslant 1$ if and only if $\begin{pmatrix} e_n & s \\ s^* & e_n \end{pmatrix} \in M_2(M_n(\mathcal{S}))^+$. Since we shall need this fact often, it is worthwhile to write it out in tensor notation. Thus, we have that $\|\sum_{i,j=1}^n E_{i,j} \otimes s_{i,j}\| \leqslant 1$ if and only if $E_{1,1} \otimes e_n + E_{2,2} \otimes e_n + E_{1,2} \otimes s + E_{2,1} \otimes s^* = \sum_{i=1}^n (E_{1,1} + E_{2,2}) \otimes E_{i,i} \otimes e + \sum_{i,j=1}^n (E_{1,2} \otimes E_{i,j} \otimes s_{i,j} + E_{2,1} \otimes E_{i,j} \otimes s_{j,i}^*)$ is in $(M_2 \otimes M_n \otimes \mathcal{S})^+ = M_{2n}(\mathcal{S})^+$.

Since operator systems are also operator spaces, it is important to understand the relationship between operator system tensor products and operator space tensor products. But first, we record some two elementary facts that will be useful throughout.

Proposition 3.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and let τ be an operator system structure on $\mathcal{S} \otimes \mathcal{T}$. If $\phi : \mathcal{S} \to M_n$ and $\psi : \mathcal{T} \to M_m$ are completely positive, then $\phi \otimes \psi : \mathcal{S} \otimes_\tau \mathcal{T} \to M_{mn}$ is completely positive.*

Proof. By [18, Exercise 6.2], there exist unital completely positive maps $\phi_1 : \mathcal{S} \to M_n$ and $\psi_1 : \mathcal{T} \to M_m$ and positive matrices $P \in M_n$, $Q \in M_m$ such that $\phi(x) = P\phi_1(x)P$ and $\psi(y) = Q\psi_1(y)Q$. Hence, $\phi \otimes \psi(x \otimes y) = (P \otimes Q)(\phi_1 \otimes \psi_1(x \otimes y))(P \otimes Q)$. By Property (T3), $\phi_1 \otimes \psi_1 : \mathcal{S} \otimes_\tau \mathcal{T} \to M_{mn}$ is completely positive, and the result follows. $\square$

The next fact is a trick that is sometimes used in the theory of “decomposable” maps.

Proposition 3.3. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and let $\gamma_{i,j} : \mathcal{S} \to \mathcal{T}$, $1 \leqslant i, j \leqslant n$ be linear maps. Define $\Gamma : \mathcal{S} \to M_n(\mathcal{T})$ by $\Gamma(x) = (\gamma_{i,j}(x))$ and $\widetilde{\Gamma} : M_n(\mathcal{S}) \to M_n(\mathcal{T})$ by $\widetilde{\Gamma}((x_{i,j})) = (\gamma_{i,j}(x_{i,j}))$. Then Γ is completely positive if and only if $\widetilde{\Gamma}$ is completely positive.*

We can now prove the main result of this section.

Proposition 3.4. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and let τ be an operator system structure on $\mathcal{S} \otimes \mathcal{T}$. Then the operator space $\mathcal{S} \otimes_\tau \mathcal{T}$ is an operator space tensor product of the operator spaces $\mathcal{S}$ and $\mathcal{T}$ in the sense of* [1]; *that is, the following two conditions hold*:

(1) *For any $s \in M_n(\mathcal{S})$ and any $t \in M_m(\mathcal{T})$ we have $\|s \otimes t\|_{M_{mn}(\mathcal{S} \otimes_\tau \mathcal{T})} \leqslant \|s\|_{M_n(\mathcal{S})} \|t\|_{M_m(\mathcal{T})}$.*
(2) *If $\phi : \mathcal{S} \to M_n$ and $\psi : \mathcal{T} \to M_m$ are completely bounded maps, then $\phi \otimes \psi : \mathcal{S} \otimes_\tau \mathcal{T} \to M_{mn}$ is completely bounded and $\|\phi \otimes \psi\|_{cb} \leqslant \|\phi\|_{cb} \|\psi\|_{cb}$.*

Proof. Let e denote the order unit of $\mathcal{S}$, and let f denote the order unit of $\mathcal{T}$. To prove the first statement, it will be enough to assume that $\|s\| \leqslant 1$ and $\|t\| \leqslant 1$, and show that $\|s \otimes t\| \leqslant 1$. But, in this case, $P = \begin{pmatrix} e_n & s \\ s^* & e_n \end{pmatrix} \in M_2(M_n(\mathcal{S}))^+ = M_{2n}(\mathcal{S})^+$ and $Q = \begin{pmatrix} f_m & t \\ t^* & f_m \end{pmatrix} \in M_2(M_m(\mathcal{T}))^+ = M_{2m}(\mathcal{T})^+$. Since τ is an operator system structure, Property (T2) implies that $P \otimes Q \in M_{4mn}(\mathcal{S} \otimes_\tau \mathcal{T})^+$. Writing this matrix in block form as a 4×4 matrix of $n \times m$ blocks, we have that

$$\begin{pmatrix} e_n\otimes f_m & e_n\otimes t & s\otimes f_m & s\otimes t\\ e_n\otimes t^* & e_n\otimes f_m & s\otimes t^* & s\otimes f_m\\ s^*\otimes f_m & s^*\otimes t & e_n\otimes f_m & e_n\otimes t\\ s^*\otimes t^* & s^*\otimes f_m & e_n\otimes t^* & e_n\otimes f_m \end{pmatrix} \in M_4\big(M_{mn}(\mathcal{S}\otimes_\tau \mathcal{T})\big)^+.$$

Compressing this block matrix to the four corner entries preserves positivity, and hence

$$\begin{pmatrix} e_n\otimes f_m & s\otimes t\\ s^*\otimes t^* & e_n\otimes f_m\end{pmatrix} \in M_2\big(M_{mn}(\mathcal{S}\otimes_\tau \mathcal{T})\big),$$

and condition (1) follows.

To prove the second property, it will be enough to consider the case where $\|\phi\|_{cb}\leqslant 1$ and $\|\psi\|_{cb}\leqslant 1$. But in this case, by [18, Theorem 8.3], there exists a completely positive map $\Phi : M_2(\mathcal{S})\to M_2(M_n)$ given by

$$\Phi\left(\begin{pmatrix} s_{1,1} & s_{1,2}\\ s_{2,1} & s_{2,2}\end{pmatrix}\right) = \begin{pmatrix} \phi_{1,1}(s_{1,1}) & \phi(s_{1,2})\\ \phi(s_{2,1}^*)^* & \phi_{2,2}(s_{2,2})\end{pmatrix} \in M_2(M_n)$$

where $\phi_{1,1}, \phi_{2,2} : \mathcal{S}\to M_n$ are unital and completely positive. Also, there exists a similar completely positive map $\Psi : M_2(\mathcal{T})\to M_2(M_m)$ with analogous properties.

Let $\Phi_0 = \Phi\circ\delta : \mathcal{S}\to M_2(M_n)$ so that $\Phi_0(s) = \begin{pmatrix} \phi_{1,1}(s) & \phi(s)\\ \phi(s^*)^* & \phi_{2,2}(s)\end{pmatrix}$ and $\Psi_0 : \mathcal{T}\to M_2(M_m)$ be defined in a similar way. By Proposition 3.3, Φ_0 and Ψ_0 are completely positive. By Proposition 3.2, $\Phi_0\otimes\Psi_0 : \mathcal{S}\otimes_\tau\mathcal{T}\to M_4(M_{mn})$ is completely positive. Again, compressing to corners yields a completely positive map $\Gamma : \mathcal{S}\otimes_\tau \mathcal{T}\to M_2(M_{mn})$ with

$$\Gamma(s\otimes t) = \begin{pmatrix} \phi_{1,1}(s)\otimes\psi_{1,1}(t) & \phi(s)\otimes\psi(t)\\ \phi(s^*)^*\otimes\psi(t^*)^* & \phi_{2,2}(s)\otimes\psi_{2,2}(t)\end{pmatrix}.$$

Since $\phi\otimes\psi$ is a compression of a unital completely positive map, it is completely contractive. This completes the proof. □

One method that we shall use to distinguish operator system tensor products is to examine a canonical tensor product that they induce on the category of operator spaces and completely contractive maps. Given an operator space X, there is a canonical operator system $\mathcal{S}_X$ that can be associated to X. If $X\subseteq\mathcal{B}(H)$, then $\mathcal{S}_X\subseteq\mathcal{B}(H\oplus H)$ is the operator system given by

$$\mathcal{S}_X = \left\{\begin{pmatrix} \lambda I_H & x\\ y^* & \mu I_H\end{pmatrix} : \lambda,\mu\in\mathbb{C},\ x,y\in X\right\}.$$

We regard $X\subseteq\mathcal{S}_X$, via the inclusion $x\to\begin{pmatrix} 0 & x\\ 0 & 0\end{pmatrix}$. Note that the unit for $\mathcal{S}_X$ is $\begin{pmatrix} I_H & 0\\ 0 & I_H\end{pmatrix}$.

Definition 3.5. Let X and Y be operator spaces and τ be an operator system structure on $\mathcal{S}_X\otimes\mathcal{S}_Y$. Then the embedding

$$X\otimes Y\subseteq \mathcal{S}_X\otimes_\tau \mathcal{S}_Y$$

endows $X\otimes Y$ with an operator space structure; we call the resulting operator space the *induced operator space tensor product of* X *and* Y and denote it by $X\otimes^\tau Y$.

Proposition 3.6. *Let X and Y be operator spaces, let τ be an operator system structure on $\mathcal{S}_X \otimes \mathcal{S}_Y$, and let $X \otimes^\tau Y$ be the induced operator space tensor product. Then $X \otimes^\tau Y$ is an operator space tensor product in the sense of* [1]; *that is, the following two conditions hold*:

(1) *If $x \in M_n(X)$ and $y \in M_m(Y)$, then*

$$\|x \otimes y\|_{M_{nm}(X \otimes^\tau Y)} \leqslant \|x\|_{M_n(X)} \|y\|_{M_m(Y)}.$$

(2) *If $\phi : X \to M_n$ and $\psi : Y \to M_m$ are completely bounded, then $\phi \otimes \psi : X \otimes^\tau Y \to M_{mn}$ is completely bounded and $\|\phi \otimes \psi\|_{cb} \leqslant \|\phi\|_{cb} \|\psi\|_{cb}$.*

Proof. The first claim follows from Proposition 3.4 and the fact that the inclusions $X \subseteq \mathcal{S}_X$ and $Y \subseteq \mathcal{S}_Y$ are complete isometries.

To prove the second condition, note that by [18, Lemma 8.1] if $\phi : X \to M_n$ is completely contractive, then the map $\Phi : \mathcal{S}_X \to M_2(M_n)$ given by

$$\Phi \begin{pmatrix} \lambda 1 & x_1 \\ x_2^* & \mu 1 \end{pmatrix} = \begin{pmatrix} \lambda I_n & \phi(x_1) \\ \phi(x_2)^* & \mu I_n \end{pmatrix}$$

is a unital completely positive map. Similarly, the completely contractive map $\psi : Y \to M_m$ yields a unital completely positive map $\Psi : \mathcal{S}_Y \to M_2(M_m)$. By Property (T3) the map $\Phi \otimes \Psi : \mathcal{S}_X \otimes_\tau \mathcal{S}_Y \to M_{4mn}$ is unital and completely positive. Noticing that $\phi \otimes \psi$ occurs in a corner block of $\Phi \otimes \Psi$, we obtain that $\phi \otimes \psi$ is completely contractive. □

Let $\mathcal{OS}p$ be the category whose objects are operator spaces and whose morphisms are completely contractive linear maps. Suppose that we are given an operator system tensor product $\tau : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$. We have that the mapping $\tilde{\tau} : \mathcal{OS}p \times \mathcal{OS}p \to \mathcal{OS}p$ given by $\tilde{\tau}(X, Y) = X \otimes^\tau Y$ is an *operator space tensor product* in the sense of [1]. We call $\tilde{\tau}$ the operator space tensor product *induced by τ*.

The proof of the following result is similar to the proof of our last proposition, and we omit it.

Proposition 3.7. *If τ is a functorial operator system tensor product then $\tilde{\tau}$ is a functorial operator space tensor product*; *that is, given any four operator spaces X_1, X_2, Y_1, and Y_2 and completely contractive maps $\phi : X_1 \to X_2$ and $\psi : Y_1 \to Y_2$, the map $\phi \otimes \psi : X_1 \otimes^\tau Y_1 \to X_2 \otimes^\tau Y_2$, is completely contractive.*

4. The minimal tensor product

In this section we construct the operator system tensor product min, which is minimal among all operator system tensor products. This section has overlaps with the work of Choi, Effros and Lance [14,15,6,3,4,7] for C*-algebras and Blecher and Paulsen [1] for operator spaces. We include this material for completeness and because we will need some of the results in later sections.

Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. For each $n \in \mathbb{N}$, we let

$$C_n^{\min} = C_n^{\min}(\mathcal{S}, \mathcal{T}) = \big\{(p_{i,j}) \in M_n(\mathcal{S} \otimes \mathcal{T}) : \big((\phi \otimes \psi)(p_{i,j})\big)_{i,j} \in M_{nkm}^+, \\ \text{for all } \phi \in S_k(\mathcal{S}),\ \psi \in S_m(\mathcal{T}) \text{ for all } k, m \in \mathbb{N}\big\}.$$

Lemma 4.1. *Let $\mathcal{S}$ be an operator system and $P \in M_n(\mathcal{S})$. If $\phi^{(n)}(P) \in M_{nk}^+$ for every $\phi \in S_k(\mathcal{S})$ and every $k \in \mathbb{N}$, then $P \in M_n(\mathcal{S})^+$.*

In what follows we will identify $M_n(\mathcal{S} \otimes \mathcal{T})$ with $M_n(\mathcal{S}) \otimes \mathcal{T}$ in the natural way.

Lemma 4.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and $P \in M_n(\mathcal{S}) \otimes \mathcal{T}$. If $(\phi^{(n)} \otimes \psi)(P) \geqslant 0$ for all $\phi \in S_\infty(\mathcal{S})$ and all $\psi \in S_\infty(\mathcal{T})$, then $(\Phi \otimes \psi)(P) \geqslant 0$ for all $\Phi \in S_\infty(M_n(\mathcal{S}))$ and all $\psi \in S_\infty(\mathcal{T})$.*

Proof. Fix $m \in \mathbb{N}$ and $\psi \in S_m(\mathcal{T})$. For each functional $\omega : M_m \to \mathbb{C}$, let $\rho_\omega : M_n(\mathcal{S}) \otimes \mathcal{T} \to M_n(\mathcal{S})$ be the mapping given by $\rho_\omega(X \otimes y) = \omega(\psi(y))X$, and $L_\omega : M_m(V) \to V$ be the slice with respect to ω. If $\eta_1, \eta_2 \in \mathbb{C}^m$, let ω_{η_1,η_2} be the functional on M_m given by $\omega_{\eta_1,\eta_2}(x) = (x\eta_1, \eta_2)$.

Suppose that $(\phi^{(n)} \otimes \psi)(P) \in M_{nkm}^+$ for all $\phi \in S_k(\mathcal{S})$, $k \in \mathbb{N}$, and let $\eta_1, \dots, \eta_r \in \mathbb{C}^m$. Since the map $(L_{\omega_{\eta_t,\eta_s}})_{s,t} : M_{nkm} \to M_{nkr}$ is completely positive, we have that $(L_{\omega_{\eta_t,\eta_s}}((\phi^{(n)} \otimes \psi)(P)))_{s,t} \in M_{nkr}^+$. Thus,

$$\phi^{(nr)}\big(\big(\rho_{\omega_{\eta_t,\eta_s}}(P)\big)_{s,t}\big) = \big(\phi^{(n)}\big(\rho_{\omega_{\eta_t,\eta_s}}(P)\big)\big)_{s,t} \geqslant 0, \quad \text{for all } \phi \in S_k(\mathcal{S}),\ k \in \mathbb{N}.$$

By Lemma 4.1, $(\rho_{\omega_{\eta_t,\eta_s}}(P))_{s,t} \in M_{nr}(\mathcal{S})^+$, and hence $\Phi^{(r)}((\rho_{\omega_{\eta_t,\eta_s}}(P))_{s,t}) \geqslant 0$ for every completely positive map $\Phi : M_n(\mathcal{S}) \to M_k$, $k \in \mathbb{N}$. Fixing such a Φ, we have that $(L_{\omega_{\eta_t,\eta_s}}((\Phi \otimes \psi)(P)))_{s,t} \geqslant 0$. Thus if $\xi_1, \dots, \xi_r \in \mathbb{C}^k$, then

$$\begin{aligned}&\left((\Phi \otimes \psi)(P)\left(\sum_{t=1}^r \xi_t \otimes \eta_t\right), \left(\sum_{s=1}^r \xi_s \otimes \eta_s\right)\right)\\ &\quad = \big(\big(L_{\omega_{\eta_t,\eta_s}}\big((\Phi \otimes \psi)(P)\big)\big)_{s,t}(\xi_1, \dots, \xi_r)^{\mathrm{t}}, (\xi_1, \dots, \xi_r)^{\mathrm{t}}\big) \geqslant 0.\end{aligned}$$

It follows that $(\Phi \otimes \psi)(P) \geqslant 0$. The proof is complete. □

Lemma 4.3. *If $\phi \in S_k(\mathcal{S})$ and $\psi \in S_m(\mathcal{T})$ then $(\phi \otimes \psi)^{(n)} = \phi^{(n)} \otimes \psi$.*

Proof. It suffices to check the equality on elementary tensors of the form $P = X \otimes y$, where $X = (x_{i,j}) \in M_n(\mathcal{S})$ and $y \in \mathcal{T}$. For such a P we have that $(\phi^{(n)} \otimes \psi)(P) = (\phi(x_{i,j}))_{i,j} \otimes \psi(y)$. On the other hand,

$$(\phi \otimes \psi)^{(n)}(P) = \big((\phi \otimes \psi)(x_{i,j} \otimes y)\big)_{i,j} = \big(\phi(x_{i,j}) \otimes \psi(y)\big)_{i,j}. \qquad \square$$

Theorem 4.4. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems, and let $\iota_{\mathcal{S}} : \mathcal{S} \to \mathcal{B}(H)$ and $\iota_{\mathcal{T}} : \mathcal{T} \to \mathcal{B}(K)$ be embeddings that are unital complete order isomorphisms onto their ranges. The family $(C_n^{\min}(\mathcal{S}, \mathcal{T})_{n=1}^\infty)$ is the operator system structure on $\mathcal{S} \otimes \mathcal{T}$ arising from the embedding $\iota_{\mathcal{S}} \otimes \iota_{\mathcal{T}} : \mathcal{S} \otimes \mathcal{T} \to \mathcal{B}(H \otimes K)$.*

Proof. Let $P \in C_n^{\min}(\mathcal{S}, \mathcal{T})$. We claim that

$$Q \stackrel{def}{=} (\iota_{\mathcal{S}} \otimes \iota_{\mathcal{T}})^{(n)}(P) \in \mathcal{B}\big((H \otimes K)^n\big)^+. \tag{4.1}$$

Suppose that $Q = \sum_{r=1}^{l} X_r \otimes y_r$, where $X_r \in M_n(\iota_{\mathcal{S}}(\mathcal{S}))$ and $y_r \in \iota_{\mathcal{T}}(\mathcal{T})$ for $r = 1, \dots, l$. Let $\xi_s \in H^{(n)}$ and $\eta_s \in K$ for $s = 1, \dots, k$, and set $\zeta = \sum_{s=1}^{k} \xi_s \otimes \eta_s$. Let $\Phi : M_n(\iota_{\mathcal{S}}(\mathcal{S})) \to M_k$ be the mapping given by $\Phi(X) = ((X\xi_t, \xi_s))_{s,t}$ and let $\psi : \iota_{\mathcal{T}}(\mathcal{T}) \to M_k$ be the mapping given by $\psi(y) = ((y\eta_t, \eta_s))_{s,t}$. By the proof of Lemma 4.1, Φ and ψ are completely positive. Since $Q \in C_n^{\min}(\iota_{\mathcal{S}}(\mathcal{S}), \iota_{\mathcal{T}}(\mathcal{T}))$, Lemma 4.3 implies that $(\phi_0^{(n)} \otimes \psi_0)(Q) \in M_{nk^2}^{+}$, for all $\phi_0 \in S_k(\iota_{\mathcal{S}}(\mathcal{S}))$ and all $\psi_0 \in S_k(\iota_{\mathcal{T}}(\mathcal{T}))$. Lemma 4.2 implies that $(\Phi \otimes \psi)(Q) \in M_{nk^2}^{+}$. Let $e = (e_1, \dots, e_k)^{\mathrm{t}} \in \mathbb{C}^{k^2}$, where $\{e_j\}_{j=1}^{k}$ is the standard basis of $\mathbb{C}^k$. We then have

$$(Q\zeta, \zeta) = \sum_{r=1}^{l} \sum_{s,t=1}^{k} (X_r\xi_t, \xi_s)(y\eta_t, \eta_s)$$

$$= \sum_{r=1}^{l} \big(\big(\Phi(X_r) \otimes \psi(y_r)\big)e, e\big) = \big((\Phi \otimes \psi)(Q)e, e\big).$$

It follows that $Q \in \mathcal{B}((H \otimes K)^n)^+$ and (4.1) is established. Thus, if D_n is the cone in $M_n(\mathcal{S} \otimes \mathcal{T})$ arising from the inclusion of $\iota_{\mathcal{S}}(\mathcal{S}) \otimes \iota_{\mathcal{T}}(\mathcal{T})$ into $\mathcal{B}(H \otimes K)$, we have that $C_n^{\min}(\mathcal{S}, \mathcal{T}) \subseteq D_n$.

We now show that $D_n \subseteq C_n^{\min}(\mathcal{S}, \mathcal{T})$. Suppose that $\phi \in S_m(\mathcal{S})$ and $\psi \in S_k(\mathcal{T})$. By identifying $\mathcal{S} = \iota_{\mathcal{S}}(\mathcal{S}) \subseteq \mathcal{B}(H)$ and applying Arveson's extension theorem, we obtain a unital completely positive map $\tilde{\phi} : \mathcal{B}(H) \to M_m$ that agrees with ϕ on $\mathcal{S}$. Similarly, we obtain a unital completely positive map $\tilde{\psi} : \mathcal{B}(K) \to M_k$ that extends ψ. By C*-algebra theory, the minimal C*-tensor product $\otimes_{\mathrm{C^*\,min}}$ satisfies $\mathcal{B}(H) \otimes_{\mathrm{C^*\,min}} \mathcal{B}(K) \subseteq \mathcal{B}(H \otimes K)$ and there exists a unital completely positive map $\tilde{\phi} \otimes \tilde{\psi} : \mathcal{B}(H) \otimes_{\mathrm{C^*\,min}} \mathcal{B}(K) \to M_{mk}$. Applying Arveson's extension theorem once again, we obtain a unital completely positive map $\gamma : \mathcal{B}(H \otimes K) \to M_{mk}$. Therefore, if $P = (p_{i,j}) \in D_n \subseteq \mathcal{B}((H \otimes K)^n)^+$, then $(\phi \otimes \psi(p_{i,j})) = (\gamma(p_{i,j})) \in M_{nmk}^{+}$. Hence, $D_n = C_n^{\min}(\mathcal{S}, \mathcal{T})$.

It follows that $C_n^{\min}(\mathcal{S}, \mathcal{T})$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ with an Archimedean matrix unit $1 \otimes 1$, where 1 denotes the units for both $\mathcal{S}$ and $\mathcal{T}$. $\square$

Definition 4.5. We call the operator system $(\mathcal{S} \otimes \mathcal{T}, (C_n^{\min}(\mathcal{S}, \mathcal{T}))_{n=1}^{\infty}, 1 \otimes 1)$ the *minimal tensor product* of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{\min} \mathcal{T}$.

Theorem 4.6. *The mapping* min : $\mathcal{O} \times \mathcal{O} \to \mathcal{O}$ *sending* $(\mathcal{S}, \mathcal{T})$ *to* $\mathcal{S} \otimes_{\min} \mathcal{T}$ *is an injective, associative, symmetric, functorial operator system tensor product.*

Moreover, if $\mathcal{S}$ and $\mathcal{T}$ are operator systems and τ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$, then τ is larger than min.

Proof. By Theorem 4.4, the mapping min is an injective functorial operator system tensor product. Suppose that $\mathcal{S}_j$ is an operator system and that $\iota_j : \mathcal{S}_j \to B(H_j)$ is a complete order embedding, $j = 1, 2, 3$. By the associativity of the Hilbert space tensor product, we may identify $(H_1 \otimes H_2) \otimes H_3$ with $H_1 \otimes (H_2 \otimes H_3)$. This identification yields a complete order isomorphism of $(\mathcal{S}_1 \otimes_{\min} \mathcal{S}_2) \otimes_{\min} \mathcal{S}_3$ with $\mathcal{S}_1 \otimes_{\min} (\mathcal{S}_2 \otimes_{\min} \mathcal{S}_3)$, and hence min is associative. We see similarly that min is symmetric.

By (T3), we have that if τ is any operator system structure on $\mathcal{S} \otimes \mathcal{T}$, then $M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+ \subseteq C_n^{\min}(\mathcal{S}, \mathcal{T})$ and hence min is the minimal among all operator system structures on $\mathcal{S} \otimes \mathcal{T}$. $\square$

Remark 4.7. It was shown in [1] that the minimal operator space tensor product, the spatial operator space tensor product, and the injective operator space tensor product all coincide. For operator spaces X and Y, we will let $X\check{\otimes}Y$ denote this tensor product, and choose to refer to it as the minimal operator space tensor product.

The following corollaries are immediate.

Corollary 4.8. *Let X and Y be operator spaces. Then the induced tensor product $X \otimes^{\min} Y$ (see Definition 3.5) coincides with the minimal operator space tensor product $X\check{\otimes}Y$.*

Corollary 4.9. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. Then the identity map is a complete isometry between the operator spaces $\mathcal{S} \otimes_{\min} \mathcal{T}$ and $\mathcal{S}\check{\otimes}\mathcal{T}$.*

Corollary 4.10. *Let A and B be C^*-algebras. Then the minimal operator system tensor product $A \otimes_{\min} B$ is completely order isomorphic to the image of $A \otimes B$ inside the minimal C^*-algebraic tensor product $A \otimes_{\mathrm{C}^* \min} B$.*

We close this section with a result which relates the minimal tensor product of operator systems with the minimal operator system structure on an AOU space studied in [21]. We recall from [21] that if (V, V^+) is an AOU space, $\mathrm{OMIN}(V)$ denotes the minimal operator system whose underlying ordered $*$-vector space is (V, V^+).

Proposition 4.11. *Let V and W be AOU spaces. Equip the tensor product $V \otimes W$ with the cone*

$$Q_{\min} = \{u \in V \otimes W\colon\ (f \otimes g)(u) \geqslant 0,\ \textit{for all } f \in S(V),\ g \in S(W)\}.$$

Then $\mathrm{OMIN}(V) \otimes_{\min} \mathrm{OMIN}(W) = \mathrm{OMIN}(V \otimes W)$.

Remark 4.12. Given two AOU spaces V and W, which are also often called **function systems**, Effros [6] (see also Namioka and Phelps [16]) defines their minimal tensor product $V \otimes_{MIN} W$. The cone $Q_{\min}$ from Proposition 4.11 coincides with the set of positive elements of $V \otimes_{MIN} W$. Thus, Proposition 4.11 says that $\mathrm{OMIN}(V) \otimes_{\min} \mathrm{OMIN}(W) = \mathrm{OMIN}(V \otimes_{MIN} W)$.

5. The maximal tensor product

In this section we construct the maximal operator system tensor product and explore its properties. Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems whose units will both be denoted by 1. For each $n \in \mathbb{N}$, we let

$$\begin{aligned} D_n^{\max} &= D_n^{\max}(\mathcal{S}, \mathcal{T}) \\ &= \{\alpha(P \otimes Q)\alpha^*\colon\ P \in M_k(\mathcal{S})^+,\ Q \in M_m(\mathcal{T})^+,\ \alpha \in M_{n,km},\ k, m \in \mathbb{N}\}. \end{aligned}$$

Lemma 5.1. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and $\{D_n\}_{n=1}^\infty$ be a compatible collection of cones, where $D_n \subseteq M_n(\mathcal{S} \otimes \mathcal{T})$, satisfying Property (T2). Then $D_n^{\max} \subseteq D_n$ for each $n \in \mathbb{N}$.*

Proof. If $P \in M_k(\mathcal{S})^+$ and $Q \in M_m(\mathcal{T})^+$, Property (T2) implies that $P \otimes Q \in D_{km}$. The compatibility of $\{D_n\}_{n=1}^\infty$ implies that $\alpha(P \otimes Q)\alpha^* \in D_n$ for every $\alpha \in M_{n,km}$. Thus $D_n^{\max} \subseteq D_n$. $\square$

Lemma 5.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems, $P = (P_{i,j})_{i,j} \in M_k(M_n(\mathcal{S}))^+$, and $Q = (q_{i,j})_{i,j} \in M_k(\mathcal{T})^+$. Then $\sum_{i,j=1}^k P_{i,j} \otimes q_{i,j} \in D_n^{\max}$.*

Proof. Let I_n be the identity matrix in M_n, and $X = (X_1, X_2, \dots, X_{k^2}) \in M_{n,nk^2}$, where $X_l \in M_n$ for $l = 1, \dots, k^2$, with

$$X_1 = X_{k+2} = X_{2k+3} = \cdots = X_{k^2} = I_n$$

and $X_l = 0$ if $l \notin \{1, k+2, 2k+3, \dots, k^2\}$. Then

$$\sum_{i,j=1}^{k} P_{i,j} \otimes q_{i,j} = X(P \otimes Q)X^* \in D_n^{\max}. \qquad \square$$

The following proposition can be easily verified, we omit the proof.

Proposition 5.3. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. The family $\{D_n^{\max}(\mathcal{S}, \mathcal{T})\}_{n=1}^{\infty}$ is a matrix ordering on $\mathcal{S} \otimes \mathcal{T}$ with order unit $1 \otimes 1$.*

Definition 5.4. Let $C_n^{\max} = C_n^{\max}(\mathcal{S}, \mathcal{T})$ be the Archimedeanization of the matrix ordering $\{D_n^{\max}(\mathcal{S}, \mathcal{T})\}_{n=1}^{\infty}$. We call the operator system

$$\big(\mathcal{S} \otimes \mathcal{T}, \big\{C_n^{\max}(\mathcal{S}, \mathcal{T})\big\}_{n=1}^{\infty}, 1 \otimes 1\big)$$

the *maximal operator system tensor product* of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{\max} \mathcal{T}$.

By [21, Remark 3.19], we have that $P \in C_n^{\max}(\mathcal{S}, \mathcal{T})$ if and only if $re_n + P \in D_n^{\max}(\mathcal{S}, \mathcal{T})$ for every $r > 0$.

Theorem 5.5. *The mapping* $\max : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ *sending $(\mathcal{S}, \mathcal{T})$ to $\mathcal{S} \otimes_{\max} \mathcal{T}$ is a symmetric, associative, functorial operator system tensor product. Moreover, if τ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$, then* max *is larger than τ.*

Proof. Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. By its definition, the family $\{C_n^{\max}\}_{n=1}^{\infty}$ satisfies Property (T1) and Property (T2). Since $C_n^{\max}(\mathcal{S}, \mathcal{T}) \subseteq C_n^{\min}(\mathcal{S}, \mathcal{T})$, it follows from Theorem 4.6 that $\mathcal{S} \otimes_{\max} \mathcal{T}$ satisfies Property (T3). Suppose that $\phi \in \mathrm{UCP}(\mathcal{S}_1, \mathcal{S}_2)$ and $\psi \in \mathrm{UCP}(\mathcal{T}_1, \mathcal{T}_2)$, and let $P \in M_k(\mathcal{S}_1)^+$, $Q \in M_m(\mathcal{T}_1)^+$, and $\alpha \in M_{n,km}$. Then $\phi^{(k)}(P) \in M_k(\mathcal{S}_2)^+$ and $\psi^{(m)}(Q) \in M_m(\mathcal{T}_2)^+$. Hence

$$(\phi \otimes \psi)^{(n)}\big(\alpha(P \otimes Q)\alpha^*\big) = \alpha\big(\phi^{(k)}(P) \otimes \psi^{(m)}(Q)\big)\alpha^* \in M_n(\mathcal{S}_2 \otimes_{\max} \mathcal{T}_2)^+.$$

It follows that $(\phi \otimes \psi)^{(n)}(D_n^{\max}(\mathcal{S}_1, \mathcal{T}_1)) \subseteq D_n^{\max}(\mathcal{S}_2, \mathcal{T}_2)$. Lemma 2.5 now implies that Property (T4) is satisfied.

Suppose that $P \in M_k(\mathcal{S})^+$ and $Q \in M_m(\mathcal{T})^+$. Recall that the map $\theta : \mathcal{S} \otimes \mathcal{T} \to \mathcal{T} \otimes \mathcal{S}$ is given by $\theta(x \otimes y) = y \otimes x$. We have that, after conjugation with a permutation matrix, $\theta^{(km)}(P \otimes Q) = Q \otimes P$. It follows that if $\alpha \in M_{n,km}$, then

$$\theta^{(n)}\big(\alpha(P \otimes Q)\alpha^*\big) = \alpha\theta^{(km)}(P \otimes Q)\alpha^* = \alpha(Q \otimes P)\alpha^*.$$

Thus $\theta : \mathcal{S} \otimes_{\max} \mathcal{T} \to \mathcal{T} \otimes_{\max} \mathcal{S}$ is a complete order isomorphism and hence max is symmetric.

The fact that max is the maximal operator system tensor product follows from Lemma 5.1.

We leave the proof of associativity to the interested reader. □

Definition 5.6. Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. A bilinear map $\phi : \mathcal{S} \times \mathcal{T} \to \mathcal{B}(H)$ is called *jointly completely positive* if $\phi^{(n,m)}(P, Q)$ is a positive element of $M_{nm}(\mathcal{B}(H))$, for all $P \in M_n(\mathcal{S})^+$ and all $Q \in M_m(\mathcal{T})^+$.

The following result from [14] gives a useful characterization of jointly completely positive maps. Given a bounded bilinear map $\phi : \mathcal{S} \times \mathcal{T} \to \mathbb{C}$ we can define $\mathcal{L}(\phi) : \mathcal{S} \to \mathcal{T}^d$ (respectively, $\mathcal{R}(\phi) : \mathcal{T} \to \mathcal{S}^d$) by $\mathcal{L}(\phi)(s)(t) = \phi(s, t)$ (respectively, $\mathcal{R}(\phi)(t)(s) = \phi(s, t)$).

Lemma 5.7. *(See [14, Lemma 3.2].) Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and let $\phi : \mathcal{S} \times \mathcal{T} \to \mathbb{C}$ be a bilinear map. Then the following are equivalent:*

(i) *ϕ is jointly completely positive.*
(ii) *$\mathcal{L}(\phi) : \mathcal{S} \to \mathcal{T}^d$ is completely positive.*
(iii) *$\mathcal{R}(\phi) : \mathcal{T} \to \mathcal{S}^d$ is completely positive.*

The next theorem characterizes the maximal operator system tensor product in terms of a certain universal property.

Theorem 5.8. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems.*

(i) *If $\phi : \mathcal{S} \times \mathcal{T} \to \mathcal{B}(H)$ is a jointly completely positive map, then its linearization ϕ_L: $\mathcal{S} \otimes \mathcal{T} \to \mathcal{B}(H)$ is completely positive on $\mathcal{S} \otimes_{\max} \mathcal{T}$.*
(ii) *If $\psi : \mathcal{S} \otimes_{\max} \mathcal{T} \to \mathcal{B}(H)$ is completely positive, then the map $\phi : \mathcal{S} \times \mathcal{T} \to \mathcal{B}(H)$ given by $\phi(x, y) = \psi(x \otimes y)$, for $x \in \mathcal{S}$ and $y \in \mathcal{T}$, is jointly completely positive.*
(iii) *If τ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ with the property that the linearization of every unital jointly completely positive map $\phi : \mathcal{S} \times \mathcal{T} \to \mathcal{B}(H)$ is completely positive on $\mathcal{S} \otimes_\tau \mathcal{T}$, then $\mathcal{S} \otimes_\tau \mathcal{T} = \mathcal{S} \otimes_{\max} \mathcal{T}$.*
(iv) *For every $n \in \mathbb{N}$, we have that*

$$C_n^{\max}(\mathcal{S}, \mathcal{T}) = \big\{u \in M_n(\mathcal{S} \otimes \mathcal{T})\colon\ \phi_L^{(n)}(u) \geqslant 0,\ \textit{for all jointly completely positive } \phi : \mathcal{S} \times \mathcal{T} \to \mathcal{B}(H) \textit{ and all Hilbert spaces } H\big\}.$$

Proof. Fix operator systems $\mathcal{S}$ and $\mathcal{T}$.

(i) Let $\phi : \mathcal{S} \times \mathcal{T} \to \mathcal{B}(H)$ be a jointly completely positive map. If $P \in M_k(\mathcal{S})^+$ and $Q \in M_m(\mathcal{T})^+$, then $\phi_L^{(km)}(P \otimes Q) = \phi^{(k,m)}(P, Q) \geqslant 0$. Thus if $\alpha \in M_{n,km}$, then

$$\phi_L^{(n)}\big(\alpha(P \otimes Q)\alpha^*\big) = \alpha\phi_L^{(km)}(P \otimes Q)\alpha^* \geqslant 0,$$

and hence $\phi_L^{(n)}(D_n^{\max}) \subseteq M_n(\mathcal{B}(H))^+$. By Lemma 2.5, we have ϕ_L is completely positive.

(ii) If $P \in M_k(\mathcal{S})^+$ and $Q \in M_m(\mathcal{T})^+$, then $\phi^{(k,m)}(P, Q) = \psi^{(km)}(P \otimes Q) \geqslant 0$.

(iii) By Lemma 5.1, max is larger than τ, and hence every unital completely positive map on $\mathcal{S} \otimes_\tau \mathcal{T}$ is completely positive on $\mathcal{S} \otimes_{\max} \mathcal{T}$. By hypothesis, $\mathrm{UCP}(\mathcal{S} \otimes_\tau \mathcal{T}, \mathcal{B}(H)) = \mathrm{UCP}(\mathcal{S} \otimes_{\max} \mathcal{T}, \mathcal{B}(H))$ for every Hilbert space H. By Lemma 2.6, we have $\mathcal{S} \otimes_\tau \mathcal{T} = \mathcal{S} \otimes_{\max} \mathcal{T}$.

(iv) Let $C_n \subseteq M_n(\mathcal{S} \otimes \mathcal{T})$ be the set defined by the right-hand side of the displayed equation, and check that $\{C_n\}_{n=1}^\infty$ is an operator system structure, say τ, on $\mathcal{S} \otimes \mathcal{T}$. The result now follows by observing that τ satisfies the hypotheses of (iii). $\square$

If X and Y are operator spaces, then we let $X \hat{\otimes} Y$ denote the operator space projective tensor product. We refer the reader to [1] and [8] for the definition and properties of this tensor product.

Theorem 5.9. *Let X and Y be operator spaces. Then $X \otimes^{\max} Y$ coincides with the operator space projective tensor product $X \hat{\otimes} Y$.*

Proof. Let $e = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$ denote the identity of $\mathcal{S}_X$ and let $f = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$ denote the identity of $\mathcal{S}_Y$, so that $e \otimes f$ is the identity of $\mathcal{S}_X \otimes \mathcal{S}_Y$. Let $U = (u_{r,s}) \in M_p(X \otimes^{\max} Y)$ with $\|U\|^{\max} < 1$. We must prove that the norm $\|U\|$ of U as an element of $M_p(X \hat{\otimes} Y)$ does not exceed 1.

We have that

$$\begin{pmatrix} \|U\|^{\max}(e \otimes f)_p & U \\ U^* & \|U\|^{\max}(e \otimes f)_p \end{pmatrix} \in M_2\big(M_p(\mathcal{S}_X \otimes_{\max} \mathcal{S}_Y)\big)^+ = C_{2p}^{\max}(\mathcal{S}_X, \mathcal{S}_Y)$$

and hence

$$\begin{pmatrix} (e \otimes f)_p & U \\ U^* & (e \otimes f)_p \end{pmatrix} = \big(1 - \|U\|^{\max}\big) \begin{pmatrix} (e \otimes f)_p & 0 \\ 0 & (e \otimes f)_p \end{pmatrix} + \begin{pmatrix} \|U\|^{\max}(e \otimes f)_p & U \\ U^* & \|U\|^{\max}(e \otimes f)_p \end{pmatrix}$$

is in $D_{2p}^{\max}(\mathcal{S}_X, \mathcal{S}_Y)$.

Thus, there exist $P = (P_{i,j}) \in M_n(\mathcal{S}_X)^+$, $Q = (Q_{i,j}) \in M_m(\mathcal{S}_Y)^+$ and a $2p \times mn$ matrix $T = \begin{pmatrix} A \\ B \end{pmatrix}$ where $A = (a_{r,(i,k)})$, $B = (b_{r,(i,k)})$ are $p \times mn$ matrices, such that

$$\begin{pmatrix} (e \otimes f)_p & U \\ U^* & (e \otimes f)_p \end{pmatrix} = T(P \otimes Q)T^*.$$

This leads to the equations $(e \otimes f)_p = A(P \otimes Q)A^*$, $U = A(P \otimes Q)B^*$, $U^* = B(P \otimes Q)A^*$, and $(e \otimes f)_p = B(P \otimes Q)B^*$.

Recall that each element of $\mathcal{S}_X$ and $\mathcal{S}_Y$ is itself a 2×2 matrix and let $P_{i,j} = \begin{pmatrix} \alpha_{i,j} e_1 & x_{i,j} \\ w_{i,j}^* & \beta_{i,j} e_2 \end{pmatrix} \in \mathcal{S}_X$, where $\alpha_{i,j}, \beta_{i,j} \in \mathbb{C}$ and $x_{i,j}, w_{i,j} \in X$. Similarly, let $Q_{k,l} = \begin{pmatrix} \gamma_{k,l} f_1 & y_{k,l} \\ z_{k,l}^* & \delta_{k,l} f_2 \end{pmatrix} \in \mathcal{S}_Y$, where $\gamma_{k,l}, \delta_{k,l} \in \mathbb{C}$ and $y_{k,l}, z_{k,l} \in Y$. Finally, set $R_1 = (\alpha_{i,j})$, $R_2 = (\beta_{i,j})$, $S_1 = (\gamma_{k,l})$, $S_2 = (\delta_{k,l})$, $\mathcal{X} = (x_{i,j})$, and $\mathcal{Y} = (y_{k,l})$.

Since P and Q are positive we have that R_1, R_2, S_1, and S_2 are positive scalar matrices, that $(w_{i,j}^*) = \mathcal{X}^*$, $(z_{k,l}^*) = \mathcal{Y}^*$, and that for every $r > 0$, $\|(R_1 + rI_n)^{-1/2} \mathcal{X} (R_2 + rI_n)^{-1/2}\| \leqslant 1$ in $M_n(X)$ and $\|(S_1 + rI_m)^{-1/2} \mathcal{Y} (S_2 + rI_m)^{-1/2}\| \leqslant 1$ in $M_m(Y)$ (see [18, p. 99]).

Let R_1e_1 denote the matrix $(\alpha_{i,j}e_1)$ with similar definitions for R_2e_2, S_1f_1, S_2f_2. Recalling that the equation $(e \otimes f)_p = A(P \otimes Q)A^*$ takes place in $\mathcal{S}_X \otimes \mathcal{S}_Y$, which is represented by 4×4 block matrices, we see that it yields $(e_i \otimes f_j)_p = A(R_ie_i \otimes S_jf_j)A^*$ for $i, j = 1, 2$. Thus, $I_p = A(R_i \otimes S_j)A^*$. Similarly, $I_p = B(R_i \otimes S_j)B^*$.

Recall that we have identified x with $\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$ and y with $\left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right)$, so that U only occurs in the $(1,4)$ block of the 4×4 block matrix, with the remaining entries equal to zero. Thus, the equation $U = A(P \otimes Q)B^*$ in $\mathcal{S}_X \otimes \mathcal{S}_Y$ yields $U = A(\mathcal{X} \otimes \mathcal{Y})B^*$ in $X \otimes Y$.

In the case that all scalar matrices R_1, R_2, S_1 and S_2 are invertible, let $A_1 = A(R_1 \otimes S_1)^{1/2}$ and let $B_1 = B(R_2 \otimes S_2)^{1/2}$, so that $U = A_1(R_1 \otimes S_1)^{-1/2}(\mathcal{X} \otimes \mathcal{Y})(R_2 \otimes S_2)^{-1/2}B_1^* = A_1[(R_1^{-1/2}\mathcal{X}R_2^{-1/2}) \otimes (S_1^{-1/2}\mathcal{Y}S_2^{-1/2})]B_1^*$. Since $A_1A_1^* = I_p$ and $B_1B_1^* = I_p$, we have that $\|R_1^{-1/2}\mathcal{X}R_2^{-1/2}\| \leqslant 1$ and $\|S_1^{-1/2}\mathcal{Y}S_2^{-1/2}\| \leqslant 1$, and we have obtained $U = A_1(\mathcal{X}_1 \otimes \mathcal{Y}_1)B_1^*$, where $\mathcal{X}_1 = R_1^{-1/2}\mathcal{X}R_2^{-1/2}$, $\mathcal{Y}_1 = S_1^{-1/2}\mathcal{Y}S_2^{-1/2}$ and all matrices $A_1, \mathcal{X}_1, \mathcal{Y}_1, \mathcal{B}_1$ have norm at most one. This implies that $\|U\| \leqslant 1$.

When the scalar matrices are not all invertible, one needs to first add rI_n and rI_m $(r > 0)$ to the corresponding matrices, set $A_1 = A[(R_1 + rI_n) \otimes (S_1 + rI_m)]^{1/2}$, $B_1 = B[(R_2 + rI_n) \otimes (S_2 + rI_m)]^{1/2}$, and conclude that $\|U\| \leqslant 1 + Cr$ where C is a constant independent of r. Since this inequality holds for all $r > 0$, we again obtain that $\|U\| \leqslant 1$. □

Remark 5.10. Given two operator systems $\mathcal{S}$ and $\mathcal{T}$, Choi and Effros define in [3] an ordered $*$-vector space, which they call the maximal tensor product of $\mathcal{S}$ and $\mathcal{T}$, using a scalar version of Theorem 5.8 (iv) to define its positive cone. Let A and B be C*-algebras. Then $C_n^{\min}(A, B)$ can be canonically identified with $C_1^{\min}(M_n(A), B)$ and any bilinear map $\phi : M_n(A) \times B \to \mathbb{C}$ can be identified with a bilinear map $\tilde{\phi} : A \times B \to M_n$. Using techniques of Lance [15] and these identifications, one can show that $u \in C_n^{\min}(A, B)$ if and only if $\phi_L^{(n)}(u) \geqslant 0$ for all H and for all $\phi : A \times B \to \mathcal{B}(H)$ with ϕ jointly completely positive and of finite rank. (We say that a bounded bilinear map $\phi : A \times B \to \mathcal{B}(H)$ is of finite rank if the induced map $\mathcal{L}(\phi) : A \to \mathcal{B}(B, \mathcal{B}(H))$ has finite rank.) This fails for general operator systems, as we shall now show. If $\mathcal{S}$ is a finite-dimensional operator system, then for any operator system $\mathcal{T}$, every bilinear map $\phi : \mathcal{S} \times \mathcal{T} \to \mathcal{B}(H)$ is of finite rank. Thus, if Lance's result held for operator systems, it would imply that the minimal and maximal tensor products on $\mathcal{S} \otimes \mathcal{T}$ are equal whenever $\mathcal{S}$ is finite-dimensional. Applying this fact to operator systems of the form $\mathcal{S}_X$ and using Corollary 4.9 and Theorem 5.9 would yield that $X \hat{\otimes} Y$ is completely isometric to $X \otimes_{\min} Y$ whenever X is a finite-dimensional operator space. But this is known to be false, see [1]. Thus, the analogue of this result of Lance fails for operator systems. In particular, we see that there exist finite-dimensional operator systems that are not (min, max)-nuclear. Thus, the characterization due to [12] and [5] of nuclearity of C*-algebras via the completely positive approximation property (CPAP) does not hold for operator systems.

Even for matrix algebras, the maximal operator space cross-norm is larger than the operator space norm induced by the maximal operator system tensor product. In fact, it can be shown that the cb-norm of $\mathrm{id} : M_n \otimes_{\max} M_n \to M_n \hat{\otimes} M_n$ tends to $+\infty$ as $n \to +\infty$. One way to prove this is to use Theorem 5.12 below to see that $M_n \otimes_{\max} M_n = M_{n^2}$, up to a unital complete order isomorphism, use the fact that the norm on $M_n \hat{\otimes} M_n$ is larger than the Haagerup tensor norm [1] and compare these two norms for the element $U = \sum_{i=1}^n E_{1,i} \otimes E_{i,1}$.

The following result characterizes when these two tensor products yield completely isomorphic operator spaces.

Proposition 5.11. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. The following are equivalent:*

(i) *The identity map $\psi : \mathcal{S} \otimes_{\max} \mathcal{T} \to \mathcal{S}\hat{\otimes}\mathcal{T}$ is completely bounded.*
(ii) *There exists $C > 0$ such that for every jointly completely contractive map $\phi : \mathcal{S} \times \mathcal{T} \to \mathcal{B}(H)$ there exist jointly completely positive maps $\phi_i : \mathcal{S} \times \mathcal{T} \to \mathcal{B}(H)$ such that $\|\phi_i(e_{\mathcal{S}}, e_{\mathcal{T}})\| \leqslant C$, $i = 1, 2, 3, 4$, and $\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4)$.*

Proof. (i) $\Rightarrow$ (ii). By assumption, the identity map $\psi : \mathcal{S} \otimes_{\max} \mathcal{T} \to \mathcal{S}\hat{\otimes}\mathcal{T}$ is completely bounded; let C be its cb-norm. Let $\phi : \mathcal{S} \times \mathcal{T} \to \mathcal{B}(H)$ be a jointly completely contractive map. Then its linearization $\tilde{\phi} : \mathcal{S}\hat{\otimes}\mathcal{T} \to \mathcal{B}(H)$ is completely contractive and hence $\tilde{\phi} \circ \psi$: $\mathcal{S} \otimes_{\max} \mathcal{T} \to \mathcal{B}(H)$ is completely bounded with cb-norm not exceeding C. By the Wittstock Decomposition Theorem, there exist completely positive maps $\tilde{\phi}_i : \mathcal{S} \otimes_{\max} \mathcal{T} \to \mathcal{B}(H)$ for $i = 1, 2, 3, 4$, with norm not exceeding C and such that $\tilde{\phi} = (\tilde{\phi}_1 - \tilde{\phi}_2) + i(\tilde{\phi}_3 - \tilde{\phi}_4)$. If ϕ_i is the bilinear map corresponding to $\tilde{\phi}_i$ then ϕ_i ($i = 1, 2, 3, 4$) is jointly completely positive by Theorem 5.8(ii); clearly, $\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4)$.

(ii) $\Rightarrow$ (i). Let $\iota : \mathcal{S}\hat{\otimes}\mathcal{T} \to \mathcal{B}(H)$ be a complete isometry. By assumption, $\iota = (\tilde{\phi}_1 - \tilde{\phi}_2) + i(\tilde{\phi}_3 - \tilde{\phi}_4)$, where $\tilde{\phi}_i$ is the linearization of a jointly completely positive map $\phi_i : \mathcal{S} \times \mathcal{T} \to \mathcal{B}(H)$ for $i = 1, 2, 3, 4$. By Theorem 5.8(i), $\tilde{\phi}_i : \mathcal{S} \otimes_{\max} \mathcal{T} \to \mathcal{B}(H)$ is completely positive, and hence completely bounded. It follows that the identity map $\mathrm{id} : \mathcal{S} \otimes_{\max} \mathcal{T} \to \mathcal{B}(H)$ is completely bounded, and therefore $\mathcal{S} \otimes_{\max} \mathcal{T}$ is completely boundedly isomorphic to $\mathcal{S}\hat{\otimes}\mathcal{T}$. $\square$

The following result rests on the deep work of Choi, Effros, and Lance (see [3–5,7]). The main details to be checked are that the way we have defined the maximal via minimal cones coincides with their definition via a universal object for jointly completely positive maps and the details of the "completely" need some verification.

Theorem 5.12. *Let A and B be* C*-*algebras. Then the operator system $A \otimes_{\max} B$ is completely order isomorphic to the image of $A \otimes B$ inside the maximal* C*-*algebraic tensor product of A and B.*

Proof. Let $\mathcal{C} = A \otimes_{\mathrm{C}^* \max} B$ denote the maximal C*-algebraic tensor product of A and B. We claim that the faithful inclusion $A \otimes B \subseteq \mathcal{C}$ endows $A \otimes B$ with an operator system structure. Indeed, (T1) and (T2) are trivial and (T3) follows since it holds for the minimal C*-tensor product, which is a quotient of $\mathcal{C}$. We let $A \otimes_\tau B \subseteq \mathcal{C}$ denote this operator system.

For each $n \in \mathbb{N}$, let $D_n = M_n(A \otimes_\tau B)^+ = M_n(A \otimes B) \cap M_n(\mathcal{C})^+$. Lemma 5.1 implies that $A \otimes_{\max} B$ is larger than $A \otimes_\tau B$, and hence $C_n^{\max}(A, B) \subseteq D_n$.

We next show that the AOU spaces $(M_n(A \otimes B), C_n^{\max}(A, B))$ and $(M_n(A \otimes B), D_n)$ have the same state space. In view of the last inclusion, it suffices to show that if $f : A \otimes B \to \mathbb{C}$ and $f(C_n^{\max}(A, B)) \subseteq \mathbb{R}^+$ then $f(D_n) \subseteq \mathbb{R}^+$. So, let us fix an f with $f(C_n^{\max}(A, B)) \subseteq \mathbb{R}^+$. Suppose that $X = \sum_{i=1}^k a_i \otimes b_i$, with $a_i \in M_n(A)$ and $b_i \in B$. Then

$$XX^* = \sum_{i,j=1}^k a_i a_j^* \otimes b_i b_j^*.$$

Let $P = (a_i a_j^*)_{i,j}$ and $Q = (b_i b_j^*)_{i,j}$; then $P \in M_k(M_n(A))^+$ and $Q \in M_k(B)^+$. It follows from Lemma 5.2 that $XX^* \in C_n^{\max}(A, B)$ and hence $f(XX^*) \geqslant 0$. On the other hand, by the associativity of the C*-algebraic tensor product and the fact that M_n is a nuclear C*-algebra, we have a natural identification $M_n(\mathcal{C}) = M_n(A) \otimes_{\mathrm{C}^*\max} B$. By the definition of the set of states on the C*-algebraic tensor product [15, p. 381], we have that $f(D_n) \subseteq \mathbb{R}^+$.

Now let $u \in D_n$ and $f : M_n(A \otimes_{\max} B) \to \mathbb{C}$ be positive, that is, $f(C_n^{\max}(A, B)) \subseteq \mathbb{R}^+$. By the previous paragraph, $f(u) \geqslant 0$. By [22, Proposition 3.13], $u \in C_n^{\max}(A, B)$ and the proof is complete. $\square$

For the next proposition, we recall that if (V, V^+) is an AOU space, $\mathrm{OMAX}(V)$ denotes the maximal operator system whose underlying ordered $*$-vector space is (V, V^+) [21].

Proposition 5.13. *Let (V, V^+) and (W, W^+) be AOU spaces. Equip the tensor product $V \otimes W$ with the Archimedeanization of the cone*

$$Q_{\max} = \left\{ \sum_{i=1}^{k} v_i \otimes w_i \colon\ v_i \in V^+,\ w_i \in W^+,\ \textit{and}\ k \in \mathbb{N} \right\}.$$

Then $\mathrm{OMAX}(V) \otimes_{\max} \mathrm{OMAX}(W) = \mathrm{OMAX}(V \otimes W)$.

Remark 5.14. If V and W are AOU spaces, Effros defines in [6] their "maximal tensor product" $V \otimes_{MAX} W$ by using bilinear maps that are "jointly positive". (Effros actually uses lower case notation "max" for this tensor product, but we have adopted an upper case to avoid confusion.) Our jointly completely positive maps are the "complete" analogue of these maps. In a recent preprint [11], Han also defines a maximal tensor product $V \otimes_\pi W$ in the category of AOU spaces whose cone of positive elements coincides with our set $Q_{\max}$. Combining [6] with [11] (or just using [11]) one sees that these two definitions of the maximal tensor product in the category of AOU spaces coincide. Thus Proposition 5.13 shows that for any two AOU spaces V and W we have $\mathrm{OMAX}(V) \otimes_{\max} \mathrm{OMAX}(W) = \mathrm{OMAX}(V \otimes_{MAX} W)$. This maximal tensor product of AOU spaces is also considered in Namioka and Phelps [16].

Remark 5.15. Let A be a unital C*-algebra. Then A is nuclear if and only if A is (min, max)-nuclear; that is, if and only if $A \otimes_{\min} \mathcal{S} = A \otimes_{\max} \mathcal{S}$ for every operator system $\mathcal{S}$. Thus, the family of (min, max)-nuclear operator systems contains the family of nuclear C*-algebras. Although it is possible to prove this now, using the Choi–Effroe characterization on nuclear C*-algebras [5], we defer a proof until the next section where we can prove this fact without recourse to the Choi–Effros result.

By Proposition 5.15, every finite-dimensional C*-algebra is (min-max)-nuclear. Unlike C*-algebras, finite-dimensional operator systems do not have to be (min, max)-nuclear, as we have observed in Remark 5.10. We now exhibit an operator system that is "nuclear" when tensored with any C*-algebra, but is not (min, max)-nuclear and is also not (completely order isomorphic to) a C*-algebra. The operator system defined in Theorem 5.16 will be fixed for the rest of this section.

Theorem 5.16. *Let $\mathcal{S} = span\{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}, E_{2,3}, E_{3,2}, E_{3,3}\} \subseteq M_3$. Then $\mathcal{S} \otimes_{\min} A = \mathcal{S} \otimes_{\max} A$ for every* C*-*algebra A, and $\mathcal{S}$ is not completely order isomorphic to a* C*-*algebra.*

Proof. By the injectivity of the minimal tensor product, we have that $\mathcal{S} \otimes_{\min} A \subseteq M_3 \otimes_{\min} A = M_3(A)$. Thus, to show that $C_n^{\max}(\mathcal{S}, A) = C_n^{\min}(\mathcal{S}, A)$, after identifying $M_n(\mathcal{S} \otimes A) = \mathcal{S} \otimes M_n(A)$, it will suffice to show that if

$$P = \begin{pmatrix} P_{1,1} & P_{1,2} & 0 \\ P_{2,1} & P_{2,2} & P_{2,3} \\ 0 & P_{3,2} & P_{3,3} \end{pmatrix} \in M_3\big(M_n(A)\big)^+,$$

then $P \in C_n^{\max}$.

For every $r > 0$ we have that $rI_n + P_{i,i} > 0$ and that

$$rI_{3n} + P = \begin{pmatrix} rI_n + P_{1,1} & P_{1,2} & 0 \\ P_{2,1} & P_{2,1}(rI_n + P_{1,1})^{-1}P_{1,2} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & rI_n + P_{2,2} - P_{2,1}(rI_n + P_{1,1})^{-1}P_{1,2} & P_{2,3} \\ 0 & P_{3,2} & rI_n + P_{3,3} \end{pmatrix}.$$

Moreover, by the Cholesky algorithm both block matrices appearing in the sum are positive.

By the nuclearity of M_2 and Theorem 5.12, these matrices belong to $C_n^{\max}(\mathcal{S}, A)$.

To finish the proof we need to show that $\mathcal{S}$ is not completely order isomorphic to a C*-algebra. Assume, by way of contradiction, that $\mathcal{S}$ is completely order isomorphic to a C*-algebra. Since $\dim(\mathcal{S}) = 7$, it must be completely order isomorphic to either $M_2 \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ or $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$. Since these C*-algebras are injective, $\mathcal{S}$ is injective. This implies the existence of a completely positive projection Ψ from M_3 onto $\mathcal{S}$. The map Ψ fixes the algebra $\mathcal{D}_3$ of diagonal matrices and is hence a $\mathcal{D}_3$-bimodule map. But such bimodule maps are given by Schur products with 3×3 matrices. It follows that Ψ is given by Schur product against the matrix $R = \begin{pmatrix} 1&1&0 \\ 1&1&1 \\ 0&1&1 \end{pmatrix}$. However, a Schur product map corresponding to a matrix S is completely positive if and only if the matrix S is positive. Since R is not a positive matrix, we obtain a contradiction which shows that $\mathcal{S}$ can not be completely order isomorphic to a C*-algebra. □

We would like to point out that the fact that $\mathcal{S}$ is not completely order isomorphic to a C*-algebra can also be deduced from Theorem 5.18 and Theorem 5.12, but the above argument avoids duality considerations.

We now wish to develop some further properties of the above operator system and of its dual. To this end, set

$$G = \big\{(1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,3)\big\},$$

so that $\mathcal{S} = \operatorname{span}\{E_{i,j}\colon (i,j) \in G\}$. Let $f_{i,j} : \mathcal{S} \to \mathbb{C}$, $i, j = 1, 2, 3$, be the dual functionals given by $f_{i,j}(E_{k,l}) = \delta_{(i,j),(k,l)}$, where $\delta_{p,q}$ is the usual Kronecker delta function. Then $\mathcal{S}^d = \operatorname{span}\{f_{i,j}\colon (i,j) \in G\}$.

If $\mathcal{T}$ is an operator system and $f \in \mathcal{T}^d$ is a positive linear functional which is a matrix order unit for $\mathcal{T}^d$ it is easily seen that f is Archimedean. Thus, by [4, Theorem 4.4], $(\mathcal{T}^d, \{M_n(\mathcal{T}^d)^+\}_{n=1}^\infty, f)$ is (completely order isomorphic to) an operator system. It is shown in [4, Corollary 4.5] that whenever $\mathcal{T}$ is finite-dimensional, then such a functional f exists and thus $\mathcal{S}^d$ is an operator system. Below we give a concrete representation for $\mathcal{S}^d$.

Proposition 5.17. *Let $\mathcal{S}$ and $\mathcal{S}^d$ be as above, and let $A_{i,j} \in M_n$, $(i,j) \in G$. Then $\sum_{(i,j)\in G} A_{i,j} \otimes f_{i,j} \in M_n(\mathcal{S}^d)^+$ if and only if $\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \in M_2(M_n)^+$ and $\left(f \begin{smallmatrix} A_{2,2} & A_{2,3} \\ A_{3,2} & A_{3,3} \end{smallmatrix}\right) \in M_2(M_n)^+$. Consequently, the linear map $\Gamma : \mathcal{S}^d \to M_2 \oplus M_2$ defined by*

$$\Gamma\left(\sum_{(i,j)\in G} a_{i,j} f_{i,j}\right) = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \oplus \begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix},$$

is a complete order isomorphism onto its range.

Proof. We have that $\sum_{(i,j)\in G} A_{i,j} \otimes f_{i,j}$ is in $M_n(\mathcal{S}^d)^+$ if and only if the map $\Phi : \mathcal{S} \to M_n$ defined by $\Phi(E_{i,j}) = A_{i,j}$ is completely positive.

If we assume that Φ is completely positive, then the restriction of Φ to $\operatorname{span}\{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}\} = M_2$ is completely positive. By a result of Choi, we have that $\begin{pmatrix} \Phi(E_{1,1}) & \Phi(E_{1,2}) \\ \Phi(E_{2,1}) & \Phi(E_{2,2}) \end{pmatrix} \in M_2(M_n)^+$. In other words, $\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \in M_2(M_n)^+$. Similarly, $\begin{pmatrix} A_{2,2} & A_{2,3} \\ A_{3,2} & A_{3,3} \end{pmatrix}$ can be seen to be positive by restricting to $\operatorname{span}\{E_{2,2}, E_{2,3}, E_{3,2}, E_{3,3}\}$.

Conversely, if we assume that $\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$ and $\begin{pmatrix} A_{2,2} & A_{2,3} \\ A_{3,2} & A_{3,3} \end{pmatrix}$ are positive, then by the positive completion results of [20], there exist $A_{1,3}, A_{3,1} \in M_n$, such that $(A_{i,j})_{i,j=1}^3 \in M_3(M_n)^+$. If we define $\Psi : M_3 \to M_n$, via $\Psi(E_{i,j}) = A_{i,j}$, then we will have that $(\Psi(E_{i,j})) \in M_3(M_n)^+$ and so again by Choi's result, Ψ is completely positive. Hence Φ is completely positive, since it is the restriction of Ψ to an operator subsystem of M_3. $\square$

Theorem 5.18. *The following hold for the operator system $\mathcal{S}$ and its dual $\mathcal{S}^d$:*

(1) *If $A \subseteq B$ are unital C^*-algebras and $\phi : A \to \mathcal{S}^d$ is completely positive, then ϕ possesses a completely positive extension $\psi : B \to \mathcal{S}^d$.*
(2) *The identity map $\mathrm{id} : \Gamma(\mathcal{S}^d) \to \Gamma(\mathcal{S}^d)$ is a completely positive map that has no completely positive extension to a map from $M_2 \oplus M_2$ to $\Gamma(\mathcal{S}^d)$.*
(3) $\mathrm{id} : \Gamma(\mathcal{S}^d) \otimes_{\min} \mathcal{S} \to \Gamma(\mathcal{S}^d) \otimes_{\max} \mathcal{S}$ *is not completely positive.*
(4) $\mathcal{S}$ *is not* (min, max)*-nuclear.*

Proof. By Theorem 5.16 and the fact that min and max are symmetric, $A \otimes_{\max} \mathcal{S} = A \otimes_{\min} \mathcal{S} \subseteq B \otimes_{\min} \mathcal{S} = B \otimes_{\max} \mathcal{S}$, completely order isomorphically. Hence every jointly completely positive map defined on $A \times \mathcal{S}$ can be extended to a jointly completely positive map defined on $B \times \mathcal{S}$. Part (1) now follows by identifying $\phi : A \to \mathcal{S}^d$ with a jointly completely positive map into $\mathbb{C}$, extending it to a jointly completely positive map from $B \times \mathcal{S}$ into $\mathbb{C}$, and letting $\psi : B \to \mathcal{S}^d$ be the corresponding linear map (see Lemma 5.7).

To prove (2), suppose that the identity map on $\Gamma(\mathcal{S}^d)$ had a completely positive extension $\Phi : M_2 \oplus M_2 \to \Gamma(\mathcal{S}^d)$. Then Φ would be a completely positive projection onto $\Gamma(\mathcal{S}^d)$. We identify $M_2 \oplus M_2$ with the algebra of block diagonal matrices in M_4. Under this identification, $\Gamma(f_{1,1}) = E_{1,1}$, $\Gamma(f_{1,2}) = E_{1,2}$, $\Gamma(f_{2,1}) = E_{2,1}$, $\Gamma(f_{2,2}) = E_{2,2} + E_{3,3}$, $\Gamma(f_{2,3}) = E_{3,4}$, $\Gamma(f_{3,2}) = E_{4,3}$, and $\Gamma(f_{3,3}) = E_{4,4}$. Thus, $\mathcal{D} = \operatorname{span}\{E_{1,1}, E_{2,2} + E_{3,3}, E_{4,4}\}$ would be a C^*-algebra fixed by Φ, and hence Φ would be a $\mathcal{D}$-bimodule map (see [18, Corollary 3.19]). Since $\Phi(E_{2,2}) \in \Gamma(\mathcal{S}^d)$ and $(E_{2,2} + E_{3,3})\Phi(E_{2,2}) = \Phi(E_{2,2}) = \Phi(E_{2,2})(E_{2,2} + E_{3,3})$, we would have that $\Phi(E_{2,2}) = t(E_{2,2} + E_{3,3})$ for some $t \geqslant 0$. Similarly, $\Phi(E_{3,3}) = r(E_{2,2} + E_{3,3})$ for some $r \geqslant 0$, and it would follow that $t + r = 1$. But since $0 \leqslant J_1 = E_{1,1} + E_{1,2} + E_{2,1} + E_{2,2}$, we have that $0 \leqslant \Phi(J_1) = E_{1,1} + E_{1,2} + E_{2,1} + tE_{2,2}$, and hence $t = 1$. Similarly, considering $J_2 = E_{3,3} + E_{3,4} + E_{4,3} + E_{4,4}$ yields that $r = 1$, contradicting the fact that $r + t = 1$.

To see (3), suppose that the identity map is completely positive. Then we have that $\Gamma(\mathcal{S}^d) \otimes_{\max} \mathcal{S} = \Gamma(\mathcal{S}^d) \otimes_{\min} \mathcal{S} \subseteq (M_2 \oplus M_2) \otimes_{\min} \mathcal{S} = (M_2 \oplus M_2) \otimes_{\max} \mathcal{S}$, where the identifications and inclusions are in the complete order sense. These inclusions imply that every jointly completely positive map on $\Gamma(\mathcal{S}^d) \times \mathcal{S}$ extends to a jointly completely positive map on $(M_2 \oplus M_2) \times \mathcal{S}$. Thus every completely positive map from $\Gamma(\mathcal{S}^d)$ into $\mathcal{S}^d = \Gamma(\mathcal{S}^d)$ extends to a completely positive map from $M_2 \oplus M_2$ to $\Gamma(\mathcal{S}^d)$, which contradicts (3).

(4) is a direct consequence of (3). □

The above results show that even though $A \otimes_{\min} \mathcal{S} = A \otimes_{\max} \mathcal{S}$ for every C*-algebra, neither $\mathcal{S}$ nor $\mathcal{S}^d$ is injective.

Remark 5.19. A graph G on n vertices can be identified with a subset $G \subseteq \{1, \dots, n\} \times \{1, \dots, n\}$ satisfying the properties that $(i, j) \in G$ whenever $(j, i) \in G$ and that $(i, i) \in G$ for $i = 1, \dots, n$. To such a graph one can associate an operator system $\mathcal{S}(G) = \operatorname{span}\{E_{i,j}\colon (i, j) \in G\} \subseteq M_n$. One can show that if the graph G is chordal, then $\mathcal{S}(G) \otimes_{\min} A = \mathcal{S}(G) \otimes_{\max} A$ for every C*-algebra A. The proof is similar to that of Theorem 5.16 and uses the fact that chordal graphs have a "perfect vertex elimination scheme" and the techniques of [19] and [20], where it is shown that whenever one has a perfect vertex elimination scheme, then one can carry out a Cholesky-type algorithm as above to decompose strictly positive matrices in $\mathcal{S}(G) \otimes_{\min} A$ as encountered in the proof of Theorem 5.16. We do not present this argument here though, since this result also follows more readily from results in the next section.

We note that the operator system $\mathcal{S}$ of Theorem 5.16 is the operator system associated to the following chordal graph:

$$\bullet \!-\!-\!-\! \bullet \!-\!-\!-\! \bullet$$

6. The commuting tensor product

In this section we introduce another operator system tensor product which agrees with the max tensor product for all pairs of C*-algebras, but does not agree with the max tensor product on all pairs of operator systems. Thus, this new operator system tensor product gives a different extension of the maximal C*-algebraic tensor product from the category of C*-algebras to the category of operator systems. In contrast with the maximal operator system tensor product, but in analogy with the minimal one, this tensor product is defined by specifying a collection of completely positive maps rather than specifying the matrix ordering.

Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. Set

$$\begin{aligned} \mathrm{cp}(\mathcal{S}, \mathcal{T}) = \bigl\{(\phi, \psi)\colon\ & H \text{ is a Hilbert space}, \phi \in \mathrm{CP}\bigl(\mathcal{S}, \mathcal{B}(H)\bigr), \\ & \psi \in \mathrm{CP}\bigl(\mathcal{T}, \mathcal{B}(H)\bigr), \text{ and } \phi(\mathcal{S}) \text{ commutes with } \psi(\mathcal{T})\bigr\}. \end{aligned}$$

Given $(\phi, \psi) \in \mathrm{cp}(\mathcal{S}, \mathcal{T})$, let $\phi \cdot \psi : \mathcal{S} \otimes \mathcal{T} \to \mathcal{B}(H)$ be the map given on elementary tensors by $(\phi \cdot \psi)(x \otimes y) = \phi(x)\psi(y)$.

For each $n \in \mathbb{N}$, define a cone $P_n \subseteq M_n(\mathcal{S} \otimes \mathcal{T})$ by letting

$$P_n = \bigl\{u \in M_n(\mathcal{S} \otimes \mathcal{T}):\ (\phi \cdot \psi)^{(n)}(u) \geqslant 0,\ \text{for all } (\phi, \psi) \in \mathrm{cp}(\mathcal{S}, \mathcal{T})\bigr\}.$$

Proposition 6.1. *The collection $\{P_n\}_{n=1}^{\infty}$ is a matrix ordering on $\mathcal{S} \otimes \mathcal{T}$ with Archimedean matrix unit $1 \otimes 1$.*

Proof. It is clear that P_n is a cone. If $\alpha \in M_{n,m}$ and $u \in P_m$ then

$$(\phi \cdot \psi)^{(n)}\bigl(\alpha u \alpha^*\bigr) = \alpha(\phi \cdot \psi)^{(m)}(u)\alpha^* \geqslant 0,$$

and hence the family $\{P_n\}_{n=1}^{\infty}$ is compatible. Let $\phi \in S_k(\mathcal{S})$ and $\psi \in S_m(\mathcal{T})$, and define $\tilde{\phi} : \mathcal{S} \to M_k \otimes 1_m$ (respectively, $\tilde{\psi} : \mathcal{T} \to 1_k \otimes M_m$) by $\tilde{\phi}(x) = \phi(x) \otimes 1_m$ (respectively, $\tilde{\psi}(y) = 1_k \otimes \psi(y)$). Then $(\tilde{\phi}, \tilde{\psi}) \in \mathrm{cp}(\mathcal{S}, \mathcal{T})$ and hence

$$(\phi \otimes \psi)^{(n)}(u) = (\tilde{\phi} \cdot \tilde{\psi})^{(n)}(u) \geqslant 0 \quad \text{for each } u \in P_n.$$

Thus $P_n \subseteq C_n^{\min}$ for each $n \in \mathbb{N}$. It now follows that $P_n \cap (-P_n) = \{0\}$ and that $1 \otimes 1$ is an matrix order unit for $\{P_n\}_{n=1}^{\infty}$.

Suppose that $r(1 \otimes 1)_n + u \in P_n$ for each $r > 0$. Then $(\phi \cdot \psi)^{(n)}(r(1 \otimes 1)_n + u) \geqslant 0$ for all $(\phi, \psi) \in \mathrm{cp}(\mathcal{S}, \mathcal{T})$ and all $r > 0$. Thus $rI_H + (\phi \cdot \psi)^{(n)}(u) \geqslant 0$ for all $(\phi, \psi) \in \mathrm{cp}(\mathcal{S}, \mathcal{T})$ and all $r > 0$, which implies that $u \in P_n$. Hence, $1 \otimes 1$ is an Archimedean matrix order unit, and the proof is complete. $\square$

Definition 6.2. We let $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T}$ denote the operator system $(\mathcal{S} \otimes \mathcal{T}, \{P_n\}_{n=1}^{\infty}, 1 \otimes 1)$.

The proof of the following result is similar to earlier proofs so we omit the details.

Theorem 6.3. *The mapping $\mathrm{c} : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ sending the pair $(\mathcal{S}, \mathcal{T})$ to the operator system $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T}$ is a symmetric, functorial operator system tensor product.*

We recall that for an operator system $\mathcal{S}$, there exists a unital C*-algebra $C_u^*(\mathcal{S})$ introduced in [13] (called either the **universal C*-algebra of** $\mathcal{S}$ or the **maximal C*-algebra of** $\mathcal{S}$) and a unital completely positive map $\iota : \mathcal{S} \to C_u^*(\mathcal{S})$ with the properties that $\iota(\mathcal{S})$ generates $C_u^*(\mathcal{S})$ as a C*-algebra, and that for every unital completely positive map $\phi : \mathcal{S} \to \mathcal{B}(H)$ there exists a unique $*$-homomorphism $\pi : C_u^*(\mathcal{S}) \to \mathcal{B}(H)$ such that $\pi \circ \iota = \phi$. We refer to [13] for the details of how this algebra is constructed.

Theorem 6.4. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. The operator system arising from the inclusion of $\mathcal{S} \otimes \mathcal{T}$ into $C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T})$ coincides with $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T}$.*

Proof. Let τ be the operator system structure on $\mathcal{S} \otimes \mathcal{T}$ arising from the inclusion $\mathcal{S} \otimes \mathcal{T} \subseteq C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T})$. Suppose that $u \in M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+$ and let $(\phi, \psi) \in \mathrm{cp}(\mathcal{S}, \mathcal{T})$. By the universal properties of $C_u^*(\mathcal{S})$ and $C_u^*(\mathcal{T})$, there exist (unique) $*$-homomorphisms $\pi : C_u^*(\mathcal{S}) \to \mathcal{B}(H)$ and $\rho : C_u^*(\mathcal{T}) \to \mathcal{B}(H)$ extending ϕ and ψ, respectively. Since $\mathcal{S}$ (respectively, $\mathcal{T}$) generates $C_u^*(\mathcal{S})$

(respectively, $C_u^*(\mathcal{T})$) as a C*-algebra, we have that the ranges of π and ρ commute. It follows that

$$(\phi \cdot \psi)^{(n)}(u) = (\pi \cdot \rho)^{(n)}(u) \geqslant 0,$$

and hence $u \in M_n(\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T})$.

Conversely, suppose that $u \in M_n(\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T})^+$. To show that u is in the positive cone of $M_n(C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T}))$, it suffices by Lemma 4.1 to prove that $\eta^{(n)}(u) \geqslant 0$ for each completely positive map $\eta : C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T}) \to \mathcal{B}(H)$. By Stinespring's Theorem, we may moreover assume that η is a $*$-homomorphism. By Theorem 5.12 and the universal property of the maximal tensor product of C*-algebras, each such η is equal to $\pi \cdot \rho$, where $\pi : C_u^*(\mathcal{S}) \to \mathcal{B}(H)$ and $\rho : C_u^*(\mathcal{T}) \to \mathcal{B}(H)$ are $*$-homomorphisms with commuting ranges. Since the restrictions of π to $\mathcal{S}$ and the restriction of ρ to $\mathcal{T}$ are each completely positive, we have that $\eta(u) \geqslant 0$. □

We obtain the following consequence of Theorem 6.4.

Corollary 6.5. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. A linear map $f : \mathcal{S} \otimes_{\mathrm{c}} \mathcal{T} \to \mathcal{B}(H)$ is a unital completely positive map if and only if there exist a Hilbert space K, $*$-homomorphisms π: $C_u^*(\mathcal{S}) \to \mathcal{B}(K)$ and $\rho : C_u^*(\mathcal{T}) \to \mathcal{B}(K)$ with commuting ranges, and an isometry $V : H \to K$ such that $f(x \otimes y) = V^*\pi(x)\rho(y)V$ for all $x \in \mathcal{S}$ and all $y \in \mathcal{T}$.*

Proof. Suppose that K, V, π, and ρ are as in the statement. Since $\pi \cdot \rho$ is completely positive on $C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T})$, Theorem 6.4 implies that the restriction of $\pi \cdot \rho$ to $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T}$ is completely positive. Hence the map $u \to V^*\pi \cdot \rho(u)V$ on $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T}$ is completely positive.

Conversely, suppose that $f : \mathcal{S} \otimes_{\mathrm{c}} \mathcal{T} \to \mathcal{B}(H)$ is completely positive. By Theorem 6.4, f has a completely positive extension $\tilde{f} : C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T}) \to \mathcal{B}(H)$. Stinespring's Theorem implies the existence of a Hilbert space K, an isometry $V : H \to K$, and a *-homomorphism $\eta : C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T}) \to \mathcal{B}(K)$ such that $\tilde{f}(u) = V^*\eta(u)V$ for all $u \in C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T})$. By the universal property of the maximal C*-algebraic tensor product, $\eta = \pi \cdot \rho$ for some $*$-homomorphisms $\pi : C_u^*(\mathcal{S}) \to \mathcal{B}(K)$ and $\rho : C_u^*(\mathcal{T}) \to \mathcal{B}(K)$. □

The next result, Theorem 6.6, can be deduced as a corollary of the following Theorem 6.7, but we present a separate proof because it is a considerably more elementary result.

Theorem 6.6. *If A and B are unital* C*-*algebras, then $A \otimes_{\mathrm{c}} B = A \otimes_{\max} B$.*

Proof. By Theorem 5.5, $M_n(A \otimes_{\max} B)^+ \subseteq M_n(A \otimes_{\mathrm{c}} B)^+$. Conversely, suppose that $u \in M_n(A \otimes_{\mathrm{c}} B)^+$. By Theorem 5.12, $A \otimes_{\max} B$ is completely order isomorphic to the image of $A \otimes B$ inside $A \otimes_{\mathrm{C}^*\max} B$, the maximal C*-algebraic tensor product of A and B. Now let $i_A : A \to A \otimes_{\mathrm{C}^*\max} B$ be given by $i_A(a) = a \otimes 1_B$ and let $i_B : B \to A \otimes_{\mathrm{C}^*\max} B$ be given by $i_B(b) = 1_A \otimes b$. Clearly, i_A and i_B are completely positive and have commuting ranges. Theorem 5.12 implies that $u \in M_n(A \otimes_{\max} B)^+$ if and only if $(i_A \cdot i_B)^n(u)$ is positive. But the latter is true by the definition of the commuting tensor product. Thus the result follows. □

The following result gives another characterization of the c tensor product. We prove that, in a certain precise sense, c is the minimal extension of C^* max from the category of C*-algebras to the category of operator systems.

Theorem 6.7. *If A is a unital* C*-*algebra and $\mathcal{S}$ is an operator system, then $A \otimes_{\mathrm{c}} \mathcal{S} = A \otimes_{\max} \mathcal{S}$. Moreover, if $\alpha : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ is any symmetric, functorial operator system tensor product such that $A \otimes_\alpha B = A \otimes_{\max} B$ for every pair of unital* C*-*algebras A, then $\mathrm{c} \leqslant \alpha$, i.e., for every pair of operator systems $\mathcal{S}$ and $\mathcal{T}$, the identity map $id_{\mathcal{S}} \otimes id_{\mathcal{T}} : \mathcal{S} \otimes_\alpha \mathcal{T} \to \mathcal{S} \otimes_{\mathrm{c}} \mathcal{T}$ is completely positive.*

Proof. By defining $a_1 \cdot (a \otimes s) \cdot a_2 = (a_1 a a_2) \otimes s$, the algebraic tensor product $A \otimes \mathcal{S}$ becomes an A-bimodule. We claim that $A \otimes_{\max} \mathcal{S}$ is an *operator A-system* in the sense of [18, Chapter 15]; that is, if $U \in M_n(A \otimes_{\max} \mathcal{S})^+$ and $B \in M_{n,k}(A)$, then $B^* \cdot U \cdot B$ is in $M_k(A \otimes_{\max} \mathcal{S})^+$. To show this, we may assume that U is in $D_n^{\max}$. Indeed, suppose that the assertion is true in this case. Given $V \in C_n^{\max}$, we know that $V + \epsilon I_n \in D_n^{\max}$ for every $\epsilon > 0$. We have that $B^* \cdot (V + \epsilon I_n) \cdot B = B^* \cdot V \cdot B + \epsilon B^* \cdot I_n \cdot B = B^* \cdot V \cdot B + \epsilon B^* B \otimes (1_{\mathcal{S}})$ is in $C_n^{\max}$ for every $\epsilon > 0$. So the result follows from the fact that $C_n^{\max}$ is closed.

Let $U \in D_n^{\max}$ have the form $U = \alpha(P \otimes Q)\alpha^*$, where $P \in M_p(A)^+$, $Q = (s_{ij}) \in M_q(\mathcal{S})^+$ and $\alpha \in M_{n,pq}$. Note that $B^* \cdot \alpha(P \otimes Q)\alpha^* \cdot B = (\alpha^* B)^* \cdot (P \otimes Q) \cdot (\alpha^* B)$. Thus we may assume that $U = P \otimes Q$, where $P \in M_p(A)^+$ and $Q = (s_{ij}) \in M_q(\mathcal{S})^+$ with $pq = n$. Let $B = (B_1\, B_2 \ldots B_q)^{\mathrm{t}}$, where each B_i is a $p \times k$ matrix. Then

$$B^* \cdot (P \otimes Q) \cdot B = \left(B_1^*\, B_2^* \,\ldots\, B_q^*\right) \cdot \begin{pmatrix} P \otimes s_{11} & \cdots & P \otimes s_{1q} \\ \vdots & \ddots & \vdots \\ P \otimes s_{q1} & \cdots & P \otimes s_{qq} \end{pmatrix} \cdot \begin{pmatrix} B_1 \\ \vdots \\ B_q \end{pmatrix}$$

$$= \sum_{i,j=1}^{q} \left(B_i^* P B_j\right) \otimes s_{ij}.$$

Let $C = (B_i^* P B_j)_{i,j=1}^q$, so that $C \in M_{kq}(A)^+$. Let $X = (e_1 \otimes I_k \,\ldots\, e_q \otimes I_k)^{\mathrm{t}}$, where $e_i \otimes I_k = (0 \,\ldots\, I_k \,\ldots\, 0)^{\mathrm{t}}$ is a $qk \times k$ scalar matrix. Then

$$B^* \cdot (P \otimes Q) \cdot B = X^*(C \otimes Q)X \in C_n^{\max}.$$

Thus we have shown that $A \otimes_{\max} \mathcal{S}$ is an operator A-system. By [18, Theorem 15.12], the map $\pi : A \to \mathcal{I}(A \otimes_{\max} \mathcal{S})$ given by $\pi(a) = a \otimes 1_{\mathcal{S}}$ is a unital $*$-homomorphism. In this case, π is also injective and hence an isometry.

Let $i : \mathcal{S} \to \mathcal{I}(A \otimes_{\max} \mathcal{S})$ be given by $i(s) = 1_A \otimes s$. Then i is a complete order isomorphism onto its range. Note that $\pi(A)$ commutes with $i(\mathcal{S})$ since $(a \otimes 1_{\mathcal{S}})(1_A \otimes s) = a \cdot (1_A \otimes 1_{\mathcal{S}}) \times (1_A \otimes s) = a \cdot (1_A \otimes s) = a \otimes s = (1_A \otimes s) \cdot a = (1_A \otimes s)(1_A \otimes 1_{\mathcal{S}}) \cdot a = (1_A \otimes s) \times (a \otimes 1_{\mathcal{S}})$. Thus $\pi : A \to \mathcal{I}(A \otimes_{\max} \mathcal{S})$ and $i : \mathcal{S} \to \mathcal{I}(A \otimes_{\max} \mathcal{S})$ are completely positive and have commuting ranges. This means that $\pi \cdot i : A \otimes_{\mathrm{c}} \mathcal{S} \to \mathcal{I}(A \otimes_{\max} \mathcal{S})$ is completely positive with range $A \otimes_{\max} \mathcal{S}$. Note that $\pi \cdot i(a \otimes s) = a \otimes s$, which implies that the identity map from $A \otimes_{\mathrm{c}} \mathcal{S}$ to $A \otimes_{\max} \mathcal{S}$ is completely positive. Thus $A \otimes_{\mathrm{c}} \mathcal{S} = A \otimes_{\max} \mathcal{S}$ by the maximality of max.

Finally, to see the last claim assume that α is as above and let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. By the functoriality of α we have that the inclusion maps $\mathcal{S} \to C_u^*(\mathcal{S})$ and $\mathcal{T} \to C_u^*(\mathcal{T})$ induce a completely positive map $\mathcal{S} \otimes_\alpha \mathcal{T} \to C_u^*(\mathcal{S}) \otimes_\alpha C_u^*(\mathcal{T}) = C_u^*(\mathcal{S}) \otimes_{max} C_u^*(\mathcal{T})$. But we also have that these inclusion maps induce a complete order isomorphism of $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T}$ into $C_u^*(\mathcal{S}) \otimes_{\max} C_u^*(\mathcal{T})$ and the result follows. □

We can now give the promised proof of Remark 5.15.

Corollary 6.8. *Let A be a unital C^*-algebra. Then A is a nuclear C^*-algebra if and only if $A \otimes_{\min} \mathcal{S} = A \otimes_{\max} \mathcal{S}$ for every operator system $\mathcal{S}$.*

Proof. We have that $A \otimes_{\max} \mathcal{S} = A \otimes_{\mathrm{c}} \mathcal{S} \subseteq A \otimes_{\max} C_u^*(\mathcal{S}) = A \otimes_{\min} C_u^*(\mathcal{S})$ and $A \otimes_{\min} \mathcal{S} \subseteq A \otimes_{\min} C_u^*(\mathcal{S})$, where both containments are complete order isomorphisms. Thus, the result follows. $\square$

We now define a tensor product for operator spaces that is related to the μ tensor product of Oikhberg and Pisier [17]. Let X and Y be operator spaces. For $u \in X \otimes Y$, let

$$\|u\|_{\mu^*} = \sup\bigl\{\bigl\|(f \cdot g)(u)\bigr\|:\ f : X \to \mathcal{B}(H) \text{ and } g : Y \to \mathcal{B}(H) \text{ are completely contractive maps with the property that } f(x) \text{ commutes with } \bigl\{g(y), g(y)^*\bigr\} \text{ for all } x \in X \text{ and } y \in Y\bigr\}.$$

We define norms on $M_n(X \otimes Y)$ in a similar fashion. It is easily checked that this gives an operator space structure to $X \otimes Y$, and we denote the resulting operator space $X \otimes_{\mu^*} Y$. If the mappings f and g satisfy the properties in the definition of $\|\cdot\|_{\mu^*}$, we say that their ranges are $*$-commuting.

Proposition 6.9. *Let X and Y be operator spaces. Then the identity map is a completely isometric isomorphism betwen $X \otimes^{\mathrm{c}} Y$ and $X \otimes_{\mu^*} Y$.*

Proof. Given unital completely positive maps $\Phi : \mathcal{S}_X \to \mathcal{B}(K)$ and $\Psi : \mathcal{S}_Y \to \mathcal{B}(K)$ with commuting ranges, define $f : X \to \mathcal{B}(K)$ and $g : Y \to \mathcal{B}(K)$ via $f(x) = \Phi\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$ and $g(y) = \Psi\left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right)$. Then f and g are completely contractive maps whose ranges are $*$-commuting. This shows that the norm on $X \otimes_{\mu^*} Y$ is greater than the norm on $X \otimes^{\mathrm{c}} Y$.

Conversely, given completely contractive commuting maps $f : X \to \mathcal{B}(H)$ and $g : Y \to \mathcal{B}(H)$ as in the above definition define completely positive maps $\Phi : \mathcal{S}_X \to \mathcal{B}(H \oplus H \oplus H \oplus H)$ and $\Psi : \mathcal{S}_Y \to \mathcal{B}(H \oplus H \oplus H \oplus H)$ by

$$\Phi\begin{pmatrix} \lambda & x_1 \\ x_2^* & \mu \end{pmatrix} = \begin{pmatrix} \lambda I_H & f(x_1) & 0 & 0 \\ f(x_2)^* & \mu I_H & 0 & 0 \\ 0 & 0 & \lambda I_H & f(x_1) \\ 0 & 0 & f(x_2)^* & \mu I_H \end{pmatrix}$$

and

$$\Psi\begin{pmatrix} \alpha & y_1 \\ y_2^* & \beta \end{pmatrix} = \begin{pmatrix} \alpha I_H & 0 & g(y_1) & 0 \\ 0 & \alpha I_H & 0 & g(y_1) \\ g(y_2)^* & 0 & \beta I_H & 0 \\ 0 & g(y_2)^* & 0 & \beta I_H \end{pmatrix}.$$

The maps Φ and Ψ are readily seen to be unital completely positive and to have commuting ranges. This shows that the norm on $X \otimes_{\mu^*} Y$ does not exceed the norm on $X \otimes^{\mathrm{c}} Y$, and hence the two norms are equal. $\square$

Corollary 6.10. *The operator system tensor products* max *and* c *are distinct.*

Proof. It will be enough to show that the induced operator space tensor products $\otimes^{\max}$ and $\otimes^{c}$ are different. In [17] Oikhberg and Pisier introduce a tensor norm $\otimes_{\mu}$ on operator spaces by considering the supremum over all pairs of commuting (but not necessarily $*$-commuting) completely contractive maps, and prove that this tensor norm is strictly smaller than the projective operator space tensor norm. Clearly, our $\|\cdot\|_{\mu^*} = \|\cdot\|_{c}$ is dominated by $\|\cdot\|_{\mu}$ and since, by Theorem 5.9, $\|\cdot\|_{\max}$ coincides with the operator projective tensor norm, the result follows. □

For the next result we need to recall the operator systems associated with graphs that were introduced in Remark 5.19

Proposition 6.11. *Let $G \subseteq \{1,\dots,k\} \times \{1,\dots,k\}$ be a graph on k vertices and let $\mathcal{S}(G) \subseteq M_k$ be the operator system of the graph. If G is a chordal graph, then $\mathcal{S}(G) \otimes_{c} \mathcal{T} = \mathcal{S}(G) \otimes_{\min} \mathcal{T}$ for every operator system $\mathcal{T}$, and so $\mathcal{S}(G)$ is* (min, c)*-nuclear.*

Proof. Let $\{E_{i,j}\}$ be the canonical matrix units in M_k. Suppose that $\phi : \mathcal{S}(G) \to \mathcal{B}(H)$ and $\psi : \mathcal{T} \to \mathcal{B}(H)$ are completely positive maps with commuting ranges. Let $T_{i,j} = \phi(E_{i,j})$, $(i, j) \in G$. For every complete subgraph $G_0 \subseteq G$ (that is, a subset G_0 of G of the form $G_0 = J \times J$ for some $J \subseteq \{1,\dots,k\}$), we have that $\phi|_{\mathcal{S}(G_0)} : \mathcal{S}(G_0) \to \mathcal{B}(H)$ is completely positive. It follows by Choi's characterization [2] that the matrix $(T_{i,j})_{(i,j)\in G_0}$ is positive.

Thus, the partially defined matrix $(T_{i,j})_{(i,j)\in G}$ is *partially positive* in the sense of [20]. It follows from [20] that this operator matrix has a positive completion in the von Neumann algebra $\phi(\mathcal{S}(G))''$; that is, there exist $T_{i,j} \in \phi(\mathcal{S}(G))''$ for $(i, j) \notin G$, such that the (fully defined) operator matrix $(T_{i,j})_{i,j=1}^{k}$ is positive. Another application of Choi's Theorem implies that the mapping $\tilde{\phi} : M_k \to \mathcal{B}(H)$ sending a matrix $(\lambda_{i,j})$ to the operator $\sum_{i,j=1}^{k} \lambda_{i,j} T_{i,j}$ is completely positive. Thus, $\tilde{\phi}$ is a completely positive extension of ϕ. Clearly the ranges of $\tilde{\phi}$ and ψ commute.

It follows from the previous paragraph that $\mathcal{S}(G) \otimes_{c} \mathcal{T} \subseteq M_k \otimes_{c} \mathcal{T}$ as operator systems. However, M_k is a nuclear C*-algebra, and hence Theorem 5.15 implies that $M_k \otimes_{c} \mathcal{T} = M_k \otimes_{\min} \mathcal{T}$. On the other hand, $\mathcal{S}(G) \otimes_{\min} \mathcal{T} \subseteq M_k \otimes_{\min} \mathcal{T}$ by the injectivity of the minimal operator system tensor product. It follows that $\mathcal{S}(G) \otimes_{c} \mathcal{T} = \mathcal{S}(G) \otimes_{\min} \mathcal{T}$. □

Combining this proposition with Theorem 6.7, we have that when G is a chordal graph and A is a C*-algebra, then

$$\mathcal{S}(G) \otimes_{\min} A = \mathcal{S}(G) \otimes_{c} A = \mathcal{S}(G) \otimes_{\max} A,$$

which is the result claimed in Remark 5.19.

It follows from Proposition 6.11 that the 7-dimensional operator system of Theorem 5.18 is (min, c)-nuclear but not (min, max)-nuclear.

7. The lattice of tensor products

In this section we examine the collection of all operator system tensor products, show that it is a lattice, and introduce some tensor products that can also be characterized via this lattice. These tensor products appear to have important categorical roles and are natural analogues of some of the tensor products that appear in Grothendieck's programme. We then relate preservation of these tensor products to certain important properties of C*-algebras. First we will need a preliminary result.

Proposition 7.1. *The collection of all operator system tensor products is a complete lattice with respect to the order introduced in Section* 3. *The collection of all functorial operator system tensor products is a complete sublattice of this lattice.*

Proof. Let $\{\tau_j\}_{j\in J}$ be a collection of operator system tensor products, where J is a non-empty set. It suffices to show that $\{\tau_j\}_{j\in J}$ possesses a greatest lower bound. Fix operator systems $\mathcal{S}$ and $\mathcal{T}$. For each $n\in\mathbb{N}$, let $P_n=\bigcap_{j\in J}M_n(\mathcal{S}\otimes_{\tau_j}\mathcal{T})^+$. Since $P_n\subseteq M_n(\mathcal{S}\otimes_{\tau_{j_0}}\mathcal{T})^+$ for each $j_0\in J$, it follows that $P_n\cap(-P_n)=\{0\}$. It is trivial to check that the family $\{P_n\}_{n=1}^\infty$ is compatible and that it satisfies $M_n(\mathcal{S})^+\otimes M_m(\mathcal{T})^+\subseteq P_{mn}$. Hence $(P_n-P_n)+i(P_n-P_n)=M_n(\mathcal{S}\otimes\mathcal{T})$. Thus $\{P_n\}_{n=1}^\infty$ is a matrix ordering on $\mathcal{S}\otimes\mathcal{T}$. We shall denote this matrix-ordered space by $\mathcal{S}\otimes_\tau\mathcal{T}$.

Since $M_n(\mathcal{S}\otimes_{\tau_j}\mathcal{T})^+\subseteq M_n(\mathcal{S}\otimes_{\min}\mathcal{T})^+$ for every $j\in J$, it follows that $P_n\subseteq M_n(\mathcal{S}\otimes_{\min}\mathcal{T})^+$, $n\in\mathbb{N}$. Since $1\otimes 1$ is a matrix order unit for $\mathcal{S}\otimes_{\min}\mathcal{T}$, it follows that $1\otimes 1$ is a matrix order unit for $\mathcal{S}\otimes_\tau\mathcal{T}$. Also, since $1\otimes 1$ is Archimedean for each $\mathcal{S}\otimes_{\tau_j}\mathcal{T}$, it follows that $1\otimes 1$ is Archimedean for $\mathcal{S}\otimes_\tau\mathcal{T}$. Hence $\mathcal{S}\otimes_\tau\mathcal{T}$ is an operator system, that is, Property (T1) holds. The fact that Property (T2) holds follows from the fact that $M_n(\mathcal{S}\otimes_{\max}\mathcal{T})^+\subseteq M_n(\mathcal{S}\otimes_\tau\mathcal{T})^+$. Property (T3) holds because it holds min and $M_n(\mathcal{S}\otimes_\tau\mathcal{T})^+\subseteq M_n(\mathcal{S}\otimes_{\min}\mathcal{T})^+$.

Finally, if every τ_j is functorial and $\phi_i:\mathcal{S}_i\to\mathcal{T}_i$ for $i=1,2$ are unital completely positive maps, then $\phi_1\otimes\phi_2:\mathcal{S}_1\otimes_{\tau_j}\mathcal{T}_1\to\mathcal{S}_2\otimes_{\tau_j}\mathcal{T}_2$ is a unital completely positive map for every $j\in J$. Since the positive cones for $\mathcal{S}_1\otimes_\tau\mathcal{T}_1$ are smaller than the positive cones for $\mathcal{S}_1\otimes_{\tau_j}\mathcal{T}_1$, we have that $\phi_1\otimes\phi_2:\mathcal{S}_1\otimes_\tau\mathcal{T}_1\to\mathcal{S}_2\otimes_{\tau_j}\mathcal{T}_2$ is a unital completely positive map for every $j\in J$. From this it follows that $\phi_1\otimes\phi_2:\mathcal{S}_1\otimes_\tau\mathcal{T}_2\to\mathcal{S}_2\otimes_\tau\mathcal{T}_2$ is a unital completely positive map, and the functoriality of τ follows. $\square$

There is a general way to induce operator system structures from inclusions. Let α be an operator system tensor product. If $\mathcal{S}_i$ and $\mathcal{T}_i$, $i=1,2$, are operator systems with $\mathcal{S}_1\subseteq\mathcal{S}_2$ and $\mathcal{T}_1\subseteq\mathcal{T}_2$, let $\{C_n\}_{n=1}^\infty$ be the matrix ordering on $\mathcal{S}_1\otimes\mathcal{T}_1$ given by

$$C_n=M_n(\mathcal{S}_2\otimes_\alpha\mathcal{T}_2)^+\cap M_n(\mathcal{S}_1\otimes\mathcal{T}_1),\quad n\in\mathbb{N}.$$

We call $\{C_n\}_{n=1}^\infty$ the operator system structure on $\mathcal{S}_1\otimes\mathcal{T}_1$ **induced by** α and the pair $(\mathcal{S}_2,\mathcal{T}_2)$. We note that this is not an operator system tensor product in the sense of definition given in Section 3; it is defined "locally" for every quadruple of operator systems $\mathcal{S}_1\subseteq\mathcal{S}_2$ and $\mathcal{T}_1\subseteq\mathcal{T}_2$.

A tensor product α on the category of operator systems is called **left injective** if for all operator systems $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{T}$ with $\mathcal{S}_1\subseteq\mathcal{S}_2$, the inclusion of $\mathcal{S}_1\otimes_\alpha\mathcal{T}$ into $\mathcal{S}_2\otimes_\alpha\mathcal{T}$ is a complete order isomorphism. Equivalently, α is left injective if the operator system structure of $\mathcal{S}_1\otimes_\alpha\mathcal{T}$ coincides with the one induced by α and $(\mathcal{S}_2,\mathcal{T})$ for all operator systems $\mathcal{S}_2$ with $\mathcal{S}_1\subseteq\mathcal{S}_2$, and all operator systems $\mathcal{T}$. We define a **right injective** operator system tensor product similarly. An operator system tensor product is **injective** if it is both left and right injective or, equivalently, if the inclusion of $\mathcal{S}_1\otimes_\alpha\mathcal{T}_1$ into $\mathcal{S}_2\otimes_\alpha\mathcal{T}_2$ is a complete order injection whenever $\mathcal{S}_1\subseteq\mathcal{S}_2$ and $\mathcal{T}_1\subseteq\mathcal{T}_2$. For example, min is an injective tensor product.

Given an operator system $\mathcal{S}$ we let $I(\mathcal{S})$ denote its injective envelope. There is a precise sense in which $I(\mathcal{S})$ is the "smallest" injective operator system that contains $\mathcal{S}$. See [18, Chapter 15] for a detailed development of this concept.

Definition 7.2. Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. We let $\mathcal{S} \otimes_{\mathrm{el}} \mathcal{T}$ (respectively, $\mathcal{S} \otimes_{\mathrm{er}} \mathcal{T}$) be the operator system with underlying space $\mathcal{S} \otimes \mathcal{T}$ whose matrix ordering is induced by the inclusion $\mathcal{S} \otimes \mathcal{T} \subseteq I(\mathcal{S}) \otimes_{\max} \mathcal{T}$ (respectively, $\mathcal{S} \otimes \mathcal{T} \subseteq \mathcal{S} \otimes_{\max} I(\mathcal{T})$).

Likewise, we let $\mathcal{S} \otimes_{\mathrm{e}} \mathcal{T}$ be the operator system with underlying space $\mathcal{S} \otimes \mathcal{T}$ whose matrix ordering is induced by the inclusion $\mathcal{S} \otimes \mathcal{T} \subseteq I(\mathcal{S}) \otimes_{\max} I(\mathcal{T})$.

The proof of the following is routine, so we omit it.

Theorem 7.3. *The mappings* $\mathrm{el} : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ $\mathrm{er} : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ *and* $\mathrm{e} : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ *sending the pair* $(\mathcal{S}, \mathcal{T})$ *to the operator system* $\mathcal{S} \otimes_{\mathrm{el}} \mathcal{T}$*,* $\mathcal{S} \otimes_{\mathrm{er}} \mathcal{T}$ *and* $\mathcal{S} \otimes_{\mathrm{e}} \mathcal{T}$ *are functorial operator system tensor products.*

Lemma 7.4. *Let* $\mathcal{S}$*,* $\mathcal{S}_1$*, and* $\mathcal{T}$ *be operator systems with* $\mathcal{S} \subseteq \mathcal{S}_1$*, and let* τ *be the operator system structure induced by the inclusion* $\mathcal{S} \otimes \mathcal{T} \subseteq \mathcal{S}_1 \otimes_{\max} \mathcal{T}$*. Then* $\mathcal{S} \otimes_\tau \mathcal{T}$ *is greater than* $\mathcal{S} \otimes_{\mathrm{el}} \mathcal{T}$*.*

Proof. Let $\phi : \mathcal{S}_1 \to I(\mathcal{S})$ be a unital completely positive map extending the inclusion $\iota : \mathcal{S} \to I(\mathcal{S})$. By the functoriality of the maximal operator system tensor product, we have that $\phi \otimes \mathrm{id} : \mathcal{S}_1 \otimes_{\max} \mathcal{T} \to I(\mathcal{S}) \otimes_{\max} \mathcal{T}$ is completely positive. Since $\phi \otimes \mathrm{id}$ coincides on $\mathcal{S} \otimes \mathcal{T}$ with the identity map, the conclusion follows. □

We now show the role that these tensor products play within the family of all operator system tensors.

Theorem 7.5. *The operator system tensor product* el *is left injective. Moreover, if* $\alpha : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ *is a left injective functorial operator system tensor product then* el *is greater than* α*. Similarly,* er *is the largest right injective and* e *is the largest injective functorial operator system tensor products.*

Proof. We only prove the first statement.

Suppose that $\mathcal{S} \subseteq \mathcal{S}_1$. Let $\mathcal{S} \otimes_\tau \mathcal{T}$ denote the operator system induced by the inclusion $\mathcal{S} \otimes \mathcal{T} \subseteq I(\mathcal{S}_1) \otimes_{\max} \mathcal{T}$. By Lemma 7.4, $\mathcal{S} \otimes_\tau \mathcal{T}$ is greater than $\mathcal{S} \otimes_{\mathrm{el}} \mathcal{T}$. On the other hand, the inclusion $\mathcal{S} \subseteq I(\mathcal{S}_1)$ gives rise to a unital completely positive map $\phi : I(\mathcal{S}) \to I(\mathcal{S}_1)$. By functoriality, the map $\phi \otimes \mathrm{id} : I(\mathcal{S}) \otimes_{\max} \mathcal{T} \to I(\mathcal{S}_1) \otimes_{\max} \mathcal{T}$ is completely positive. Restricting to the subspace $\mathcal{S} \otimes \mathcal{T}$ implies that the corresponding map $\phi \otimes \mathrm{id} : \mathcal{S} \otimes_{\mathrm{el}} \mathcal{T} \to \mathcal{S} \otimes_\tau \mathcal{T}$ is completely positive. Since this map coincides with the identity map, we have that $\mathcal{S} \otimes_{\mathrm{el}} \mathcal{T}$ is greater than $\mathcal{S} \otimes_\tau \mathcal{T}$, and hence $\mathcal{S} \otimes_\tau \mathcal{T} = \mathcal{S} \otimes_{\mathrm{el}} \mathcal{T}$. Thus the inclusion $\mathcal{S} \otimes_{\mathrm{el}} \mathcal{T} \subseteq \mathcal{S}_1 \otimes_{\mathrm{el}} \mathcal{T}$ is completely isometric. It is thus shown that el is injective.

Suppose now that $\alpha : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ is a left injective operator system tensor product. If $\mathcal{S}$ and $\mathcal{T}$ are operator systems, then $\mathcal{S} \otimes_\alpha \mathcal{T} \subseteq I(\mathcal{S}) \otimes_\alpha \mathcal{T}$ completely order isomorphically. By the maximality property of max, we have that the identity map $\mathrm{id} \otimes \mathrm{id} : I(\mathcal{S}) \otimes_{\max} \mathcal{T} \to I(\mathcal{S}) \otimes_\alpha \mathcal{T}$ is completely positive. Hence its restriction to $\mathcal{S} \otimes \mathcal{T}$ maps the positive cones of $\mathcal{S} \otimes_{\mathrm{el}} \mathcal{T}$ into those of $\mathcal{S} \otimes_\alpha \mathcal{T}$. Thus el is greater than α. □

We summarize the order relations between the particular tensor products studied in this paper:

$$\min \leqslant \mathrm{e} \leqslant \mathrm{el}, \mathrm{er} \leqslant \mathrm{c} \leqslant \max .$$

Since el and er play central roles in the family of all tensor products, it is interesting to know if their relationship to important properties of C*-algebras. The following results provide partial answers to these questions.

Proposition 7.6. *Let A be a unital* C*-*algebra. The following are equivalent*:

(i) *A possesses the weak expectation property* (*WEP*);
(ii) $A \otimes_{\mathrm{el}} B = A \otimes_{\max} B$ *for every* C*-*algebra B.*

Proof. (i) $\Rightarrow$ (ii) By Lance's characterization of WEP (see [15]), the inclusion of $A \otimes_{\max} B$ into $I(A) \otimes_{\max} B$ is a complete order isomorphism onto its range. However, $A \otimes_{\mathrm{el}} B$ is by definition obtained by restricting the matrix order structure of $I(A) \otimes_{\max} B$ to $A \otimes B$. It follows that $A \otimes_{\max} B = A \otimes_{\mathrm{el}} B$.

(ii) $\Rightarrow$ (i) Suppose that A_1 and B are C*-algebras such that $A \subseteq A_1$. By Lemma 7.4, the matrix ordering on $A \otimes B$ induced by its inclusion in $A_1 \otimes_{\max} B$ is (set-theoretically) contained in that of $A \otimes_{\mathrm{el}} B = A \otimes_{\max} B$. However, it is trivial that the matrix ordering of $A \otimes_{\max} B$ is contained in the former matrix ordering since $A \otimes_{\max} B$ is the largest matrix ordering on $A \otimes B$. Thus, $A \otimes_{\max} B \subseteq A_1 \otimes_{\max} B$ (as C*-algebras). It follows from [15] that A has WEP. $\square$

Proposition 7.6 shows that WEP can be thought of as a nuclearity property with respect to el, which is an operator system structure on the tensor products bigger than the minimal one. The next observation characterizes nuclearity in terms of the right injective tensor product er.

Proposition 7.7. *Let A be a unital* C*-*algebra. The following are equivalent*:

(i) *A is nuclear,*
(ii) $A \otimes_{\mathrm{er}} B = A \otimes_{\max} B$*, for every unital* C*-*algebra B.*

Proof. (i) $\Rightarrow$ (ii) If A is nuclear then $A \otimes_{\min} B = A \otimes_{\max} B$ sits completely order isomorphically in $A \otimes_{\min} I(B) = A \otimes_{\max} I(B)$, and hence $A \otimes_{\max} B = A \otimes_{\mathrm{er}} B$.

(ii) $\Rightarrow$ (i) Let B and B_1 be unital C*-algebras with $B \subseteq B_1$. Let $\phi : B_1 \to I(B)$ be a completely positive extension of the inclusion $B \to I(B)$. Suppose that $u \in M_n(A \otimes B) \cap M_n \times (A \otimes_{\max} B_1)^+$. Using the identifications $M_n(A \otimes B) \equiv M_n(A) \otimes B$ and $M_n(A \otimes_{\max} B_1) \equiv M_n(A) \otimes_{\max} B_1$ and the functoriality of the maximal tensor product, we have that

$$(\mathrm{id}_{M_n(A)} \otimes \phi)(u) \in \big(M_n(A) \otimes_{\max} I(B)\big)^+ \equiv M_n\big(A \otimes_{\max} I(B)\big)^+.$$

Since $u \in M_n(A \otimes B)$ and ϕ coincides with the identity mapping on B, we have that $u \in M_n(A \otimes_{\max} I(B))^+$. By assumption, $u \in M_n(A \otimes_{\max} B)^+$. We thus showed that the inclusion $A \otimes_{\max} B \to A \otimes_{\max} B_1$ is a complete order isomorphism onto its range. It follows from [15, Theorem A] that A is nuclear. $\square$

Proposition 7.7 allows one to establish the nuclearity of a C*-algebra by comparing the maximal tensor product with er, which is a priori bigger than the minimal tensor product.

Propositions 7.6 and 7.7 have the following consequence.

Corollary 7.8. *The tensor product* el *is not symmetric.*

Proof. By [15], there exists a C*-algebra A which is not nuclear and possesses the weak expectation property. By Propositions 7.6 and 7.7, there exists a unital C*-algebra B such that $A \otimes_{\mathrm{er}} B \neq A \otimes_{\max} B = A \otimes_{\mathrm{el}} B$.

Suppose that the map $\theta : A \otimes B \to B \otimes A$ given by $\theta(x \otimes y) = y \otimes x$ was a complete order isomorphism of $A \otimes_{\mathrm{el}} B$ onto $B \otimes_{\mathrm{el}} A$. Since A has WEP, Proposition 7.6 implies that $\theta : A \otimes_{\max} B \to I(B) \otimes_{\max} A$ is a complete order isomorphism onto its range. Since max is symmetric, the restriction of the mapping $\theta^{-1} : I(B) \otimes_{\max} A \to A \otimes_{\max} I(B)$ to $B \otimes A$ is a complete order isomorphism onto its range. It follows that the inclusion $A \otimes_{\max} B \to A \otimes_{\max} I(B)$ is a complete order isomorphism onto its range, and hence $A \otimes_{\max} B = A \otimes_{\mathrm{er}} B$, a contradiction with the choice of B. $\square$

Remark 7.9. Arguments similar to those given above show that if X and Y are operator spaces, then the inclusion $X \otimes Y \subseteq I(X)\hat{\otimes} I(Y)$ induces an operator space tensor product $X \otimes_{\hat{\mathrm{e}}} Y$ that is the largest injective tensor product in the operator space category. We claim that the operator space structure on $X \otimes_{\hat{\mathrm{e}}} Y$ is distinct from the one on $X \otimes^{\mathrm{e}} Y$ (recall that $X \otimes^{\mathrm{e}} Y$ arises from the embedding $X \otimes Y \subseteq \mathcal{S}_X \otimes_{\mathrm{e}} \mathcal{S}_Y$—or, equivalently, from the embedding $X \otimes Y \subseteq I(\mathcal{S}_X) \otimes_{\max} I(\mathcal{S}_Y)$). To see this, let $X = Y = M_{m,n}$. Then $I(\mathcal{S}_X) = M_{m+n,m+n}$ and by the nuclearity of $M_{m+n,m+n}$ we have that $I(\mathcal{S}_X) \otimes_{\max} I(\mathcal{S}_Y) = M_{(m+n)^2,(m+n)^2}$. However, $M_{m,n}\hat{\otimes} M_{m,n}$ is distinct from M_{m^2,n^2}. Hence, $X \otimes_{\hat{\mathrm{e}}} Y \neq X \otimes^{\mathrm{e}} Y$ in this case. As a corollary we obtain the following.

Corollary 7.10. *There exists a functorial injective operator space tensor product that is not induced by a functorial injective operator system tensor product.*

In our last proposition, we characterize the norm $\|\cdot\|_{\mathrm{e}}$ induced by the operator system structure e introduced in Definition 7.2.

Proposition 7.11. *Let A and B be unital* C*-*algebras and $u \in A \otimes B$. Then*

$$\|u\|_{\mathrm{e}} = \inf\big\{\|u\|_{A_1 \otimes_{\max} B_1}\colon\ A_1 \textit{ and } B_1 \textit{ are C*-algebras} \\ \textit{with } 1_A \in A \subseteq A_1 \textit{ and } 1_B \in B \subseteq B_1\big\}.$$

Proof. Fix $u \in A \otimes B$ and denote the quantity on the right-hand side by δ. By the definition of e and δ, we have that $\delta \leqslant \|u\|_{\mathrm{e}}$.

Let A_1 and B_1 be C*-algebras with $1_A \in A \subseteq A_1$ and $1_B \in B \subseteq B_1$. Let $\phi : A_1 \to I(A)$ and $\psi : B_1 \to I(B)$ be completely positive extensions of the inclusion maps $A \to I(A)$ and $B \to I(B)$, respectively. By functoriality, $\phi \otimes \psi$ is a unital completely positive, and hence completely contractive, map from $A_1 \otimes_{\max} B_1$ into $I(A) \otimes_{\max} I(B)$. It follows that

$$\|u\|_{\mathrm{e}} = \big\|(\phi \otimes \psi)(u)\big\|_{I(A)\otimes_{\max} I(B)} \leqslant \|u\|_{A_1 \otimes_{\max} B_1},$$

and hence $\delta = \|u\|_{\mathrm{e}}$. $\square$

Remark 7.12. Pisier [23, p. 350] defines a tensor product $\otimes_M$ on operator spaces $X \subseteq B(H)$ and $Y \subseteq B(K)$ by identifying $X \otimes_M Y$ with the subspace $X \otimes Y \subseteq B(H) \otimes_{\max} B(K)$, and argues that this tensor product is independent of the particular completely isometric inclusions of X and Y into $B(H)$ spaces. It is not difficult to see that this tensor product is identical with our tensor product $\otimes^e$. We make this precise in the following.

Recall that every operator system is also an operator space. Thus, we may form the operator system $\mathcal{S} \otimes_e \mathcal{T}$ and the operator space $\mathcal{S} \otimes^e \mathcal{T}$.

Proposition 7.13. *Let X and Y be operator spaces and let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. Then $X \otimes^e Y = X \otimes_M Y$ and $\mathcal{S} \otimes_e \mathcal{T} = \mathcal{S} \otimes^e \mathcal{T}$, completely isometrically.*

Proof. First let $I_H \in A \subseteq \mathcal{B}(H)$ and $I_K \in \mathcal{B} \subseteq \mathcal{B}(K)$ be unital, injective C*-subalgebras. Then there exists unital completely positive projections $\phi : \mathcal{B}(H) \to A$ and $\psi : \mathcal{B}(K) \to B$. This implies that the map $\phi \otimes \psi : \mathcal{B}(H) \otimes_{\max} \mathcal{B}(K) \to A \otimes_{\max} B$, is a unital completely positive map. Hence it follows that the operator subsystem $A \otimes B \subseteq \mathcal{B}(H) \otimes_{\max} \mathcal{B}(K)$ is completely order isomorphic to $A \otimes_{\max} B$.

Thus if we are given operator spaces X and Y and we embed $I(S_X) \subseteq \mathcal{B}(H)$ and $I(S_Y) \subseteq \mathcal{B}(K)$, then the subspaces $X \otimes^e Y \subseteq I(S_X) \otimes_{\max} I(S_Y)$ and $X \otimes_M Y \subseteq \mathcal{B}(H) \otimes_{\max} \mathcal{B}(K)$, will be completely isometric.

If $\mathcal{S} \subseteq I(\mathcal{S}) \subseteq \mathcal{B}(H)$ and $\mathcal{T} \subseteq I(\mathcal{T}) \subseteq \mathcal{B}(K)$ are operator systems, then the previous paragraph shows that $\mathcal{S} \otimes^e \mathcal{T} = \mathcal{S} \otimes_M \mathcal{T}$ completely isometrically. But $\mathcal{S} \otimes_M \mathcal{T}$ can be completely isometrically identified with the subspace $\mathcal{S} \otimes \mathcal{T} \subseteq \mathcal{B}(H) \otimes_{\max} \mathcal{B}(K)$, and we also have the completely isometric identification $I(\mathcal{S}) \otimes_{\max} I(\mathcal{T}) \subseteq \mathcal{B}(H) \otimes_{\max} \mathcal{B}(K)$. Hence we have that $\mathcal{S} \otimes_M \mathcal{T} \subseteq I(\mathcal{S}) \otimes_{\max} I(\mathcal{T})$ is a completely isometric inclusion and so $\mathcal{S} \otimes_M \mathcal{T} = \mathcal{S} \otimes_e \mathcal{T}$. □

In contrast, recall that even if A and B are unital C*-algebras, then $A \otimes^{\max} B = A \hat{\otimes} B$, which is not completely isometrically equal to $A \otimes_{\max} B$.

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