The Noncommutative Residue for Manifolds with Boundary

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We construct a trace on the algebra of classical elements in Boutet de Monvel’s calculus on a compact manifold with boundary of dimension n ≥ 2. This trace coincides with Wodzicki’s noncommutative residue if the boundary is reduced to the empty set. Moreover, we show that it is the unique continuous trace on this algebra up to multiplication by a constant.

INTRODUCTION

Let M be a closed compact manifold. Denote by \( \Psi^{-\infty} \) the algebra of all classical pseudodifferential operators on M with integral order and by \( \Psi^{-\infty} \) its ideal of smoothing pseudodifferential operators. The noncommutative residue is a trace on the algebra \( \mathcal{A} = \Psi^{-\infty}/\Psi^{-\infty} \), i.e., a surjective linear map

\[
\text{res}: \mathcal{A} \rightarrow C,
\]

where \( C \) is the algebra of complex-valued continuous functions.
which vanishes on all commutators: \( \text{res}(PQ - QP) = 0 \) for all \( P, Q \in \mathfrak{A} \).

In fact, this trace, \( \text{res} \), turns out to be the unique trace on this algebra, up to a multiplicative constant, provided that the manifold is connected and of dimension higher than 1. It is given as an integral over a local density, \( \text{res}_x \), on \( M \).

For one-dimensional manifolds, the noncommutative residue was discovered by Manin [10] and Adler [1] in connection with geometric aspects of nonlinear partial differential equations. In this situation, the algebra \( \mathfrak{A} \) can be viewed as the algebra of formal Laurent series in the covariable \( \xi \in R \), and the local density \( \text{res}_x \), indeed takes the form of a classical residue: it is the coefficient of \( \xi^{-1} \). For arbitrary closed compact \( n \)-dimensional manifolds, the noncommutative residue was introduced by Wodzicki in [14] using the theory of zeta functions of elliptic pseudodifferential operators. Later, Wodzicki gave a more geometric account based on the theory of homogeneous forms on symplectic cones [15]. In this framework of symplectic cones, Guillemin had independently discovered the noncommutative residue as an important ingredient of his so-called “soft” proof of Weyl’s formula on the asymptotic distribution of eigenvalues [6].

Meanwhile, the noncommutative residue has found many applications in both mathematics and mathematical physics. A detailed introduction to the noncommutative residue together with its mathematical consequences was given by Kassel in [8]. For applications in physics, cf. e.g., Connes [4], Radul [11], and Kravchenko and Khesin [9].

In the present paper, we introduce a noncommutative residue for the operators in Boutet de Monvel’s algebra on manifolds with boundary. More precisely, let \( M \) be a compact connected manifold with boundary of dimension \( n > 1 \). Denote by \( \mathcal{B}^\infty \) the algebra of all operators in Boutet de Monvel’s calculus (with integral order) and by \( \mathcal{B}^{-\infty} \) the ideal of smoothing operators. We then construct a trace on the quotient \( \mathcal{B} = \mathcal{B}^\infty / \mathcal{B}^{-\infty} \). Moreover, we show that this trace is the only continuous trace on \( \mathcal{B} \).

As in [15] we work directly at the symbol level. We avoid zeta function techniques, primarily, because the results known in this direction are not sufficient for our purposes. Computations of the Hochschild and cyclic homologies of \( \mathcal{B}^\infty / \mathcal{B}^{-\infty} \) in the spirit of Brylinski and Getzler [3] will be the subject of a forthcoming article.

The paper is organized as follows. In Section 1 we give a simplified proof (see Theorem 1.4 and formula (1.8)) for the existence and uniqueness of the noncommutative residue on a compact manifold without boundary. Section 2 starts with a short review of Boutet de Monvel’s algebra \( \mathcal{B} \). We then consider two natural subalgebras of \( \mathcal{B} \): the algebra \( \mathcal{B}_0 \) of all operators with vanishing interior pseudodifferential symbol, and the subalgebra \( \mathcal{B}_\infty \) of all operators whose interior pseudodifferential symbol stabilizes in a
neighborhood of the boundary, i.e. is independent of the normal coordinate $x_n$. We define (see formula (2.13)) an analogue of the noncommutative residue on $\mathcal{B}_0$ and show that it is the unique continuous trace on $\mathcal{B}_0$, cf. Proposition 2.3. We can extend this trace to $\mathcal{B}$; this extension, however, will no longer be the only trace on $\mathcal{B}_0$. In Section 3 we finally treat the general case. We define the noncommutative residue on $\mathcal{B}$ (see formula (2.23)). We prove in Theorem 3.1 that it is a trace; in Theorem 3.2, we show that it is the unique continuous trace up to a multiplicative constant. This is achieved by using the uniqueness property in $\mathcal{B}_0$ and by proving that there is no trace on $\mathcal{B}/\mathcal{B}_0$.

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1. WODZICKI’S RESIDUE ON A CLOSED COMPACT MANIFOLD

In this section we recall the construction of the noncommutative residue for a closed compact manifold. We will need some lemmata on homogeneous functions on $\mathbb{R}^n \setminus \{0\}$. Let $\mathbb{R}^n$ be the standard oriented Euclidean space, $n > 1$, with coordinates $\xi_1, \xi_2, ..., \xi_n$. A smooth function $p(\xi)$ on $\mathbb{R}^n \setminus \{0\}$ is homogeneous of degree $\lambda \in \mathbb{R}$ if for any $t > 0$

$$p(t\xi) = t^\lambda p(\xi).$$ (1.1)

Euler’s theorem for homogeneous functions is the following statement:

$$\sum_{j=1}^n \xi_j \frac{\partial p}{\partial \xi_j} = i\lambda p;$$ (1.2)

this follows by differentiating (1.1) with respect to $t$ and setting $t = 1$.

Consider the $n - 1$ form

$$\sigma = \sum_{j=1}^n (-1)^{j+1} \xi_j d\xi_1 \wedge ... \wedge \hat{d\xi_j} \wedge ... \wedge d\xi_n,$$

where the hat indicates that the corresponding factor has been omitted. Clearly, $d\sigma = n d\xi_1 \wedge d\xi_2 \wedge ... \wedge d\xi_n$. Restricted to the unit sphere $S = S^{n-1}$, $\sigma$ gives the volume form on $S^{n-1}$.

**Lemma 1.1.** For any function $p_{-\lambda}(\xi)$ which is homogeneous of degree $-\lambda$, the form $p_{-\lambda} \sigma$ is closed.
**Proof.** We have

\[
d(p_{-n}\sigma) = \sum_{j=1}^{n} \frac{\partial p_{-n}}{\partial \xi_j} d\xi_j \wedge \sigma + p_{-n} d\sigma
\]

\[
= \sum_{j=1}^{n} \frac{\partial p_{-n}}{\partial \xi_j} \xi_j d\xi_1 \wedge d\xi_2 \wedge \cdots \wedge d\xi_n + np_{-n} d\xi_1 \wedge \cdots \wedge d\xi_n = 0
\]

by Euler’s theorem. □

We will consider the integral

\[
\int_S p_{-n}\sigma
\]

(1.3)

over the unit sphere oriented by the outer normal. For a bounded domain \(D \subset \mathbb{R}^n\) containing the origin,

\[
\int_{\partial D} p_{-n}\sigma = \int_S p_{-n}\sigma
\]

(1.4)

since the form \(p_{-n}\sigma\) is closed. Here we suppose that \(\partial D\) is also oriented by the outer normal, otherwise we have to change the sign in (1.4).

Consider the behavior of (1.3) under a linear change of variables. Let \(g\) be a linear map, and let \(\eta = g\xi\). Using (1.4) with the proper sign, we get

\[
\int_S p_{-n}(\eta) \sigma_\eta = \pm \int_S g^*(p_{-n}(\eta) \sigma_\eta) = \pm \int_S (g^* p_{-n}(\eta) \sigma_\eta)
\]

\[
= \pm \int_S p_{-n}(g\xi) (g^* \sigma)_\xi = |\det g| \int_S p_{-n}(g\xi) \sigma_\xi,
\]

(1.5)

since, under the linear change of variables above,

\[(g^* \sigma)_\xi = |\det g| \sigma_\xi.\]

Equality (1.4) also holds for some unbounded domains \(D\). We will need the case when \(D\) is the cylinder \(\{\xi \in \mathbb{R}^n : |\xi| < 1, \xi_n \in \mathbb{R}\}\). (Here, \(\xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}\)). Denoting by \(S'\) and \(\sigma'\) the \(n-2\)-dimensional unit sphere and the corresponding \(n-2\)-form we obtain from (1.4)

\[
\int_S p_{-n}\sigma = \int_{S'} \left( \int_{-\infty}^{\infty} p_{-n}(\xi', \xi_n) d\xi_n \right) \sigma'.
\]

(1.6)

The orientation of \(S'\) is completely defined by this equality.
Lemma 1.2. Let $p_{-n}$ be a derivative

$$p_{-n} = \frac{\partial}{\partial \zeta_k} p_{-(n-1)},$$

where $p_{-(n-1)}$ is a smooth homogeneous function on $\mathbb{R}^n \setminus \{0\}$ of degree $-(n-1)$. Then

$$\int_S p_{-n} \sigma = 0.$$  

Proof. Without loss of generality take $k = n$. Then by (1.6)

$$\int_S p_{-n} \sigma = \int_S \left( \int_{-\infty}^{\infty} \frac{\partial p_{-(n-1)}}{\partial \zeta_n} d\zeta_n \right) \sigma' = 0,$$

since the inner integral is equal to

$$p_{-(n-1)}(\zeta', \infty) - p_{-(n-1)}(\zeta', -\infty)$$

and $p_{-(n-1)}$ vanishes at infinity.

Lemma 1.2 raises the question as to whether a homogeneous function may be represented as a sum of derivatives.

Lemma 1.3. Let $p$ be a homogeneous function on $\mathbb{R}^n \setminus \{0\}$. Each of the following conditions is sufficient for $p$ to be a sum of derivatives:

(i) $\deg p \neq -n$

(ii) $\deg p = -n$ and $\int_S p \sigma = 0$.

(iii) $p = \xi^\alpha \partial^\beta q$ where $q$ is a homogeneous function and $|\beta| > |\alpha|$.

Proof. (i) If $\deg p = \lambda \neq -n$, then

$$\sum_{j=1}^{n} \frac{\partial}{\partial \zeta_j} (\xi^\alpha p) = \sum_{j=1}^{n} \xi_j \frac{\partial p}{\partial \zeta_j} + np = (n + \lambda) p$$

by Euler’s theorem.

(ii) On the unit sphere $S$ consider the equation

$$A_S q = p |_S$$

where $A_S$ is the Laplace–Beltrami operator, and $p |_S$ denotes the restriction of $p$ to $S$. This equation has a solution since $p |_S$ is orthogonal to $\text{Ker} A_S$ which consists of the constants.
Denoting $|\xi|$ by $r$ and applying the Laplace operator

$$A = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} d_s$$

to the function $(1/r^{n-2}) q$ we obtain

$$\frac{1}{r^{n-2}} A_s q = \frac{1}{r^2} p_s q = p_s |_{s = s},$$

from which the second statement follows.

(iii) Let $\partial^\beta = (\partial/\partial \xi_j) \partial^\gamma$. Then

$$p = \xi^\alpha \partial/\partial \xi_j \partial^\gamma q = \xi^\alpha (\xi^\beta \partial^\gamma q) \partial/\partial \xi_j,$$

and the case of multi-indices $\alpha, \beta$ is reduced to the case $\alpha - \{ j \}$, $\gamma$ with $|\gamma| = |\beta| - 1$. Hence, the third statement follows by induction.

Now let $M$ be a closed compact manifold of dimension $n > 1$. Consider the algebra $\mathcal{A} = \Psi^\infty / \Psi^{-\infty}$, where $\Psi^\infty$ denotes the algebra of all classical scalar-valued pseudodifferential operators on $M$ and $\Psi^{-\infty}$ its ideal of smoothing operators. We assume all orders to be integers. Let $U$ be an arbitrary local coordinate chart. An operator $P \in \Psi^\infty$ of order $m \in \mathbb{Z}$ on $U \subset M$ is defined up to a smoothing operator by its “symbol”

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x, \xi);$$

(1.7)

this is an infinite formal sum of functions $p_j(x, \xi)$ on $U \times (\mathbb{R}^n \setminus \{0\})$, which are homogeneous in $\xi$ of degree $k$, $k = m, m-1, ...$. The present version of symbol differs from the usual notion of the complete symbol only by a term of order $-\infty$, but it is sufficiently precise for our purposes.

The form $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ defines an orientation of $U$ and induces the orientation of $\mathbb{R}^n$ given by $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. For $P \in \mathcal{A}$ with symbol $p$ we define the local density $\text{res}_x P$, $x \in M$, by

$$\text{res}_x P = \left( \int_S p_{-n}(x, \xi) \sigma \right) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$ (1.8)

The definition of the noncommutative residue and its properties are given in the following theorem.
Theorem 1.4 (Wodzicki). Expression (1.8) is a density on M not depending on the local representation of the symbol, so that

$$\text{res } P = \int_M \text{res}_x P$$

(1.9)

is well-defined.

For any $P, Q \in \mathcal{A}$

$$\text{res}[P, Q] = 0,$$

(1.10)

hence the noncommutative residue is a trace on the algebra $\mathcal{A}$.

Any trace defined on the algebra $\mathcal{A}$ coincides with the trace res up to multiplication by a constant.

The case of pseudodifferential operators with values in sections of vector bundles over $M$ is an easy consequence, cf. Remark 1, below.

Note. Formula (1.9) can be recast as an integral on the cosphere bundle of $M$: the local density $p \wedge dx_1 \wedge \cdots \wedge dx_n$ on the cosphere bundle $S^*U$ of $U$ can be patched to define a global density $\Omega_p$, and $\text{res } P = \int_{S^*M} \Omega_p$.

Denoting by $\omega$ the canonical symplectic form on $T^*M$ and by $\rho$ the Euler vector field one has

$$p \wedge dx_1 \wedge \cdots \wedge dx_n = (-1)^{n(n-1)/2} \frac{1}{n!} (p \rho \cdot \omega^n)_0,$$

(1.11)

where $(\cdots)_0$ is the homogeneous component of degree 0 in an asymptotic expansion of $p \rho \cdot \omega^n$ into homogeneous forms. (The notation $v \cdot \omega$ stands for the contraction—or interior product—of the form $\omega$ with the vector field $v$).

Proof. Under a change of variables $x = f(y)$ the symbol $p(x, \xi)$ transforms to a symbol $\tilde{p}$ according to the formula

$$\tilde{p}(y, \xi) = \sum_{|\alpha| \leq |x|/2} \partial_{\xi}^{\alpha} p(f(y), \xi) \varphi_\alpha(y, \xi),$$

(1.12)

where $\varphi_\alpha(y, \xi)$ are polynomials in $\xi$ of degree $\leq |x|/2$ and $\varphi_0 = 1$ (see Hörmander [7], formula (18.1.30)). Using (1.5) and (1.12) we get

$$\int_S \tilde{p}(y, \eta) \sigma_n = |\det f'(y)| \int_S \tilde{p}(y, f'(y) \xi) \sigma_\xi$$

$$= |\det f'(y)| \sum_{|\alpha| \leq |x|/2} \int_S (\partial_{\xi}^{\alpha} p(f(y), \xi) \varphi_\alpha(y, \xi)) \sigma_\xi$$

$$= |\det f'(y)| \int_S p(x, f(y), \xi) \sigma_\xi$$

(1.13)
since the terms with $|x| > 0$ do not contribute to the integral in virtue of Lemma 1.3(iii). The transformation law (1.13) shows that expression (1.8) is indeed a density on $M$, so that (1.9) is well-defined.

We may proceed considering the operators whose symbols have supports in a fixed coordinate chart. The general case may be reduced to this special one using a partition of unity since the density $\text{res}_x P$ does not depend on the choice of local coordinates.

To prove (1.10), consider two operators $P, Q$ with symbols $p$ and $q$ supported in a coordinate chart $U$. Without loss of generality, we shall assume that $U$ is diffeomorphic to an open ball of $\mathbb{R}^n$. The symbol of $[P, Q]$ is given by

$$\sum_{|\alpha| \neq 0} \frac{(-i)^{|\alpha|}}{\alpha!} (\partial^{\alpha}_x p \partial^*_x q - \partial^{\alpha}_x q \partial^*_x p).$$

(1.14)

This expression may be represented as a sum of derivatives

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} A_j + \frac{\partial}{\partial x_j} B_j,$$

(1.15)

where $A_j$ and $B_j$ are bilinear expressions in $p$ and $q$ and their derivatives. In particular, they have compact supports contained in $U$. Thus, the integrals over $S$ of $(\partial/\partial x_j) A_j$ vanish by Lemma 1.2, while the integrals of $(\partial/\partial x_j) B_j$ over $U$ vanish, since all $B_j$ have compact support in $U$. This proves (1.10).

We will need the explicit expressions of $A_j$ and $B_j$ for the terms in (1.14) with $|\alpha| = 1$, that is

$$-i \sum_{k=1}^{n} \frac{\partial p}{\partial x_k} \frac{\partial q}{\partial x_k} \frac{\partial q}{\partial x_k} \frac{\partial p}{\partial x_k} = -i \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left( p \frac{\partial q}{\partial x_k} - q \frac{\partial p}{\partial x_k} \right).$$

(1.16)

Finally, to prove uniqueness, consider an operator $P$ with symbol $p$ supported in a coordinate chart $U$ and let $\xi_j, \bar{\xi}_j$ denote any symbols with supports in $U$ coinciding with $x_j$ and $\bar{x}_j$ on the support of $p$. Then, taking $q = \bar{x}_j$ or $q = x_j$ in (1.16), we obtain

$$[p, \bar{x}_j] = -i \frac{\partial p}{\partial \bar{x}_j}; \quad [p, x_j] = i \frac{\partial p}{\partial x_j}$$

(1.17)

Given a trace $\tau$ on the whole algebra of complete symbols the equalities (1.17) imply that

$$\tau \left( \frac{\partial p}{\partial \xi_j} \right) = \tau \left( \frac{\partial p}{\partial x_j} \right) = 0$$

(1.18)
since the trace must vanish on commutators. Let \( p = \sum_{k=m}^{n} p_k \in \mathcal{P}^\infty/\mathcal{P}^{-\infty} \)
and define \( \overline{p_{-\sigma}(x)} = (1/\text{vol } S) \int_S p_{-\sigma}(x, \xi) \sigma_\xi. \) Applying Lemma 1.3(i) to \( p_k \)
for all \( k \neq -n, \) there exist \( n \) functions \( q^{(j)}_k(x, \xi), 1 \leq j \leq n, \) homogeneous of
degree \( k + 1 \) in \( \xi \) such that \( p_k = \sum_{1 \leq j \leq n} \partial_\xi q^{(j)}_k. \) Define, for all \( 1 \leq j \leq n, \)
\( b_j(x, \xi) = \sum_{k=m, k \neq -n} q^{(j)}_k. \) One has
\[
p(x, \xi) - \overline{p_{-\sigma}(x)} |_{\xi}^{-m} = \sum_{j=1}^{n} \partial_\xi b_j(x, \xi) + (p_{-\sigma}(x, \xi) - \overline{p_{-\sigma}(x)} |_{\xi}^{-m}).
\]
Since
\[
\int_S (p_{-\sigma}(x, \xi) - \overline{p_{-\sigma}(x)} |_{\xi}^{-m}) \sigma_\xi = 0,
\]
Lemma 1.3(ii) shows that the expression \( (p_{-\sigma}(x, \xi) - \overline{p_{-\sigma}(x)} |_{\xi}^{-m}) \) is a
(finite) sum of derivatives in the variable \( \xi. \) Putting this together, it follows that
\[
\tau(p) = \tau(\overline{p_{-\sigma}(x)} |_{\xi}^{-m})).
\]
Now, the map \( C_0^\infty(U) \ni f \mapsto \mu(f) = \tau(f |_{\xi}^{-m}) \) defines a \( C \)-linear form on
\( C_0^\infty(U); \) it follows from (1.17) above that \( \mu(\partial_\xi f) = 0 \) for all \( 1 \leq j \leq n \) and
\( f \in C_0^\infty(U). \) Hence, since \( U \) is diffeomorphic to an open ball of \( \mathbb{R}^n, \) there
exists \( c \in C \) such that \( \mu(f) = c \int_U f(x) \, dx \) for all \( f \in C_0^\infty(U). \)

Remark 1. The theorem remains valid for pseudodifferential operators
acting on sections of vector bundles over \( M, \) if we replace \( p_{-\sigma} \) be the
matrix trace \( \text{Tr} \ p_{-\sigma}. \) Thus, in the general case, the definition of the non-
commutative residue will read
\[
res \, P = \int_M \int_S \text{Tr} \ p_{-\sigma}(x, \xi) \sigma_\xi \, dx_1 \wedge \cdots \wedge dx_n.
\]
(1.19)

Remark 2. As may be seen from the proof, no continuity condition is
required for the uniqueness of the noncommutative residue.

2. BOUTET DE MONVEL’S ALGEBRA

Let \( M \) be a compact manifold with boundary \( \partial M \) and dimension
\( \dim M = n \geq 2. \) In a neighborhood of the boundary we consider local coordinates
\( x', x_n \) where \( x' = (x_1, \ldots, x_{n-1}) \) are coordinates on \( \partial M \) and \( x_n \) is the
godesic distance to \( \partial M \) in some Riemannian metric. So, any boundary
cordinate chart \( U \) is diffeomorphic to the closed half-space \( R_n^- = \{ x_n \geq 0 \}, \) and transition diffeomorphisms change \( x' \in R^{n-1} = \partial R_n^- \) only,
while $x_n$ remains unchanged. Finally, we shall denote by $\theta$ the indicator function of the open half-line $]0, +\infty[ \subset R$.

For a detailed introduction to Boutet de Monvel's algebra see Boutet de Monvel [2], Grubb [5], Rempel–Schulze [12] or Schrohe–Schulze [13]. In the following we will give a review of some basic fact we need.

By $\mathcal{B}^\infty$ let us denote the algebra of all operators in Boutet de Monvel’s calculus of arbitrary (integer) order and type; by $\mathcal{B}^{-\infty}$ denote the ideal of all regularizing elements of arbitrary type in $\mathcal{B}^\infty$. We will be interested in the quotient $\mathcal{B} = \mathcal{B}^\infty / \mathcal{B}^{-\infty}$. While in the pseudodifferential calculus, an element of $\Psi^\infty / \Psi^{-\infty}$ is given by its pseudodifferential symbol, an element of $\mathcal{B}$ is described by a symbol tuple $(p_i, p_b)$ consisting of the interior symbol $p_i$ and the boundary symbol $p_b$. The interior symbol is a classical pseudodifferential symbol which in addition satisfies the so-called transmission condition at the boundary. This is a technical condition that will not be of major importance here. It is required so that the various compositions can be performed within the calculus; moreover, it guarantees that functions which are smooth up to the boundary are mapped to functions that are smooth up to the boundary.

The boundary symbol on the other hand is a family of Wiener–Hopf type operators on the half-line $R_+$, parametrized by the cotangent bundle of the boundary. In fact, the boundary symbols behave like operator-valued classical pseudodifferential symbols on the boundary. If one denotes by $H^+$ the set of Fourier transforms of functions of the form $\theta u$ with $u \in \mathcal{S}(R)$, these symbols take values in an algebra of operators on $H^+ \otimes C^k$ that are of the form “Wiener–Hopf+compact”, see also the remark before Definition 2.2.

The elements with vanishing interior symbol form an ideal $\mathcal{B}_0$ of $\mathcal{B}$. Our construction starts from the following observation. The usual noncommutative residue restricted to the interior symbols will not produce a trace: applying it to a commutator will give rise to (in general) nonvanishing remainders at the boundary. In fact, there is no trace on the interior symbols alone—this will play a decisive role in the uniqueness part of our proof in Theorem 3.2 and Lemma 3.3. Hence any construction of a noncommutative residue must involve the boundary symbols.

We shall therefore first focus on the algebra $\mathcal{B}_0$. Using a trace “tr” on the Wiener–Hopf type algebra in which the boundary symbols take their values, we construct a trace on $\mathcal{B}_0$ essentially by composing “tr” with the $n-1$-dimensional noncommutative residue at the boundary. Indeed, up to scalar multiples, this is the only way to obtain a continuous trace on $\mathcal{B}_0$ as we show in Proposition 2.3.

The next task then is to prove that the noncommutative residue on the interior symbols can be combined with this new trace on $\mathcal{B}_0$ to a trace on $\mathcal{B}$. This is the technical part of the proof that also involves a healthy dose
of computations. In order to make it more transparent we proceed in two steps. We first consider the subalgebra $B_s$ of $B$ consisting of those elements of $B$ whose interior symbol is independent of $x_n$ close to the boundary. We next show that our construction furnishes a trace on $B_s$. The decisive identities are (2.21), (2.22), and (2.23), in particular the last one: it shows how, in a commutator, the contribution from the boundary symbol cancels the remainder from the interior symbol. In Section 3 we will eventually treat the full algebra $B$.

In order to introduce the interior and boundary symbols we need some preparations. Recalling that $H^+$ is the space of all Fourier transforms of functions of the form $\theta u$, $u \in \mathcal{S}(R)$, it is easily seen that $H^+$ consists precisely of all functions $h \in C^\infty(R)$ which have an analytic extension to the lower complex half-plane $\{ \text{Im} \, \zeta < 0 \}$ such that for all nonnegative integer $l$

$$
\frac{d^l}{d\zeta^l}(\zeta) \sim \sum_{k=0}^l \frac{d^l}{d\zeta^k}(c_k) \\
(2.1)
$$

as $|\zeta| \to \infty$, $\text{Im} \, \zeta \leq 0$.

Similarly, $H^-_0$ is the space of all Fourier transforms of functions of the form $(1 - \theta) u$, with $u \in \mathcal{S}(R)$. It can be characterized as the space of all functions in $C^\infty(R)$ that have an analytic extension into the upper half-plane $\{ \text{Im} \, \zeta > 0 \}$ and an asymptotic expansion (2.1) as $|\zeta| \to \infty$, $\text{Im} \, \zeta \geq 0$, cf. [12], Section 2.1.1.1.

Finally, $H'$ is the space of all polynomials. We let $H^{-} = H^-_0 \oplus H'$ and $H = H^+ \oplus H^-$. By $\Pi^+$ (resp. $\Pi^-$) we denote the projections onto $H^+$ (resp. $H^-$) parallel to $H^-$ (resp. $H^+$). For calculations it is convenient to think of $H$ as a space of rational functions having no poles on the real axis (these functions form a dense set in the topology of $H$). On these functions, the projectors $\Pi^\pm$ may be represented by Cauchy integrals

$$
(\Pi^\pm h)(\xi_n) = \pm \frac{1}{2\pi i} \oint_{\Gamma^\mp} \frac{h(\eta_n)}{\eta_n - \xi_n \pm i0} \, d\eta_n,
$$

where $\Gamma^\pm$ is a contour consisting of a segment of the real axis and of a half-circle surrounding all the singularities of $h$ in the upper half-plane. Introduce also the functional $\Pi'$ on $H$, defined by

$$
\Pi' h = \lim_{x_n \to 0^+} (\mathcal{F}^{-1} h)(x_n) = \frac{1}{2\pi i} \oint_{\Gamma^+} h(\xi_n) \, d\xi_n.
$$

For rational functions the contour $\Gamma^+$ may be shifted slightly into the upper (lower) half-plane. Clearly, $\Pi'$ vanishes on the subspace $H^-$. For functions $h \in H \cap L^1(R)$ the integral may be taken over the real axis instead
of $T^*$. Moreover, if $h \in H^+ \cap L^1(R)$, then $\Pi'h = 0$, because $h$ is holomorphic in the lower half-plane with an estimate $|h(\xi_n)| = O(1/|\xi_n|^2)$.

Let us now focus first on the interior symbols. The interior symbols are classical pseudodifferential symbols in the sense of Section 1, cf. (1.7), except that the coordinate $x$ varies in $\mathbb{R}^n$ for boundary charts. Moreover, they have the transmission property. For a classical pseudodifferential symbol $p$ with an asymptotic expansion $p \sim \sum p_i$ into homogeneous terms $p_i$ of degree $f$ this means that in every boundary chart we have

$$D_\xi^k D_\xi^l p_i(x', 0, 0, +1) = e^{int - |\xi|^2} D_\xi^k D_\xi^l p_i(x', 0, 0, -1)$$

for every multi-index $\alpha$ and all $k \in \mathbb{N}$, cf. [12], Section 2.2.2.3. In particular, we will then have $D_\xi^k D_\xi^l p_i(x', 0, \xi', \xi_n) \in H$ as a function of $\xi_n$ for all fixed $(x', \xi') \in T^*\partial M \setminus \{0\}$.

We make the following observations.

In a boundary chart the composition formula for interior symbols on $M$ takes the form

$$p_1(x', x_n, \xi', \xi_n) \circ p_2(x', x_n, \xi', \xi_n) = \sum_{|\alpha| \leq 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_{\xi_n} p_1 \circ \partial^\alpha_{\xi_n} p_2,$$  \hspace{1cm} (2.2)

where $\circ$ means the composition of symbols on $\partial M$ and $x_n$ as well as $\xi_n$ are regarded as parameters, that is

$$p_1 \circ p_2 = \sum_{|\alpha| \geq 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_{\xi_n} p_1 \circ \partial^\alpha_{\xi_n} p_2.$$  \hspace{1cm} (2.3)

**Lemma 2.1.** One has the following formula for a commutator in a boundary chart:

$$[p_1, p_2] = \sum_{k=0}^\infty \frac{(-i)^k}{k!} \left[ \partial^k_{\xi_n} p_1, \partial^k_{\xi_n} p_2 \right] + \partial_{\xi_n} A_n + \frac{\partial}{\partial x_n} B_n.$$  \hspace{1cm} (2.4)

Here $\left[ \cdot, \cdot \right]$ denotes the commutator with respect to $\circ$,

$$A_n = \sum_{m, j \geq 0} \frac{(-i)^{m+j+1}}{(m+j+1)!} \partial^m_{\xi_n} \partial^j_{\xi_n} p_1 \circ \partial^m_{\xi_n} \partial^j_{\xi_n} p_2$$  \hspace{1cm} (2.5)

and

$$B_n = -\sum_{m, j \geq 0} \frac{(-i)^{m+j+1}}{(m+j+1)!} \partial^m_{\xi_n} \partial^j_{\xi_n} p_1 \circ \partial^m_{\xi_n} \partial^j_{\xi_n} p_2.$$  \hspace{1cm} (2.6)

**Proof.** This is an immediate consequence of the composition formula (2.2). The details are left to the reader. \hfill $\blacksquare$
In particular, the terms in (2.5) and (2.6) with \( j = m = 0 \) give (1.16).

Next we describe the boundary symbols. The boundary symbol is a family of operators parametrized by \( T^* \partial M \setminus \{0\} \). In a local chart on \( \partial M \) it has the form of a \( 2 \times 2 \) matrix

\[
\begin{pmatrix}
  b(x', \xi', D_n) & k(x', \xi', D_n) \\
  t(x', \xi', D_n) & q(x', \xi')
\end{pmatrix}, \quad (x', \xi') \in T^* \partial M \setminus \{0\}.
\]

(2.7)

It acts on pairs of the form \((\xi, \eta)\), where \( \eta \in H^+ \) is in general vector-valued, and \( \nu \) is a vector in \( C^k \). The entries of the above matrix (2.7) are operators given again by symbols \( b = b(x', \xi', \xi_a, \eta_a) \), \( k = k(x', \xi', \xi_a) \), \( t = t(x', \xi', \xi_a) \), and \( q = q(x', \xi') \), respectively. First of all, these symbols are formal sums of jointly homogeneous smooth functions with respect to all the variables except for \( x' \). So,

\[
\begin{align*}
  b(x', \xi', \xi_a, \eta_a) &\sim \sum_{-\infty < l \leq m} b_l(x', \xi', \xi_a, \eta_a), \\
  k(x', \xi', \xi_a) &\sim \sum_{-\infty < l \leq m} k_l(x', \xi', \xi_a), \\
  t(x', \xi', \xi_a) &\sim \sum_{-\infty < l \leq m} t_l(x', \xi', \xi_a), \\
  q(x', \xi') &\sim \sum_{-\infty < l \leq m} q_l(x', \xi').
\end{align*}
\]

where, for \( \lambda > 0 \),

\[
\begin{align*}
  b_l(x', \lambda \xi', \lambda \xi_a, \lambda \eta_a) &= \lambda^l b_l(x', \xi', \xi_a, \eta_a), \\
  k_l(x', \lambda \xi', \lambda \xi_a) &= \lambda^l k_l(x', \xi', \xi_a), \\
  t_l(x', \lambda \xi', \lambda \xi_a) &= \lambda^l t_l(x', \xi', \xi_a), \\
  q_l(x', \lambda \xi') &= \lambda^l q_l(x', \xi').
\end{align*}
\]

Under a change of variables on \( \partial M \) they obey the same rule (1.12) as symbols on \( \partial M \), that is with respect to the variables \( x', \xi' \), the extra variables \( \xi_a, \eta_a \) can be considered as parameters. In order to state the additional properties of the symbols and the way they act we consider them separately.

1. \( \) The symbol \( b \) is called the singular Green symbol. For every \( l \) and fixed \( x', \xi' \)

\[
b_l(x', \xi', \xi_a, \eta_a) \in H^+ \hat{\Theta}_a H^-
\]
(where as usual \(\widehat{\Theta}_a\) denotes Grothendieck’s completion of the algebraic tensor product). The operator \(b(x', \xi', D_a): H^+ \to H^+\) is given by

\[
[b(x', \xi', D_a) h](\xi_n) = \Pi_{\xi_n}(b(x', \xi', \xi_n, \eta_n) h(\eta_n)).
\]

Singular Green symbols on \(\partial M\) form an algebra under composition; this algebra is denoted by \(\mathcal{G}\).

(2) For fixed \(x', \xi'\), each component \(k_j(x', \xi', \xi_n)\) of the potential symbol \(k(x', \xi', \xi_n)\) belongs to \(H^+ \otimes (C^k)^*\) with respect to \(\xi_n\). The operator \(k(x', \xi', D_a): C^k \to H^+\) acts on \(v \in C^k\) by multiplication \(v \mapsto k(x', \xi', \xi_n) v \in H^+\).

(3) For fixed \(x', \xi'\), each component \(t_{ij}(x', \xi', \xi_n)\) of the trace symbol \(t(x', \xi', \xi_n)\) belongs to \(H^- \otimes C^k\) with respect to \(\xi_n\). The operator \(t(x', \xi', D_a): H^+ \to C^k\) acts by

\[
t(x', \xi', D_a) h = \Pi'(t(x', \xi', \xi_n) h(\xi_n)).
\]

(4) The symbol \(q = q(x', \xi')\) is simply a classical pseudodifferential symbol on \(\partial M\) in the sense of (1.7) with values in \(\mathcal{L}(C^k)\); \(q(x', \xi')\) and \(q_l(x', \xi')\) act by matrix multiplication on \(C^k\).

Given two operators in \(\mathcal{B}\) with symbols \((p_{i1}, p_{j1})\) and \((p_{i2}, p_{j2})\) the composition is again an operator in \(\mathcal{B}\). It has the symbol \((p_i, p_j)\), where \(p_i = p_{i1} + p_{i2}\) simply is the composition of the pseudodifferential symbols in the sense of (2.2); it again satisfies the transmission condition. The resulting boundary symbol has the form

\[
p_{b1 \cdot \cdot b2} = \\
+ \left( L(p_{i1}, p_{i2}) + p_{i1}^* (D_a) b_{j1}(D_a) + b_{j1}(D_a) p_{i1}^* (D_a) \right) p_{j2}^* (D_a) k_{j2}^* (D_a) t_{ij}(D_a) \right)
\]

Here, \(p_{b1 \cdot \cdot b2}\) is the pseudodifferential composition of \(p_{b1}\) and \(p_{b2}\) with respect to the variables \((x', \xi')\), cf. (2.3), together with composition of the operator-valued matrices (2.7). The terms in the second summand come from the interior symbols. There, the composition is the pseudodifferential composition for operator-valued symbols with respect to \((x', \xi')\) cf. (2.3). We have denoted the entries of \(p_{b_j}, j = 1, 2\) by \(b_j, k_j,\) and \(t_{ij}\) and omitted the variables \((x', \xi')\) for better legibility.

(i) \(L(p_{i1}, p_{i2})\) is the so-called leftover term. It is induced by the particular way the action of a pseudodifferential operator \(P\) on the manifold with boundary \(M\) is defined: we assume that \(M\) is embedded in a manifold

\[
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\]
\[
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\]
without boundary and that $P$ extends to it. Given a function or distribution on $M$ of sufficiently high regularity we first extend it by zero to the full manifold, then apply $P$ and finally restrict to $M$, in other words we apply the operator $P_M = r_M P e_M$; here $e_M$ denotes extension by zero and $r_M$ restriction to $M$. Given two pseudodifferential operators $P_1$ and $P_2$ with interior symbols $p_{i1}$ and $p_{i2}$ respectively, the difference $[P_1]_M [P_2]_M - [P_1 P_2]_M$ turns out to be a singular Green operator with an associated singular Green boundary symbol operator $L(p_{i1}, p_{i2})$. The asymptotic expansion of the associated singular Green symbol can be computed from the knowledge of $p_{i1}$, $p_{i2}$, and their derivatives at the boundary. Obviously it is zero if either $p_{i1}$ or $p_{i2}$ is zero.

(ii) Given an interior symbol $p_i$, the operator $p_i^+ (x', \xi', D_n): H^+ \to H^+$ is induced from the action of the interior symbol in the normal direction for fixed $(x', \xi')$. More precisely, understanding $p_i$ as a full classical symbol rather than the associated formal sum, one lets

$$p_i^+ (x', \xi', D_n) h = \mathcal{F} (u),$$

where

$$u(x_n) = \frac{1}{2\pi} \int e^{ix_n \cdot \xi} p_i (x', x_n, \xi', \xi_n) h(\xi_n) \, d\xi_n.$$

The last integral should be understood as an oscillatory integral. It is a consequence of the transmission property that $u|_{R_+} \in \mathscr{S}'(R_+)$, so that $\mathcal{F} (u_h) \in H^+$. This, however, is of minor importance here. We shall mainly be interested in the (nontrivial) fact that for a singular Green boundary symbol operator $b(x', \xi', D_n)$ both compositions

$$p_i^+ (x', \xi', D_n) b(x', \xi', D_n) \quad \text{and} \quad b(x', \xi', D_n) p_i^+ (x', \xi', D_n)$$

are singular Green boundary symbol operators and that the asymptotic expansion of the corresponding symbols can be computed from the knowledge of $p_i$ and its derivatives at the boundary.

More information will be given later when we need it.

Remark. For simplicity of the exposition we have given a slightly non-standard definition of the boundary symbol operators. In the usual terminology, the boundary symbol is the operator

$$\begin{pmatrix} p_i^+ (x', 0, \xi', D_n) + b(x', \xi', D_n) & k(x', \xi', D_n) \\ t(x', \xi', D_n) & q(x', \xi') \end{pmatrix}, \quad (x', \xi') \in T^* \partial M \setminus \{0\}.$$

Here, $p_i^+ (x', 0, \xi', D_n)$ is the operator constructed from the symbol $p(x', 0, \xi)$. Noting that the only operator which is induced by both a classical
pseudodifferential symbol and a classical singular Green symbol is 0, we can tell the effect of $p_+^+(x', 0, \xi', D_n)$ from that of $g(x', \xi', D_n)$ in the upper left corner. So our definition yields a neat separation of the two symbol levels and makes the presentation easier.

For fixed $x', \xi'$, the boundary symbol operator

$$
\begin{pmatrix}
  p_+^+(x', 0, \xi', D_n) + b(x', \xi', D_n) \\
  t(x', \xi', D_n)
\end{pmatrix}
\begin{pmatrix}
  H^+ \\
  C_k
\end{pmatrix} \rightarrow
\begin{pmatrix}
  H^+ \\
  \mathbb{C}^k
\end{pmatrix}
$$

is an operator of the type Wiener–Hopf + compact: for one thing, $p_+^+(x', 0, \xi', D_n) = \Pi^+ p_0(x', 0, \xi', D_n)$ is Wiener–Hopf like, while the part not involving the pseudodifferential symbol is compact. The composition formula (2.8) shows that we have a commutative algebra modulo compacts. For more information on this point of view see [2, Section 1] and [12, Section 2.1.2.2].

Now we give the following definition:

**Definition 2.2.** (a) $\mathcal{B}_0$ is the subalgebra of $\mathcal{B}$ consisting of all elements with zero interior symbol.

(b) $\mathcal{B}_s$ is the subalgebra of $\mathcal{B}$ where the interior symbol stabilizes near the boundary: $p_0(x', x_n, \xi) = p_0(x', 0, \xi)$ for small $x_n$.

Clearly $\mathcal{B}_0 \subseteq \mathcal{B}_s \subseteq \mathcal{B}$, and $\mathcal{B}_0$ is an ideal in $\mathcal{B}$, since both the resulting pseudodifferential part and the leftover term in any composition will be zero.

**Note.** The following facts are well-known but important:

(i) the identity is not singular Green operator; in particular, $\mathcal{B}_0$ is a nonunital algebra;

(ii) the operator induced by an interior symbol $p_+^+$ is not a singular Green operator unless it is zero; however composing $p_+^+$ with a singular Green operator on the right or on the left yields a singular Green operator.

For fixed $x', \xi'$, and a singular Green boundary symbol operator $b(x', \xi', D_n): H^+ \rightarrow H^+$

acting by

$$
[b(x', \xi', D_n) h](\xi_n) = \Pi_{\eta_n}^+ [b(x', \xi', \xi_n, \eta_n) h(\eta_n)]
$$

$$
= \frac{1}{2\pi i} \int_{\Gamma^+} b(x', \xi', \xi_n, \eta_n) h(\eta_n) d\eta
$$
we define a trace similarly to the trace of usual integral operators

$$\text{tr} \ b(x', \xi', D_n) = \Pi'_{\text{out}} b(x', \xi', \xi_n, \xi_n') = \frac{1}{2\pi} \int_{\Gamma'} b(x', \xi', \xi_n, \xi_n') \, d\xi_n'. \quad (2.9)$$

In case $b(x', \xi', D_n)$ is matrix-valued we additionally take the matrix trace under the integral. We clearly have the trace property:

$$\text{tr}(b_1(x', \xi', D_n) b_2(x', \xi', D_n)) = \text{tr}(b_2(x', \xi', D_n) b_1(x', \xi', D_n)).$$

As indicated by the missing $\xi'$, the composition is with respect to the $x_n$-action only. Moreover, taking the trace of a singular Green boundary symbol operator $b(x', \xi', D_n)$ yields a symbol $\tilde{b}(x', \xi')$ on $\partial M$:

$$\tilde{b}(x', \xi') = \text{tr} b(x', \xi', D_n) = \Pi'_{\text{out}} b(x', \xi', \xi_n, \xi_n') = \frac{1}{2\pi} \int_{\Gamma'} b(x', \xi', \xi_n, \xi_n') \, d\xi_n. \quad (2.10)$$

is a sum of homogeneous component; the component $\tilde{b}_k(x', \xi')$ of degree $k$ is obtained from $\tilde{b}_{k-1}(x', \xi', \eta_n)$. Indeed,

$$\tilde{b}_k(x', t\xi') = \frac{1}{2\pi} \int_{\Gamma'} \tilde{b}_{k-1}(x', t\xi', \xi_n, \xi_n') \, d\xi_n' \quad (2.11)$$

$$= \frac{t}{2\pi} \int_{\Gamma'} \tilde{b}_{k-1}(x', \xi', \eta_n, \eta_n) \, d\eta_n$$

$$= \frac{t^k}{2\pi} \int_{\Gamma'} \tilde{b}_{k-1}(x', \xi', \eta_n, \eta_n) \, d\eta_n$$

$$= t^k \tilde{b}_k(x', \xi'). \quad (2.12)$$

Since the change of variables acts on $x', \xi'$, and does not affect the variables $\xi_n, \eta_n$, of the Green symbol $b(x', \xi', \xi_n, \eta_n)$, $\tilde{b}(x', \xi')$ is indeed a symbol on $\partial M$.

**Proposition 2.3.** Assume that $\partial M$ is connected. For the boundary symbol we use the notation of (2.7). Then the functional

$$\text{res}_{\partial M} p_b = \int_{\partial M} \text{res}_{\xi'} \left\{ \text{tr} \ b(x', \xi', D_n) + \text{Tr} \ q(x', \xi') \right\}$$

$$= \int_{\partial M} \text{res}_{\xi'} (\tilde{b} + \text{Tr} \ q) \quad (2.13)$$
is the unique continuous trace functional on the subalgebra $\mathcal{B}_0 \subset \mathcal{B}$ up to multiplication by a constant.

As before, $\text{Tr}$ denotes the matrix trace on $L(C^k)$.

In the proof of Proposition 2.3, we shall use the following simple argument that we have isolated below in the following:

**Lemma 2.4.** Let $\tau: \mathcal{B}_0 \to C$ be a $C$-linear form. Then $\tau$ is a trace on $\mathcal{B}_0$ if and only if there exists a trace $\tau_1$ on $\mathcal{B}$ (the algebra of singular Green symbols) and a trace $\tau_2$ on $\mathcal{P}^\infty(\partial M)/\mathcal{P}^\infty(\partial M)$ such that for all trace operators $t$ and all potential operators $k$

$$\tau_1(kt) = \tau_2(tk)$$  \hspace{1cm} (2.14)

and

$$\tau \begin{bmatrix} b & k \\ t & q \end{bmatrix} = \tau_1(b) + \tau_2(q).$$ \hspace{1cm} (2.15)

**Proof.** Given a trace $\tau$ on $\mathcal{B}_0$, define $\tau_1(b) = \tau \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\tau_2(q) = \tau \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The identities

$$0 = \tau \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \tau \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};$$

$$0 = \tau \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \tau \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};$$

$$0 = \tau \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \tau \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};$$

imply the assertion. Conversely, given traces $\tau_1$ and $\tau_2$ satisfying the compatibility condition (2.14), it is easily checked that (2.15) defines a trace.

**Proof of Proposition 2.3.** Correctness and the trace property (1.10) follow from Theorem 1.4 applied to $\mathcal{B}_0$ considered as an operator-valued symbol algebra on $\partial M$ whose coefficient trace is

$$\text{tr} b(x', \xi', D_a) + \text{Tr} q(x', \xi'), \quad (x', \xi') \text{ fixed}. \hspace{1cm} (2.16)$$

Conversely, let $\tau$ be a trace on $\mathcal{B}_0$. Applying Lemma 2.4, there exist traces $\tau_1$ and $\tau_2$ such that

$$\tau_1(b) = \tau \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}, \quad \tau_2(q) = \tau \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix}. \hspace{1cm} (2.17)$$
The functional \( \tau_2 \) is a trace on \( \mathcal{P}^+(\partial M)/\mathcal{P}^-(\partial M) \); hence, Wodzicki's result shows that there exists a constant \( c \in \mathbb{C} \) such that \( \tau_2 = c \text{ res} \). Since by assumption \( \tau \) is a trace on \( \mathcal{B}_0 \), (2.14) holds and shows that, for all trace symbols \( t \) and potential symbols \( k \), \( \tau_{\partial}(kt) = c \text{ res}(tk) \). Hence the functional \( \tau = c \text{ res}_{\partial M} \) vanishes on all elements of the form

\[
\left( \sum_{i=1}^{m} k_i t_i \right) \quad \text{for all potential symbols } k_i, k, \text{ all trace symbols } t_i, t \text{ and all pseudodifferential symbol } q.
\]  

(2.18)

for all potential symbols \( k_i, k \), all trace symbols \( t_i, t \) and all pseudodifferential symbol \( q \). Since the set of such elements is dense in \( \mathcal{B}_0 \) (see properties (1)-(2)-(3) after Lemma 2.1) and since the trace \( \tau \) is continuous, one has

\[
\tau = c \text{ res}_{\partial M}.
\]

**Remark.** In case \( \partial M \) consists of finitely many components, the preceding proof shows that we may pick a constant factor for each component. Moreover, it shows that the trace on the singular Green operators, hence that on \( \mathcal{B}_0 \) is uniquely determined by the Wodzicki residue at the boundary. Since the latter is independent of the choice of the metric as we saw in Section 1, so is our construction. The geodesic coordinates are a merely technical tool in this construction.

Next we consider the subalgebra \( \mathcal{B}_s \subseteq \mathcal{B} \).

We use the notation of (2.7) and set for a pair \( P = \{ p_i(x, \zeta), p_b(x', \zeta') \} \in \mathcal{B}_s \)

\[
\text{res } P = \int_{\partial M} \text{res } p_i(x, \zeta) - 2\pi \int_{\partial M} \text{res } (\text{tr } b(x', \zeta', D_n) + \text{Tr } q(x', \zeta')).
\]

(2.19)

**Theorem 2.5.** For any \( P, Q \in \mathcal{B}_s \)

\[ \text{res}[P, Q] = 0. \]

**Proof.** Write \( P = \{ p_i, p_b \}, Q = \{ q_i, q_b \} \). Note that, in general, neither of the two terms in (2.19) vanishes on commutators.

First let us compute the contribution of the interior symbols. In view of Wodzicki's theorem it is sufficient to prove Theorem 2.5 for symbols supported on a boundary chart. Using Lemma 2.1 and formula (1.6) we obtain after a straightforward computation

\[
\int_{\partial M} \text{res } [p_i, q_j] = - \int_{\partial M} \int_S \left( i^{\nu} \sum_{k \cdot \nu} d_{\nu} d_{\nu} \right) d\sigma \wedge \cdots \wedge d\sigma_{n-1},
\]
where, as in Section 1, $S'$ is the $(n-1)$-sphere with volume form $\sigma'$. Since the symbols $p_\nu$, $q_\nu$ do not depend on $x_n$ the only nonvanishing term of (2.6) is that with $m = j = 0$, so

$$
\int_{M} \text{res}_x [p_\nu, q_\nu] = -i \int_{\partial M} \left( \int_{-\infty}^{\infty} dx_n \right) \left. \left( p_\nu \frac{\partial q_\nu}{\partial x_n} \right) \right|_{x_n = 0} \sigma' \times dx_1 \land \cdots \land dx_{n-1}. \tag{2.20}
$$

Now for the boundary symbol part. Denote the entries of the matrix $p_\nu$ by $b_1, k_1, t_1, q_1$, those of $q_\nu$ by $b_2, k_2, t_2, q_2$. Considering the representation (2.8) for the composition of the boundary symbol operators and the fact that $\text{res}$ is a trace on $\partial M$, we conclude that the contribution by $p_\nu$ $q_\nu$ $p_\nu$ is zero. The only terms that may contribute are those in the difference $g_1(x', \xi', D_n) - g_2(x', \xi', D_n)$ where

$$
g_1(D_n) = L(p_\nu, q_\nu) + b_1(D_n) q_\nu(D_n) + b_2(D_n) q_\nu(D_n)
$$

and

$$
g_2(D_n) = L(q_\nu, p_\nu) + q_\nu(D_n) b_1(D_n) + b_2(D_n) p_\nu(D_n).
$$

We will prove the following identities:

$$
\int_{\partial M} \text{res}_x \{ p_\nu(D_n) b_2(D_n) - b_2(D_n) p_\nu(D_n) \} = 0, \tag{2.21}
$$

$$
\int_{\partial M} \text{res}_x \{ b_1(D_n) q_\nu(D_n) - q_\nu(D_n) b_1(D_n) \} = 0, \tag{2.22}
$$

$$
\int_{\partial M} \text{res}_x \{ L(p_\nu, q_\nu) - L(q_\nu, p_\nu) \} = \frac{1}{2\pi} \int_{\partial M} \text{res}_x [p_\nu, q_\nu]. \tag{2.23}
$$

Let us start with (2.21). We have the following asymptotic expansion formulas for the symbols $c_1$ and $c_2$ of the compositions $p_\nu(x', \xi', D_n)$ and $b_2(x', \xi', D_n)$. (2.19), cf. [5, Theorem 2.7.4, 20].

$$
c_1(x', \xi', \xi_n, \eta_n) \sim \sum_{j=0}^\infty \frac{i^j}{j!} \Pi_{\nu=1}^\nu \{ \partial_{x_n}^j p(x', 0, \xi', \xi_n) \}
$$

$$
c_2(x', \xi', \xi_n, \eta_n) \sim \sum_{j=0}^\infty \frac{(-i)^j}{j!} \Pi_{\nu=1}^\nu \{ \partial_{x_n}^j b(x', \xi', \xi_n, \eta_n) \}. \tag{2.24}
$$

$$
c_1(x', \xi', \xi_n, \eta_n) \sim \sum_{j=0}^\infty \frac{i^j}{j!} \Pi_{\nu=1}^\nu \{ \partial_{x_n}^j p(x', 0, \xi', \eta_n) \}
$$

$$
c_2(x', \xi', \xi_n, \eta_n) \sim \sum_{j=0}^\infty \frac{(-i)^j}{j!} \Pi_{\nu=1}^\nu \{ \partial_{x_n}^j b(x', \xi', \xi_n, \eta_n) \}. \tag{2.25}
$$
Since $p_1$ is independent of $x_n$ close to the boundary, this reduces to
\[
c_1(x', \xi', \xi_n, \eta_n) \sim \Pi_{-n}^{+} \{ p_1(x', 0, \xi', \xi_n) \cdot b_2(x', \xi', \eta_n) \}
\]
and
\[
c_2(x', \xi', \xi_n, \eta_n) \sim \Pi_{-n}^{+} \{ b_2(x', \xi', \xi_n, \eta_n) \cdot p_1(x', 0, \xi', \eta_n) \}
\]
respectively. For the associated symbols $\tilde{c}_1$ and $\tilde{c}_2$, cf. (2.10), we then have
\[
\tilde{c}_1(x', \xi') = \Pi_{-n}^{+} \{ p_1(x', 0, \xi', \xi_n) \cdot b_2(x', \xi', \xi_n, \eta_n) \} \big|_{\xi_n = \xi'}
\]
\[
= \Pi_{-n}^{+} \{ p_1(x', 0, \xi', \xi_n) \cdot b_2(x', \xi', \xi_n, \eta_n) \} \big|_{\eta_n = \xi'},
\]
since $\Pi_{-n}^{+} \{ p_1(x', 0, \xi', \xi_n) \cdot b_2(x', \xi', \xi_n, \eta_n) \} \big|_{\eta_n = \xi'} \in H^{-}$, where $H^\prime$ vanishes. In particular, we have, according to (2.11),
\[
(\tilde{c}_1(x', \xi'))_{-(n-1)} = \Pi_{-n}^{+} \{ p_1(x', 0, \xi', \xi_n) \cdot b_2(x', \xi', \xi_n, \eta_n) \} \big|_{\eta_n = \xi'} \cdot -n. \quad (2.26)
\]
Similarly,
\[
(\tilde{c}_2(x', \xi'))_{-(n-1)} = \Pi_{-n}^{+} \{ b_2(x', \xi', \xi_n, \eta_n) \cdot p_1(x', 0, \xi', \eta_n) \} \big|_{\eta_n = \xi'} \cdot -n. \quad (2.27)
\]
This time the reason is that $\Pi_{-n}^{+} \{ b_2(x', \xi', \xi_n, \eta_n) \cdot p_1(x', 0, \xi', \eta_n) \} \big|_{\eta_n = \xi'} \in H^{+} \cap L_1(R)$, where $H^\prime$ also vanishes (recall that $n > 1$ and that the subscript $-n$ denotes the component of homogeneity $-n$). We therefore have
\[
\int_{\mathcal{D}(M)} \underset{\mathcal{D}(M)}{\text{res}} \{ \text{tr} \ c_1(x', \xi', D_n) - \text{tr} \ c_2(x', \xi', D_n) \}
\]
\[
= \Pi_{-1}^{+} \int_{\mathcal{D}(M)} \underset{\mathcal{D}(M)}{\text{res}} \{ p_1(x', 0, \xi', \xi_n) \cdot b_2(x', \xi', \xi_n, \eta_n) \}
\]
\[
- b_2(x', \xi', \xi_n, \eta_n) \cdot p_1(x', 0, \xi', \xi_n) \} = 0
\]
by Theorem 1.4. This proves (2.21). It also proves (2.22), since the situation there is completely analogous.

Hence consider (2.23). The leftover terms depend only on the behavior of the interior symbols near the boundary. In view of the fact that those stabilize near the boundary by assumption, we may as well assume that both $p_1$ and $q$ are independent of $x_n$. Then the action of $p_1^+(x', \xi', D_n)$: $H^{+} \to H^{+}$ is simply given by
\[
[p_1^+(x', \xi', D_n) h](\xi_n) = \Pi_{-1}^{+} \{ p_1(x', 0, \xi', \xi_n) h(\xi_n) \}, \quad (2.28)
\]
similarly for \( q_i \). Moreover,

\[
L(p_i, q_i) = p_i^+(D_n) q_i^+(D_n) - (p_i q_i)^+(D_n) = -p_i^+(D_n) q_i^-(D_n).
\]

Again, we have omitted \( x', \zeta' \), and denoted by \( q_i^-(D_n) \): \( H^+ \rightarrow H^- \) the operator given by \( q_i^-(D_n) h = \Pi^-(g\partial h) \), analogously to (2.28). The standard expression for the symbol of this leftover term, cf. [12] or [2], is not helpful for our purposes. We therefore shall now derive a new representation. Keeping \( (x', \zeta') \) fixed, we let \( p_i = p_i(x', 0, \zeta', \zeta_n) \) and \( q_i = q_i(x', 0, \zeta', \zeta_n) \) be the projections of the interior symbols on \( H^+ \) and \( H^- \), respectively, not to be confused with the operators \( p_i^+(D_n) \) and \( q_i^+(D_n) \).

Applying the operator \( q_i^-(D_n) \) to a function \( h \in H^+ \) and observing that \( \Pi^-(q^+ h) = 0 \) we obtain

\[
\Pi_i^-(q_i(x', 0, \zeta', \zeta_n) h(\zeta_n)) = \Pi^-(q^- h(\zeta_n))
\]

\[
= \frac{1}{2\pi i} \int_{\zeta_n} q^-(-\zeta) \frac{h(\zeta)}{\zeta - \zeta_n - i0} d\zeta
\]

\[
= \frac{1}{2\pi i} \int_{\zeta_n} q^-(-\zeta) - q^-(-\zeta_n) \frac{h(\zeta)}{\zeta - \zeta_n} d\zeta,
\]

since

\[
\frac{q^-(-\zeta_n)}{2\pi i} \int_{\zeta_n} \frac{1}{\zeta - \zeta_n - i0} h(\zeta) d\zeta = 0,
\]

noting that \( h \in H^+ \). A similar argument applied to the function \( v = \Pi^-(q^- h) \in H^- \) yields

\[
\Pi_i^-(p_i(x', 0, \zeta', \zeta_n) v(\zeta_n)) = \Pi^-(p^+ v)(\zeta_n)
\]

\[
= - \frac{1}{2\pi i} \int_{\zeta_n} p^+(\eta_n) \frac{v(\eta_n)}{\eta_n - \zeta_n - i0} d\eta_n
\]

\[
= - \frac{1}{2\pi i} \int_{\zeta_n} p^+(\eta_n) - p^+(\zeta_n) \frac{v(\eta_n)}{\eta_n - \zeta_n} d\eta_n.
\]

Thus, the singular Green boundary symbol operator \( L(p_i, q_i) \) is given by the symbol

\[
b_{p_i q_i}(\zeta_n, \eta_n) = \frac{1}{2\pi i} \int_{\zeta_n} \frac{p^+(-\zeta) - p^+(-\zeta_n)}{\zeta - \zeta_n} \frac{q^-(-\eta_n) - q^-(-\zeta)}{\eta_n - \zeta} d\zeta, \tag{2.29}
\]
\[ b_{p;0} = \frac{1}{(2\pi)^2} \int_{\Gamma^+} d\bar{\xi}_m \int_{\Gamma^+} \frac{(p^+(\xi) - p^+(\xi_n)) \cdot (q^-(\xi) - q^-(\xi_n))}{(\xi - \xi_n)^2} d\xi. \tag{2.30} \]

Considering the inner integral shift the contour \( \Gamma^+ \) to a contour \( \Gamma' \) in the upper half-plane so that \( \Gamma' \) is inside \( \Gamma^+ \). Then the inner integral is equal to

\[ \int_{\Gamma'} p^+(\xi) \cdot (q^-(\xi) - q^-(\xi_n)) (\xi - \xi_n)^2 d\xi = p^+(\xi_n) \cdot \int_{\Gamma'} q^-(\xi) - q^-(\xi_n) (\xi - \xi_n)^2 d\xi \]

and the second term vanishes being analytic inside \( \Gamma' \). Integrating the first term over \( \bar{\xi}_n = \xi \) yielding

\[ b_{p;0} = \frac{1}{(2\pi)^2} \int_{\Gamma'} p^+(\xi) d\xi \cdot \int_{\Gamma'} q^-(\xi) - q^-(\xi_n) (\xi - \xi_n)^2 d\xi_n = \frac{1}{2\pi i} \int_{\Gamma'} p^+(\xi) \frac{\partial q^-(\xi)}{\partial \xi} d\xi = -i \Pi' \left\{ p^+(\xi_n) \cdot \frac{\partial q^-(\xi_n)}{\partial \xi_n} \right\}. \]

Similarly an integration by parts gives

\[ b_{q;0} = i \Pi' \left\{ q^+(\xi_n) \cdot \frac{\partial p^-(\xi_n)}{\partial \xi_n} \right\} = i \Pi' \left\{ \frac{\partial q^+(\xi_n)}{\partial \xi_n} \cdot p^-(\xi_n) \right\}. \]

Thus,

\[ \int_{\partial M} \text{res}_{\xi'} (b_{p;0} - b_{q;0}) \cdot (\partial_{\xi_n} - 1) = \frac{i}{2\pi} \int_{\partial M} \text{res}_{\xi'} \left\{ \left( p^+(\xi_n) \cdot \frac{\partial q^-(\xi_n)}{\partial \xi_n} + p^-(\xi_n) \cdot \frac{\partial q^+(\xi_n)}{\partial \xi_n} \right) \cdot (\partial_{\xi_n} - 1) \right\} d\xi_n \]

\[ = \frac{i}{2\pi} \int_{\partial M} \text{res}_{\xi'} \left\{ p^+ (\xi', 0, \xi_n) \cdot \frac{\partial q^- (\xi', 0, \xi_n, \xi_n)}{\partial \xi_n} \right\} d\xi_n \tag{2.31} \]

since

\[ \Pi' \left( p^+ \cdot \frac{\partial q^+}{\partial \xi_n} \right) = \Pi' \left( p^- \cdot \frac{\partial q^-}{\partial \xi_n} \right) = 0. \]

Notice that we could interchange the order of \( p^- \) and \( \partial_{\xi_n} q^+ \) as a consequence of Theorem 1.4. Using formulas (2.24), (1.6) and (1.8), one obtains at once (2.28). This completes the proof of Theorem 2.5.
For the algebra \( B \), there is no uniqueness property of the noncommutative residue. Examples of trace functionals not coinciding with \( \text{res} \, P \) may be constructed as follows. For \( P = \{ p_\iota, p_\j \} \) take \( p_\iota \big|_{x_\iota = 0} = p_\iota(x', \zeta', \xi_\iota) \). The variable \( \xi_\iota \) is globally defined so that for any \( k = 0, 1, 2, \ldots \) we have a symbol on \( \partial M \) defined by

\[
\alpha_k = \left. \frac{\partial^k}{\partial \xi_\iota^k} (p_\iota \big|_{x_\iota = 0}) \right|_{\zeta_\iota = 0}
\]

and we may define

\[
\text{Tr}_k \, P = \text{res}_{\partial M} \alpha_k \tag{2.32}
\]

taking the noncommutative residue of the symbols \( \alpha_k \) on \( \partial M \). It is easy to verify that these functionals are really traces on \( B \). The reason is that the restriction map

\[
\{ p_\iota, p_\j \} \mapsto p_\iota \big|_{x_\iota = 0}
\]

is an algebra homomorphism from \( B \) to the algebra of classical pseudo-differential operators on \( \partial M \) (under the assumption that \( p_\iota \) does not depend on \( x_\iota \) near the boundary), and any trace on the restricted algebra will serve as a trace for the whole algebra \( B \).

3. THE NONCOMMUTATIVE RESIDUE ON BOUTET DE MONVEL'S ALGEBRA

Now we consider the full algebra \( \mathcal{B} \) of pairs \( \{ p_\iota, p_\j \} \), consisting of the interior symbol \( p_\iota \) and the boundary symbol \( p_\j \) as introduced in Section 2. We define the noncommutative residue by the same formula (2.19) as in the case of \( B \). As before \( \text{res}_x \, p_\iota \) and \( \text{res}_\xi \, p_\j \) are densities on \( M \) and \( \partial M \), respectively. The latter is a consequence of the fact that in a neighborhood of the boundary, \( x_\iota \) has been chosen as the geodesic distance to the boundary, so that a change of variables on a boundary chart does not affect the variables \( x - n \) and \( \zeta_\iota \). Note, however, that the choice of geodesic coordinates merely is a technical tool. The noncommutative residue we define is independent of the metric, because neither Wodzicki’s noncommutative residue depends on it nor does the noncommutative residue on \( B_0 \) as we pointed out in the remark after the proof of Proposition 2.3.

**Theorem 3.1.** The residue (2.19) is a trace on the algebra \( \mathcal{B} \).

**Proof.** Let \( P = \{ p_\iota, p_\j \} \) and \( Q = \{ q_\iota, q_\j \} \) be elements of \( \mathcal{B} \). In order to show that \( \text{res}[P, Q] \) vanishes we will use the same set-up and notations as
in the proof of Theorem 2.5; we suppose that all symbols are supported in
a boundary chart and that the boundary symbols \( p_b \) and \( q_b \) are given by
matrices as in (2.7) whose entries we denote by \( b_1, k_1, t_1, q_1 \) and \( b_2, k_2, t_2, q_2 \),
respectively.

Of course we can rely on what we showed in the proof of 2.5. Since now
\( p_i \) and \( q_i \) may depend on \( x_n \), we will have to revise (2.20). On the other
hand we know that the noncommutative residue is a trace on the ideal
\( \mathcal{B}_0 \subset \mathcal{B} \) of all operators with zero interior symbol. So the contribution of
\( p_b \circ q_b - q_b \circ p_b \) will vanish again, and it will suffice to show the identities
(2.21), (2.22), and (2.23) using the composition formulas (2.24) and (2.25),
plus an asymptotic expansion formula for the symbol of the leftover term
in the \( x_n \)-dependent case, cf. (3.1), below.

In analogy to (2.26) we get

\[
(\tilde{c}_1(x', \xi'))_{-(n-1)} \sim \sum_{j=0}^{\infty} \frac{(-\partial)^j}{j!} \Pi^j \{ \{ \partial_{\xi_n} b_2(x', \xi', \xi_n, \eta_n) \} \mid_{\eta_n - \xi_n} \}_{-n}
\]

\[
\sim \sum_{j=0}^{\infty} \frac{(-\partial)^j}{j!} \Pi^j \{ \{ \partial_{\xi_n} b_2(x', \xi', \xi_n, \eta_n) \} \mid_{\eta_n - \xi_n} \}_{-n}
\]

\[
\partial^j \{ \partial_{\xi_n} b_2(x', \xi', \xi_n, \eta_n) \} \mid_{\eta_n - \xi_n} \}_{-n}
\]

Here we have made use of an induction on \( j \) and the following identity

\[
\frac{\partial}{\partial \xi_n} a(\xi_n, \eta_n) = \frac{\partial}{\partial \eta_n} a(\xi_n, \eta_n) \bigg|_{\eta_n - \xi_n} + \frac{\partial}{\partial \eta_n} a(\xi_n, \eta_n) \bigg|_{\eta_n - \xi_n}
\]

together with the fact that \( \Pi^j \{ (\partial/\partial \xi_n) a(\xi_n, \xi_n) \} \mid_{\eta_n - \xi_n} = 0 \).

Similarly,

\[
(\tilde{c}_2(x', \xi'))_{-(n-1)} \sim \sum_{j=0}^{\infty} \frac{(-\partial)^j}{j!} \Pi^j \{ \{ \partial_{\xi_n} b_2(x', \xi', \xi_n, \eta_n) \} \mid_{\eta_n - \xi_n} \}_{-n}
\]

\[
\partial^j \{ \partial_{\xi_n} b_2(x', \xi', \xi_n, \eta_n) \} \mid_{\eta_n - \xi_n} \}_{-n}
\]

so that

\[
\int_{\partial M} (\text{res}_c \tilde{c}_1 - \text{res}_c \tilde{c}_2) = 0
\]

by Theorem 1.4 for \( \partial M \).
Now for the leftover terms. In the $x_n$-dependent case we will have to replace (2.29) according to [5, Theorem 2.7.7] by

$$ b_{p,q}(\xi_n, \eta_n) \sim \sum_{j,l,m=0}^{\infty} \frac{(-1)^m j^{j+m} l^j}{j! l! m!} \partial_{\xi_n}^j \partial_{\eta_n}^m b_{p,q}\big|_{x_n=0} \partial_{\xi_n}^j \partial_{\eta_n}^m b_{p,q\big|_{x_n=0}}(\xi_n, \eta_n). $$

(3.1)

Abbreviating $a = \partial_{\xi_n}^j p\big|_{x_n=0}$ and $b = \partial_{\eta_n}^m q\big|_{x_n=0}$, each term $b_{ab}$ under the summation in (3.1) denotes the singular Green symbol obtained from the $x_n$-independent symbols $a$ and $b$ by (2.29). Writing $a^\pm = \Pi^\pm a$, $b^\pm = \Pi^\pm b$ we have

$$ b_{ab} = \frac{1}{(2\pi)^2} \int_{\Gamma^+} d\xi_n \int_{\Gamma^+} d\eta_n \frac{a^+(\xi) - a^+(\xi_n)}{\xi - \xi_n} \partial_{\xi_n}^j \frac{b^-(\eta_n) - b^-(\xi)}{\eta_n - \xi} d\eta_n. $$

With the aim of eventually computing $b_{p,q}$ we put $\eta_n = \xi_n$, multiply by $(2\pi)^{-1}$, and integrate over $\xi_n \in \Gamma^+$. We obtain, using integration by parts,

$$ b_{ab} = \frac{1}{(2\pi)^2} \int_{\Gamma^+} d\xi_n \int_{\Gamma^+} d\eta_n \frac{a^+(\xi) - a^+(\xi_n)}{\xi - \xi_n} \partial_{\xi_n}^j \frac{b^-(\eta_n) - b^-(\xi)}{\eta_n - \xi} d\eta_n. $$

This integral may be simplified similarly to (2.30). For the integration over $\xi$ shift the contour $\Gamma^+$ to a contour $\Gamma^+_1$ inside $\Gamma^+$ and note that for fixed $\xi_n$, the function $\partial_{\xi_n}^m (b^-(\xi_n) - b^-(\xi))/(\xi_n - \xi)$ is analytic in the upper half plane $\{ \text{Im} \, \xi > 0 \}$. We get

$$ b_{ab} = \frac{1}{(2\pi)^2} \int_{\Gamma^+_1} d\xi_n \int_{\Gamma^+_1} d\eta_n \frac{a^+(\xi) - a^+(\xi_n)}{\xi - \xi_n} \partial_{\xi_n}^j \frac{b^-(\eta_n) - b^-(\xi)}{\eta_n - \xi} d\eta_n. $$

For the second equality we have interchanged the order of integration and applied Cauchy's theorem for fixed $\xi$. The identity is most easily checked using that

$$ \partial_{\xi_n}^m \frac{b^-(\xi_n) - b^-(\xi)}{\xi_n - \xi} = \partial_{\xi_n}^m \int_{0}^{1} (\partial_{\xi_n} b^-)(\xi_n + \theta(\xi_n - \xi)) d\theta. $$
Thus,

\[
\bar{b}_{p,q} = \frac{1}{2\pi i} \int \frac{b_{p,q}(\xi_a, \xi_n)}{z_n} d\xi_n
\]

\[
\simeq \sum_{j,l,m=0}^{\infty} \frac{i^{j+l+m+1}(-1)^{m+j+1}}{m! l! (m+j+1)} \frac{1}{2\pi i} \int \partial^l_{\xi_n} p_{j}^+(x', 0, \xi', \xi_n) \partial^m_{\xi_n} q_{l}^-(x', 0, \xi', \xi_n) d\xi_n.
\]

The notation should be obvious: we let \( p_{j}^+(x', 0, \xi', \xi_n) = \Pi_{\xi_n}^j p_j(x', 0, \xi', \xi_n) \)
and \( q_{l}^-(x', 0, \xi', \xi_n) = \Pi_{\xi_n}^l q_l(x', 0, \xi', \xi_n) \).

We need to calculate the sum

\[
\sum_{m+l=k} \frac{(-1)^m}{m! l! (m+j+1)}
\]

To this end consider the binomial formula

\[(1-t)^k = k! \sum_{m+l=k} \frac{(-1)^m}{m! l!} t^m.
\]

Multiplying by \( t^j \) and integrating over \([0,1]\) we obtain

\[
k! \sum_{m+l=k} \frac{(-1)^m}{m! l! (m+j+1)} = \int_0^1 t'(1-t)^k \, dt = B(j+1, k+1)
\]

\[
= \frac{k!}{(k+j+1)!}.
\]

Substituting this result we get

\[
\bar{b}_{p,q} = \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k+1}}{(j+k+1)!} \Pi^j_{\xi_n} \partial^j_{\xi_n} p_{j}^+(x', 0, \xi', \xi_n)
\]

\[
\partial^k_{\xi_n} q_{k}^-(x', 0, \xi', \xi_n))
\]

\[
= \sum_{j,k=0}^{\infty} \frac{(-1)^{j+k+1}}{(j+k+1)!} \Pi^j_{\xi_n} \partial^j_{\xi_n} p_{j}^+(x', 0, \xi', \xi_n)
\]

\[
\partial^k_{\xi_n} q_{k}^-(x', 0, \xi', \xi_n))
\]
Similarly,

\[
\bar{b}_{\varphi, r} = \sum_{j, k = 0}^{\infty} \frac{(-i)^{j+k+1}}{(j+k+1)!} \mathcal{H}_{\mathcal{G}} \left( \partial_{x_{ij}} \partial_{x_{ik}} q_{i}^+ (x', 0, \xi', \xi_n) \right) \\
\quad \cdot \partial_{x_{ij}} \partial_{x_{ik}} p_i^- (x', 0, \xi', \xi_n)
\]

\[= - \sum_{j, k = 0}^{\infty} \frac{(-i)^{j+k+1}}{(j+k+1)!} \mathcal{H}_{\mathcal{G}} \left( \partial_{x_{ij}} \partial_{x_{ik}} q_{i}^+ (x', 0, \xi', \xi_n) \right) \\
\quad \cdot \partial_{x_{ij}} \partial_{x_{ik}} p_i^- (x', 0, \xi', \xi_n).
\]

Thus

\[
\int_{\partial \mathcal{M}} (\text{res}_x \bar{b}_{p \varphi} - \text{res}_x \bar{b}_{\varphi, p})
\]

\[= \frac{1}{2\pi} \int_{\partial \mathcal{M}} \int_{\partial S} \int_{-\infty}^{\infty} \sum_{j, k = 0}^{\infty} \frac{(-i)^{j+k+1}}{(j+k+1)!} \left( \partial_{x_{ij}} \partial_{x_{ik}} q_{i}^+ (x', 0, \xi', \xi_n) \right) \\
\quad \cdot \partial_{x_{ij}} \partial_{x_{ik}} p_i^- (x', 0, \xi', \xi_n) + \partial_{x_{ij}} \partial_{x_{ik}} q_{i}^+ (x', 0, \xi', \xi_n) \\
\quad \cdot \partial_{x_{ij}} \partial_{x_{ik}} p_i^- (x', 0, \xi', \xi_n) \right) \cdot d\xi_n \sigma' dx_1 \wedge \cdots \wedge dx_{n-1}.
\]

In the last expression we may interchange the order of \(\partial_{x_{ij}} \partial_{x_{ik}} q_{i}^+ (x', 0, \xi', \xi_n)\)
and \(\partial_{x_{ij}} \partial_{x_{ik}} p_i^- (x', 0, \xi', \xi_n)\) as a consequence of Theorem 1.4. With the considerations justifying (2.31) we then conclude that

\[
\int_{\partial \mathcal{M}} (\text{res}_x \bar{b}_{p \varphi} - \text{res}_x \bar{b}_{\varphi, p})
\]

\[= \frac{1}{2\pi} \int_{\partial \mathcal{M}} \int_{\partial S} \int_{-\infty}^{\infty} \sum_{j, k = 0}^{\infty} \frac{(-i)^{j+k+1}}{(j+k+1)!} \left( \partial_{x_{ij}} \partial_{x_{ik}} q_{i}^+ (x', 0, \xi', \xi_n) \right) \\
\quad \cdot \partial_{x_{ij}} \partial_{x_{ik}} p_i^- (x', 0, \xi', \xi_n) \right) \cdot d\xi_n \sigma' dx_1 \wedge \cdots \wedge dx_{n-1} \tag{3.2}
\]

since the ++ and --- parts vanish after integration with respect to \(\xi_n\).

Now the representation of the commutator \(p_i, q_j - q_j, p_i\) in (2.6) together with (1.6) shows that (3.2) coincides precisely with

\[\frac{1}{2\pi} \int_{\partial \mathcal{M}} \int_{\partial S} \int_{-\infty}^{\infty} (B_n)_{-n} \cdot d\xi_n \sigma' dx_1 \wedge \cdots \wedge dx_{n-1}
\]

\[= \frac{1}{2\pi} \int_{\partial \mathcal{M}} \text{res}_x [p_i, q_j].
\]

Hence the sum of both is zero, and we have proven the theorem. \(\blacksquare\)
Unlike in the case of $B$, the non-commutative residue is the unique continuous trace on the full algebra $B$.

**Theorem 3.2.** Denote by $B$ Boutet de Monvel's algebra on $M$ as introduced in Section 2. Then any continuous trace on $B$ is a scalar multiple of the noncommutative residue $\text{res}$.

**Proof.** Let $\text{Tr}$ be a continuous trace functional on $B$. Choose a boundary chart $U$ that intersects only one component of $\partial M$. Denote by $B^U \subseteq B$ the ideal of those elements whose interior symbol has support in $U$ and whose boundary symbol has support in $U \cap \partial M$. By $B^U_0$ denote the subset of those elements with zero interior symbol. Restricted to $B^U_0$ the trace $\text{Tr}$ must coincide with $c_U \text{res}$ for a suitable constant $c_U$. This is a consequence of the considerations for the uniqueness part in the proof of Theorem 1.4 together with the fact that there is only one continuous trace on the algebra of boundary symbol operators, established in the proof of Proposition 2.3. Then $\text{Tr}' = \text{Tr} - c_U \text{res}$ is a trace functional on $B^U$ vanishing on the subalgebra $B^U_0$.

Clearly, $B^U_0$ is a two-sided ideal in $B^U$, so $\text{Tr}'$ is actually defined on the algebra $B^U/B^U_0$. This quotient is understood purely algebraically (no topology on $B^U/B^U_0$ is required); moreover, it obviously can be identified with the algebra of all interior symbols supported in $U$. Without loss of generality we may assume that $U$ is an interval in $\mathbb{R}^n$. It therefore follows from the lemma, below, that any trace functional on this algebra is trivial. This yields the assertion of the theorem. □

**Lemma 3.3.** Let $U = ]-1, 1[^{n-1} \times [0, 1[ \subset \mathbb{R}_+^n$. Denote by $\mathcal{E}$ the algebra of all classical pseudodifferential symbols with $x$-support in $U$ that satisfy the transmission condition at $x_n = 0$. Then any trace on $\mathcal{E}$ vanishes as a consequence of the following three assertions:

(a) We have $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}$, where $\mathcal{E}_0$ denotes the subalgebra of all elements of $\mathcal{E}$ vanishing identically in a neighborhood of $\{x_n = 0\}$.

(b) Let $\text{Tr}'$ be a trace on $\mathcal{E}$. Then $\text{Tr}' = c \text{res}$ for a suitable constant $c$.

(c) The constant in (b) is necessarily 0.

**Proof.** (a) Let $p$ be an arbitrary classical symbol with the transmission property. We may confine ourselves to the case where $p(x, \xi)$ vanishes for $x$ outside $[-1/4, 1/4[^{n-1} \times [0, 1/4]$. Choose a smooth function $\alpha \geq 0$ on $[0, \infty[$ with $\alpha(t) = 1$ for $0 \leq t \leq 1/3$ and $\alpha(t) = 0$ for $t \geq 1/2$. By $\xi_n$ denote a symbol with the transmission property which is equal to $\xi_n$ for $x \in [-1/2, 1/2[^{n-1} \times [0, 1/2]$ and vanishes for $x$ outside a compact set in $U$. 

Let \( q(x', \xi) = \pi(x_n) \int_0^\infty p(x', t, \xi) \, dt \). Then \( q \) is a classical symbol with the transmission property. The symbol of the commutator \([i_\xi^n, q]\) is

\[
\partial_{x_n} q(x, \xi) = p(x, \xi) + \partial_{x_n} \pi(x_n) \int_0^\infty p(x', t, \xi) \, dt
\]

(3.3)

This gives the desired decomposition.

(b) Let \( Tr' \) be a trace on \( \mathcal{E} \). The restriction of \( Tr' \) to \( \mathcal{E}_0 \) is a trace, and according to the considerations in the proof of Theorem 1.4 it coincides with \( c \, res \) for a suitable constant \( c \). We conclude from (3.3) and the fact that \( Tr' \partial_{x_n} q = 0 \) that

\[
Tr' p = -c \, \text{res} \left( \partial_{x_n} \pi(x_n) \int_0^\infty p(x', t, \xi) \, dt \right)
\]

\[
= -c \int_0^\infty \partial_{x_n} \pi(x_n) \, dx_n \iint_{S^1} \int_0^\infty p(x', t, \xi) \, dt \, dx_1 \cdots dx_{n-1} \]

\[
= c \, \text{res} \, p.
\]

(c) In order to see that \( c \) vanishes, choose a homogeneous function \( h(\xi) \) of degree \( -n \) with \( \int_0^1 h(\xi) \sigma_\xi \neq 0 \) that satisfies the transmission condition; it is well-known that such functions exist, cf. [12, Section 2.3.2.4]. Then pick \( \beta \in C_0^\infty([0, 1]; [0, 1]) \), not identically zero, and let \( p(x', x_n, \xi) = \pi(x_n) \beta(x') h(\xi) \) with the function \( \pi \) introduced in (a). Define

\[
q(x, \xi) = -\int_{x_n} p(x', t, \xi) \, dt.
\]

Clearly, the symbol \( q \) satisfies the transmission condition and, in the notation of (a), \([i_\xi^n, q] = \partial_{x_n} q = p\). This implies that \( Tr'p = 0 \), while, by construction, \( \text{res} \, p \neq 0 \). Hence \( c = 0 \).}

Remark. What we have implicitly used in the proof of Theorem is, of course, the fact that the first Čech cohomology group with compact support \( H^1_{\text{compact}}([0, \infty), R) \) is \( \{0\} \).

REFERENCES


