Positive solutions for a weakly coupled nonlinear Schrödinger system

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Abstract

Existence of a nontrivial solution is established, via variational methods, for a system of weakly coupled nonlinear Schrödinger equations. The main goal is to obtain a positive solution, of minimal action if possible, with all vector components not identically zero. Generalizations for nonautonomous systems are considered.
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1. Introduction

In recent years a large amount of work has been devoted to the study of the propagation of pulses in nonlinear optical fiber. In a single-mode optical fiber, when third order nonlinear effects are included, pulse propagation is described by the nonlinear Schrödinger equation. This equation has been extensively studied (see [10,22,26,29]) and it has been proved the existence and uniqueness of a soliton ground state solution, i.e., a positive, radial, symmetric solution whose
level energy is the minimal among the ones of every possible solutions. However, a single-mode optical fiber is not exactly single mode, it is actually bimodal due to the presence of birefringence. Birefringence tends to split a pulse into two pulses in the two polarization directions, but nonlinear effects can trap them together against splitting. Menyuk [25] showed that the two polarization components in a birefringence optical fiber are governed by the two following coupled nonlinear Schrödinger equations:

$$\begin{cases}
    i\phi_t + \phi_{xx} + (|\phi|^2 + b|\psi|^2)\phi = 0, \\
i\psi_t + \psi_{xx} + (|\psi|^2 + b|\phi|^2)\psi = 0,
\end{cases} \quad (1.1)$$

where $b$ is a real positive constant which depends on the anisotropy of the fiber. Looking for standing wave solutions of (1.1), i.e., solutions of the form

$$\phi(x, t) = e^{i\omega^2 t} u(x) \quad \text{and} \quad \psi(x, t) = e^{i\omega^2 t} v(x), \quad (1.2)$$

and performing a rescaling of variables, one obtains that $u$ and $v$ satisfy the following system

$$\begin{cases}
-u_{xx} + u = |u|^2 u + b|v|^2 u \quad \text{in} \mathbb{R}, \\
v_{xx} + \omega^2 v = |v|^2 v + b|u|^2 v \quad \text{in} \mathbb{R},
\end{cases} \quad (1.3)$$

where $\omega^2 = \omega_2^2/\omega_1^2$. The existence of vector solitary waves (1.2) as solutions to (1.1), i.e., waves which are localized pulses that propagate without change of shape, has been investigated by theoretical and numerical means, as reviewed in [34]. If $b = 0$ the equations in (1.3) are two copies of a single nonlinear Schrödinger equations which is integrable; when $b = 1$ (1.3) is known as the Manakov system (see [24]) which is also integrable. In all the other cases the situation is much more complicated from different points of view. The existence of a ground state solution $(u, v) \neq (0, 0)$ of (1.3) has been proved by means of concentration compactness methods in [16]. Notice that, if $u, v$ are solutions respectively of the equations

$$-u_{xx} + u = |u|^2 u, \quad -v_{xx} + \omega^2 v = |v|^2 v,$$

then the pairs $(u, 0)$ and $(0, v)$ solve (1.3). These are known as scalar solitary waves. Because of the presence of these particular nontrivial solutions it becomes important to study whether or not a solution found is a really vector soliton, that is if both $u$ and $v$ are nontrivial. To this end, many authors studied, by numerical and by asymptotic analysis arguments, the existence of a vector positive solution. In [9,15,35] some different families of positive solutions have been found for the frequencies ratio $\omega$ belonging in a range value in dependence on $b$.

Our first purpose here is to consider $q$ such that $1 < q < N/(N - 2)$ for $N \geq 3$ and $q > 1$ for $N = 1, 2$ and to study the following weakly coupled nonlinear elliptic system

$$\begin{cases}
-\Delta u + u = |u|^{2q-2} u + b|v|^q |u|^{q-2} u \quad \text{in} \mathbb{R}^N, \\
-\Delta v + \omega^2 v = |v|^{2q-2} v + b|u|^q |v|^{q-2} v \quad \text{in} \mathbb{R}^N,
\end{cases} \quad (1.4)$$

which reduces to problem (1.3) for $q = 2$ and $N = 1$. This nonlinear Schrödinger system has peculiarities as of (1.3). Indeed, also for (1.4) there exist scalar solutions $(u_0, 0)$ or $(0, v_0)$ where $u_0$ and $v_0$ are the unique positive radial solutions respectively of the following equations

$$-\Delta u + u = |u|^{2q-2} u, \quad -\Delta v + \omega^2 v = |v|^{2q-2} v \quad \text{in} \mathbb{R}^N. \quad (1.5)$$
The existence of least energy solutions for (1.4) can be obtained following several different arguments present in literature.

For a general class of autonomous systems which contains (1.4) the existence of a solution \((u, v) \neq (0, 0)\) has been proved by Brezis and Lieb in [13] using a constrained minimization method. Furthermore, they have shown that, among the nontrivial solutions, there is one that minimizes the associated action. Systems as (1.4) has been also studied in [17] by concentration compactness arguments, and they prove the existence and the regularity of a ground state solution \((u, v) \neq (0, 0)\). Also the ideas and results contained in [31] and in [11], which use symmetrization arguments, can be employed to prove the existence of a nontrivial solution of minimum action (see Section 4 for details).

Our main aim here is to search for a purely vector ground state for (1.4), i.e., a minimal action solution \((u, v)\) with both \(u, v\) nontrivial. In order to do this, we tackle the problem of the existence by different arguments. We will follow in fact the ideas of Rabinowitz in [28] where the case of a single semilinear equation is studied. This approach allows us to give sufficient conditions on the parameter \(b\) to assure the existence of a ground state solution \((u, v)\) with \(u, v > 0\), by comparing the level set (or the Morse index) of \((u_0, 0)\) and \((0, v_0)\) than those of \((u, v)\). What will happen will be that, for \(b\) sufficiently large (see condition (2.4) in Theorem 2.3 and condition (2.7) in Theorem 2.8), the solution we find has a smaller critical level than the one of \((u_0, 0)\) and \((0, v_0)\).

To our knowledge, this is the first result concerning the existence of a purely vector ground state. This result seems to suggest that in order to find a vector ground state the parameter \(b\) has to be sufficiently large. This is precisely proved for \(\omega = 1\) and \(q \geq 2\) (see Theorem 2.5 and Corollary 2.6). Indeed, in this model case we prove that there exists a ground state with both \(u \neq 0\) and \(v \neq 0\) if and only if \(b \geq 2^{q-1} - 1\). Moreover, for \(q = 2\) and \(N = 1, 2, 3\) we can improve our sufficient (see Theorem 2.8) and necessary (see Theorem 2.9) conditions.

Then, we consider the nonautonomous system where \(b\) is a positive function of the variable \(x\) and the coupling is modeled by a function \(F\)

\[
\begin{align*}
-\Delta u + u &= |u|^{2q-2}u + b(x)F_u(u, v) \quad \text{in } \mathbb{R}^N, \\
-\Delta v + \omega^2 v &= |v|^{2q-2}v + b(x)F_v(u, v) \quad \text{in } \mathbb{R}^N.
\end{align*}
\]

A main difficulty in treating the nonautonomous problem in \(\mathbb{R}^N\) is the possible lack of compactness. Besides, for this general problem we do not have any monotonicity property on the function \(F\). Nevertheless, by following an argument in [21] we can still prove that there exists a least energy solutions \((u, v) \neq (0, 0)\) under reasonable hypotheses on the function \(b(x)\) and on the nonlinearity \(F\) (see Theorem 5.3). Moreover, for the model case \(F(u, v) = |uv|^{q/q}\) we may again find solutions which are positive in both components and of minimal action, using some comparison argument with the autonomous problem at infinity (see Theorem 5.5 and Remark 5.6).

Finally, we give local bifurcation type results. We, in fact, study problem (1.6) with \(\varepsilon b(x)\) in the place of \(b(x)\) (where \(\varepsilon\) is a small parameter) and \(F(u, v) = |uv|^{q/q}\). Since we are interested in finding a solution with both nontrivial components, we study, following the approach of [1–3], the local bifurcation from a particular manifold of solutions of the unperturbed problem, i.e., the problem for \(\varepsilon = 0\). Indeed, we will consider the manifold generated from \((u_0, v_0)\) (solutions of (1.5)) by the translation invariance of the problem and we will prove the existence of solutions “close to” \((u_0, v_0)\) (see Theorem 6.5).

The paper is organized as follows. In Section 2 we state our main results for the autonomous system. The definitions and preliminary results, preparatory to the proofs, are presented in Sec-
tion 3. In Section 4 we give the proofs of our existence results for the autonomous problem. In Section 5 we study the nonautonomous weakly coupled system with a general coupling nonlinearity, obtaining a general existence result. Finally, in Section 6 we study the nonautonomous system in the perturbative case.

2. Translation-invariant system

In this section we will study the following autonomous system

\[
\begin{cases}
-\Delta u + u = |u|^{2q-2}u + b|v|^q |u|^{q-2}u & \text{in } \mathbb{R}^N, \\
-\Delta v + \omega^2 v = |v|^{2q-2}v + b|u|^q |v|^{q-2}v & \text{in } \mathbb{R}^N, \\
u(x) \to 0, & v(x) \to 0 \quad \text{as } |x| \to \infty,
\end{cases}
\]

(2.1)

where \(\omega, b > 0\) are constants and \(q\) is such that

\[
2 < 2q < 2^* = \begin{cases}
+\infty & \text{if } N = 1, 2, \\
\frac{2N}{N-2} & \text{if } N \geq 3.
\end{cases}
\]

(2.2)

In order to find a solution of problem (2.1) we will use variational methods. To that end, we consider the following Hilbert space

\[
\begin{align*}
E &= H^1(\mathbb{R}^N) \times E_\omega, & |w|_E^2 &= \|(u, v)\|_E^2 = \|u\|^2 + \|v\|_\omega^2, \\
E_\omega &= H^1(\mathbb{R}^N), & \text{with } (v|v)_\omega &= \|v\|_\omega^2 = \|\nabla v\|_2^2 + \omega^2 \|v\|_2^2,
\end{align*}
\]

(2.3)

where \(|\cdot|\) stands for the norm in \(H^1(\mathbb{R}^N)\), \(||\cdot||_p\) denotes the standard norm in \(L^p(\mathbb{R}^N)\) and \(||\cdot||_p^p + \||\cdot||_p^p\|^{1/p}\) is the norm of a vector in \(L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)\). We will study the functional \(I : E \to \mathbb{R}\) defined by

\[
I(w) = I(u, v) = \frac{1}{2} \|(u, v)\|_E^2 - \frac{1}{2q} \|(u, v)\|_2^{2q} - \frac{b}{q} \|uv\|_q^q,
\]

for every \(w = (u, v)\) in \(E\). The functional \(I\) is of class \(C^1(E, \mathbb{R})\) and its differential is given by

\[
\langle I'(u, v), (\varphi, \psi) \rangle = (u|\varphi) + (v|\psi)_w - \int_{\mathbb{R}^N} \left[ |u|^{2q-2}u\varphi + |v|^{2q-2}v\psi \right] \\
- b \int_{\mathbb{R}^N} \left[ |v|^q |u|^{q-2}u\varphi + |u|^q |v|^{q-2}v\psi \right].
\]

Hence, the critical points of \(I\) in \(E\) are the weak solutions of (2.1) and by standard regularity theory are, in fact, classical solutions. We will prove the following results.

**Theorem 2.1.** Assume (2.2). Then, for every \(b > 0\) there exists a least energy solution (ground state) \(w = (u, v) \neq (0, 0)\) of problem (2.1), with \(u \geq 0, v \geq 0\) and both \(u\) and \(v\) radial.
Remark 2.2. The previous result can be deduced from several results present in literature (see Section 4 for some details). Here we will follow [28], as the arguments in this paper will be useful when proving the following theorems.

Theorem 2.3. Assume (2.2) and suppose that

\[
b \geq \begin{cases} 
\frac{1}{2} f(\omega) - 1 & \text{if } \omega \geq 1, \\
\frac{1}{2} f(1/\omega) - 1 & \text{if } \omega \leq 1,
\end{cases}
\]

where

\[
f(\omega) = \left[ 1 + \frac{N}{2} \left( 1 - \frac{1}{q} \right) + \frac{1}{\omega^2} \left( 1 - \frac{N}{2} \left( 1 - \frac{1}{q} \right) \right) \right]^q \omega^{2q - N(q-1)}. \tag{2.5}
\]

Then there exists a least energy solution \( w = (u, v) \) with \( u > 0 \) and \( v > 0 \).

Remarks 2.4.

(1) Notice that if \((u, v)\) are solution of system (2.1), then the pair \((1/\omega v(x/\omega), 1/\omega u(x/\omega))\) is a solution of (2.1) where \(\omega\) is replaced by \(1/\omega\). As a consequence of this symmetry property, we obtain symmetric conditions ((2.4), (2.7)) on \(b\) (or \(\omega\)).

(2) In all our results we consider \(\omega\) fixed and \(b\) as a parameter, so that we find conditions on \(b\) in dependence on \(\omega\). Note that, since \(b\) is a constant depending on the fiber and \(\omega\) is the frequencies ratio of the pulse that propagates along the fiber, conditions (2.4) and (2.7) have to be read as bounds on \(\omega\) in dependence of \(b\).

The preceding result is a sufficient condition which guarantees the existence of vector ground state of (1.4), however we are able to prove also a necessary condition.

Theorem 2.5. Assume condition (2.2) and \(q \geq 2\). If there exists a least energy solution of (1.4) with both nontrivial components then

\[
b \geq 2^q - 1. \tag{2.6}
\]

From Theorems 2.3 and 2.5 we immediately obtain the following result.

Corollary 2.6. Assume (2.2) and \(q \geq 2\) and \(\omega = 1\). There exists a least energy solution of (1.4) with both nontrivial components if and only if \(b\) satisfies (2.6).

Remarks 2.7.

(1) Note that \(f(\omega) \geq 2^q\) for \(\omega \geq 1\) and \(f(1/\omega) \geq 2^q\) for \(\omega < 1\), so that Theorems 2.3 and 2.5 are not in contradiction.

(2) Assuming condition (2.2) and \(q \geq 2\) implies that Theorem 2.5 and Corollary 2.6 holds only for dimension \(N = 1, 2, 3\). Even though these values are the more relevant from a physicist point of view, it would be interesting to find a necessary and sharp condition for every \(N\).

(3) In the light of Theorems 2.3 and 2.5 we have that for \(b\) small, the least energy solution are of the form \((u_0, 0)\) (for \(\omega \geq 1\)) or \((0, v_0)\) (if \(\omega \leq 1\)). While, for \(b\) large, the least energy solution has both nontrivial components. The problem is completely solved only when \(\omega = 1\) as stated.
in Corollary 2.6. It should be interesting to prove a necessary and sufficient condition on $b$ in the general case.

In the model case, i.e., $q = 2$, we can prove also the following sufficient condition.

**Theorem 2.8.** Assume $N = 1, 2, 3$, $q = 2$ and suppose that

$$b \geq \frac{4 - N}{4} \max\left\{ \frac{1}{\omega^2}, \omega^2 \right\} + \frac{N}{4},$$

then there exists a least energy solution $w = (u, v)$ with $u > 0$ and $v > 0$.

For $q = 2$ Theorem 2.5 state that if there exists a vector ground state then $b \geq 1$. In fact, in this case we can improve this condition, as the following result shows.

**Theorem 2.9.** Assume $N = 1, 2, 3$ and $q = 2$. If there exists a least energy solution with both nontrivial components, then the following condition is satisfied

$$b \geq \max\left\{ \omega^{(4-N)/2}, \frac{1}{\omega^{(4-N)/2}} \right\}.$$

**Remark 2.10.** Note that the preceding results hold only for $N = 1, 2, 3$, since the exponent $q = 2$ is critical (or supercritical) if $N \geq 4$.

**Remark 2.11.** If $N = 3$ and $q = 2$, sufficient conditions (2.4), (2.7) become respectively (for $\omega \geq 1$)

$$b \geq h(\omega) = \frac{\omega}{32} \left( 7 + \frac{1}{\omega^2} \right)^2 - 1, \quad b \geq \omega^2 + \frac{3}{4}.$$

These curves intersect in an unique point $\omega_0$. So that, (2.4) give a better result than (2.7) if and only if $\omega \geq \omega_0$ (a symmetric remark holds for $\omega < 1$).

The conclusions of Theorems 2.3, 2.5, 2.8 and 2.9 are collected in Fig. 1.
3. Qualitative properties

In this section we will prove some results that we will use in the sequel.

In this first part a slightly more general functional, still called $I$, will be studied, and some important qualitative properties concerning the Mountain Pass level will be proved. We will follow [28,33] where these results are proved in the case of a single equation.

Let us consider a measurable function $b : \mathbb{R}^N \to \mathbb{R}$, such that the following hypotheses are satisfied

$$
\begin{cases}
  b(x) = b_1(x) + b_2(x), & b_1 \in L^\infty(\mathbb{R}^N), \ b_2 \in L^m(\mathbb{R}^N), \\
  m = \frac{N}{N - q(N - 2)} \quad \text{if } N \geq 3, & m > 1 \quad \text{if } N = 2, \quad m \geq 1 \quad \text{if } N = 1,
\end{cases}
$$

and let us define the functional $I : E \to \mathbb{R}$ by

$$
I(u, v) = \frac{1}{2} \| (u, v) \|_E^2 - \frac{1}{2q} \| (u, v) \|_{2q}^{2q} - \frac{1}{q} \int_{\mathbb{R}^N} b(x)|u|^q|v|^q.
$$

First notice that hypotheses (2.2) and (3.1) imply that $I$ is of class $C^1$, so that we can define the Nehari manifold

$$
\mathcal{N} := \{ w \in E \setminus \{0\} : \langle I'(w), w \rangle = 0 \}.
$$

In [28] the following result is proved for a single equation.

**Lemma 3.1.** For every $w \in E \setminus \{0\}$ there exists a unique $\tilde{t}(w) > 0$ such that $\tilde{t}(w)w \in \mathcal{N}$. The maximum of $I(\tilde{t}w)$ for $\tilde{t} \geq 0$ is achieved at $\tilde{t} = \tilde{t}(w)$. The function

$$
E \setminus \{0\} \mapsto (0, +\infty) : w \mapsto \tilde{t}(w)
$$

is continuous and the map $w \mapsto \tilde{t}(w)w$ defines a homeomorphism of the unit sphere of $E$ with $\mathcal{N}$.

The proof is analogous to the one done for the single equation in [28]. We include the details in our case for the sake of clearness.

**Proof of Lemma 3.1.** The functional $I$ has the following geometrical properties:

(i) $(0, 0)$ is a strict local minimum.
(ii) $I(Tw) < 0$ for every $w \in E$ and for $T > 0$ sufficiently large.

Indeed, if we define

$$
F(u, v) = \frac{1}{2q} [\|u\|^{2q} + \|v\|^{2q}] + \frac{1}{q} b(x)|u|^q|v|^q,
$$

then...
then (3.1) implies that $F$ satisfies the following conditions

$$\lim_{\|(u,v)\|_E \to (0,0)} \frac{1}{\|(u,v)\|_E^2} \int_{\mathbb{R}^N} F(u,v) = 0,$$

(3.6)

$$F(u,v) \geq \frac{1}{2q} \left( |u|^{2q} + |v|^{2q} \right).$$

(3.7)

Thus, (3.6) implies (i). From (3.7) we deduce that for every $t > 0$ the following inequality holds

$$I(tu,tv) \leq t^{2q} \left[ \frac{\|(u,v)\|_E^2}{2t^{2q-2}} - \frac{1}{2q} (\|u\|^{2q}_{2q} + \|v\|^{2q}_{2q}) \right],$$

so that (ii) follows. Now, for any $w = (u,v) \in E \setminus \{0\}$ and $t > 0$, let

$$g(t) := I(tw) = I((tu,tv)),$$

(3.8)

from (i) and (ii) we deduce that there exists $\bar{t} = \bar{t}(w) > 0$ such that

$$g(\bar{t}) = \max_{t > 0} g(t).$$

(3.9)

Moreover, every positive critical point $t$ of $g$ satisfies the following equation

$$\|(u,v)\|_E^2 - t^{2q-2} \left[ \|u\|^{2q}_{2q} + \int_{\mathbb{R}^N} b(x)|u|^q |v|^q \right] = 0,$$

(3.10)

so that, as $q > 1$, the point $\bar{t} = \bar{t}(w)$ is the unique value of $t > 0$ at which $\bar{t}(w)w \in \mathcal{N}$. Thus, $\mathcal{N}$ is radially homeomorphic to the unit sphere in $E$. Finally, (2.2), (3.1) and (3.10) imply that the application $w \mapsto \bar{t}(w)$ is continuous.

Let us define

$$c_{\mathcal{N}} := \inf_{\mathcal{N}} I(w),$$

(3.11)

$$c_1 := \inf_{w \in E \setminus \{0\}} \max_{t \geq 0} I(tw),$$

(3.12)

$$c := \inf_{\Gamma} \max_{[0,1]} I(\gamma(t)),$$

(3.13)

where

$$\Gamma = \{ \gamma : [0, 1] \to E. \, \gamma \text{ is continuous and } \gamma(0) = 0, \, I(\gamma(1)) < 0 \}.$$  

(3.14)

The following result is proved in [28] (see also [33]) for functionals associated to a single equation.

**Lemma 3.2.** One has $c_{\mathcal{N}} = c_1 = c$. 

Proof. We first notice that, from Lemma 3.1, it follows
\[ c_1 = \inf_E I(\bar{t}(w)w) = \inf_{\mathcal{N}} I(z) = c_{\mathcal{N}}. \]
Moreover, since \( I(tw) < 0 \) for every \( t \) large, we get that \( c \leq c_1 \). Finally, notice that every \( \gamma \in \Gamma \) intersects \( \mathcal{N} \), so that \( c \geq c_{\mathcal{N}} \). Therefore, the conclusion follows. \( \square \)

In order to find a solution \((u, v)\) with \( u \geq 0 \) and \( v \geq 0 \), the following result will be used.

Lemma 3.3. Let \( w \in \mathcal{N} \) and \( I(w) = c \), where \( c \) is defined in (3.13). Then, \( w \) is a critical point of \( I \).

Proof. The proof follows easily by the arguments of [33, Theorem 4.3]. \( \square \)

4. Proofs of the main results

In this section we give the proofs of our existence results for the autonomous problem (2.1). The proof of Theorem 2.1 can be obtained as a consequence of different results present in literature. See [13] for general systems and [17] where concentration compactness methods are used for the system (1.4). Theorem 2.1 can also be deduced following the ideas and results in [31] and [11] for scalar problems. In the former the existence of a nontrivial solution is obtained via a constrained minimization on the Nehari manifold, the latter uses the Mountain Pass theorem [6]. In both these papers these arguments are combined with Schwartz symmetrization to recover some compactness.

In order to prove Theorem 2.3 the key point is the equality in Lemma 3.2, which is proved in [28]. Thus, just for the sake of clearness, we will include a proof of Theorem 2.1 following [28].

Proof of Theorem 2.1. The functional \( I \) satisfies conditions (i) and (ii) (defined in the proof of Lemma 3.1). Then, we fix \( T > 0 \) and \( w_1 \) such that \( I(Tw_1) < 0 \) and we define \( \Gamma \) as in (3.14) with \( \gamma(1) = Tw_1 \) for every \( \gamma \in \Gamma \). Moreover, we define \( c \) as in (3.13) and observe that (i) gives that \( c > 0 \). The Ekeland variational principle (see [19] or [33]) implies that there exists a sequence \( w_n = (u_n, v_n) \) such that
\[
I(w_n) \to c, \tag{4.1}
\]
\[
I'(w_n) \to 0 \quad \text{strongly in } E'. \tag{4.2}
\]
By computing \( 2qI(w_n) - \langle I'(w_n), w_n \rangle \) and applying (2.2), (4.1) and (4.2), we get that \( w_n \) is bounded in \( E \). Then there exists \( w = (u, v) \in E \) such that, up to a subsequence
\[
(u_n, v_n) \to (u, v) \quad \text{weakly in } E, \tag{4.3}
\]
\[
(u_n, v_n) \to (u, v) \quad \text{strongly in } L^p_{\text{loc}}(\mathbb{R}^N) \times L^p_{\text{loc}}(\mathbb{R}^N) \quad \forall p \in [1, 2^*), \tag{4.4}
\]
\[
(u_n, v_n) \to (u, v) \quad \text{almost everywhere in } \mathbb{R}^N. \tag{4.5}
\]
From (4.3) and (4.4) it follows that \((u, v)\) is a critical point of \(I\), so that it is a weak solution of (1.4). Since also \((0, 0)\) is a critical point of \(I\), we still need to show \((u, v) \neq (0, 0)\). In order to do this, we first show that there exist \(y_k \in \mathbb{R}^N\), \(\beta, R \in \mathbb{R}^+\) such that, up to a further subsequence

\[
\int_{B_R(y_k)} |w_k|^2 \geq \beta, \quad \forall k \in \mathbb{N}.
\]  

(4.6)

If not, it results

\[
\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |w_n|^2 \to 0.
\]

Then, condition (2.2) allows us to use the argument of [23, Lemma I.1] (see also [33]) to deduce that

\[
w_n \to 0 \quad \text{strongly in } L^{2q}(\mathbb{R}^N) \times L^{2q}(\mathbb{R}^N).
\]  

(4.7)

On the other hand, from (4.1) and (4.2) we get

\[
c = I(u_n, v_n) - \frac{1}{2}(I'(u_n, v_n), (u_n, v_n)) + o(1)
\]

\[
= \frac{1}{2} \left(1 - \frac{1}{q}\right) \int_{\mathbb{R}^N} [u_n^{2q} + v_n^{2q}] + b(1 - \frac{1}{q}) \int_{\mathbb{R}^N} |u_n|^q|v_n|^q + o(1)
\]

\[
\leq \frac{1}{2} \left(1 - \frac{1}{q}\right)(1 + b)[\|u_n\|_{2q}^{2q} + \|v_n\|_{2q}^{2q}] + o(1),
\]

which contradicts (4.7), proving that (4.6) holds. Now, we define

\[
\bar{w}_k(z) = (\bar{u}_k(z), \bar{v}_k(z)) = (u_k(z + y_k), v_k(z + y_k)).
\]  

(4.8)

From the translation invariance of the functional, we get that also \(\bar{w}_k\) is a Palais–Smale sequence at level \(c\). So that, by arguing as we did for \(w_k\), we deduce that, up to a subsequence, there exists a weak limit of \(\bar{w}_k\). Moreover, (4.6) implies that

\[
\liminf_{k \to \infty} \int_{B_R(0)} |\bar{w}_k(z)|^2 \geq \beta > 0.
\]  

(4.9)

Indeed, from (4.8) it follows

\[
\int_{B_R(0)} |\bar{w}_k(z)|^2 = \int_{B_R(0)} |w_k(z + y_k)|^2 = \int_{B_R(y_k)} |w_k|^2,
\]
and (4.9) follows easily from (4.6). Since $w_k \to w$ strongly in $L^2_{\text{loc}}(\mathbb{R}^N) \times L^2_{\text{loc}}(\mathbb{R}^N)$, we pass to the limit in (4.9) and we obtain

\[
\int_{B_R(0)} |w|^2 \geq \beta,
\]

which implies that $w \not\equiv 0$.

Now we will show that, in fact, $I(\bar{u}, \bar{v}) = c$, even though we do not know if $(\bar{u}_k, \bar{v}_k)$ converges strongly in $E$.

Since $(\bar{u}, \bar{v}) \neq (0, 0)$ is a critical point of $I$, we deduce that $(\bar{u}, \bar{v}) \in \mathcal{N}$ ($\mathcal{N}$ is defined in (3.4)), so that from Lemma 3.2 we deduce that $I(\bar{u}, \bar{v}) \geq c$. On the other hand, from the Fatou lemma we get

\[
c = \lim_{k \to \infty} \left[ I(\bar{u}_k, \bar{v}_k) - \frac{1}{2} I'(\bar{u}_k, \bar{v}_k), (\bar{u}_k, \bar{v}_k) \right] \geq \left( 1 - \frac{1}{q} \right) \left\{ \frac{1}{2} (\|\bar{u}\|_{2q}^2 + \|\bar{v}\|_{2q}^2) + b\|\bar{u}\bar{v}\|_q^q \right\}
\]

\[
= I(\bar{u}, \bar{v}) - \frac{1}{2} \left\langle I'(\bar{u}, \bar{v}), (\bar{u}, \bar{v}) \right\rangle = I(\bar{u}, \bar{v}).
\]

Thus, Lemma 3.2 implies that $I(\bar{u}, \bar{v}) = c$ and $(\bar{u}, \bar{v})$ is a least energy solution.

Now observe that

\[
I(|\bar{u}|, |\bar{v}|) = I(\bar{u}, \bar{v}) = c, \quad \left\langle I'(|\bar{u}|, |\bar{v}|), (|\bar{u}|, |\bar{v}|) \right\rangle = \left\langle I'(\bar{u}, \bar{v}), (\bar{u}, \bar{v}) \right\rangle = 0.
\]

Then, Lemma 3.3 implies that $(|\bar{u}|, |\bar{v}|)$ is a critical point of $I$ at the same level of $(\bar{u}, \bar{v})$. Hence, we have found a least energy solution $w = (u, v) = (|\bar{u}|, |\bar{v}|)$, whose components are nonnegative functions.

If $u$ and $v$ are both nontrivial functions, the strong maximum principle implies that they are positive functions, so we apply the result of [14] to deduce that $u$ and $v$ are radial functions (up to translations) and that decay to zero exponentially as $|x| \to +\infty$. Otherwise, if one between $u$ and $v$ is the zero function, the other component of the solution is radial, positive and decays to zero exponentially, as showed in [22]. In both cases the theorem is proved.

Proof of Theorem 2.3. We have to prove that $\bar{u}$ and $\bar{v}$ are nontrivial, assuming (2.4). First, notice that if $\bar{v} \equiv 0$, then $w = (\bar{u}, 0)$ with $\bar{u}$ a solution of the following problem

\[
\begin{cases}
-\Delta u + u = u^{2q-1} & \text{in } \mathbb{R}^N, \\
u > 0, & u \in H^1(\mathbb{R}^N).
\end{cases}
\]

(4.10)

It is well known that Eq. (4.10) has a unique (up to translation), radial solution (see [10,22]), which we denote $u_0$. Whereas, if $\bar{u} \equiv 0$, then $w = (0, \bar{v})$ with $\bar{v}$ a solution of the problem

\[
\begin{cases}
-\Delta v + \omega^2 v = v^{2q-1} & \text{in } \mathbb{R}^N, \\
v > 0, & v \in H^1(\mathbb{R}^N).
\end{cases}
\]

(4.11)
The function $v_0(x) := \omega^{1/(q-1)} u_0(\omega x)$ is the unique (up to translation) solution of the problem (4.11). Taking into account these informations, it remains to prove that

$$c = I(\bar{u}, \bar{v}) < \min \{ I(u_0, 0), I(0, v_0) \}. \quad (4.12)$$

Since $u_0$ and $v_0$ are the solutions of (4.10) and (4.11), it holds

$$\|u_0\|_2^2 = \|u_0\|_{2q}^{2q}, \quad \|v_0\|_{\omega\omega}^2 = \|v_0\|_{2q}^{2q}. \quad (4.13)$$

moreover, by Pohozaev identity (see [10,20] for the scalar case and [12] for the case of systems or see Proposition 5.2 in Section 5) it follows that

$$\frac{N - 2}{2} \|\nabla v_0\|_2^2 + \frac{N}{2} \omega \|v_0\|_{\omega\omega}^2 = \frac{N}{2q} \|v_0\|_{2q}^{2q}, \quad (4.14)$$

and a corresponding identity holds for $u_0$ when $\omega = 1$. Collecting the preceding identities, we can write

$$\|u_0\|_2^2 = \left[ 1 - \frac{N}{2} + \frac{N}{2q} \right] \|u_0\|_{2q}^{2q}, \quad \omega \|v_0\|_{\omega\omega}^2 = \left[ 1 - \frac{N}{2} + \frac{N}{2q} \right] \|v_0\|_{2q}^{2q}. \quad (4.15)$$

The previous equalities together with (4.13) give the following relations

$$C := I(u_0, 0) = \frac{1}{2} \left( 1 - \frac{1}{q} \right) \|u_0\|_{2q}^{2q} = \frac{1}{N} \|\nabla u_0\|_2^2, \quad (4.15)$$

$$I(0, v_0) = \omega^{\frac{2q}{q-1} - N} C = \frac{1}{2} \left( 1 - \frac{1}{q} \right) \|v_0\|_{2q}^{2q} = \frac{1}{N} \|\nabla v_0\|_2^2. \quad (4.16)$$

On the other hand, Lemma 3.2 implies that

$$c = I(\bar{u}, \bar{v}) \leq \max_{t \geq 0} I(t(\phi, \psi)) \quad \forall (\phi, \psi) \in E. \quad (4.17)$$

Then, from (4.15) and (4.16) it follows that it is enough to find $(\phi, \psi) \in E$, $\phi \not\equiv 0$ and $\psi \not\equiv 0$, such that

$$c \leq \max_{t \geq 0} I(t(\phi, \psi)) \leq \min \{ C, \omega^{\frac{2q}{q-1} - N} C \}, \quad (4.17)$$

where $C$ is defined in (4.15).

Given $(\phi, \psi) \in E$, we consider the function $g(t)$ defined in (3.8) and its maximum value $J(\phi, \psi)$ defined by

$$J(\phi, \psi) = \max_{t > 0} g(t) = \frac{1}{2} \left( 1 - \frac{1}{q} \right) \left[ \frac{\|\phi\|^2 + \|\psi\|^{2q}}{\|\phi\|_{2q}^2 + \|\psi\|_{2q}^2 + 2b \|\psi\|_{q}^q} \right]^{\frac{1}{q-1}}. \quad (4.18)$$
Let us first consider the case in which $\omega \geq 1$. Hypothesis (2.2) implies that $2q/(q - 1) - N > 0$, so that $\min\{C, \omega^{2q/(q - 1) - N}C\} = C$. Thus, we have to find a pair $(\varphi, \psi)$, $\varphi \not\equiv 0$ and $\psi \not\equiv 0$, such that

$$c \leq J(\varphi, \psi) \leq I(u_0, 0) = C. \quad (4.19)$$

Choosing $(\varphi, \psi) = (v_0, v_0)$, from (4.13), (4.16) and (4.18) we have

$$J(v_0, v_0) = \frac{1}{2} \left(1 - \frac{1}{q}\right) \left[ \frac{\|v_0\|_{\omega}^{2q} (1 + c(\omega))^q}{2(1 + b)} \right]^{\frac{1}{q-1}},$$

where

$$c(\omega) = \frac{N}{2} \left(1 - \frac{1}{q}\right) + \frac{1}{\omega^2} \left(1 - \frac{N}{2} \left(1 - \frac{1}{q}\right)\right).$$

(4.15) yields

$$J(v_0, v_0) = C \omega^{\frac{2q}{q-1} - N} \left[ \frac{(1 + c(\omega))^q}{2(1 + b)} \right]^{\frac{1}{q-1}}.$$ 

Now it is clear that (4.19) is equivalent to

$$\omega^{\frac{2q}{q-1} - N} \left[ \frac{(1 + c(\omega))^q}{2(1 + b)} \right]^{\frac{1}{q-1}} \leq 1,$$

that is satisfied whenever (2.4) is assumed.

When $\omega < 1$, $\min\{C, \omega^{\frac{2q}{q-1} - N} C\} = \omega^{\frac{2q}{q-1} - N} C$, so that we lead to look for a pair $(\varphi, \psi)$, $\varphi \not\equiv 0$ and $\psi \not\equiv 0$, with

$$c \leq J(\varphi, \psi) \leq I(0, v_0) = \omega^{\frac{2q}{q-1} - N} C. \quad (4.20)$$

If we choose $(\varphi, \psi) = (u_0, u_0)$ and proceed as above, we find

$$J(u_0, u_0) = \frac{1}{2} \left(1 - \frac{1}{q}\right) \|u_0\|^2 \left[ \frac{(1 + c(1/\omega))^q}{2(1 + b)} \right]^{\frac{1}{q-1}},$$

and again condition (2.4) implies that (4.20) is satisfied. So the theorem is proved. □

**Proof of Theorem 2.5.** Suppose that there exists $(u_1, v_1)$ a least energy solution of (1.4) with $u_1 \not\equiv 0$ and $v_1 \not\equiv 0$, then

$$c = \inf_{(u, v) \neq (0, 0)} J(u, v) \leq \inf_{u \neq 0, v \neq 0} J(u, v) \leq J(u_1, v_1) = c,$$

so that

$$c = J(u_1, v_1) = \inf_{(u, v) \neq (0, 0)} J(u, v) = \inf_{u \neq 0, v \neq 0} J(u, v). \quad (4.21)$$
Notice that, given \( u \neq 0 \) and \( v \neq 0 \), Hölder’s inequality yields
\[
\frac{(||u||^2 + ||v||_2^2)^q}{||u||_{2q}^{2q} + ||v||_{2q}^{2q} + 2b||uv||_q^q} \geq \frac{(||u||^2 + ||v||_2^2)^q}{||u||_{2q}^{2q} + ||v||_{2q}^{2q} + 2b||uv||_q^q}.
\]
(4.22)

Since \( u \neq 0 \) and \( v \neq 0 \), then
\[
eq \begin{cases} 
 1 & \text{if } r_1 \geq 1, \\
 2 & \text{otherwise}.
\end{cases}
\]

In the first case it follows that
\[
\frac{(||u||^2 + ||v||_2^2)^q}{||u||_{2q}^{2q} + ||v||_{2q}^{2q} + 2b||uv||_q^q} = \frac{||v||_{2q}^{2q} (1 + r_1^2 s^2)^q}{||u||_{2q}^{2q} (1 + s^2 q + 2b s^q)} \geq \frac{||v||_{2q}^{2q} (1 + s^2 q + 2b s^q)}{||v||_{2q}^{2q} (1 + s^2 q + 2b s^q)},
\]
where \( s = ||u||_{2q} / ||v||_{2q} \).

Otherwise it holds
\[
\frac{(||u||^2 + ||v||_2^2)^q}{||u||_{2q}^{2q} + ||v||_{2q}^{2q} + 2b||uv||_q^q} = \frac{||u||_{2q}^{2q} (1 + r_2^2 s^2)^q}{||u||_{2q}^{2q} (1 + s^2 q + 2b s^q)} \geq \frac{||u||_{2q}^{2q} (1 + s^2 q + 2b s^q)}{||u||_{2q}^{2q} (1 + s^2 q + 2b s^q)},
\]
where \( s = ||v||_{2q} / ||u||_{2q} \).

Then \( J(u, v) \) defined in (4.18) satisfies
\[
J(u, v) \geq \frac{1}{2} \left( 1 - \frac{1}{q} \right) \left[ \frac{||v||_{2q}^{2q} (1 + s^2 q)}{||u||_{2q}^{2q} (1 + s^2 q + 2b s^q)} \right]^{\frac{1}{q-1}} \quad \text{if } r_1 \geq 1,
\]
\[
J(u, v) \geq \frac{1}{2} \left( 1 - \frac{1}{q} \right) \left[ \frac{||u||_{2q}^{2q} (1 + s^2 q)}{||u||_{2q}^{2q} (1 + s^2 q + 2b s^q)} \right]^{\frac{1}{q-1}} \quad \text{otherwise}.
\]

This gives
\[
\inf_{u \neq 0, v \neq 0} J(u, v) = \inf_{u \neq 0, s > 0} \frac{1}{2} \left( 1 - \frac{1}{q} \right) \left[ \frac{||v||_{2q}^{2q} (1 + s^2 q)}{||u||_{2q}^{2q} (1 + s^2 q + 2b s^q)} \right]^{\frac{1}{q-1}}
\]
\[
\inf_{u \neq 0, v \neq 0} J(u, v) = \inf_{u \neq 0, s > 0} \frac{1}{2} \left( 1 - \frac{1}{q} \right) \left[ \frac{||u||_{2q}^{2q} (1 + s^2 q)}{||u||_{2q}^{2q} (1 + s^2 q + 2b s^q)} \right]^{\frac{1}{q-1}}.
\]

From (4.16) and the results in [18] we have for every \( \omega > 0 \)
\[
\inf_{v \neq 0} \frac{1}{2} \left( 1 - \frac{1}{q} \right) \frac{||v||_{2q}^{2q}}{||v||_{2q}^{2q}} = \frac{||v_0||_{2q}^{2q}}{||v_0||_{2q}^{2q}} = \omega^{\frac{2q}{q-1} - N} C.
\]
(4.23)
Using this information in the previous inequalities gives
\[
\inf_{u \neq 0, v \neq 0} J(u, v) \geq \min \left\{ \omega^{\frac{2q}{q-1}} C, C \right\} \inf_{s > 0} L(s),
\]
(4.24)
where \( L(s) = (1 + s^2)^q / (1 + s^2q + 2bs^q) \).

From (4.15), (4.16), (4.21) and (4.24), we deduce that if there exists a vector ground state it follows that
\[
\inf_{s > 0} L(s) < 1 = L(0) = \lim_{s \to +\infty} L(s),
\]
(4.25)
so that we have to study the behaviour of the function \( L \).

If \( 1 < q < 2 \), \( s = 0 \) is always a local strict maximum (\( L''(0) < 0 \)) for \( L \), so that (4.25) is always satisfied and we cannot get any necessary condition on \( b \).

For \( q \geq 2 \), \( s = 0 \) is a local strict minimum and it is possible to show that \( \inf L(s) = \min\{L(0), L(1)\} \), so that we get \( b > 2^{q-1} - 1 \). \( \square \)

**Proof of Theorem 2.8.** Consider first the case \( \omega > 1 \). As above it follows that \( I(u_0, 0) \leq I(0, v_0) \), so we have to prove that \( (u_0, 0) \) is not a ground state. In order to show this, we will prove that \( i(u_0, 0) \geq 2 \) (\( i \) stands for the Morse index of the solution), since it is well known that the Morse index of a ground state is less than or equal to 1.

We compute the quadratic form \( I''(u_0, 0)(\phi, \psi) \) for every \( (\phi, \psi) \in E \),
\[
I''(u_0, 0)(\phi, \psi) = \begin{pmatrix} L \phi & 0 \\ 0 & T \psi \end{pmatrix},
\]
where the linear operators \( L : H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N) \), \( T : E_\omega \to E'_\omega \) (where \( E'_\omega \) stands for the dual space of \( E_\omega \)) are defined by
\[
L \phi = -\Delta \phi + \phi - 3u_0^2 \phi, \quad T \psi = -\Delta \psi + \omega^2 \psi - bu_0^2 \psi.
\]
In [26] it is proved that the linear equation
\[
-\Delta \phi + \phi - 3u_0^2 \phi = 0
\]
is solved only by the partial derivatives of \( u_0 \). Then \( \lambda = 0 \) is an eigenvalue of \( L \) with multiplicity equal to 3, that is, it is not simple. Then \( \lambda_1 \), the first eigenvalue, has to be negative. This implies that \( i(u_0, 0) \geq 1 \).

In order to show that \( i(u_0, 0) \geq 2 \) it is left to prove that
\[
\inf_{\psi \neq 0} \langle T \psi, \psi \rangle = \inf_{\psi \neq 0} \int_{\mathbb{R}^N} (|\nabla \psi|^2 + \omega^2 \psi^2 - bu_0^2 \psi^2) < 0.
\]
(4.26)
Notice that (4.13)–(4.15) imply
\[
\|u_0\|^4_2 = 4C, \quad \|u_0\|^2_2 = (4 - N)C, \quad \|\nabla u_0\|^2_2 = NC,
\]
so that, if we choose \( \psi = u_0 \) in (4.26) we obtain
\[
\langle Tu_0, u_0 \rangle = NC + (4 - N)\omega^2 C - 4bC.
\]
Now it is clear that (4.26) holds if the last term above is nonpositive, this occurs if (2.7) is satisfied. The case \( \omega < 1 \) follows in an analogous way, we have only to point out that the weak formulation of Eq. (4.11), Proposition 5.2 and (4.16) imply
\[
\|v_0\|_4^4 = 4\omega C, \quad \|v_0\|_2^2 = (4 - N)\frac{C}{\omega}, \quad \|\nabla v_0\|_2^2 = 3N\omega C.
\]

**Proof of Theorem 2.9.** Consider first the case \( \omega < 1 \). If \((u,v)\) is a least energy solution with \(u \not\equiv 0\) and \(v \not\equiv 0\), it follows that, given any \(\lambda\) and \(\mu\) in \(\mathbb{R}\),
\[
\inf_{(\varphi,\psi) \neq (0,0)} J(\lambda\varphi, \mu\psi) \leq \omega^{4-N} C = J(0, v_0)
\]
(see (4.20)). This is equivalent to
\[
\inf_{(\varphi,\psi) \neq (0,0)} Q(\lambda\varphi, \mu\psi) \leq 0,
\]
where
\[
Q(\lambda\varphi, \mu\psi) = \lambda^4 [\|\varphi\|^4 - 4\omega^{4-N} C \|\varphi\|_2^4] + \mu^4 [\|\psi\|^4 - 4\omega^{4-N} C \|\psi\|_2^4]
\]
\[
+ 2\lambda^2 \mu^2 [\|\varphi\|^2 \|\psi\|_2^2 - 4\omega^{4-N} C b \|\varphi \psi\|_2^2].
\]
From (4.23) we obtain for \(0 < \omega < 1\)
\[
\|\varphi\|^4 - 4\omega^{4-N} C \|\varphi\|_2^4 > 4C(1 - \omega^{4-N}) \|\varphi\|_2^4 > 0, \quad \|\psi\|^4 - 4\omega^{4-N} C \|\psi\|_2^4 \geq 0.
\]
So the last coefficient of the form \(Q\) is nonpositive, that is
\[
\inf_{(\varphi,\psi) \neq (0,0)} [\|\varphi\|^2 \|\psi\|_2^2 - 4\omega^{4-N} C b \|\varphi \psi\|_2^2] \leq 0.
\]
This and the Hölder inequality yield
\[
\inf_{(\varphi,\psi) \neq (0,0)} \frac{\|\varphi\|^2 \|\psi\|_2^2}{\|\varphi\|_2^4} \leq \inf_{(\varphi,\psi) \neq (0,0)} \frac{\|\varphi\|^2 \|\psi\|_2^2}{\|\varphi \psi\|_2^2} \leq 4b\omega^{4-N} C,
\]
and from (4.23) we deduce that \(b \geq \frac{1}{\omega^{(4-N)/2}}\). The case in which \(\omega > 1\) can be handled in an analogous way. □

5. Nonautonomous systems

In this section, we consider a more general problem
\[
\begin{aligned}
-\Delta u + u &= |u|^{p_0 - 2} u + b_1(x) F_u(u, v) \quad \text{in } \mathbb{R}^N, \\
-\Delta v + \omega^2 v &= |v|^{q_0 - 2} v + b_1(x) F_v(u, v) \quad \text{in } \mathbb{R}^N, \\
u, v &\in H^1(\mathbb{R}^N),
\end{aligned}
\]
(5.1)
where \( n \geq 3 \), and \( p_0 \) and \( q_0 \) are such that

\[
2 < p_0, q_0 < 2^*,
\]

with \( 2^* \) defined in (2.2) and \( b_1(x) \) belongs to \( L^\infty(\mathbb{R}^N) \). We assume that \( F \in C^1(\mathbb{R}^2) \) is such that

\[
F(u, v) = F(-u, v) = F(u, -v) = F(-u, -v)
\]

for any \((u, v) \in \mathbb{R}^2\). Moreover, we suppose that there exist positive constants \( A \) and \( \mu \) such that the following conditions hold for every \((u, v) \in \mathbb{R}^2\)

\[
\left| F_u(u, v) \right| \leq A\left( |u|^{r-1} + |v|^{r-1} \right), \quad \left| F_v(u, v) \right| \leq A\left( |u|^{s-1} + |v|^{s-1} \right), \quad 2 < r, s < 2^*,
\]

\[
F(u, v) > 0 \quad \text{if} \quad uv \neq 0 \quad \text{and} \quad F(u, 0) = F(0, v) = 0,
\]

\[
F_u(u, v)u + F_v(u, v)v - \mu F(u, v) > 0 \quad \text{with} \quad \mu > 2.
\]

**Remark 5.1.** Note that we do not assume any monotonicity condition on the nonlinearity as it was supposed in problem (2.1). However, we can still prove the existence of a nontrivial ground state solution of problem (5.1).

The functional \( I : E \to \mathbb{R} \) associated to (5.1) is defined by

\[
I(w) = \frac{1}{2} \| (u, v) \|^2_E - \frac{1}{p_0} \| u \|^{p_0}_{p_0} - \frac{1}{q_0} \| v \|^{q_0}_{q_0} - \int_{\mathbb{R}^N} b_1(x)F(u, v),
\]

for every \( w = (u, v) \) in \( E \). \( I \) is well defined and of class \( C^1 \) via (5.4) and (5.5) and its differential is

\[
\{ I'(u, v), (\phi, \psi) \} = (u | \phi) + (v | \psi)_{\omega} - \int_{\mathbb{R}^N} \left[ |u|^{p_0-2}u\phi + |v|^{q_0-2}v\psi \right]
\]

\[
- \int_{\mathbb{R}^N} b_1(x)\left[ F_u(u, v)\phi + F_v(u, v)\psi \right],
\]

so that every critical point of \( I \) is a weak solution of system (5.1). Also for this problem it will be useful to study the functional “at infinity”

\[
I_\infty(u, v) = \frac{1}{2} \| (u, v) \|^2_E - \frac{1}{p_0} \| u \|^{p_0}_{p_0} - \frac{1}{q_0} \| v \|^{q_0}_{q_0} - b_\infty \int_{\mathbb{R}^N} F(u, v).
\]

Before stating our existence result, let us recall the Pohozaev identity for systems and general nonlinearities (see [27] or [12]).

**Proposition 5.2.** Let \( b_\infty \) be a positive constant. If (5.2)–(5.5) hold, every critical point of \( I_\infty \) satisfies
\[
\frac{N-2}{2} \left[ \|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right] \\
= -\frac{N}{2} \left[ \|u\|_2^2 + \omega^2 \|v\|_2^2 \right] + N \left[ \frac{1}{p_0} \|u\|_{p_0}^{p_0} + \frac{1}{q_0} \|v\|_{q_0}^{q_0} + b_{\infty} \int_{\mathbb{R}^N} F(u, v) \right]. 
\] (5.9)

**Proof.** Identity (5.9) can be easily obtained from the Pohozaev identity on a bounded domain [12,27], by arguing as in the proof of Proposition 1 in [10]. \(\square\)

**Theorem 5.3.** Suppose that \(b_1 \in L^\infty(\mathbb{R}^N)\) satisfies the following conditions:

\[
\lim_{|x| \to \infty} b_1(x) =: b_{\infty} > 0, \quad (5.10)
\]

\[
b_1(x) \geq b_{\infty}, \quad b_1(x) \neq b_{\infty}. \quad (5.11)
\]

If (5.2), (5.4)--(5.6) hold, then problem (5.1) possesses a nontrivial ground state solution.

**Proof.** We first notice that from (5.4) and (5.5) it follows that there exists a positive constant \(B\) such that

\[
F(u, v) \leq B \left( \|u\|_r + \|v\|_s \right). \quad (5.12)
\]

Indeed, integrating the inequalities in (5.4), using condition (5.5) and Young’s inequality, one obtains

\[
F(u, v) \leq |F(0, v)| + C_1 \left( \|u\|_r + \|v\|_{r-1} \|u\|_r \right) \leq C_2 \|v\|_s + C_3 \|u\|_r, \quad (5.13)
\]

i.e., (5.12). This inequality together with (5.2) and (5.5) imply that \((0, 0)\) is a strict local minimum of \(I\). Condition (5.6) together with assumption (5.2) imply \(I(tw) \to -\infty\) as \(t \to +\infty\) for any \(w \neq 0\) in \(E\), thus \(I\) has the geometry of the Mountain Pass theorem. We consider the set \(\Gamma\) defined by (3.14) and the critical level \(c\) defined by (3.13). From now on we will divide the proof into two steps.

**Step 1.** We will first prove, following the ideas in [21], that there exists a nontrivial critical point \(w\) of \(I\) such that \(I(w) \leq c\).

As in the proof of Theorem 2.3 we obtain a sequence \(\{w_n\}\) such that \(I(w_n) \to c\) and \(I'(w_n) \to 0\) strongly in \(E'\). Without loss of generality, we may assume that the constant \(\mu\) in \(F_3\) satisfies \(2 < \mu \leq \min\{p_0, q_0\}\). We compute \(I(w_n) - \frac{1}{n} \langle I'(w_n), w_n \rangle\) and obtain that \(\{w_n\}\) is bounded in \(E\).

This implies that there exists \((u, v)\) in \(E\) for which (4.3)--(4.5) are satisfied. Thus \((u, v)\) is a critical point of \(I\). Since \((0, 0)\) is a solution, we still need to prove that \((u, v)\) is not trivial. Suppose now that the critical point \((u_n, v_n) = (0, 0)\). We claim that in this case \((u_n, v_n)\) is a Palais–Smale sequence at level \(c\) also for the functional \(I_\infty : E \to \mathbb{R}\) defined in (5.8) In fact, as \(n \to \infty\),

\[
I_\infty(w_n) - I(w_n) = \int_{\mathbb{R}^N} (b_\infty - b_1(x)) F(u_n, v_n) \to 0,
\]

since \(w_n\) is bounded and \(u_n \to (0, 0)\) in \(L^p_{\text{loc}}(\mathbb{R}^N) \times L^p_{\text{loc}}(\mathbb{R}^N)\) for any \(p \in [1, 2^*)\) and as (5.10) holds, we obtain
\[
\sup_{\|\psi\| \leq 1} \left| \left( I'_\infty (w_n) - I'(w_n), (\phi, \psi) \right) \right| \\
= \sup_{\|\psi\| \leq 1} \left| \int_{\mathbb{R}^N} \left( b_{\infty} - b_1(x) \right) F_u(u_n, v_n) \phi + F_v(u_n, v_n) \psi \right| \to 0.
\]

Since \( I_\infty \) is translation invariant, arguing as in the proof of Theorem 2.3 we may construct a new \((PS)\)-sequence \( \overline{w}_k \) for \( I_\infty \), given by translating \( w_k \). This new \((PS)\)-sequence \( \overline{w}_k \) weakly converges to a function \( \overline{w} = (\overline{u}, \overline{v}) \neq (0, 0) \), which is a critical point of \( I_\infty \). Suppose that \((\overline{u}, \overline{v})\) has one trivial component, for example \( \overline{v} \equiv 0 \), then (5.5) implies that \( I(\overline{u}, 0) = I_\infty(\overline{u}, 0) \) and \( I'(\overline{u}, 0) = I'_\infty(\overline{u}, 0) \) so that \((\overline{u}, 0)\) is a nontrivial critical point of \( I \). Moreover, by computing \( I_\infty(\overline{w}_k) - 1/2( I'_\infty(\overline{w}_k), \overline{w}_k) \), and by using (5.6) and the Fatou lemma, we deduce that

\[
I_\infty(\overline{w}) \leq c, \quad (5.14)
\]

and as \( I(\overline{w}) = I_\infty(\overline{w}) \), the claim is proved.

Otherwise, we have found \((\overline{u}, \overline{v})\) critical point of \( I_\infty \) with both nontrivial components. Let us set

\[
\gamma(t) = \overline{w}\left( \frac{x}{t} \right) \quad \text{for } t > 0, \quad \gamma(0) = 0.
\]

By direct calculations it is easy to prove that

\[
\| \gamma(t) \|^2 = t^{N-2} \| \nabla \overline{w} \|^2 + t^N \| \overline{w} \|^2, \quad (5.15)
\]

so that \( \gamma \in C([0, \infty), E) \) and it satisfies

\[
I_\infty(\gamma(t)) = \frac{t^{N-2}}{2} \| \nabla \overline{w} \|^2 - t^N \left[ -\frac{1}{2} \| \overline{w} \|^2 + \frac{1}{p_0} \| \overline{u} \|_{p_0}^2 + \frac{1}{q_0} \| \overline{v} \|_{q_0}^2 \right] - t^N b_{\infty} \int_{\mathbb{R}^N} F(\overline{u}, \overline{v}). \quad (5.16)
\]

Moreover, the Pohozaev identity (5.9) implies

\[
\left\{ -\frac{1}{2} \| (\overline{u}, \overline{v}) \|^2 + \frac{1}{p_0} \| \overline{u} \|_{p_0}^2 + \frac{1}{q_0} \| \overline{v} \|_{q_0}^2 + \int_{\mathbb{R}^N} b_{\infty} F(\overline{u}, \overline{v}) \right\} > 0,
\]

\[
\frac{d}{dt} I_\infty(\gamma(t)) > 0 \quad \text{for } t \in (0, 1) \quad \text{and} \quad \frac{d}{dt} I_\infty(\gamma(t)) < 0 \quad \text{for } t > 1.
\]

If we take a constant \( L > 1 \), sufficiently large, the path \( \gamma_L(t) = \gamma(Lt) \) will be such that \( \gamma_L \in C([0, 1], E) \), \( \overline{w}(x) \in \gamma_L([0, 1]) \), \( I_\infty(\gamma_L(1)) < 0 \) and

\[
\max_{t \in [0, 1]} I_\infty(\gamma_L(t)) = I_\infty(\overline{w}).
\]

Assumptions (5.10), (5.11) and (5.5) imply

\[
I(\gamma_L(t)) < I_\infty(\gamma_L(t)) \quad \forall t \in [0, 1].
\]
Since the functions \( \bar{u} \) and \( \bar{v} \) are both nontrivial, from (5.5), (5.14) and (5.16) we deduce
\[
c \leq \max_{t \in [0, 1]} I(\gamma_L(t)) < \max_{t \in [0, 1]} I_\infty(\gamma_L(t)) = I_\infty(\bar{w}) \leq c,
\]
giving a contradiction. Then, \((u, v)\) is a nontrivial critical point of \( I \). Finally, conditions (5.5) and (5.6) allow us to use Fatou’s lemma and get that
\[
c = \lim_{n \to \infty} I(w_n) = \lim_{n \to \infty} \left[ I(w_n) - \frac{1}{2}(I'(w_n), w_n) \right] \geq I(w)
\]
and the claim is proved also in this case.

**Step 2.** Now, we will show that there exists \((u, v) \neq 0\) with \( u, v \geq 0 \) a least energy critical point of \( I \).

We will first use an argument similar to the proof of Theorem 4.5 in [21] in order to show that \( m \), defined as
\[
m = \inf \{ I(w) : w \neq 0 \text{ and } I'(w) = 0 \},
\]
is attained.

Observe that \( 0 \leq m \leq c \), for \( c \) defined by (3.13). Indeed, take \( w \) a critical point of \( I \); by computing \( I(w) = I(w) - \frac{1}{2}(I'(w), w) \) and using (5.5) and (5.6) we obtain that \( I(w) \geq 0 \), so that \( m \geq 0 \). Furthermore, when we take a nontrivial solution \( w \) obtained as the weak limit of a Palais–Smale sequence at level \( c \) we can argue as in the previous step to deduce that \( I(w) \leq c \), so that \( m \leq c \).

In order to show that \( m \) is attained, let \( w_k \) be a sequence of critical points of \( I \) such that \( I(w_k) \to m \). Thus, \( w_k \) is a Palais–Smale sequence for \( I \) at level \( m \). Reasoning as before, we may conclude that \( w_k \) is bounded, and that there exists \( w \) a nontrivial critical point of \( I \) with \( I(w) \leq m \), and as \( m \) is the lowest critical level it has to be \( I(w) = m \).

Finally, notice that if \((u, v)\) is a nontrivial critical point at level \( m \), (5.3) implies that also the pair \((|u|, |v|)\) is such that \( I(|u|, |v|) = m \) and \( \langle I'(|u|, |v|), (|u|, |v|) \rangle = 0 \). Arguing as in the proof of Lemma 3.3 we deduce that also \((|u|, |v|)\) is a critical point of \( I \) at the same level \( m \). So we can assume that the functions are nonnegative, and the strong maximum principle implies that they are positive, if nontrivial. \( \square \)

**Remark 5.4.** Theorem 5.3 still holds when (3.1) holds with \( b_2 \geq 0 \) and \( b_1 \) satisfying (5.10) and \( b(x) \geq b_\infty \), \( b(x) \neq b_\infty \). Indeed, the integral with \( b_2(x) \) defines a compact operator and \( b_2 \geq 0 \) is a sufficient condition to compare \( I \) with \( I_\infty \).

In the model case
\[
F(u, v) = \frac{1}{q} |uv|^q \quad \text{for } 1 < q < \frac{N}{N - 2}
\]  \hspace{1cm} (5.17)
we can prove also the following result.

**Theorem 5.5.** Assume (5.10), (5.11), (5.17). Moreover, suppose that \( b_\infty \) satisfies (2.4). Then, there exists a least energy solution with both nontrivial components.
Proof. From (5.11) we deduce that $I(w) < I_{\infty}(w)$ for every $w \in E$, then the critical values $c, c_{\infty}$ defined in (3.13) for $I$ and $I_{\infty}$ respectively satisfy the following relation

$$c < c_{\infty}. \quad (5.18)$$

Moreover, we can use Theorem 2.3 to find a vectorial ground state $(u_{\infty}, v_{\infty})$ such that

$$c_{\infty} = I_{\infty}(u_{\infty}, v_{\infty}) < \min\{I_{\infty}(u_0, 0), I_{\infty}(0, v_0)\} = \min\{I(u_0, 0), I(0, v_0)\}. \quad (5.19)$$

Since (5.17) holds the Mountain Pass critical level $c$ satisfies the conclusion of Lemma 3.2 so that from Theorem 5.3 we deduce that there exists $(u, v) \neq (0, 0)$ nontrivial critical point of $I$ at level $c$. Finally, (5.18) and (5.19) give

$$I(u,v) = c < \min\{I(u_0, 0), I(0, v_0)\},$$

which implies that $u \neq 0, v \neq 0$. $\square$

Remark 5.6. Actually, if $F$ satisfies (5.17), we can prove a better result of Theorem 5.5. More precisely, following the arguments of Theorem 2.3 we can obtain a vector soliton if $b(x)$ satisfies

$$\frac{1}{\|u_0\|^2} \int_{\mathbb{R}^N} b(x)|u_0|^{2q} > \frac{2q-1}{\omega^{2q-N(q-1)}} - 1 \quad \text{if } \omega < 1,$$

$$\frac{1}{\|v_0\|^2} \int_{\mathbb{R}^N} b(x)|v_0|^{2q} > 2q-1 \omega^{2q-N(q-1)} - 1 \quad \text{if } \omega > 1,$$

where $u_0$ and $v_0$ are defined by (4.10) and (4.11), respectively.

6. Perturbation results

In this last section we will search for perturbative type results. More precisely, we consider a small parameter $\varepsilon > 0$ and we study a very special case of (5.1), that is the following problem

$$\begin{cases}
-\Delta u + u = |u|^{2q-2}u + \varepsilon b(x)|v|^q|u|^{q-2}u & \text{in } \mathbb{R}^N, \\
-\Delta v + \omega^2 v = |v|^{2q-2}v + \varepsilon b(x)|u|^q|v|^{q-2}v & \text{in } \mathbb{R}^N, \\
u, v \in H^1(\mathbb{R}^N). 
\end{cases} \quad (6.1)$$

where $\omega \in \mathbb{R}$ and $b$ is a measurable function, such that $b$ satisfies (3.1). As we will see in the sequel (Theorem 6.5), in this case we are able to handle a more general function $b$.

In order to tackle the problem we consider the Hilbert space $E$ defined in (2.3). Then, a solution $w = (u, v)$ of (6.1) is a critical point of the functional $I_{\varepsilon} : E \to \mathbb{R}$ defined by $I_{\varepsilon}(w) =$
\[ I_0(w) - \varepsilon G(w), \text{ where } I_0 \text{ and } G(w) \text{ are defined by} \]
\[
I_0(u, v) = \frac{1}{2} \| (u, v) \|_E^2 - F(u, v), \quad F(u, v) = \frac{1}{2q} \| (u, v) \|_q^{2q}, \tag{6.2}
\]
\[
G(u, v) = \frac{1}{q} \int_{\mathbb{R}^N} b(x)|u|^q|v|^q, \quad \forall (u, v) \in E. \tag{6.3}
\]

In order to get our existence results we will apply a perturbation method that has been developed in [1–3]. We will need the following preliminary results.

**Proposition 6.1.** Assume conditions (2.2) and (3.1). Then the functional \( I_0 \) has the following properties:

(\( h_1 \)) \( I_0 \in C^2(\mathbb{E}, \mathbb{R}) \) and it has a 2N-dimensional \( C^2 \) manifold \( Z \) of critical points.

(\( h_2 \)) For all \( z = (u, v) \) in \( Z \) and for every \( (\phi, \psi) \) in \( \mathbb{E} \) the linear operator \( (\phi, \psi) \mapsto F''(u, v)[\phi, \psi] \) is compact.

(\( h_3 \)) For all \( z \in Z \) one has \( T_z Z = \text{Ker} I'_0(z) \), where \( T_z Z \) denotes the tangent space to \( Z \) in \( z \).

Moreover, the functional \( G(u, v) \) defined in (6.3) is of class \( C^1 \).

**Proof.** Hypothesis (3.1) and Sobolev imbedding theorem imply that \( I_0 \) is well defined and of class \( C^2 \) on \( E \). Moreover, \( z = (u, v) \) is a critical point of \( I_0 \) iff \( (u, v) \) are solutions of the following uncoupled system

\[
\begin{cases}
-\Delta u + u = u^{2q-1} \quad &\text{in } \mathbb{R}^N, \\
-\Delta v + \omega^2 v = v^{2q-1} \quad &\text{in } \mathbb{R}^N, \\
u, v \geq 0, \quad u, v \neq 0, \quad u, v \in H^1(\mathbb{R}^N).
\end{cases} \tag{6.4}
\]

Then, \( u = u_0 \) and \( v = v_0 \) are the unique (up to translation), radial solutions of (4.10) and (4.11), respectively (see [10,22]). Thus, \( I_0 \) has a 2N-dimensional manifold of critical points

\[
Z_{\theta, \mu} := Z_\theta \times Z_\mu = \{(u_\theta, v_\mu) = (u_0(x + \theta), v_0(x + \mu)) : \theta, \mu \in \mathbb{R}^N\}.
\]

In order to prove \( h_2 \) consider \( F''(z) = F''(u_\theta, v_\mu) \) which is defined by

\[
F''(u_\theta, v_\mu)[\psi, \phi] = (2q - 1) \begin{pmatrix}
\int_{\mathbb{R}^N} u_\theta^{2q-2} \phi \\
0
\end{pmatrix} \begin{pmatrix}
0 \\
\int_{\mathbb{R}^N} v_\mu^{2q-2} \psi
\end{pmatrix}
\]

for every \( (\phi, \psi) \in \mathbb{E} \). Then, \( h_2 \) easily follows from the exponential decay of \( u_0 \) and \( v_0 \). Moreover, \( w = (u, v) \in \text{Ker} I_0''(u_\theta, v_\mu) \) if and only if \( u \) is a solution of the linearized equation (4.10) and \( v \) is a solution of the linearized equation (4.11). Since it is well known (see [26,32]) that the only solutions of the linearized equation of problems (4.10) and (4.11) are the partial derivatives of \( u_\theta \) and \( v_\theta \), respectively, we deduce that also \( h_3 \) holds. Finally, the regularity properties of the functional \( G \) are an easy consequence of (2.2) and (3.1). □
For every $\varepsilon_0$ and $R > 0$, let us define

$$M(\varepsilon_0, R) = \{ (\varepsilon, \theta, \mu) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N : |\varepsilon| < \varepsilon_0, |\theta| < R, |\mu| < R \}.$$ 

Under hypotheses (h1)–(h3) in [1] (see also [7]) the following result is proved.

**Lemma 6.2.** Suppose that hypotheses (2.2) and (3.1) hold. Then, given $R > 0$, there exists $\varepsilon_0 > 0$ and a function $w : M(\varepsilon_0, R) \to E$ such that

1. $w$ is of class $C^1$ with respect to $(\varepsilon, \theta, \mu)$ and $w(0, \theta, \mu) = 0$,
2. $I_\varepsilon'(z + w(\varepsilon, \theta, \mu)) \in T_zZ_{\theta, \mu}$ for $z = (u_{\theta}, v_{\mu}), \forall (\varepsilon, \theta, \mu) \in M(\varepsilon_0, R)$,
3. $w(\varepsilon, \theta, \mu)$ is orthogonal to $T_zZ_{\theta, \mu} \forall (\varepsilon, \theta, \mu) \in M(\varepsilon_0, R)$.

Then, we have constructed the following manifold

$$Z_{\varepsilon} := \{ z + w(\varepsilon, z) : z = (u_{\theta}, v_{\mu}) \text{ with } (\varepsilon, \theta, \mu) \in M(\varepsilon_0, R) \}. \quad (6.5)$$

Moreover, from (ii) we deduce that any constrained critical point of $I_\varepsilon$ on $Z_{\varepsilon}$ is a critical point without constraint, so that we are lead to search for critical points of $I_\varepsilon$ on $Z_{\varepsilon}$. In [1] it is proved that $I_\varepsilon$ has a convenient Taylor expansion on $Z_{\varepsilon}$, so that the following abstract existence results hold (for a proof see [1]).

**Theorem 6.3.** Assume hypotheses (2.2) and (3.1). Moreover, suppose that $G$ has a proper local minimum (or maximum) at some $z = (u_{\theta}, v_{\mu}) \in Z$, then $I_\varepsilon$ has a critical point $w_\varepsilon = z + w(\varepsilon, \theta, \mu)$.

**Remark 6.4.** Notice that from the choice of $Z_{\theta, \mu}$ and from the continuous dependence of $w$ on $\varepsilon$ we deduce that the critical points we will find $(u_\varepsilon, v_\varepsilon) = (u_{\theta_0} + w_1(\varepsilon, \theta_0, \mu_0), v_{\mu_0} + w_2(\varepsilon, \theta_0, \mu_0))$ is such that $u_\varepsilon, v_\varepsilon \not\equiv 0$. For more details see [1].

Theorem 6.3 leads us to study the function $\Gamma : \mathbb{R}^{2N} \to \mathbb{R}$ defined by

$$\Gamma(\theta, \mu) = \frac{1}{q} \int_{\mathbb{R}^N} b(x)u_0^q(x)\nu_0^q(x) = \frac{1}{q} \int_{\mathbb{R}^N} b(x)u_0^q(x + \theta)\nu_0^q(x + \mu). \quad (6.6)$$

We can now state our existence result for problem (6.1)

**Theorem 6.5.** Assume (3.1). Moreover, suppose that one of the following conditions holds

$$\lim_{|x| \to \infty} b_1(x) = 0 \quad \text{and} \quad b(x) > 0 \quad \text{or} \quad b(x) < 0. \quad (6.7)$$

Then, there exists at least a critical point $w$ of $I_\varepsilon$. Moreover, $w = (u_1, v_1)$ with $u_1 = u_{\theta_0} + w_1(\varepsilon, \theta_0, \mu_0), v_1 = v_{\mu_0} + w_2(\varepsilon, \theta_0, \mu_0))$, with $u_1, v_1 \not\equiv 0$. 

Proof. In order to get the conclusion we apply Theorem 6.3. Hence, it is enough to show that $\Gamma$, defined in (6.6), has a global minimum or maximum. We observe that if we assume (6.7) we get
\[
\lim_{|\theta, \mu| \to \infty} \Gamma(\theta, \mu) = 0.
\] (6.8)

Indeed, if $|\theta, \mu| \to \infty$ then $b(x)u_0^q(x)v_0^q(x) \to 0$ almost everywhere, and (3.1) and (6.7) allow us to apply Lebesgue dominated convergence theorem to get (6.8). Moreover, we have
\[
\Gamma(0, 0) = \frac{1}{q} \int_{\mathbb{R}^N} b(x)u_0^q(x)v_0^q(x) \neq 0,
\]
so that there exists at least a minimum or a maximum depending on the sign of $b$. \hfill \square

Remark 6.6. After this work was completed, we became aware of [4,5] in which there are some results related to ours and of [8,30] also concerning this subject.

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