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On an Alternative Representation for a Wide Class of Binary Relations

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It is shown that any antireflexive binary relation which is either transitive or symmetric can be represented by a specific tree structure whose nodes are of maximum order 3. The relationship between this representation and an analogous representation for a narrower class of relations (previously called "splitting") is discussed. Algorithms are developed for obtaining the tree structures from a matrix representation.

1. INTRODUCTION

The motivation for the discussion in this paper can perhaps best be given in terms of Fig. 1.

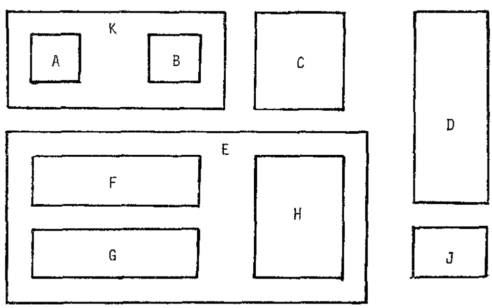

FIG. 1. Relations "left of," "above," and "includes" as represented in Figs. 2–5.

One way of describing this picture would be to say (perhaps in the form of 3 relation matrices) that A is to the left of B, that these two as well as F, G, and K are to the left of C and H and all these as well as E are to the left of D and J. A, B, C, and K are above E, F, G, H, and J; F and D are above G and J. A and B are contained in K; F, G, and H are contained in E.

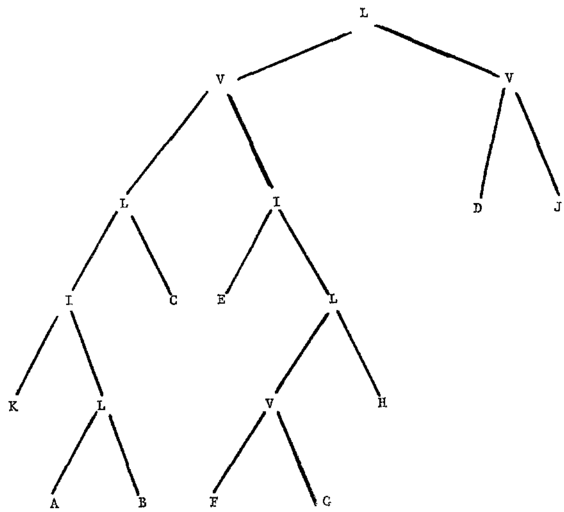

FIG. 2. Minsky's representation of Fig. 1.

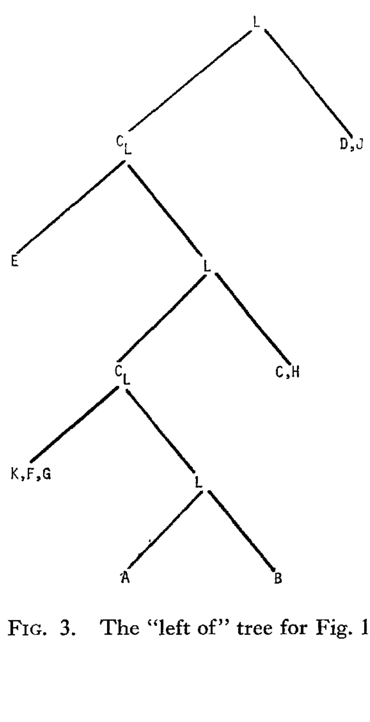

FIG. 3. The "left of" tree for Fig. 1.

Minsky (1960) once suggested that these above facts could also be stored in the form of a tree as in Fig. 2. A considerable saving of memory is obtained this way.

The trouble with this representation is that there is no way of obtaining from the tree that F is to the left of C or D is above G.

I suggested in a previous paper (1973) that the information be preserved in 3 trees—one for the horizontal, one for the vertical, and one for the inclusion relationships. The trees for the "left of" and "includes" relationships are shown in Figs. 3 and 4, respectively. It will be noted that the trees have very similar structure. The abstract basis for this similarity, as well as an algorithm for the construction of the trees from the relation matrix was described by us in the previous paper.

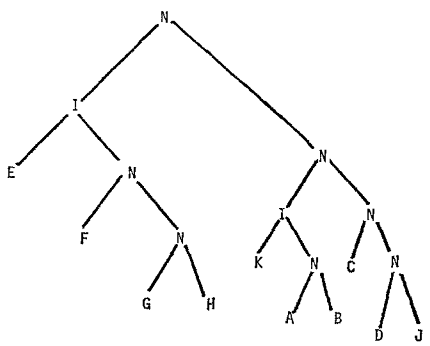

FIG. 4. The "incloses" tree for Fig. 1. It is really the "neither incloses nor is inclosed" tree.

The techniques of that paper, however, cannot be applied to the "above" relation in Fig. 1. It will be noted that in the horizontal relationships in Fig. 1, if any block was neither to the right nor to the left of the other, then one of them straddled the other one. The technique depended heavily on this property which we call the property of "splitting." Unfortunately, in Fig. 1, the block D and block E "slide" with respect to one another—neither is above the other, nor does one "straddle" the other.

In the present paper the technique of the previous paper has been generalized to be applicable to any relation which is antireflexive and either transitive or symmetric. To this end we shall first develop certain properties of such relations in an axiomatic manner.

2. Binary Relations and Dichotomies

In what follows we shall consider a binary relation R on an abstract set S. For the purposes of intuitive discussion S can be taken to be the lettered rectangles of Fig. 1 and R the relation "above" which we have denoted by V in Fig. 5.

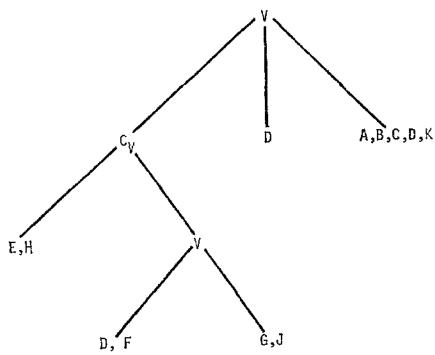

Fig. 5. The "above" or "vertical" tree for Fig. 1. Note how the top node has three branches since "above" does not split.

Definition 2.1. Given $R \subseteq S \times S$ we define a function f mapping S into subsets of S as

$$f(s) = \{s' \mid s \not R s' \ \&\ s' \not R s\},$$

where $s \not R s'$ denotes that sRs' is false.

Definition 2.2. Given R, S, and f as above we define $C_R \subseteq S \times S$ as $sC_Rs' \equiv f(s') \subseteq f(s)$.

Proposition 2.1. *C_R is a reflexive, transitive relation (i.e., a preorder). Hence $E_R = C_R \cap C_R^{-1}$ is an equivalence and consequently the quotient relation $C_R' = C_R/E_R$ is a partial order.*

The proof is immediate from definitions. However, Proposition 2.1 is not of great use unless C_R is sufficiently rich (for instance, is not just the diagonal). Fortunately, we can, without losing too much generality, assume that no element is related to itself. It will be noticed that when we say "A is above F" in Fig. 1 we mean, "the bottom edge of A is above the top edge of F." So, since no figure can have its bottom edge above its top edge (or its right edge to the left of its left edge), our restriction will apply to the kind of relations we are dealing with.

DEFINITION 2.3. $R \subseteq S \times S$ will be called antireflexive if for all $s \in S$, $s \not R s$.

We have two immediate results.

PROPOSITION 2.2. *If R is antireflexive then for each $s \in S$, $s \in f(s)$, where f is as defined above. Moreover, for each s, $s' \in S$, $sC_R s' \rightarrow s \not R s'$ and $s' \not R s$.*

Proof. The first result follows immediately from definition. The second follows since for all $x \in S$, $x \not R s'$ and $s' \not R x$ implies $x \not R s$ and $s \not R x$ while $s' \not R s'$ from antireflexivity. Q.E.D.

The C_R defined above is distinct from the one defined in our previous paper. However, the distinction is not of major importance. Notice that both the relations V and L in our example are transitive. Also the relation "does not contain nor is contained in" discussed in Section 1, though not transitive, is symmetric by its nature. The rest of our discussion will therefore assume explicitly that R is either transitive or symmetric. The next proposition establishes a close tie with the definition of C_R in our previous paper, in addition to being useful for the main result in this paper.

PROPOSITION 2.3. *If R is antireflexive and either symmetric or transitive then $sC_R s'$ if and only if for all $x \in S$, xRs implies xRs' and sRx implies $s'Rx$.*

Proof. If $sC_R s'$ then for $x \in S$, $x \not R s$ and $s \not R x$ whenever $x \not R s'$ and $s'Rx$. Contrapositively if xRs or if sRx, then we must have either xRs' or $s'Rx$. Concentrating on the case xRs, if xRs' there is nothing to prove. If $s'Rx$ then again xRs' follows if R is symmetric. If R is transitive then $s'Rx$ is impossible since with xRs it implies $s'Rs$ which violates Proposition 2.2. In any case xRs implies xRs'. It can be similarly proved that sRx implies $s'Rx$.

To prove the converse, let it be assumed that for any $x \in S$, if xRs then xRs' and if sRx then $s'Rx$. Let now $x \in f(s')$, i.e., $x \not R s'$ and $s' \not R x$. Hence $x \not R s$ and $s \not R x$, i.e., $x \in f(s)$. Hence $f(s') \subseteq f(s)$, i.e., $sC_R s'$. Q.E.D.

We can now establish the formal bases for the construction of Fig. 5 from Fig. 1.

DEFINITION 2.4. Given $R \subseteq S \times S$, a pair A, B of subsets of S is called an R-dichotomy of S if and only if $A \neq \varnothing$, $B \neq \varnothing$, $A \cap B = \varnothing$, $A \cup B = S$ and for $x \in S$, $\{x\} \times B \subseteq R$ iff $x \in A$.

A reference to Fig. 5 will clarify the definition. Concentrating on the boxes D, E, F, G, H, and J, and calling it S', we note that D, F, G, and J, are such that both E and H straddle all of them and no other elements of S' do. Hence $((E, H), (D, F, G, J))$ is a C_V dichotomy of S'. Similarly $((F, D), (G, J))$ form a V-dichotomy of the set (D, F, G, J). In this example, when we say V here, we mean V restricted to S'.

Unfortunately, this happy state of affairs breaks down when we look at the entire picture. Any V-dichotomy we try gets "obstructed" by D or E and J. To handle this phenomenon we need another definition.

DEFINITION 2.5. Given $R \subseteq S \times S$, a pair A, B of subsets of S is called an R-trichotomy of S if and only if $A - B \neq \varnothing$, $B - A \neq \varnothing$, $A \cup B = S$, and $(A - B) \times (B - A) \subseteq R$.

We are now in a position to prove the main theorem of this section.

THEOREM 2.1. *Let $R \subseteq S \times S$ be antireflexive and either transitive or symmetric. Then S either has a C_R-dichotomy or an R-trichotomy or else R is empty.*

Proof. Let x_0 be any element of S. Consider the set $[x_0]^{E_R}$, the equivalence class of E_R containing x_0.

If $[x_0]^{E_R} = S$ then for all y, $x_0 C_R y$ and $y C_R x_0$. By Proposition 2.2 $x_0 \not R y$ & $y \not R x_0$. Again, for any $x \in S$, xRy & $y C_R x_0$ implies, by Proposition 2.3, that xRx_0, which has just been shown false. Hence for all x and y, $x \not R y$, i.e., R is empty.

If R is not empty then $[x_0]^{E_R} \neq S$. Consider the two sets

$$M = \{y \mid x_0 \not R y\},$$
$$N = \{y \mid y \not R x_0\}.$$

If both the sets are equal to S then, consider the pair of sets $([x_0]^{E_R}, S - [x_0]^{E_R})$. By hypothesis, neither of these two sets is empty. Also $x \in [x_0]^{E_R}$ implies $f(x) = f(x_0) = S$. Hence for any $y \in S$ (and a fortiori any $y \in S - [x_0]^{E_R}$) $f(y) \subseteq S = f(x_0)$. So $x C_R y$. Conversely, if $x \notin [x_0]^{E_R}$, one has a $y \in S = f(x_0)$ such that $y \notin f(x)$. Hence xRy or yRx. In either case $x \not C_R y$ and since $x \notin f(y)$ & $x \in S = f(x_0)$, $y \notin [x_0]^{E_R}$.

Now we come to the case where either M or N is a proper subset of S. If M be a proper subset of S, then we shall claim that $(M, f(S - M))$ is an R-trichotomy. If N is a proper subset of S, we shall claim that $(f(S - N), N)$ is an R-trichotomy. In what follows we establish only the first claim. The second follows similarly.

By antireflexivity $S - M \subseteq f(S - M)$ and hence $M \cup f(S - M) = S$. To see that $M - f(S - M) \neq \varnothing$ note that $x_0 \in M$ by antireflexivity and that for each $y \in S - M$, $x_0 R y$ so that $x_0 \notin f(S - M)$. To see that

$f(S - M) - M \neq \varnothing$ note that since $M \neq S$, there is an $s \in S - M$. But since $s \in f(s)$, $s \in f(S - M)$ whence $f(S - M) - M$ is nonempty. The three preliminary conditions for an R-trichotomy are thus shown valid.

Let now $x \in M - f(S - M)$ and $y \in f(S - M) - M$. To show xRy we note that since $y \notin M$ we have x_0Ry. Assume that $x \not R y$. Now either yRx or $y \not R x$. If the latter, then with $x \not R y$ we have $x \in f(y)$. Since $y \notin M$, this yields $x \in f(S - M)$ which contradicts $x \in M - f(S - M)$. Hence yRx. But this contradicts $x \not R y$ if R is symmetric. It also contradicts $x \in M$ if R is transitive since x_0Ry and yRx implies x_0Rx. Hence in all cases $x \not R y$ is false. Q.E.D.

The above theorem yields a mechanical procedure for dividing a set into either an R-trichotomy or a C_R-dichotomy, as long as the relation is nonempty. In the case of an R-trichotomy, a part of the relation R is very succinctly stored by storing the sets A and B. In the case of the C_R-dichotomy, Proposition 2.2 gives a succint expression for the complement of R.

Note that once a C_R-dichotomy or R-trichotomy is found in the set, a part of the relation R can be represented in the tree forms shown in Fig. 6.

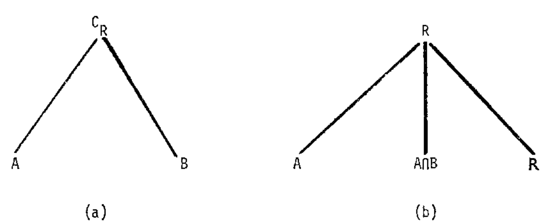

FIG. 6. Trees constructed on the basis of Theorem 2.1.

However, no information about R can be obtained among the elements of A or B. So the process indicated in Theorem 2.1 has to be applied recursively. The next theorem indicates that this process can be applied recursively to obtain the required refinement.

THEOREM 2.2. *Under the hypothesis of Theorem 2.1, if (A, B) is a C_R-dichotomy of S, then A has no C_R-dichotomy or R-trichotomy and B has at least one R-trichotomy.*

(In this statement when we talk about the relation R on the sets A and B we will mean the restriction of R. C_R will refer to the relation as defined by the restricted R as opposed to the restriction of C_R itself.)

Proof. We will show first that the restriction of R is empty on A and nonempty on B. If $x \in A$, $y \in A$, and xRy then note that for all $z \in B$, xC_Rz. However, by Proposition 2.3 we have zRy. But this, by Proposition 2.2, yields $y\not C_Rz$ contradicting that (A, B) is a C_R-dichotomy.

If the restriction of R on B is empty, then for any $x \in B$, $f(x) \supseteq B$. Also since for all $y \in A$, yC_Rx we have by Proposition 2.2 that $A \subseteq f(x)$ whence $f(x) = S$ so that xC_Rz for all $z \in B$. Hence $x \in A$ which contradicts $x \in B$.

Hence there is an $x \in B$ and a $y \in B$ such that xRy. In this case $(B - \{y\}$, $B - \{x\})$ form an R-trichotomy of B. Note that by antireflexivity $x \neq y$.

Q.E.D.

On the basis of these two theorems we can set up the following procedure for representing a binary antireflexive relation in a tree provided it is either transitive or symmetric. The input of the tree is the relation R on the set S (using any representation—we are not concerned with the efficiency of this algorithm) and the name of a node on a tree which we shall call S'. The procedure will be recursive, with the call TREE(S, R, S').

1. If R is empty, place a list for S in S' and exit.
2. Choose an element $x_0 \in S$ and form the sets M and N.
3. If $M \neq S$, place the symbol "R" at S', attach to S' a left pointer to the name S_1' and a right pointer to the name S_2'. Divide the set S into two sets, $S_1 = M$ and $S_2 = f(S - M)$. Call TREE(S_1, $R \cap S_1 \times S_1$, S_1'). Call TREE(S_2, $R \cap S_2 \times S_2$, S_2'). Exit.
4. If $N \neq S$ place the symbol "R" at S', attach to S' a left pointer to the name S_1' and a right pointer to the name S_2'. Divide S into two sets $S_1 = f(S - N)$ and $S_2 = N$. Call TREE(S_1, $R \cap S_1 \times S_1$, S_1'). Call TREE(S_2, $R \cap S_2 \times S_2$, S_2'). Exit.
5. Form the set $[x_0]^{E_R}$. Attach the symbol C_R to S'. Attach to S' a left pointer to S_1' and a right pointer to S_2'. Divide S into two sets $S_1 = [x_0]^{E_R}$ and $S_2 = S - S_1$. Call TREE(S_2, $R \cap S_2 \times S_2$, S_2'). Attach to S_1' a list for S_1. Exit.

The tree constructed by the algorithm for the "above" relation for Fig. 1 is shown in Fig. 5.

3. Concluding Remarks

The tree representation originally suggested by Minsky was suggested for A.I. considerations. However, even from the point of view of memory saving it has justification. In one extreme case, where all the trichotomies are disjoint, the maximum memory requirement is of the order of $2n$, where n is the number of elements in S. In the other extreme, when the R-trichotomies have only one element in the two symmetric differences, the requirement may go as high as $n^2 + n$. In one case the saving is considerable and in the other the loss is not much. One can, therefore, expect some saving of memory.

If the relation is "splitting"—the case discussed in our previous paper—then the $2n$ case occurs; i.e., there is a way of assuring that all the R-trichotomies are disjoint. This can be assured for splitting relations if, in our algorithm, x_0 is always chosen to be a maximal element of C_R , i.e., such that for any x, xC_Rx_0 only if x_0C_Rx. Also, many of the steps in the algorithm, like the calculation of $[x_0]^{E_R}$ and $f(S - M)$, are simplified considerably. The reader is referred to our previous paper in case he is interested. That paper also explains how the complement of the inclusion relation, being antireflexive, symmetric and splitting, yields a tree similar to the tree of the "left-of" relation in Fig. 1.

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