Advances in Mathematics **140**, 323–366 (1998)
Article No. AI981775

An Algebraic Framework for Group Duality

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Received December 11, 1996; accepted April 30, 1998

A Hopf algebra is a pair (A, Δ) where A is an associative algebra with identity and Δ a homomorphism form A to $A \otimes A$ satisfying certain conditions. If we drop the assumption that A has an identity and if we allow Δ to have values in the so-called multiplier algebra $M(A \otimes A)$, we get a natural extension of the notion of a Hopf algebra. We call this a multiplier Hopf algebra. The motivating example is the algebra of complex functions with finite support on a group with the comultiplica-

tionals exist, they are unique (up to a scalar) and faithful. For a regular multiplier Hopf algebra (A, Δ) (i.e., with invertible antipode) with invariant functionals, we construct, in a canonical way, the dual $(\hat{A}, \hat{\Delta})$. It is again a regular multiplier Hopf algebra with invariant functionals. It is also shown that the dual of $(\hat{A}, \hat{\Delta})$ is canonically isomorphic with the original multiplier Hopf algebra (A, Δ). It is possible to generalize many aspects of abstract harmonic analysis here. One can define the Fourier transform; one can prove Plancherel's formula. Because any finite-dimensional Hopf algebra is a regular multiplier Hopf algebra and has invariant functionals, our duality theorem applies to all finite-dimensional Hopf algebras. Then it coincides with the usual duality for such Hopf algebras. But our category of multiplier Hopf algebras also includes, in a certain way, the discrete (quantum) groups and the compact (quantum) groups. Our duality includes the duality between discrete quantum groups and compact quantum groups. In particular, it includes the duality between compact abelian groups and discrete abelian groups. One of the nice features of our theory is that we have an extension of this duality to the non-abelian case, but within one category. This is shown in the last section of our paper where we introduce the algebras of compact type and the algebras of discrete type. We prove that also these are dual to each other. We treat an example that is sufficiently general to illustrate most of the different features of our theory. It is also possible to construct the quantum double of Drinfel'd within this category. This provides a still wider class of examples. So, we obtain many more than just the compact and discrete quantum within this setting.

1. INTRODUCTION

Let (A, Δ) be a finite-dimensional Hopf algebra (over $\mathbb{C}$). The dual space A' of linear functionals on A can again by made into a Hopf algebra if the product and coproduct on A' are defined by

0001-8708/98 $25.00

$$(\omega_1\omega_2)(x) = (\omega_1 \otimes \omega_2)\,\Delta(x)$$

$$\Delta(\omega)(x \otimes y) = \omega(xy)$$

whenever $x, y \in A$ and $\omega, \omega_1, \omega_2 \in A'$. It is obvious from the construction that the bidual again gives the original Hopf algebra.

This duality generalizes the duality between a finite abelian group G and its dual group. Take for A the algebra of complex functions on G (with pointwise operations) and define a comultiplication Δ on A in the usual way by $\Delta(f)(p,q) = f(pq)$ whenever $f \in A$ and $p, q \in G$. Of course, we identify $A \otimes A$ with complex functions on $G \times G$ in the natural way. This gives a Hopf algebra. The dual is the group algebra, i.e. the space of complex functions on G, now with the convolution product. The coproduct here is given by $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$, where we denote by λ_p the function that is 1 on p and 0 elsewhere. If the group is abelian, the finite-dimensional Fourier transform will yield an isomorphism of this dual Hopf algebra with the Hopf algebra of complex functions on the dual group $\hat{G}$ (with pointwise operations). So indeed, the duality of finite-dimensional Hopf algebras, includes the duality theory for finite abelian groups in a very natural way.

For a general (non finite-dimensional) Hopf algebra (A, Δ), it is still possible to make the dual space A' of linear functionals into an associative algebra (by the same formula $(\omega_1\omega_2)(x) = (\omega_1 \otimes \omega_2)\,\Delta(x)$). But, at least in general, the natural candidate for $\Delta(\omega)$ lies in $(A \otimes A)'$ but not necessarily in $A' \otimes A'$. This last space is strictly smaller than $(A \otimes A)'$. So, in general, one cannot make A' into a Hopf algebra.

There have been various attemps to overcome this problem. One possibility is to look for elements $\omega \in A'$ such that $\Delta(\omega) \in A' \otimes A'$. Another possibility is to work with topological versions (e.g. taking a topological algebra A with a continuous coproduct, considering only continuous linear functionals A^* and taking comultiplications with values in some completed tensor products $A \bar{\otimes} A$ and $A^* \bar{\otimes} A^*$ respectively). All these methods only work in special situations.

In this paper, we propose another framework to extend the duality of finite-dimensional Hopf algebras. It is motivated by the following example.

Let G be any group and let A be the group algebra over $\mathbb{C}$. So A is the space of complex functions on G, now with finite support, and convolution product

$$(f * g)(p) = \sum_{q \in G} f(q)\, g(q^{-1}p)$$

whenever $f, g \in A$ and $p \in G$. If the comultiplication is defined as before by $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$, then also here, A becomes a Hopf algebra. The dual space

A' can be identified in a natural way with the space of all complex functions by means of the pairing

$$\langle f, \lambda_p \rangle = f(p).$$

And it is clear that the product in A' is just the pointwise product of functions. The coproduct Δ on A' would be defined by

$$\Delta(f)(p, q) = f(pq)$$

for all $p, q \in G$ and all complex functions f on G. Now, instead of looking for functions f so that $\Delta(f) \in A' \otimes A'$ or for suitable topologies and working with continuous maps and completed tensor product, our approach is motivated by the following observations.

Denote by B the space $K(G)$ of complex functions with finite support in G. Identify $B \otimes B$ with $K(G \times G)$ and consider $K(G \times G)$ as a subspace of $C(G \times G)$, the space of all complex functions on $G \times G$. We make these vector spaces into algebras now by using pointwise multiplication. For $f \in B$, we also consider the function $\Delta(f)$ in $C(G \times G)$ defined as before by $\Delta(f)(p, q) = f(pq)$. Then

$$\Delta(f)(1 \otimes g) \qquad \text{and} \qquad (f \otimes 1)\, \Delta(g)$$

are in $B \otimes B$ for all $f, g \in B$ and the linear maps from $B \otimes B$ to $B \otimes B$, defined by

$$f \otimes g \to \Delta(f)(1 \otimes g)$$

$$f \otimes g \to (f \otimes 1)\, \Delta(g)$$

are bijective. This follows from the fact that G is a group and that these maps are dual to the maps $(p, q) \to (pq, q)$ and $(p, q) \to (p, pq)$ of $G \times G$ to itself.

This brings us to a natural extension of the notion of a Hopf algebra, namely a multiplier Hopf algebra (see [16]). We will recall the precise definition in Section 2. But briefly, a multiplier Hopf algebra is a pair (A, Δ) where A is an associative algebra over $\mathbb{C}$, possibly without identity, and where Δ is a comultiplication on A. This is a homomorphism of A into the multiplier algebra $M(A \otimes A)$. In the group case, where $A = K(G)$ and $A \otimes A = K(G \times G)$, this multiplier algebra is $C(G \times G)$. It is also assumed that

$$\Delta(a)(1 \otimes b) \qquad \text{and} \qquad (a \otimes 1)\, \Delta(b)$$

are in $A \otimes A$ and that the linear maps on $A \otimes A$, defined by

$$a \otimes b \to \Delta(a)(1 \otimes b)$$

$$a \otimes b \to (a \otimes 1)\,\Delta(b)$$

are bijective.

So, the algebra of functions $K(G)$ as above, with the comultiplication also given above is a multiplier Hopf algebra and it can be considered as dual to the Hopf algebra with the group algebra as the underlying algebra and the usual comultiplication.

This duality essentially contains the duality between compact abelian groups and discrete abelian groups. Indeed, if G is abelian, then the group algebra of G is, using the Fourier transform, isomorphic with a dense subalgebra A of the algebra of continuous complex functions on the compact dual group $\hat{G}$ (with pointwise operations). The comultiplication on the group algebra in G is hereby transformed to the usual comultiplication given by $\Delta(f)(r, s) = f(rs)$ when $r, s \in \hat{G}$ and f is a continuous complex function in A.

In this paper, we will generalize this duality. We consider a certain class of multiplier Hopf algebras and we will define a dual within this class. The category that we will consider is the category of multiplier Hopf algebras, with invertible antipode and admitting left and right invariant functionals. It turns out to be possible to generalize many aspects of abstract harmonic analysis. We can define convolution products, the Fourier transform. We can prove the Plancherel formula, ... In [20], we develop the notion of an action of a multiplier Hopf algebra on an algebra (extending the notion of an action of a group). We define the analogue of the crossed (smashed) product. For multiplier Hopf algebras with invariant functionals, we can also prove a duality for actions (extending the duality for actions of finite-dimensional Hopf algebras) (again see [20]).

Similar theories have been developed before. In [6], Enock and Schwartz obtain a duality for Kac algebras (in the von Neumann algebra framework). The main drawback here is the assumption that the square of the antipode is one. The examples of Jimbo and Drinfel'd (see e.g. [4]) do not have this property. In [8], Masuda and Nakagami extend this theory so that also these examples can fit in. And more recently, Masuda, Nakagami and Woronowicz (in [9]) have obtained a new formulation in the C^*-algebra framework.

Our theory is purely algebraic and in many respects, much more simple. Most of the proofs are easy. Still, all the different aspects fit very nicely together. However, it is more restrictive in the sense that e.g. our category does not contain all the locally compact groups. On the other hand, it also includes e.g. some of the root of unity algebras that do not fit into the operator algebra approach. Moreover, we can construct the quantum double of Drinfel'd within our category. This is done in [3].

We also treat the *-algebra case briefly. If the invariant functionals are e.g. positive, then the same is true for the invariant functionals on the dual. The quantum double in this case is also a *-algebra with positive invariant functionals. This makes it possible to give also a C*-version. In fact, it turned out that many more nice aspects come into the picture in this C*-formulation (see [7]). Because of all this, it is our hope that our purely algebraic theory will have a topological version, including also all the locally compact groups and the cases studied by Masuda *et al.*

As in the theories of Enock and Schwartz and Masuda, Nakagami and Woronowicz, we also assume the existence of an invariant functional (Haar measure). It would be nice, as is the case for locally compact groups, to formulate conditions so that the existence of such a functional can be proven. We have some results of this type for multiplier Hopf algebras (not included in this paper—see e.g. [19]). On the other hand, uniqueness of the invariant functionals (which we can prove), is perhaps more important than being able to prove existence. In most examples, the invariant functional "is just there". The existence problem is therefore a theorical problem (and in that sense still important).

The paper is organized as follows. In Section 2, we first recall the theory of multiplier Hopf algebras as developed in [16]. We give the main definitions and formulate the main results (like the existence and uniqueness of the counit and the antipode). We also prove some results that were not proven in [16]. At the end of this section, we fix some notations and make some conventions.

In Section 3, we discuss the invariant functionals. We give a precise definition. We prove that, if they exist, then they are unique (up to a scalar), and they are faithful. We also prove the existence of the "modular element", relating the left and the right invariant functional (the equivalent of the modular function for a non-unimodular locally compact group). We show that these functionals satisfy what we call, the weak K.M.S.-condition. The name comes from physics (K.M.S. stands for Kubo Martin Schwinger) and it is like the K.M.S.-condition for positive functionals on a C*-algebra (see e.g. [10]). It helps to overcome the problems that arise from the non-abelianness of the algebra. Remark that in [18], where we announced these results, the weak K.M.S. property was assumed. Later, we discovered that it followed automatically in all cases.

Section 4 contains the main results of our paper. Given any multiplier Hopf algebra (A, Δ), with an invertible antipode and with non-zero invariant functionals, we construct the dual $(\hat{A}, \hat{\Delta})$ which is a multiplier Hopf algebra of the same type. And we show that the dual of $(\hat{A}, \hat{\Delta})$ is canonically isomorphic with (A, Δ). In this section, we also give a small account of the harmonic analysis in this category.

In Section 5, we discuss some special cases and examples. We introduce the notions compact and discrete type for our multiplier Hopf algebras here and we show that they are dual to each other. This generalizes the duality between discrete quantum groups (as defined in [5] and [17]) and compact quantum groups (as defined in [2], [21] and [22]). Of course, it then also contains the duality between discrete abelian and compact abelian groups. Multiplier Hopf algebras of discrete type are studied more in detail in a paper with Y. Zhang ([19]). In this section, we also treat an example of the duality between these compact and discrete type algebras, which is a root of unity case that exhibits almost all the different aspects of our theory.

Recall that the quantum double is treated in a separate paper with B. Drabant ([3]). There, also pairings between multiplier Hopf algebras are studied in more generality. The C*-case is extensively worked out in a paper with J. Kustermans ([7]). And actions of multiplier Hopf algebras and duality for actions is treated in a paper with B. Drabant and Y. Zhang ([20]).

2. MULTIPLIER HOPF ALGEBRAS

We will first recall some of the theory of multiplier Hopf algebras. For details, see [16].

Let A be an associative algebra over $\mathbb{C}$. We assume that the product in A is non-degenerate. This means that, if $ab=0$ for all a, then $b=0$ and similarly, if $ab=0$ for all b, then $a=0$.

A left multiplier of A is a linear map $\rho: A \to A$ such that $\rho(ab)=\rho(a)\,b$ for all $a, b \in A$. A right multiplier of A is a linear map $\rho: A \to A$ such that $\rho(ab)=a\rho(b)$ for all $a, b \in A$. A multiplier of A is a pair (ρ_1, ρ_2) of a left and right multiplier so that $\rho_2(a)\,b=a\rho_1(b)$ for all $a, b \in A$. When ρ_1 and ρ_2 are linear maps on A satisfying $\rho_2(a)\,b=a\rho_1(b)$ for all a, b, then already (ρ_1, ρ_2) is a multiplier. Indeed, the properties $\rho_1(ab)=\rho_1(a)\,b$ and $\rho_2(ab)=a\rho_2(b)$ will follow (using the non-degeneracy of the product). We will use this a few times in the paper to construct multipliers. If $c=(\rho_1, \rho_2)$ is a multiplier of A we will write $ca=\rho_1(a)$ and $ac=\rho_2(a)$. To define a multiplier c, we have to define both ac and ca and prove that $(ac)\,b=a(cb)$.

The set of multipliers of A is, in a natural way, an associative algebra. We denote it by $M(A)$. It always contains a unit (denoted by 1). And there is a natural imbedding (because the product in A is non-degenerate) of A in $M(A)$. In fact, A is an essential (dense) two-sided ideal in $M(A)$. Moreover, $M(A)$ can be characterized as the largest algebra with identity in which A sits as an essential two-sided ideal. Of course $A = M(A)$ if and only if A already has an identity.

Now, consider the tensor product $A \otimes A$ of A with itself. It can be made into an associative algebra in a natural way. Also here the product will be non-degenerate. Hence we can also consider the multiplier algebra $M(A \otimes A)$. It is clear that we have natural imbeddings

$$A \otimes A \rightarrow M(A) \otimes M(A) \rightarrow M(A \otimes A),$$

compatible with the imbedding of $A \otimes A$ in $M(A \otimes A)$.

Now, we are ready to recall the notion of a comultiplication on A.

2.1. Definition. A comultiplication on A is a homomorphism $\Delta: A \rightarrow M(A \otimes A)$ such that

(i) $\Delta(a)(1 \otimes b)$ and $(a \otimes 1)\, \Delta(b)$ are in $A \otimes A$ for all $a, b \in A$,

(ii) $(a \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(b)(1 \otimes c)) = (\iota \otimes \Delta)((a \otimes 1)\, \Delta(b))(1 \otimes 1 \otimes c)$

for all $a, b, c \in A$.

Throughout this paper ι will denote the identity map on A. We need (i) to give a meaning to the formula in (ii). The property (ii) is called the coassociativity of Δ.

Before we continue, let us consider the motivating example.

2.2. Example. Let G be any (discrete) group and let A be the algebra $K(G)$ of complex functions on G with finite support and pointwise operations. Then $M(A)$ can be naturally identified with the algebra $C(G)$ of all complex functions on G (with pointwise operations). Also $A \otimes A = K(G \times G)$ and $M(A \otimes A) = C(G \times G)$. The product in G gives rise to a coproduct Δ on A defined by

$$\Delta(f)(p, q) = f(pq)$$

whenever $f \in A$ and $p, q \in G$. Remark that

$$(\Delta(f)(1 \otimes g))(p, q) = f(pq)\, g(q)$$

$$((f \otimes 1)\, \Delta(g))(p, q) = f(p)\, g(pq)$$

when $f, g \in A$ and $p, q \in G$ and that indeed $\Delta(f)(1 \otimes g)$ and $(f \otimes 1)\, \Delta(g)$ are in $A \otimes A$ (using the group property of G). The coassociativity of Δ is a consequence of the associativity of the product in G.

From now on, let us assume that A is an algebra over $\mathbb{C}$ with a non-degenerate product and that Δ is a comultiplication on A.

Then, we can define linear maps T_1, T_2 from $A \otimes A$ to $A \otimes A$ by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b)$$

$$T_2(a \otimes b) = (a \otimes 1)\, \Delta(b).$$

In the case of the example, these maps are given by

$$(T_1 f)(p, q) = f(pq, q)$$

$$(T_2 f)(p, q) = f(p, pq)$$

when $f \in C(G \times G)$ and $p, q \in G$. Hence, in this case, these maps are bijective because these maps are dual to the maps

$$(p, q) \to (pq, q)$$

$$(p, q) \to (p, pq)$$

from $G \times G$ to $G \times G$ and these have inverses (again because G is assumed to be a group). This is the motivation for the following definition.

2.3. Definition. A pair (A, Δ) of an algebra A over $\mathbb{C}$ (with non-degenerate product) and a comultiplication Δ is called a multiplier Hopf algebra if the maps T_1, T_2 defined above, are bijective.

We saw already that $K(G)$, with its natural comultiplication, is a multiplier Hopf algebra.

But we also have the following important result.

2.4. Theorem. *If A is an algebra over $\mathbb{C}$ with identity and Δ a comultiplication on A such that (A, Δ) is a Hopf algebra, then (A, Δ) is a multiplier Hopf algebra. Conversely, if (A, Δ) is a multiplier Hopf algebra and if A has an identity, then (A, Δ) is a Hopf algebra.*

The proof of the first statement is not so difficult. If S is the antipode, the maps R_1, R_2 defined on $A \otimes A$ by

$$R_1(a \otimes b) = ((\iota \otimes S)\, \Delta(a))(1 \otimes b)$$

$$R_2(a \otimes b) = (a \otimes 1)((S \otimes \iota)\, \Delta(b))$$

are easily seen to be the inverses of T_1 and T_2.

The second statement is an easy consequence of the following result.

2.5. Proposition. *Let (A, Δ) be a multiplier Hopf algebra. There is a unique linear map $\varepsilon\colon A \to \mathbb{C}$ such that*

$$(\varepsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab$$

$$(\iota \otimes \varepsilon)((a \otimes 1)\, \Delta(b)) = ab$$

for all $a, b \in A$. This map is a homomorphism. There is also a unique linear map $S\colon A \to M(A)$ such that

$$m(S \otimes \iota)(\Delta(a)(1 \otimes b)) = \varepsilon(a)\, b$$

$$m(\iota \otimes S)((a \otimes 1)\, \Delta(b)) = \varepsilon(b)\, a$$

for all a, b (as usual, m is the multiplication map, now also considered as a map from $M(A) \otimes A$ or $A \otimes M(A)$ to A). This map is an anti-homomorphism.

We will call, as usual, ε the counit and S the antipode. Remark that, when A has a unit, the formulas of the previous proposition, are just the usual properties for the counit and the antipode. Therefore this result implies the second part of Theorem 2.4.

The theory of multiplier Hopf algebras is very similar to the theory of usual Hopf algebras. It is a very natural extension to the case where the underlying algebra has no identity. Example 2.2 makes it clear why we have to consider such algebras. But, the duality theorem that we will obtain here, will indicate even more, why these are important. We will also give more examples in Section 5. Before we have considered these, we will use the Example 2.2 (the discrete group case) to illustrate our notions and results.

In this paper, we will only deal with regular multiplier Hopf algebras. These are the ones where the antipode is a map from A to A (instead of $M(A)$) and is invertible. Let us elaborate a little more on these regular multiplier Hopf algebras. Here is the definition that we gave in [16].

2.6. Definition. A multiplier Hopf algebra (A, Δ) is called regular if also (A, Δ') is a multiplier Hopf algebra (where Δ' is the opposite comultiplication, obtained from Δ by composing it with the flip on $A \otimes A$).

For regular multiplier Hopf algebras, we will also have that $\Delta(a)(b \otimes 1)$ and $(1 \otimes a)\, \Delta(b)$ are in $A \otimes A$ for all $a, b \in A$. This is of course technically more convenient. Moreover, we have the following result.

2.7. Proposition. *If (A, Δ) is a regular multiplier Hopf algebra, then the antipode S is a map from A to A and it is bijective.*

Of course, as in the case of a usual Hopf algebra, the inverse S^{-1} of S is the antipode for the multiplier Hopf algebra (A, Δ').

We have the following converse, not contained in [16].

2.8. PROPOSITION. *If (A, Δ) is a multiplier Hopf algebra such that $S(A) \subset A$ and S is invertible, then it is regular.*

Proof. Take $a, b, c \in A$. By Lemma 4.2 of [16], we have

$$(\iota \otimes S)((c \otimes b)\, \Delta(a)) = (\iota \otimes S)((c \otimes 1)\, \Delta(a))(1 \otimes S(b))$$
$$= (c \otimes 1)\, T_1^{-1}(a \otimes S(b)).$$

So

$$(c \otimes b)\, \Delta(a) = (c \otimes 1)(\iota \otimes S^{-1})\, T_1^{-1}(\iota \otimes S)(a \otimes b).$$

It follows that $(1 \otimes b)\, \Delta(a) \in A \otimes A$ and that the map $b \otimes a \to (1 \otimes b)\, \Delta(a)$ is bijective. By a similar argument, we have $\Delta(a)(b \otimes 1) \in A \otimes A$ for all, $a, b \in A$ and that also the map $a \otimes b \to \Delta(a)(b \otimes 1)$ is bijective.

Also the next result was not formulated in [16] but is very useful to prove that an algebra with a comultiplication is a regular multiplier Hopf algebra.

2.9. PROPOSITION. *Let A be an algebra over $\mathbb{C}$ with a non-degenerate product. Assume that $\Delta: A \to M(A \otimes A)$ is a homomorphism such that*

$$\Delta(a)(1 \otimes b) \qquad (a \otimes 1)\, \Delta(b)$$
$$\Delta(a)(b \otimes 1) \qquad (1 \otimes a)\, \Delta(b)$$

all belong to $A \otimes A$. Assume that Δ is coassociative. If there is a counit ε and an invertible antipode S (mapping A into A), satisfying the formulas in Proposition 2.5, then (A, Δ) is a regular multiplier Hopf algebra.

The proof of this result is not so hard. We know, from the proof of the previous proposition (and from the remark following Theorem 2.4), what e.g. the inverse R_1 of T_1 has to be. So, now define R_1 from $A \otimes A$ to itself by

$$R_1(a \otimes b) = (\iota \otimes S)((1 \otimes S^{-1}(b))\, \Delta(a)).$$

This is possible because of our assumptions. Then it is easy to check that R_1 is an inverse for T_1. Similarly for the three other maps.

There are also three cases where regularity is automatic. The first case is when A is abelian. Then in fact $S^2 = \iota$. This will follow from the uniqueness of S. Similarly, when Δ is coabelian, i.e. $\Delta = \Delta'$, then (A, Δ) is regular and $S^2 = \iota$. The third case is that of a multiplier Hopf *-algebra. This is a multiplier Hopf algebra (A, Δ) where A is a *-algebra and Δ a *-homomorphism. Remark that $A \otimes A$ and $M(A \otimes A)$ are also *-algebras in a natural way. In this case, also using the uniqueness of the antipode, one has automatically that $S(S(a)^*)^* = a$ for all $a \in A$.

In [16], we proved the following result (Lemma 5.5) which we will use from time to time in this paper.

2.10. LEMMA. *Let (A, Δ) be a regular multiplier Hopf algebra. Suppose that a, b, a_i and b_i are all elements in A. Then, the following are equivalent*:

(i) $\Delta(a)(1 \otimes b) = \sum \Delta(a_i)(b_i \otimes 1)$,

(ii) $a \otimes S^{-1}b = \sum (a_i \otimes 1)\, \Delta(b_i)$,

(iii) $(1 \otimes a)\, \Delta(b) = \sum Sb_i \otimes a_i$.

This result can be used in different forms. We have e.g. that

$$\Delta(a)(b \otimes 1) = \sum \Delta(a_i)(1 \otimes b_i)$$

is also equivalent with

$$(a \otimes 1)\, \Delta(b) = \sum a_i \otimes S^{-1}b_i.$$

This follows from the equivalence of (i) and (ii) in the lemma.

Finally, let us fix some notations and conventions.

As we mentioned already before, ι will denote the identity map (in most cases from A to A), m will be the multiplication map (here considered as a map not only from $A \otimes A$ to A, but also extended to $M(A) \otimes A$ and $A \otimes M(A)$), ... We will always identify spaces like $A \otimes \mathbb{C}$ and $\mathbb{C} \otimes A$ with A. So, the map $\omega \otimes \iota$, where ω is a linear functional on A, will be considered as a map from $A \otimes A$ to A (a slice map) ... Finally, we will use, also in the case of multiplier Hopf algebras, $\Delta^{(2)}$ for $(\Delta \otimes \iota)\, \Delta$.

We refer to standard works as [1] and [12] for Hopf algebras. See e.g. [13] for Hopf *-algebras. We will work with the Sweedler notation when dealing with ordinary Hopf algebras. The use of the Sweedler notation can be justified, also in the case of multiplier Hopf algebras (see the discussion in [3] and [19]). But this must be done with some care. In this paper, we have avoided the use of the Sweedler notation for multiplier Hopf algebras. To make formulas precise, we have to add extra factors and this makes things often more complicated. Therefore, we will now and then make the choice of being a little less rigorous but (at least intuitively) more clear (and more transparent). In all cases, the reader can easily complete the arguments.

Sometimes, we will also extend homomorphisms and anti-homomorphisms (like ε, Δ, S) to the multiplier algebra (in these cases $M(A)$). We have made a remark about these extensions at the end of [16]. To define e.g. S on the multiplier algebra, take x in $M(A)$ and define $S(x)$ by $S(x)\,S(a) = S(ax)$ and $S(a)\,S(x) = S(xa)$ for all a. To prove that this defines a multiplier $S(x)$, just observe that

$$S(xa)\,S(b) = S(bxa) = S(a)\,S(bx)$$

(see the remark in the beginning of this section). When extending homomorphisms with range in some multiplier algebra, one also has to use that these homomorphisms are non-degenerate (see [16]).

To avoid to heavy notations, we can also use these extensions. But also here, as we have mentioned before already, one can avoid doing so by multiplying at the right place with the right elements. As an example of this, consider the formula $\Delta'(S(a)) = (S \otimes S)\,\Delta(a)$. To give a meaning to this, one has to extend $S \otimes S$ and the flip to $M(A \otimes A)$. But we can also multiply with $S(b)$ (in fact, at four different places) and rewrite this formula as e.g.

$$\Delta'(S(a))(S(b) \otimes 1) = (S \otimes S)((b \otimes 1)\,\Delta(a)).$$

3. LEFT AND RIGHT INVARIANT FUNCTIONALS

Let (A, Δ) be a regular multiplier Hopf algebra.

3.1. *Notation.* Let $a \in A$ and let ω be any linear functional on A. Then we can define an element $x \in M(A)$ by

$$xb = (\omega \otimes \iota)(\Delta(a)(1 \otimes b))$$

$$bx = (\omega \otimes \iota)((1 \otimes b)\,\Delta(a)).$$

We will write

$$x = (\omega \otimes \iota)\,\Delta(a).$$

Similarly, we can define $(\iota \otimes \omega)\,\Delta(a)$.

Using this notation, we can introduce the notion of left and right invariant functionals as follows.

3.2. DEFINITION. A linear functional φ on A is called left invariant if $(\iota \otimes \varphi)\,\Delta(a) = \varphi(a)\,1$ for all a in A. A linear functional ψ on A is called right invariant if $(\psi \otimes \iota)\,\Delta(a) = \psi(a)\,1$ for all a in A.

Let us see what this means in the example coming from a discrete group.

3.3. EXAMPLE. As in Example 2.2, let $A = K(G)$, the algebra of complex functions with finite support on a (discrete) group and let $\Delta(f)(p, q) = f(pq)$. Define $\varphi(f) = \sum f(p)$. Then

$$((\iota \otimes \varphi)((f \otimes 1)\, \Delta(g)))(q) = \sum_p f(q)\, g(qp)$$

$$= \sum_p f(q)\, g(p) = f(q)\, \varphi(g)$$

for all $f, g \in A$ and all $q \in A$. So $(\iota \otimes \varphi)\, \Delta(g) = \varphi(g)\, 1$ and therefore, φ is left invariant. In this case, φ is also right invariant because here, we have also

$$\sum_p f(pq) = \sum_p f(p)$$

for all $f \in A$ and $q \in G$. In general, this will not be the case.

If φ is a left invariant functional and $\psi = \varphi \circ S$, then ψ is right invariant. Indeed, if $a, b \in A$ we have

$$\begin{aligned}((\varphi \circ S) \otimes \iota)((1 \otimes b)\, \Delta(a)) &= (\varphi \otimes \iota)((S \otimes \iota)((1 \otimes b)\, \Delta(a))) \\ &= S^{-1}((\varphi \otimes \iota)(\Delta'(S(a))(1 \otimes S(b)))) \\ &= S^{-1}((\iota \otimes \varphi)(\Delta(S(a))(S(b) \otimes 1))) \\ &= S^{-1}(\varphi(S(a))\, S(b)) \\ &= b\varphi(S(a))\, 1\end{aligned}$$

so that $(\psi \otimes \iota)\, \Delta(a) = \psi(a)\, 1$.

Invariant functionals do not always exist. It is easy to given examples of Hopf algebras without (non-zero) invariant functionals. In certain cases however, it is possible to formulate extra assumptions on the pair (A, Δ) and prove existence of (non-zero) invariant functionals (see e.g. [15, 17, 19–22]). They always exist in the finite-dimensional case (see Section 5). But as we mentioned already in the introduction, uniqueness of these functionals is at least as important. This will be one of the main results in this section.

We first show that non-zero invariant functionals are always *faithful*. In the case of a positive functional ω on a *-algebra we usually say that ω is faithful if $\omega(a^*a) = 0$ implies $a = 0$. By the Cauchy–Schwarz inequality, this is equivalent with $\omega(ba) = 0$ for all b implies $a = 0$. So, it makes sense to call φ faithful when it satisfies the two properties in the following proposition.

3.4. PROPOSITION. *Let φ be a non-zero left invariant functional. If $a \in A$ and $\varphi(ba) = 0$ for all $b \in A$, then $a = 0$. Similarly, if $\varphi(ab) = 0$ for all b, then $a = 0$.*

Proof. Let φ be left invariant and non-zero. Let $a \in A$ and assume $\varphi(ba) = 0$ for all $b \in A$. We get

$$(\iota \otimes \varphi)((c \otimes 1)\, \Delta(b)\, \Delta(a)) = \varphi(ba)\, c = 0$$

for all $b, c \in A$. Because elements of the form $(c \otimes 1)\, \Delta(b)$ span $A \otimes A$ we also get

$$(\iota \otimes \varphi)((b \otimes c)\, \Delta(a)) = 0$$

for all $b, c \in A$. Apply Δ and multiply to the left with d to obtain

$$(\iota \otimes \iota \otimes \varphi)(((d \otimes 1)\, \Delta(b) \otimes c)(\Delta \otimes \iota)\, \Delta(a)) = 0$$

for all b, c, d. Again we can replace $(d \otimes 1)\, \Delta(b)$ by $d \otimes b$ and get

$$(\iota \otimes \iota \otimes \varphi)((d \otimes b \otimes c)(\Delta \otimes \iota)\, \Delta(a)) = 0.$$

We can cancel b and rewrite this as

$$(\iota \otimes \iota \otimes \varphi)((1 \otimes 1 \otimes c)(\iota \otimes \Delta)((d \otimes 1)\, \Delta(a))) = 0.$$

If we apply a linear functional ω and if we put $p = (\omega \otimes \iota)((d \otimes 1)\, \Delta(a))$, we get

$$(\iota \otimes \varphi)((1 \otimes c)\, \Delta(p)) = 0.$$

Take any q and write $\Delta(p)(q \otimes 1) = \sum a_i \otimes b_i$ with the a_i linearly independent. Then $\varphi(cb_i) = 0$ for all c and all i. Replace c by $cS(a_i)$ and take the sum to get $\varphi(c \sum S(a_i)\, b_i) = 0$ for all c. But

$$\sum S(a_i)\, b_i = m(S \otimes \iota)(\Delta(p)(q \otimes 1)) = S(q)\, \varepsilon(p)$$

and so $\varphi(cS(q))\, \varepsilon(p) = 0$ for all c, q.

Now, it is not hard to see that elements cq span A (e.g. using that $cq = (\iota \otimes \varepsilon)((c \otimes 1)\, \Delta(q))$ and that elements $(c \otimes 1)\, \Delta(q)$ span $A \otimes A$). As S is bijective, and $\varphi \neq 0$, we get $\varepsilon(p) = 0$. This means that $\omega(da) = 0$ for all ω and all d. This implies that $a = 0$. The argument for the second statement is similar (we have to use S^{-1} here).

Of course, by applying the antipode, we have that any non-zero right invariant functional is also faithful. This property can also be proven directly using the same techniques.

Later in this section, in the proof of Proposition 3.11, we will see that

$$S(\iota \otimes \varphi)(\Delta(b)(1 \otimes a)) = (\iota \otimes \varphi)((1 \otimes b)\, \Delta(a))$$

for all a and b. So, if a is given and $\varphi(ba) = 0$ for all b, it follows that $(\iota \otimes \varphi)((1 \otimes b)\, \Delta(a)) = 0$ for all b. This is what we have seen in the beginning of the proof of Proposition 3.4. The rest of the proof is in fact devoted to the following more general result. If ω is any linear functional on A, a any element in A and if $(\iota \otimes \omega)((1 \otimes b)\, \Delta(a)) = 0$ for all b, then either $\omega = 0$ or $a = 0$. This result is easier to prove in the case of a Hopf algebra using the Sweedler notation. Indeed, apply Δ and S to get

$$\sum \omega(ba_{(3)})\, a_{(1)} \otimes Sa_{(2)} = 0$$

for all b and replace b by $bSa_{(2)}$ to get

$$\sum \omega(bS(a_{(2)})\, a_{(3)})\, a_{(1)} = 0$$

and hence also $\omega(b)\, a = 0$ for all b.

So far about the faithfulness and the proof of this property. The following lemma is the first step to uniqueness.

3.5. Lemma. *If φ is left invariant, ψ right invariant and non-zero and if $a \in A$, then there is a $b \in A$ such that $\varphi(xa) = \psi(xb)$ for all $x \in A$. Similarly, if φ is non-zero, given $b \in A$, we have $a \in A$ so that $\varphi(xa) = \psi(xb)$ for all $x \in A$.*

Proof. Take any c in A and write

$$\Delta(a)(c \otimes 1) = \sum \Delta(p_i)(1 \otimes q_i).$$

Multiply with $\Delta(x)$ to the left and apply $\psi \otimes \varphi$. Use invariance to obtain

$$\psi(c)\, \varphi(xa) = \sum \psi(xp_i)\, \varphi(q_i).$$

Choose c so that $\psi(c) = 1$ and put $b = \sum \varphi(q_i)\, p_i$. Then we get $\varphi(xa) = \psi(xb)$ for all x.

The second statement is proved in a similar way.

As an immediate consequence of this result, we have the following important lemma.

3.6. LEMMA. *If φ_1 and φ_2 are two non-zero left invariant functionals, then the spaces of functionals*

$$\{\varphi_1(\cdot a) \mid a \in A\} \qquad \text{and} \qquad \{\varphi_2(\cdot a) \mid a \in A\}$$

are equal.

Proof. Take any right invariant functional ψ which is non-zero (e.g. take $\varphi_1 \circ S$). By the previous lemma, we have that both sets of functionals equal $\{\psi(\cdot b) \mid b \in A\}$.

This result is already a strong indication for uniqueness. In fact, we will use it to prove that this is indeed the case.

3.7. THEOREM. *Let (A, Δ) be a regular multiplier Hopf algebra. If A admits a non-zero left invariant functional, then it is unique (up to a scalar) and there is also a unique right invariant functional. These functionals are faithful.*

Proof. Take two non-zero left invariant functionals φ_1 and φ_2. Take also a non-zero right invariant functional ψ. Choose a_1 and y in A so that $\psi(ya_1) = 1$. By the lemma, we can also choose $a_2 \in A$ so that $\varphi_1(qa_1) = \varphi_2(qa_2)$ for all $q \in A$.

Now, take any $x \in A$. Write $(1 \otimes x)\,\Delta(y) = \sum (p_i \otimes 1)\,\Delta(q_i)$. If we multiply with $\Delta(a_1)$ (resp. $\Delta(a_2)$) to the right and apply $\psi \otimes \varphi_1$ (resp. $\psi \otimes \varphi_2$) we get

$$\psi(ya_1)\,\varphi_1(x) = \sum \varphi_1(q_i a_1)\,\psi(p_i)$$

$$\psi(ya_2)\,\varphi_2(x) = \sum \varphi_2(q_i a_2)\,\psi(p_i).$$

As $\psi(ya_1) = 1$ and $\varphi_1(q_i a_1) = \varphi_2(q_i a_2)$ for all i, we get

$$\varphi_1(x) = \psi(ya_2)\,\varphi_2(x)$$

for all x. So φ_1 is a scalar multiple of φ_2.

By applying S, we get the uniqueness of the right invariant functional.

Usually, invariant functionals are called integrals in Hopf algebra theory and Haar measures in the quantum group setting. We will, throughout the paper speak about invariant functionals. Also, when we state the existence, we will always assume the existence of non-zero invariant functionals. Finally, we will in general use φ for a left invariant functional and ψ for a right invariant functional.

If (A, Δ) is a Hopf algebra, φ a left invariant functional and $\varphi(1) = 1$, then it follows immediately, using invariance and $\Delta(1) = 1 \otimes 1$ that any right invariant functional ψ is a scalar multiple of φ. This easily implies uniqueness. So in this case, left an right invariant functionals coincide (up to a scalar). The same is true for the case $A = K(G)$ where G is a discrete group. If left and right invariant functionals coincide, we call (A, Δ) *unimodular*.

In the general situation, left an right invariant functionals are related by means of a multiplier (the "modular function"). This is the content of the Proposition 3.10. First, we prove the existence of this multiplier.

3.8. PROPOSITION. *There is a multiplier δ in $M(A)$ such that*

$$(\varphi \otimes \iota)\, \Delta(a) = \varphi(a)\, \delta$$

whenever φ is left invariant.

Proof. Let φ be left invariant and non-zero. For all a, we have a multiplier δ_a defined by $(\varphi \otimes \iota)\, \Delta(a) = \delta_a$. Take any element $b \in A$ and any linear functional ω. Define $\varphi_1(a) = (\varphi \otimes \omega)((1 \otimes b)\, \Delta(a)) = \omega(b\delta_a)$. It is not hard to verify that φ_1 is also left invariant. So there is a scalar λ, depending on ω and b, so that $\varphi_1(a) = \lambda\varphi(a)$ for all a. So $\omega(b\delta_a)\, \varphi(c) = \omega(b\delta_c)\, \varphi(a)$ for all a, b, c and all ω and therefore, $\varphi(c)\, \delta_a = \varphi(a)\, \delta_c$. Take c so that $\varphi(c) = 1$ and let $\delta = \delta_c$. Then $\delta_a = \varphi(a)\, \delta$.

It is not so hard to find the behaviour of Δ, ε and S on δ.

3.9. PROPOSITION. *We have δ invertible and*

$$\Delta(\delta) = \delta \otimes \delta \qquad \varepsilon(\delta) = 1 \qquad S(\delta) = \delta^{-1}.$$

Proof. As we remarked already before, at the end of Section 2, we can extend Δ, ε and S to multipliers so that these formulas make sense.

To prove the first formula, apply Δ to the defining formula in 3.8 to obtain

$$\begin{aligned}\varphi(a)\, \Delta(\delta) &= (\varphi \otimes \iota \otimes \iota)((\iota \otimes \Delta)\, \Delta(a)) \\ &= (\varphi \otimes \iota \otimes \iota)((\Delta \otimes \iota)\, \Delta(a)) \\ &= \delta \otimes (\varphi \otimes \iota)\, \Delta(a) \\ &= \varphi(a)\, \delta \otimes \delta.\end{aligned}$$

To prove $\varepsilon(\delta) = 1$, apply ε to the formula in 3.8.

Finally, to prove the last one, let us be a little more careful. Take a, b and write

$$\Delta(a\delta)(1 \otimes b) = \Delta(a)(\delta \otimes \delta b).$$

If we apply $S \otimes \iota$ and multiply, we get $\varepsilon(a\delta)\, b = S(\delta)\, \varepsilon(a)\, \delta b$. So $b = S(\delta)\, \delta b$. Similarly we get $b = b\delta S(\delta)$ for all b. This implies that δ is invertible and $S(\delta) = \delta^{-1}$.

We can do something similar for a right invariant functional ψ, but if we apply S to $(\varphi \otimes \iota)\, \Delta(a) = \varphi(a)\, \delta$ we will get immediately

$$(\iota \otimes \psi)\, \Delta(a) = \psi(a)\, \delta^{-1}.$$

This multiplier δ also relates φ and ψ in the following way.

3.10. PROPOSITION. *If φ is a left invariant functional, then $\varphi(S(a)) = \varphi(a\delta)$ for all $a \in A$.*

Proof. We will use Lemma 2.10. So, if $a, b \in A$ and

$$\Delta(a)(1 \otimes b) = \sum \Delta(a_i)(b_i \otimes 1),$$

then

$$(1 \otimes a)\, \Delta(b) = \sum Sb_i \otimes a_i.$$

If we apply $\varphi \otimes \varphi$ to the second equality, we get

$$\varphi(b)\, \varphi(a\delta) = \sum \varphi(Sb_i)\, \varphi(a_i).$$

If we apply $\varphi \circ S \otimes \varphi$ to the first equality, we get

$$\varphi(Sa)\, \varphi(b) = \sum \varphi(Sb_i)\, \varphi(a_i).$$

So $\varphi(S(a)) = \varphi(a\delta)$ for all a.

Remark that the above result implies Lemma 3.5. Indeed, if $\psi = \varphi \circ S$, and if we replace a by xa in the formula, we get $\psi(xa) = \varphi(xa\delta)$. So, with $b = a\delta$, we get the result of Lemma 3.5. In fact, it is also possible to obtain δ from this lemma.

If we apply the formula in the above proposition two times, we get

$$\varphi(S^2(a)) = \varphi(S(a)\, \delta) = \varphi(S(\delta^{-1}a)) = \varphi(\delta^{-1}a\delta)$$

for all a. By uniqueness, there is a complex number τ so that $\varphi \circ S^2 = \tau\varphi$. So $\varphi(\delta^{-1}a\delta) = \tau\varphi(a)$. In general $\tau \neq 1$. If A is abelian, of course, we have $\tau = 1$. This is also the case if A is coabelian. In that case, we have in fact $\delta = 1$. This need not be true in the abelian case.

Remark that most of the results on δ are similar to the properties of the modular function of a locally compact group, relating the left and the right

Haar measures. In this case however, because A may be not abelian, we have some new phenomena. We will illustrate these with an example in Section 5.

Now, we will prove some kind of *K.M.S. property*. We will explain the terminology later. It is a result that helps to deal with the problems that are due to the fact that our algebras are not abelian. It is essential for the construction of the dual in the next section. In [18], it was an extra assumption. Now, we discovered that it follows automatically. We prove this in the next proposition.

3.11. PROPOSITION. *Let φ be a left invariant functional on A. Then, for all $a \in A$, there exist a $b \in A$ such that $\varphi(xa) = \varphi(bx)$ for all $x \in A$.*

Proof. Take two elements p, q. First write

$$\Delta(p)(1 \otimes q) = \sum \Delta(p_i)(q_i \otimes 1).$$

Then, by Lemma 2.10 we have

$$(1 \otimes p)\, \Delta(q) = \sum Sq_i \otimes p_i.$$

If we apply $\iota \otimes \varphi$ to these formulas, we get

$$S(\iota \otimes \varphi)(\Delta(p)(1 \otimes q)) = (\iota \otimes \varphi)((1 \otimes p)\, \Delta(q)).$$

Now, we write

$$\Delta(p)(q \otimes 1) = \sum \Delta(p_i)(1 \otimes q_i).$$

Again by Lemma 2.10 (or rather the remark following this lemma), we have

$$(p \otimes 1)\, \Delta(q) = \sum p_i \otimes S^{-1}q_i.$$

Now, apply $\psi \otimes \iota$ to these formulas, with ψ right invariant. Then we get

$$S(\psi \otimes \iota)((p \otimes 1)\, \Delta(q)) = (\psi \otimes \iota)(\Delta(p)(q \otimes 1)).$$

Next, take any x. Using the two results above, we get

$$\begin{aligned}(\psi \otimes \varphi)((x \otimes p)(\iota \otimes S)\, \Delta(q)) &= \varphi(pS(\cdot))(\psi \otimes \iota)((x \otimes 1)\, \Delta(q)) \\ &= \varphi(p\, \cdot)(\psi \otimes \iota)(\Delta(x)(q \otimes 1)) \\ &= (\psi \otimes \varphi)((1 \otimes p)\, \Delta(x)(q \otimes 1))\end{aligned}$$

$$= \psi(\,\cdot\, q)(\imath \otimes \varphi)((1 \otimes p)\, \Delta(x))$$
$$= \psi(S(\,\cdot\,)\, q)(\imath \otimes \varphi)(\Delta(p)(1 \otimes x))$$
$$= (\psi \otimes \varphi)((S \otimes \imath)\, \Delta(p)(q \otimes x)).$$

If we set

$$a = (\imath \otimes \varphi)((1 \otimes p)(\imath \otimes S)\, \Delta(q))$$
$$b = (\psi \otimes \imath)((S \otimes \imath)\, \Delta(p)(q \otimes 1)),$$

we find precisely $\psi(xa) = \varphi(bx)$ for all x.

Now, given a, take c so that $\varphi(Sc) = 1$ and write

$$a \otimes c = \sum \Delta(q_i)(1 \otimes S^{-1} p_i).$$

Apply $\imath \otimes \varphi \circ S$ to get

$$a = \sum (\imath \otimes \varphi)((1 \otimes p_i)(\imath \otimes S)\, \Delta(q_i)).$$

If we then let

$$b = \sum (\psi \otimes \imath)((S \otimes \imath)\, \Delta(p_i)(q_i \otimes 1)),$$

we will again obtain $\psi(xa) = \varphi(bx)$ for all x. We can now combine this result with Lemma 3.5 and this proves the proposition.

Remark that the element b is unique by the faithfulness of φ. If we apply S, we get a similar result for ψ. For all a, we have a b so that $\psi(ax) = \psi(xb)$ for all x. If we again combine this with Lemma 3.5, we also get that for all a, there is a b so that $\varphi(ax) = \varphi(xb)$. If we put all these results together, we get the equality of the following four sets of functionals:

$$\{\varphi(a\,\cdot\,) \mid a \in A\}$$
$$\{\varphi(\,\cdot\, a) \mid a \in A\}$$
$$\{\psi(a\,\cdot\,) \mid a \in A\}$$
$$\{\psi(\,\cdot\, a) \mid a \in A\}.$$

In the next section, we will define this set of functionals as our dual $\hat{A}$.

Let us now elaborate a little more on the above result.

3.12. Proposition. *There is an automorphism σ of A such that $\varphi(ab) = \varphi(b\sigma(a))$ for all $a, b \in A$. We also have that φ is σ-invariant.*

Proof. We can define σ by this formula. Then, for all a, b, c we get

$$\varphi(abc) = \varphi(c\sigma(ab))$$

but also

$$\varphi(abc) = \varphi(bc\sigma(a)) = \varphi(c\sigma(a)\,\sigma(b)).$$

By the faithfulness of φ, we have $\sigma(ab) = \sigma(a)\,\sigma(b)$. Remark that σ is also onto by the observations given before.

Now, $\varphi(ab) = \varphi(b\sigma(a)) = \varphi(\sigma(a)\,\sigma(b)) = \varphi(\sigma(ab))$ and because $A^2 = A$, we get that φ is σ-invariant.

Remark that, in the finite-dimensional case, this result is true for all faithful linear functionals (see e.g. Section 5).

We will say that φ satisfies the *weak K.M.S. condition.*

The notion K.M.S. (Kubo Martin Schwinger) originally comes from physics and is usually formulated for a linear functional and a one-parameter group of automorphisms on an algebra. See e.g. [8]. Our notion is weaker, but clearly, it is related.

Let σ and σ' denote the automorphisms associated with φ and ψ respectively. Then we have the following.

3.13. Proposition. *The automorphisms σ and σ' satisfy $S\sigma' = \sigma^{-1}S$.*

Proof. We have $\varphi(ab) = \varphi(b\sigma(a))$ for all a and b. We also have $\varphi(S(ab)) = \varphi(S(b\sigma'(a)))$ for all a, b. Then

$$\varphi(S(b)(\sigma S\sigma')(a)) = \varphi(S(\sigma'(a))\,S(b)) = \varphi(S(b)\,S(a))$$

and so $\sigma S\sigma' = S$.

We have the following relation with Δ.

3.14. *Proposition.* For all a we have

$$\Delta(\sigma(a)) = (S^2 \otimes \sigma)\,\Delta(a)$$

$$\Delta(\sigma'(a)) = (\sigma' \otimes S^{-2})\,\Delta(a)$$

Proof. Take a, b. As we saw in the proof of Proposition 3.11, we get

$$S(\iota \otimes \varphi)(\Delta(a)(1 \otimes b)) = (\iota \otimes \varphi)((1 \otimes a)\,\Delta(b)).$$

So

$$\begin{aligned}(\iota\otimes\varphi)((1\otimes b)(S^2\otimes\sigma)\,\Delta(a)) &= S^2(\iota\otimes\varphi)(\Delta(a)(1\otimes b))\\ &= S(\iota\otimes\varphi)((1\otimes a)\,\Delta(b))\\ &= S(\iota\otimes\varphi)(\Delta(b)(1\otimes\sigma(a))).\end{aligned}$$

Applying the same formula here with a replaced by b and b by $\sigma(a)$ we get

$$(\iota\otimes\varphi)((1\otimes b)(S^2\otimes\sigma)\,\Delta(a)) = (\iota\otimes\varphi)((1\otimes b)\,\Delta(\sigma(a))).$$

This is true for all b and because φ is faithful, we get

$$\Delta(\sigma(a)) = (S^2\otimes\sigma)\,\Delta(a).$$

In a similar way, or using the relation $S\sigma' = \sigma^{-1}S$, we get the other formula.

Also remark that σ and S^2 commute. Indeed, for all a, b we have

$$\varphi(S^2(a)\,b) = \varphi(b\sigma(S^2(a))).$$

On the other hand, we have also

$$\begin{aligned}\varphi(S^2(a)\,b) &= \varphi(S^2(aS^{-2}(b))) = \tau\varphi(aS^{-2}(b))\\ &= \tau\varphi(S^{-2}(b)\,\sigma(a)) = \varphi(S^2(S^{-2}(b)\,\sigma(a))),\\ &= \varphi(bS^2(\sigma(a))).\end{aligned}$$

This holds for all b and so $\sigma(S^2(a)) = S^2(\sigma(a))$. Similarly, S^2 and σ' commute.

Now, we have the following relation of σ and the modular element δ.

3.15. Proposition. *We have* $\sigma(\delta) = \sigma'(\delta) = (1/\tau)\,\delta$ *and* $\delta\sigma(a) = \sigma'(a)\,\delta$ *for all* $a \in A$.

Proof. Because $\tau\varphi(a) = \varphi(S^2(a)) = \varphi(\delta^{-1}a\delta) = \varphi(\sigma^{-1}(a\delta)\,\delta^{-1}) = \varphi(a\delta\sigma(\delta^{-1}))$, we must have that $\delta\sigma(\delta^{-1}) = \tau 1$ and so $\sigma(\delta) = (1/\tau)\,\delta$.

On the other hand we had

$$\varphi(ab\delta) = \varphi(S(ab)) = \varphi(S(b\sigma'(a))) = \varphi(b\sigma'(a)\,\delta)$$

so that $\delta\sigma(a) = \sigma'(a)\,\delta$.

There seems to be no obvious relation between σ and ε. Except that $\varepsilon\sigma = \varepsilon\sigma'$. This can e.g. be seen by applying ε to the last relation in the previous proposition.

4. DUALITY

As before, let (A, Δ) be a regular multiplier Hopf algebra and assume that there are non-trivial invariant functionals. We will define the dual $(\hat{A}, \hat{\Delta})$. It will again be a regular multiplier Hopf algebra with non-trivial invariant functionals. So the dual $(\hat{A}, \hat{\Delta})$ belongs to the same category. Therefore we can consider the bidual, i.e., the dual of $(\hat{A}, \hat{\Delta})$ and it turns out that this is again (A, Δ). As we already mentioned in the introduction, this duality contains the duality between discrete quantum groups and compact quantum groups. We will come back to this in the next section.

In other papers on this subject, the dual is often defined by first constructing the multiplicative unitary (see e.g. [6, 8, 9]). We will follow the more direct approach and define the dual as an appropriate subspace of the dual space A' of all linear functionals on A.

4.1. *Notation.* Let φ be a left invariant functional on A. Then we denote by $\hat{A}$ the space of linear functionals on A of the form $\varphi(\,\cdot\, a)$ where $a \in A$.

We know that every element in $\hat{A}$ also has the form $\varphi(b\,\cdot\,)$ for some b. And if ψ is a right invariant functional on A, elements in $\hat{A}$ can also be written in the form $\psi(\,\cdot\, c)$ and $\psi(d\,\cdot\,)$ for some $c, d \in A$. We will need to use these different forms on different occasions.

We now make $\hat{A}$ into an associative algebra in the usual way.

4.2. Proposition. *Let $\omega_1, \omega_2 \in \hat{A}$. Then we can define a linear functional $\omega_1\omega_2$ on A by*

$$(\omega_1\omega_2)(x) = (\omega_1 \otimes \omega_2)\, \Delta(x)$$

and $\omega_1\omega_2 \in \hat{A}$. This product is non-degenerate and makes $\hat{A}$ into an associative algebra over $\mathbb{C}$.

Proof. Let $\omega_1 = \varphi(\,\cdot\, a_1)$ and $\omega_2 = \varphi(\,\cdot\, a_2)$. Then

$$(\omega_1 \otimes \omega_2)\, \Delta(x) = (\varphi \otimes \varphi)(\Delta(x)(a_1 \otimes a_2))$$

and we see that this is well defined for all $x \in A$ and gives a linear functional $\omega_1\omega_2$ on A. Now write

$$a_1 \otimes a_2 = \sum \Delta(p_i)(q_i \otimes 1).$$

Then we find

$$(\omega_1\omega_2)(x) = \sum \varphi(q_i)\, \varphi(xp_i)$$

for all x so that again $\omega_1\omega_2 \in \hat{A}$.

The associativity of the product in $\hat{A}$ is an easy consequence of the coassociativity of Δ.

To prove that the product is non-degenerate, assume that $\omega_1\omega_2 = 0$ for all ω_2. Then

$$(\varphi \otimes \varphi)(\Delta(x)(a_1 \otimes a_2)) = 0$$

for all x and all a_2. This implies that

$$(\varphi \otimes \varphi)(pa_1 \otimes q) = 0$$

for all p, q. Now, choose q so that $\varphi(q) = 1$. Then we get precisely $\omega_1 = 0$. Similarly $\omega_1\omega_2 = 0$ for all ω_1 will imply $\omega_2 = 0$.

If we define the Fourier transform $\hat{a}$ of an element a in A by $\hat{a} = \varphi(\,\cdot\, a)$, then we see from the proof of the previous proposition, that the convolution product $a_1 * a_2$ of two elements a_1 and a_2 in A has to be defined as $a_1 * a_2 = \sum \varphi(q_i)\, p_i$ when $a_1 \otimes a_2 = \sum \Delta(p_i)(q_i \otimes 1)$. Then we will have the usual formula $(a_1 * a_2)^\wedge = \hat{a}_1\hat{a}_2$ in $\hat{A}$. And it is easy to see that the above definition coincides with the usual one in the abelian case (the discrete group case).

We also want to remark that $\omega_1\omega_2$ can also be defined when either ω_1 or ω_2 is in $\hat{A}$. In fact, it can be shown that the multiplier algebra $M(\hat{A})$ of $\hat{A}$ can be identified with the space

$$\{\omega \in A' \mid (\omega \otimes \iota)\, \Delta(a) \in A \text{ and } (\iota \otimes \omega)\, \Delta(a) \in A \text{ for all } a \in A\}.$$

This was pointed out to me by J. Kustermans. The proof of this result is rather straight-forward although some care is required.

Now we also want to consider the *-algebra case.

4.3. Proposition. *If (A, Δ) is a multiplier Hopf *-algebra, then for every $\omega \in \hat{A}$ we can define ω^* in $\hat{A}$ by*

$$\omega^*(x) = \omega(S(x)^*)^-.$$

*Then $\hat{A}$ is a *-algebra.*

Proof. Let $\omega = \varphi(\cdot a)$. Then

$$\omega^*(x) = \varphi(S(x)^* a)^- = \varphi(S(xS(a)^*)^*)^- = \psi(xS(a)^*)$$

where $\psi(y) = \varphi(S(y)^*)^-$. It is not so hard to see that ψ is right invariant. So ω^* is in $\hat{A}$.

It is immediately clear that $\omega^{**} = \omega$ because $S(S(x)^*)^* = x$ for all x. The property $(\omega_1 \omega_2)^* = \omega_2^* \omega_1^*$ follows naturally from the property that Δ is a *-homomorphism and that $(S \otimes S)\, \Delta = \Delta' S$.

Let us now define the comultiplication $\hat{\Delta}$ on $\hat{A}$. We could use the obvious formula $\hat{\Delta}(\omega)(x \otimes y) = \omega(xy)$. Then we would get a map from $\hat{A}$ to $(A \otimes A)'$ in the first place. We will follow a somewhat different approach. We will define $\hat{\Delta}$ by defining the elements $\hat{\Delta}(\omega_1)(1 \otimes \omega_2)$ and $(\omega_1 \otimes 1)\, \hat{\Delta}(\omega_2)$ in $\hat{A} \otimes \hat{A}$ for ω_1, ω_2 in $\hat{A}$.

4.4. Definition. Let $\omega_1, \omega_2 \in \hat{A}$. Define

$$((\omega_1 \otimes 1)\, \hat{\Delta}(\omega_2))(x \otimes y) = (\omega_1 \otimes \omega_2)(\Delta(x)(1 \otimes y))$$
$$(\hat{\Delta}(\omega_1)(1 \otimes \omega_2))(x \otimes y) = (\omega_1 \otimes \omega_2)((x \otimes 1)\, \Delta(y)).$$

These are indeed the expected formulas: if $\varepsilon \in \hat{A}$, these would just coincide with

$$\hat{\Delta}(\omega)(x \otimes y) = \omega(xy).$$

Let us first argue that these functionals are well defined and again in $\hat{A} \otimes \hat{A}$.

4.5. Lemma. *If $\omega_1, \omega_2 \in \hat{A}$ then $\hat{\Delta}(\omega_1)(1 \otimes \omega_2)$ and $(\omega_1 \otimes 1)\, \hat{\Delta}(\omega_2)$ are in $\hat{A} \otimes \hat{A}$.*

Proof. Let $\omega_1 = \psi(a_1 \cdot)$ and $\omega_2 = \psi(a_2 \cdot)$ where now ψ is right invariant. Write

$$a_1 \otimes a_2 = \sum (1 \otimes p_i)\, \Delta(q_i).$$

Then

$$(\omega_1 \otimes \omega_2)(\Delta(x)(1 \otimes y)) = \sum (\psi \otimes \psi)((1 \otimes p_i)\, \Delta(q_i x)(1 \otimes y))$$
$$= \sum \psi(q_i x)\, \psi(p_i y)$$

for all x, y. So we see that $(\omega_1 \otimes 1)\, \hat{\Delta}(\omega_2)$ is a well defined element in $\hat{A} \otimes \hat{A}$. Similarly $\hat{\Delta}(\omega_1)(1 \otimes \omega_2) \in \hat{A} \otimes \hat{A}$ but now we use forms $\omega_1 = \varphi(\cdot a_1)$ and $\omega_2 = \varphi(\cdot a_2)$ where φ is left invariant.

To prove that this really defines a multiplier $\hat{\Delta}(\omega)$ in $M(\hat{A} \otimes \hat{A})$ for all $\omega \in \hat{A}$, we must show (according to the remark in the beginning of Section 2) that

$$((\omega_1 \otimes 1)\, \hat{\Delta}(\omega_2))(1 \otimes \omega_3) = (\omega_1 \otimes 1)(\hat{\Delta}(\omega_2)(1 \otimes \omega_3))$$

for all $\omega_1, \omega_2, \omega_3 \in \hat{A}$. This is a straightforward matter. So, $\hat{\Delta}$ is a well-defined map from $\hat{A}$ to $M(\hat{A} \otimes \hat{A})$.

4.6. PROPOSITION. *$\hat{\Delta}$ is a homomorphism.*

Proof. Take $\omega_1, \omega_2, \omega_3 \in \hat{A}$ and $x, y \in A$. Then, on the one hand, we have

$$\begin{aligned}(\hat{\Delta}(\omega_1\omega_2)(1 \otimes \omega_3))(x \otimes y) &= (\omega_1\omega_2 \otimes \omega_3)((x \otimes 1)\, \Delta(y))\\ &= (\omega_1 \otimes \omega_2 \otimes \omega_3)((\Delta(x) \otimes 1)\, \Delta^{(2)}(y)).\end{aligned}$$

On the other hand, write

$$\hat{\Delta}(\omega_2)(1 \otimes \omega_3) = \sum \omega_i' \otimes \omega_i''.$$

Then

$$\begin{aligned}&\hat{\Delta}(\omega_1)(\hat{\Delta}(\omega_2)(1 \otimes \omega_3))(x \otimes y)\\ &\quad = \sum (\hat{\Delta}(\omega_1)(\omega_i' \otimes \omega_i''))(x \otimes y)\\ &\quad = \sum (\omega_i' \otimes (\hat{\Delta}(\omega_1)(1 \otimes \omega_i'')))(\Delta'(x) \otimes y)\\ &\quad = \sum (\omega_i' \otimes \omega_1 \otimes \omega_i'')((\Delta'(x) \otimes 1)(1 \otimes \Delta(y))).\end{aligned}$$

Now, write $\Delta_{13}(y)$ for the image of $\Delta(y)$ under the map $a \otimes b \to a \otimes 1 \otimes b$. Then our expression becomes

$$\begin{aligned}&\sum (\omega_1 \otimes \omega_i' \otimes \omega_i'')((\Delta(x) \otimes 1)\, \Delta_{13}(y))\\ &\quad = (\omega_1 \otimes (\hat{\Delta}(\omega_2)(1 \otimes \omega_3)))((\Delta(x) \otimes 1)\, \Delta_{13}(y))\\ &\quad = (\omega_1 \otimes \omega_2 \otimes \omega_3)((\Delta(x) \otimes 1)\, \Delta^{(2)}(y)).\end{aligned}$$

Hence $\hat{\Delta}(\omega_1\omega_2)(1 \otimes \omega_3) = \hat{\Delta}(\omega_1)\, \hat{\Delta}(\omega_2)(1 \otimes \omega_3)$ and $\hat{\Delta}$ is a homomorphism.

The next step is to show that $(\hat{A}, \hat{\Delta})$ is a regular multiplier Hopf algebra.

4.7. PROPOSITION. *$(\hat{A}, \hat{\Delta})$ is a regular multiplier Hopf algebra. The antipode $\hat{S}$ is dual to S and the counit $\hat{\varepsilon}$ is given by evaluation in* 1.

Proof. Quite similar as in the proof of Lemma 4.5, we also have $(1 \otimes \omega_1)\,\hat{\Delta}(\omega_2)$ and $\hat{\Delta}(\omega_1)(\omega_2 \otimes 1)$ in $\hat{A} \otimes \hat{A}$. We just have to choose the appropriate representations of ω_1 and ω_2.

To prove that $(\hat{A}, \hat{\Delta})$ is a regular multiplier Hopf algebra, we must show that four maps are bijective. We do it for one of them. All the others are completely similar.

Suppose e.g. that $\omega_i', \omega_i'' \in \hat{A}$ and that

$$\sum \hat{\Delta}(\omega_i')(1 \otimes \omega_i'') = 0.$$

Then, for all $x, y \in A$ we will have

$$\sum (\omega_i' \otimes \omega_i'')((x \otimes 1)\,\Delta(y)) = 0.$$

This implies

$$\sum (\omega_i' \otimes \omega_i'')(u \otimes v) = 0$$

for all $u, v \in A$. Hence $\sum \omega_i' \otimes \omega_i'' = 0$. This gives injectivity.

On the other hand, let $\omega_1 = \varphi(\,\cdot a_1)$ and $\omega_2 = \varphi(\,\cdot a_2)$ and write

$$\Delta(a_2)(a_1 \otimes 1) = \sum p_i \otimes q_i.$$

Then, for all x, y, we have

$$\begin{aligned}(\omega_1 \otimes \omega_2)(x \otimes y) &= \varphi(xa_1)\,\varphi(ya_2) \\ &= (\varphi \otimes \varphi)((x \otimes 1)\,\Delta(ya_2)(a_1 \otimes 1)) \\ &= \sum (\varphi \otimes \varphi)((x \otimes 1)\,\Delta(y)(p_i \otimes q_i)).\end{aligned}$$

Let $\omega_i' = \varphi(\,\cdot p_i)$ and $\omega_i'' = \varphi(\,\cdot q_i)$, then

$$\begin{aligned}(\omega_1 \otimes \omega_2)(x \otimes y) &= \sum (\omega_i' \otimes \omega_i'')((x \otimes 1)\,\Delta(y)) \\ &= \sum (\hat{\Delta}(\omega_i')(1 \otimes \omega_i''))(x \otimes y).\end{aligned}$$

So, the map

$$\sum \omega_i' \otimes \omega_i'' \to \sum \hat{\Delta}(\omega_i')(1 \otimes \omega_i'')$$

is also surjective.

It is clear that

$$(\hat{\Delta}(\omega_1)(1 \otimes \omega_2))(1 \otimes x) = (\omega_1 \otimes \omega_2)(\Delta(x)) = (\omega_1 \omega_2)(x)$$

so that $\hat{\varepsilon}$ is evaluation in 1.

For the antipode, consider the usual expression

$$m(\hat{S} \otimes \iota)(\hat{\Delta}(\omega_1)(1 \otimes \omega_2))$$

where $\hat{S}$ is dual to S. Evaluation in x will give

$$\begin{aligned}(\hat{\Delta}(\omega_1)(1 \otimes \omega_2))&((S \otimes \iota)\,\Delta(x)) \\ &= (\omega_1 \otimes \omega_2)((m \otimes \iota)(S \otimes \iota \otimes \iota)\,\Delta^{(2)}(x)) \\ &= \omega_1(1)\,\omega_2(x) = \hat{\varepsilon}(\omega_1)\,\omega_2(x).\end{aligned}$$

So, $(\hat{A}, \hat{\Delta})$ is again a multiplier Hopf algebra and it is regular.

If (A, Δ) is a multiplier Hopf *-algebra, then also $(\hat{A}, \hat{\Delta})$ will be a multiplier Hopf *-algebra. For this, we must have that $\hat{\Delta}$ is a *-homomorphism. This is easy to check (and more or less standard).

The next step towards duality is to show that non-trivial invariant functionals exist on $(\hat{A}, \hat{\Delta})$.

4.8. Proposition. *If φ is a left and ψ a right invariant functional on A, then*

$$\begin{aligned}\hat{\psi}(\omega) &= \varepsilon(a) \qquad \textit{when} \quad \omega = \varphi(\,\cdot\, a) \\ \hat{\varphi}(\omega) &= \varepsilon(a) \qquad \textit{when} \quad \omega = \psi(a\,\cdot\,)\end{aligned}$$

define a right and left invariant functional on $(\hat{A}, \hat{\Delta})$.

Proof. Let $\omega_1 = \varphi(\,\cdot\, a_1)$ and $\omega_2 = \varphi(\,\cdot\, a_2)$ and write $a_1 \otimes a_2 = \sum \Delta(p_i)(q_i \otimes 1)$. Then as before

$$\begin{aligned}(\hat{\Delta}(\omega_1)(1 \otimes \omega_2))(x \otimes y) &= (\omega_1 \otimes \omega_2)((x \otimes 1)\,\Delta(y)) \\ &= \sum (\varphi \otimes \varphi)((x \otimes 1)\,\Delta(yp_i)(q_i \otimes 1)) \\ &= \sum \varphi(xq_i)\,\varphi(yp_i).\end{aligned}$$

Therefore

$$(\hat{\psi} \otimes \iota)(\hat{\Delta}(\omega_1)(1 \otimes \omega_2)) = \varphi(\,\cdot\, c)$$

where $c = \sum \varepsilon(q_i)\, p_i$. But $\sum \varepsilon(q_i)\, p_i = \varepsilon(a_1)\, a_2$. So

$$(\hat{\psi} \otimes \iota)(\hat{\Delta}(\omega_1)(1 \otimes \omega_2)) = \varepsilon(a_1)\, \omega_2 = \hat{\psi}(\omega_1)\, \omega_2.$$

This proves that $\hat{\psi}$ is right invariant.

A similar argument will give that $\hat{\varphi}$ is left invariant.

In this case, we must also see what happens for a multiplier Hopf *-algebra (A, Δ) with a positive invariant linear functional. First remark that it is shown in [7] that, if there is a positive left invariant functional, there is also a positive right invariant functional. This is obvious when the antipode S has square 1 because then S is a *-anti-homomorphism so that $\varphi(S(x^*x)) = \varphi(S(x)\, S(x)^*)$ for all x. However, in general, this result is far from obvious and the proof in [7] is quite involved. We will not use this result in this paper.

4.9. Proposition. *If φ is a positive left invariant linear functional on a multiplier Hopf *-algebra (A, Δ), then $\hat{\psi}$ is a positive right invariant linear functional on $(\hat{A}, \hat{\Delta})$. Similarly, if ψ is positive and right invariant, then $\hat{\varphi}$ is positive left invariant.*

Proof. Let $\omega = \varphi(\,\cdot\, a)$. Then

$$\begin{aligned}\omega^*(x) &= \omega(S(x)^*)^- \\ &= \varphi(S(x)^*\, a)^- \\ &= \varphi(S(xS(a)^*)^*)^- \\ &= \varphi(S(xS(a)^*)).\end{aligned}$$

Let $\psi = \varphi \circ S$. Then

$$\begin{aligned}(\omega^*\omega)(x) &= (\omega^* \otimes \omega)\, \Delta(x) \\ &= (\psi \otimes \varphi)(\Delta(x)(S(a)^* \otimes a)).\end{aligned}$$

Write $S(a)^* \otimes a = \sum \Delta(p_i)(q_i \otimes 1)$. Then $(\omega^*\omega)(x) = \sum \psi(q_i)\, \varphi(xp_i)$. So $\hat{\psi}(\omega^*\omega) = \sum \varepsilon(p_i)\, \psi(q_i) = \sum \varepsilon(p_i)\, \varphi(S(q_i))$. Now

$$\begin{aligned}\sum \varepsilon(p_i)\, S(q_i) &= \sum m(S \otimes \iota)(\Delta(p_i)(q_i \otimes 1)) \\ &= S(S(a)^*)\, a = a^*a.\end{aligned}$$

Hence $\hat{\psi}(\omega^*\omega) = \varphi(a^*a)$. Similarly for the other statement.

Remark that we get Plancherel's formula here. We see that we have the left invariant functional on (A, Δ) and the right invariant functional on the dual. Compare this with the definition of $\hat{\varphi}$ and $\hat{\psi}$ in 4.8. Also there, $\hat{\psi}$ is defined in terms of φ. If we had defined the opposite coproduct on $\hat{A}$, we would have had a Plancherel formula with left invariant functionals on the two sides.

Anyway, it is important to notice that different choices can be made. Our choices are motivated by this Plancherel formula and by the form of the biduality theorem that we obtain at the end of this section (Theorem 4.12).

We summarize our results in the following theorem.

4.10. THEOREM (Duality for regular multiplier Hopf algebras). *If (A, Δ) is a regular multiplier Hopf algebra with non-trivial invariant functionals, then the dual $(\hat{A}, \hat{\Delta})$ as defined before, is again a regular multiplier Hopf algebra with non-trivial invariant functionals. If (A, Δ) is a multiplier Hopf *-algebra with a positive left invariant functional, then $(\hat{A}, \hat{\Delta})$ is a multiplier Hopf *-algebra with a positive right invariant functional.*

We know from the general theory, that $\hat{\varphi}$ and $\hat{\psi}$ must satisfy the weak K.M.S. property. In fact, this is also an easy consequence of the following lemma. We will need this lemma to prove biduality in 4.12 below.

4.11. LEMMA. *Let $\omega = \varphi(\,\cdot\, a)$ and let ω_1 be any other element in $\hat{A}$. Then $\hat{\psi}(\omega_1 \omega) = \omega_1(S^{-1}(a))$.*

Proof. Let $\omega_1 = \varphi(\,\cdot\, a_1)$. Write $a_1 \otimes a = \sum \Delta(p_i)(q_i \otimes 1)$. Then

$$(\omega_1 \omega)(x) = \sum \varphi(q_i)\, \varphi(x p_i).$$

Therefore $\hat{\psi}(\omega_1 \omega) = \sum \varepsilon(p_i)\, \varphi(q_i)$. As we saw in the proof of the previous proposition, we get

$$\begin{aligned} \sum \varepsilon(p_i)\, q_i &= S^{-1}(S(a_1)\, a) \\ &= S^{-1}(a)\, a_1. \end{aligned}$$

So $\hat{\psi}(\omega_1 \omega) = \varphi(S^{-1}(a)\, a_1) = \omega_1(S^{-1}(a))$.

This result shows that we can essentially identify the dual of $\hat{A}$ with A. We will be more precise in the following theorem.

4.12. THEOREM (Biduality). *Let (A, Δ) be a regular multiplier Hopf algebra with non-trivial invariant functionals. Let $(\hat{A}, \hat{\Delta})$ be the dual. Define $\Gamma(a)(\omega) = \omega(a)$ whenever $a \in A$ and $\omega \in \hat{A}$. Then $\Gamma(a) \in \hat{\hat{A}}$ and Γ is an isomorphism of (A, Δ) with the dual $(\hat{\hat{A}}\, \hat{\hat{\Delta}})$ of $(\hat{A}, \hat{\Delta})$. If (A, Δ) is a multiplier Hopf *-algebra, then Γ is a *-isomorphism.*

Proof. Take $a \in A$ and define $\omega = \varphi(\cdot S(a))$. Then, from the previous lemma, we have $\hat{\psi}(\cdot \omega) = \Gamma(a)$. So $\Gamma(a) \in \hat{\hat{A}}$. It is also clear that any element in $\hat{\hat{A}}$ is of this form. If $\Gamma(a) = 0$, then $\omega(a) = 0$ for all $\omega \in \hat{A}$ and $a = 0$. So Γ is a vector space isomorphism of A with $\hat{\hat{A}}$.

It more or less follows immediately from the definition of the product and the coproduct for the dual (4.2 and 4.4—observe the symmetry in the formulas of 4.4), that Γ will be an isomorphism of multiplier Hopf algebras.

To prove the last statement, take any $a \in A$ and $\omega \in \hat{A}$ and observe that

$$\begin{aligned}
\Gamma(a)^* (\omega) &= \Gamma(a)(S(\omega)^*)^- \\
&= S(\omega)^* (a)^- \\
&= S(\omega)(S(a)^*) \\
&= \omega(S(S(a)^*)) \\
&= \omega(a^*) \\
&= \Gamma(a^*)(\omega).
\end{aligned}$$

There is another map from A to $\hat{\hat{A}}$ namely the iterated Fourier transform $a \to \hat{\hat{a}}$. This requires a choice of ψ (as we need $\hat{\varphi}$). If we take $\psi = \varphi \circ S^{-1}$, then we get $\hat{\hat{a}} = S(\sigma^{-1}(a))$, where σ is the modular automorphism for φ (introduced in Section 3).

Let us finish this section by mentioning our work on actions of multiplier Hopf algebras [20]. One of the main results in that paper is that the duality we have obtained here can be used to prove the duality for actions of this type of multiplier Hopf algebras. It is a result that extends the analogue result for finite-dimensional Hopf algebras and the duality for crossed products for group actions.

5. SPECIAL CASES AND EXAMPLES

We first consider the finite-dimensional case. It is not so hard to see that a finite-dimensional multiplier Hopf algebra must have an identity. Hence, it is a Hopf algebra. In this case, the antipode is always bijective and so regularity is automatic.

It is also known that finite-dimensional Hopf algebras have unique non-zero invariant functionals (see [1] and [12]). We like to present a simple and direct proof of this result. We use techniques, similar to those used earlier in this paper. We also obtain the invariant functionals on the dual at the same time.

5.1. PROPOSITION. *Let* (A, Δ) *be a finite-dimensional Hopf algebra. Then, there exist unique non-zero left and right invariant functionals. There also exist unique non-zero elements* h *and* k *such that* $ah = \varepsilon(a)\,h$ *and* $ka = \varepsilon(a)\,k$ *for all* $a \in A$ *(where* ε *is the counit).*

Proof. For any $a \in A$ and ω in the dual space A', we define linear maps $\pi(a)$ and $\pi(\omega)$ on A by $\pi(a)\,x = ax$ and $\pi(\omega)\,x = (\omega \otimes \iota)\,\Delta(x)$ whenever $x \in A$. It is clear that $\pi(ab) = \pi(a)\,\pi(b)$ and that $\pi(\omega\rho) = \pi(\rho)\,\pi(\omega)$ for all $a, b \in A$ and $\omega, \rho \in A'$. Now, define a linear map Γ from $A' \otimes A$ to the space $L(A)$ of linear maps from A to A by $\Gamma(\omega \otimes a) = \pi(\omega)\,\pi(a)$.

We first prove that Γ is bijective. Since $A' \otimes A$ and $L(A)$ have the same (finite) dimension, it will be enough to show that Γ is injective.

Therefore, assume that $\sum \pi(\omega_i)\,\pi(a_i) = 0$. Then

$$\sum (\omega_i \otimes \iota)(\Delta(a_i)(\Delta(x)(1 \otimes y))) = 0$$

for all $x, y \in A$ and hence

$$\sum (\omega_i \otimes \iota)(\Delta(a_i)(z \otimes 1)) = 0$$

for all $z \in A$. Now, we can proceed as in the proof of Proposition 3.4 (see remark following this proposition). Then, we get

$$\sum \omega_i(a_{i(1)} z)\, S(a_{i(2)}) \otimes a_{i(3)} = 0$$

by applying Δ and S. This holds for all z and we can replace z by $S(a_{i(2)})\,z$ to get $\sum \omega_i(z)\,a_i = 0$ for all z. This proves the injectivity of Γ.

Now, consider the map $x \to \varepsilon(x)\,1$. Write this as $\Gamma(\sum \omega_i \otimes a_i) = \sum \pi(\omega_i)\,\pi(a_i)$. Because $\varepsilon(ax) = \varepsilon(a)\,\varepsilon(x)$ and $\pi(\omega)\,1 = \omega(1)\,1$, we get that

$$\Gamma\left(\sum \omega_i \omega \otimes a_i a\right) = \omega(1)\,\varepsilon(a)\,\Gamma\left(\sum \omega_i \otimes a_i\right).$$

By the injectivity of Γ, we must have

$$\sum \omega_i \omega \otimes a_i a = \sum \omega(1)\,\omega_i \otimes \varepsilon(a)\,a_i$$

for all $a \in A$ and $\omega \in A'$.

First, take $a = 1$, choose ρ and define $\psi = \sum \omega_i \rho(a_i)$. Then, it follows that $\psi\omega = \omega(1)\,\psi$. For some ρ, we must have $\psi \neq 0$. This proves the existence of a non-zero right invariant functional. Next, take in this formula $\omega = \varepsilon$, choose b and define $k = \sum \omega_i(b)\,a_i$. Then, $ka = \varepsilon(a)\,k$. Again, for some b we must have $k \neq 0$.

To prove uniqueness of ψ and k, observe that $\pi(\psi)\,\pi(k)\,x=\varepsilon(x)\,\psi(k)\,1$ for all x and use the injectivity of Γ. We see that we also must have $\psi(k)\neq 0$.

Similarly, or by applying S, we get a unique non-zero left invariant functional φ and a unique non-zero element h such that $ah=\varepsilon(a)\,h$ for all $a\in A$.

We have seen that these functionals φ and ψ must be faithful. This follows very easily here. If $\psi(ax)=0$ for all x, then $\pi(\psi)\,\pi(a)\,x=0$ for all x and by the injectivity of Γ we must have $a=0$. This is no surprise. The injectivity of Γ and the faithfulness of the invariant functionals are proved by the same techniques.

From the proof above, we can also write down an explicit formula for $\psi\otimes k$. Let (e_i) be a basis in A and let (f_i) be a dual basis in A'. We claim that

$$\psi\otimes k=\sum (S(f_i))(S(e_{i(1)})\,\cdot\,)\otimes e_{i(2)}.$$

Indeed, if we apply Γ to the right hand side and let all this act on x, we obtain

$$\begin{aligned}\sum (S(f_i))(S(e_{i(1)})\,e_{i(2)}x_{(1)})\,e_{i(3)}x_{(2)}&=\sum (S(f_i))(x_{(1)})\,e_i x_{(2)}\\&=\sum S(x_{(1)})\,x_{(2)}=\varepsilon(x)\,1\end{aligned}$$

and this is precisely $\Gamma(\psi\otimes k)(x)$ when we assume $\psi(k)=1$.

In [14], we used this formula directly to prove the existence of invariant functionals on A and on the dual.

We also have shown in Proposition 3.11 that invariant functionals automatically satisfy the weak K.M.S. property. In the finite-dimensional case, this follows already from the faithfulness. Indeed, if ω is a faithful functional on a finite-dimensional algebra, by dimensionality, we automatically have

$$A'=\{\omega(a\,\cdot\,)\mid a\in A\}=\{\omega(\,\cdot\,a)\mid a\in A\}.$$

So, all finite-dimensional Hopf algebras belong to the class that we consider in this paper. It is also obvious that the dual, as we defined it in Section 4, is precisely the dual Hopf algebra A' in the usual sense.

In [14], we have given a simple, non-trivial example of a finite-dimensional Hopf algebra. We could use it here to illustrate the different objects ψ, φ, δ, σ, τ that we have in our theory. We will however give another example in this section which is closely related to this finite-dimensional example. This example will also be used to illustrate the two other special cases that we will treat now.

5.2. Definition. Let (A, Δ) be a regular multiplier Hopf algebra with non-trivial invariant functionals. We call (A, Δ) of *compact type* if A has an identity (i.e. when A is a Hopf algebra). We say that (A, Δ) is of *discrete type* if there is a non-zero element h in A so that $ah = \varepsilon(a)\, h$ for all $a \in A$.

It follows from the definition, that the compact quantum groups, as e.g. defined in [2] or [22], belong to our class and are of compact type. Because in this case, $\varphi = \psi$, we call them *unimodular*.

Remark that, for discrete type algebras, h is unique (see further). If we let $k = S(h)$, then $ka = \varepsilon(a)\, k$. We must have either $h = k$ or $\varepsilon(h) = \varepsilon(k) = 0$ because $kh = \varepsilon(k)\, h = \varepsilon(h)\, k$ (with the right choice of the scalars of course). We say that (A, Δ) is *counimodular* if $h = k$. Discrete quantum groups, as defined e.g. in [5] and [17] are multiplier Hopf algebras where the underlying algebra is a direct sum of full matrix algebras. They are of discrete type and counimodular.

The duality between discrete abelian (resp. quantum) groups and compact abelian (resp. quantum) groups is now generalized in the following result.

In what follows, (A, Δ) is a regular multiplier Hopf algebra with invariant functionals.

5.3. Proposition. *If (A, Δ) is of discrete type, then the dual $(\hat{A}, \hat{\Delta})$ is of compact type. If (A, Δ) is of compact type, then the dual $(\hat{A}, \hat{\Delta})$ is of discrete type.*

Proof. Assume first that (A, Δ) is of discrete type. Let h be a non-zero element such that $ah = \varepsilon(a)\, h$ for all a. Let φ be a left invariant functional on A. Then, $\varphi(xh) = \varepsilon(x)\, \varphi(h)$ for all x. As we saw already before, because φ is faithful, we must have $\varphi(h) \neq 0$. Therefore, $\varepsilon \in \hat{A}$ and $\hat{A}$ has an identity. Hence, $(\hat{A}, \hat{\Delta})$ is of compact type.

Now assume that (A, Δ) is of compact type. Let φ be a non-zero left invariant functional. Because A has an identity, $\varphi \in \hat{A}$. But left invariance means $\omega\varphi = \omega(1)\, \varphi$. Also, $\omega(1) = \hat{\varepsilon}(\omega)$. So, $(\hat{A}, \hat{\Delta})$ is of discrete type.

Let us now relate these two notions to the finite-dimensional case. It is a consequence of 5.1 that, if A is finite-dimensional, then (A, Δ) is both of compact and of discrete type. We now prove the converse.

5.4. Proposition. *If (A, Δ) is both of compact and of discrete type, then A is finite-dimensional.*

Proof. Take an element h as in the definition and a left invariant functional φ such that $\varphi(h) = 1$. Similarly as before, we now have $(a \otimes 1)\, \Delta(h) = (1 \otimes S^{-1}(a))\, \Delta(h)$ for all a. Now apply $\iota \otimes \varphi$ to obtain that

$$a = \sum h_{(1)} \varphi(S^{-1}(a)\, h_{(2)})$$

for all a. This proves that A is spanned by only finitely many elements and so, it is of finite dimension.

Before we pass to another example, let us remark that multiplier Hopf algebras of discrete type are studied in more detail in [19]. In that paper, we show that if a regular multiplier Hopf algebra has a non-zero element h such that $ah = \varepsilon(a)\, h$ for all a, then already invariant functionals exist automatically. So, the existence of such a *cointegral* is sufficient for the existence of *integrals*.

In [19] we also obtain necessary and sufficient conditions on the underlying algebra A of a regular multiplier Hopf algebra (A, Δ) to be of discrete type. One of the conditions e.g. is that A has no essential (dense) nontrivial ideals. This is the case when A is the direct sum of matrix algebras. So, in particular, in [19] we obtain the following result.

5.5. PROPOSITION. *If (A, Δ) is a regular multiplier Hopf algebra and if A is the direct sum of matrix algebras, then A has invariant functionals and is of discrete type.*

The proof of this result can essentially be found already in [17]. In our note [19], we are a little more general. Here is a sketch of the proof.

The kernel of ε is a two-sided ideal. By the structure of A, there is a nonzero element h so that $ah = ha = \varepsilon(a)\, h$ for all $a \in A$. For this element h, we have

$$\Delta(h)(a \otimes 1) = \Delta(h)(1 \otimes S(a))$$

$$(1 \otimes a)\, \Delta(h) = (S(a) \otimes 1)\, \Delta(h).$$

Using this, it is possible to show that any element $a \in A$ can uniquely be written as $(\omega \otimes \iota)\, \Delta(h)$ where ω is a linear functional supported in only finitely many components of A. This follows e.g. from the fact that such elements form a two-sided ideal. If this is not all of A, again by the structure of A, there is a non-zero element b such that

$$(\omega \otimes \iota)(\Delta(h)(1 \otimes b)) = ((\omega \otimes \iota)\, \Delta(h))\, b = 0$$

for all such ω. This implies $\Delta(h)(1 \otimes b) = 0$ and $b = 0$. Similarly, any element a has the form $(\iota \otimes \omega)\, \Delta(h)$ for some ω with support in finitely many components. Combining the two results, we get uniqueness of these representations.

Then, the left invariant functional is defined by $\varphi((\omega \otimes \iota)\, \Delta(h)) = \omega(1)$ which makes sense as ω is only supported on finitely many components. It is not so hard to see that φ is left invariant. Similarly for the right invariant functional.

In [17], only the *-algebra case is considered. In that case, we have $h^*h = \varepsilon(h^*)\, h$ and in a *good* *-algebra, this implies that $\varepsilon(h) \neq 0$ so that we are in the counimodular case. In [19] we are more general.

We will now consider a special example. It is a pair of a compact type and a discrete type multiplier Hopf algebra.

5.6. PROPOSITION. *Let $\lambda \in \mathbb{C}$ and assume that λ is a root of 1 but $\lambda \neq 1$ and $\lambda \neq -1$. Let n be the smallest natural number such that $\lambda^{2n} = 1$. Let A be the unital algebra over $\mathbb{C}$ generated by two elements a, b so that a is invertible, $ab = \lambda ba$ and $b^n = 0$. Then, A can be made into a Hopf algebra if we set*

$$\Delta(a) = a \otimes a$$

$$\Delta(b) = a \otimes b + b \otimes a^{-1}.$$

Proof. (See also [14].) To prove that this comultiplication exists, it is sufficient to verify that the elements $a \otimes a$ and $a \otimes b + b \otimes a^{-1}$ satisfy the same relations as a and b. It is clear that $a \otimes a$ is invertible and that these elements satisfy the same commutation rules as a and b. So, we have to check that $(a \otimes b + b \otimes a^{-1})^n = 0$.

Using induction on q, it is standard to show that, when $q = 1, 2, ...n$,

$$(a \otimes b + b \otimes a^{-1})^q = \sum_{k=0}^{q} C_k^q a^k b^{q-k} \otimes a^{k-q} b^k$$

where $C_0^q = C_q^q = 1$ and

$$C_k^q = \frac{r_1 r_2 \cdots r_q}{r_1 \cdots r_k r_1 \cdots r_{q-k}}$$

when $k = 1, 2...q-1$ and where

$$r_j = \frac{\lambda^j - \lambda^{-j}}{\lambda - \lambda^{-1}}$$

for $j = 1, 2, ...n$.

By definition of n, we have $r_j \neq 0$ except for $j = n$. It follows that $C_k^n = 0$ whenever $k \neq 0$ and $k \neq n$. So

$$(a \otimes b + b \otimes a^{-1})^n = a^n b^n \otimes a^{-1} b^n = 0.$$

This shows that Δ is a well-defined homomorphism of A into $A \otimes A$. It is coassociative because we have that $(\Delta \otimes \iota)\, \Delta = (\iota \otimes \Delta)\, \Delta$ on a, a^{-1} and b.

If we set $\varepsilon(a)=1$ and $\varepsilon(b)=0$, we get a counit and if we set $S(a)=a^{-1}$ and $S(b)=-\lambda^{-1}b$, we get the antipode.

In what follows, we will not always explicitly give the range of the indices that we are using. This however should be obvious from the context. When we write e.g. $a^p b^q$, it is understood that $p \in \mathbb{Z}$ and $q=0, 1, \ldots n-1$. This is the case in the next lemma.

5.7. LEMMA. *The elements $a^p b^q$ form a basis for A.*

Proof. First remark that these elements span A. So, we only have to prove that they are linearly independent. Consider the vector space V, spanned by a basis $\{e_{pq} \mid p \in \mathbb{Z},\ q=0, 1, \ldots n-1\}$. Let A act on V by

$$ae_{pq}=e_{p+1,q}$$

$$be_{pq}=\lambda^{-p}e_{p,q+1}$$

where we set $e_{pn}=0$. It is now easy to verify that this action is well-defined. Now, suppose that $\sum c_{rs}a^r b^s=0$. If we apply this on $e_{0,0}$, we get $\sum c_{rs}e_{rs}=0$. This implies $c_{rs}=0$ for all r, s.

Remark that, having Lemma 5.7, we see that in fact, we use the representation of A given by left multiplication.

We now construct the left and right invariant functional.

5.8. PROPOSITION. *Define a linear functional φ by $\varphi(a^p b^q)=0$ except for $p=-n+1$ and $q=n-1$ where we take $\varphi(a^{-n+1}b^{n-1})=1$. Similarly, let $\psi(a^p b^q)=0$ except for $\psi(a^{n-1}b^{n-1})=1$. Then, φ is left invariant and ψ is right invariant.*

Proof. Take any $p \in \mathbb{Z}$. Clearly $(\iota \otimes \varphi)\,\Delta(a^p)=\varphi(a^p)\,a^p=0=\varphi(a^p)\,1$. Now, let also $q=1, 2, \ldots n-2$. Then $\Delta(a^p b^q)$ will not contain powers of b higher than $n-2$. So again $(\iota \otimes \varphi)\,\Delta(a^p b^q)=0=\varphi(a^p b^q)\,1$.

Now,

$$\begin{aligned}(\iota \otimes \varphi)\,\Delta(a^p b^{n-1}) &= \sum_{k=0}^{n-1} C_k^{n-1}\,\varphi(a^{k+1+p-n}b^k)\,a^{p+k}b^{n-1-k} \\ &= C_{n-1}^{n-1}\varphi(a^p b^{n-1})\,a^{p+n-1}.\end{aligned}$$

If $p \neq -n+1$, we get 0, while if $p=-n+1$, we get 1. So, in all cases, again we get

$$(\iota \otimes \varphi)\,\Delta(a^p b^{n-1})=\varphi(a^p b^{n-1})\,1.$$

This proves that φ is left invariant. A similar calculation will give right invariance of ψ. But we can of course also apply S to φ to get (a scalar multiple of) ψ.

It is instructive (and not so hard) to check that any left and right invariant functional must have this form.

We see that φ and ψ are different (and $\varphi(1)=\psi(1)=0$). We get a non-unimodular (multiplier) Hopf algebra (of compact type).

In the next proposition, we get the modular element δ.

5.9. PROPOSITION. *If we set* $\delta = a^{-2n+2}$, *then we get* $(\varphi \otimes \iota)\, \Delta(x) = \varphi(x)\, \delta$ *for all* $x \in A$.

Proof. Again, we only have to consider elements x of the form $a^p b^{n-1}$. Then, we get

$$\begin{aligned}(\varphi \otimes \iota)\, \Delta(a^p b^{n-1}) &= \sum_{k=0}^{n-1} C_k^{n-1}\, \varphi(a^{p+k} b^{n-1-k})\, a^{k+1+p-n} b^k \\ &= C_0^{n-1}\, \varphi(a^p b^{n-1})\, a^{p+1-n}.\end{aligned}$$

This is 0 when $p \neq -n+1$, while if $p = -n+1$, we get a^{-2n+2}.

Let us verify the formula $\varphi(S(x)) = \varphi(x\delta)$ for all x (cf. Proposition 3.10). We only have to look at elements x of the form $a^p b^{n-1}$. Then, we get

$$\begin{aligned}a^p b^{n-1} \delta &= a^p b^{n-1} a^{-2n+2} \\ &= \lambda^{(2n-2)(n-1)} a^{p-2n+2} b^{n-1} \\ S(a^p b^{n-1}) &= (-\lambda^{-1})^{n-1}\, b^{n-1} a^{-p} \\ &= (-1)^{n-1}\, \lambda^{-n+1} \lambda^{p(n-1)} a^{-p} b^{n-1}.\end{aligned}$$

If $p \neq n-1$, we get 0 in both cases when we apply φ. So, let $p = n-1$. Then we must verify that

$$\lambda^{2(n-1)^2} = (-1)^{n-1}\, \lambda^{(n-1)^2 - (n-1)}.$$

By the choice of n, we have $\lambda^n = -1$ if n is even and then the above equality is true. If n is odd, we can have either $\lambda^n = 1$ or $\lambda^n = -1$ but also in these two cases, the equality is satisfied.

Remark also that $\Delta(\delta) = \delta \otimes \delta$.

Now, let us look at the modular automorphism σ.

5.10. PROPOSITION. *If we define* $\sigma(a) = \lambda^{n-1} a$ *and* $\sigma(b) = \lambda^{n-1} b$, *then we get the automorphism of* A *such that* $\varphi(xy) = \varphi(y\sigma(x))$ *for all* $x, y \in A$.

Proof. We only verify the non-trivial cases. We get

$$\varphi(a^{-n+1}b^{n-1}) = \lambda^{n-1}\varphi(a^{-n}b^{n-1}a)$$

proving that $\sigma(a) = \lambda^{n-1}a$ and

$$\varphi(ba^{-n+1}b^{n-2}) = \lambda^{n-1}\varphi(a^{-n+1}b^{n-1})$$

proving that $\sigma(b) = \lambda^{n-1}b$.

Let us verify the formula $(S^2 \otimes \sigma)\, \Delta(x) = \Delta(\sigma(x))$ (see Proposition 3.14). For $x = a$ we get

$$(S^2 \otimes \sigma)(a \otimes a) = \lambda^{n-1} a \otimes a.$$

For $x = b$ we get

$$\begin{aligned}(S^2 \otimes \sigma)&(a \otimes b + b \otimes a^{-1})\\ &= \lambda^{n-1} a \otimes b + (-\lambda)^{-2}\, \lambda^{-n+1} b \otimes a^{-1}\\ &= \lambda^{n-1}(a \otimes b + b \otimes a^{-1})\end{aligned}$$

as $\lambda^{-n-1} = \lambda^{n-1}$ because $\lambda^{2n} = 1$.

Finally, we get the number τ (see the remark after Proposition 3.10).

5.11. Proposition. *If $\tau = \lambda^2$, then $\varphi(S^2(x)) = \tau\varphi(x)$ for all x.*

Proof. Again, we restrict to $x = a^{-n+1}b^{n-1}$. Then

$$S^2(a^{-n+1}b^{n-1}) = (-\lambda)^{-2(n-1)} a^{-n+1}b^{n-1}$$

and we indeed get $\tau = \lambda^2$.

We now construct the dual. Given p, q, we see that $\varphi(xa^pb^q)$ will only be non-zero if x is a multiple of $a^{-n+1-p}b^{n-1-q}$. Therefore, we get the following result.

5.12. Proposition. *The dual $\hat{A}$ of (A, Δ) is spanned by the linear functionals ω_{pq} defined by $\omega_{pq}(a^r b^s) = 0$ except for $\omega_{pq}(a^p b^q) = 1$. These elements form a basis for $\hat{A}$.*

In the following proposition, we get the product rule. We will use the Kronecker δ-symbol here. We will also set $\omega_{p,q} = 0$ when $q \geqslant n$.

5.13. Proposition. *For all p, q, k, l, we get*

$$\omega_{pq}\omega_{kl} = \delta(p-k, q+l)\, C_l^{q+l}\omega_{p-l,\, q+l}.$$

Proof. For all r and s, we get

$$\begin{aligned}(\omega_{pq}\omega_{kl})(a^r b^s) &= (\omega_{pq} \otimes \omega_{kl})\, \Delta(a^r b^s) \\ &= \sum_j C_j^s \omega_{pq}(a^{r+j} b^{s-j})\, \omega_{kl}(a^{r-s+j} b^j) \\ &= \sum_j C_j^s \delta(p, r+j)\, \delta(q, s-j)\, \delta(k, r-s+j)\, \delta(l, j) \\ &= C_l^s \delta(p-l, r)\, \delta(q+l, s)\, \delta(k-l, r-s) \\ &= C_l^{q+l} \delta(k-l, p-l-q-l)\, \omega_{p-l, q+l}(a^r b^s)\end{aligned}$$

and this proves the result.

We will now look for a set of generators and relations for $\hat{A}$.

5.14. Proposition. *Let $e_p = \omega_{p,0}$. Then $e_p e_k = \delta(p, k)\, e_p$. There is also a multiplier d of $\hat{A}$ such that $e_{p+q} d^q = r_1 \cdots r_q \omega_{pq}$ when $q = 1, 2, \ldots n-1$. We have $d^n = 0$ and $de_p = e_{p+2} d$.*

Proof. If we define $e_p = \omega_{p,0}$, it is an easy consequence of the formula in the previous proposition that $e_p e_k = \delta(p, k)\, e_k$ for all p, k.

Let us now define a multiplier d of $\hat{A}$ by

$$\begin{aligned}\omega_{pq} d &= r_{q+1} \omega_{p-1, q+1} \\ d\omega_{pq} &= r_{q+1} \omega_{p+1, q+1}.\end{aligned}$$

To see hat this indeed defines a multiplier, we have to check that $(\omega_{pq} d)\, \omega_{kl} = \omega_{pq}(d\omega_{kl})$ (cf. remark in Section 2). Now

$$\begin{aligned}(\omega_{pq} d)\, \omega_{kl} &= r_{q+1} \omega_{p-1, q+1} \omega_{kl} \\ &= r_{q+1} C_l^{q+l+1} \delta(p-k-1, q+l+1)\, \omega_{p-l-1, q+l+1} \\ \omega_{pq}(d\omega_{kl}) &= r_{l+1} \omega_{pq} \omega_{k+1, l+1} \\ &= r_{l+1} C_{l+1}^{q+l+1} \delta(p-k-1, q+l+1)\, \omega_{p-l-1, q+l+1}\end{aligned}$$

and we do have

$$r_{q+1} C_l^{q+l+1} = r_{l+1} C_{l+1}^{q+l+1}.$$

Then

$$\begin{aligned}e_{p+q} d^q = \omega_{p+q, 0} d^q &= r_1 \omega_{p+q-1, 1} d^{q-1} \\ &= r_1 r_2 \omega_{p+q-2, 2} d^{q-2} = \cdots \\ &= r_1 r_2 \cdots r_q \omega_{pq}.\end{aligned}$$

Also,

$$de_p = d\omega_{p,0} = r_1 \omega_{p+1,1} = \omega_{p+2,0} d = e_{p+2} d.$$

Finally, it is clear that $d^n = 0$.

This proposition characterizes $\hat{A}$ completely. It is spanned by the elements $e_p d^q$ where $e_p e_q = \delta(p, q)\, e_p$ and $de_p = e_{p+2} d$ and $d^n = 0$.

Now, we obtain the comultiplication $\hat{\Delta}$ on $\hat{A}$.

5.15. PROPOSITION. *We have* $\hat{\Delta}(e_p) = \sum_j e_j \otimes e_{p-j}$ *and* $\hat{\Delta}(d) = d \otimes c + 1 \otimes d$ *where* $c = \sum_r \lambda^{-r} e_r$ *(in* $M(\hat{A})$*).*

Proof. First we have

$$\begin{aligned}
\hat{\Delta}(\omega_{p,0})(a^r b^s \otimes a^k b^l) &= \omega_{p,0}(a^r b^s a^k b^l) \\
&= \delta(s, 0)\, \delta(l, 0)\, \omega_{p,0}(a^{r+k}) \\
&= \delta(s, 0)\, \delta(l, 0)\, \delta(p, r+k) \\
&= \sum_j \delta(r, j)\, \delta(s, 0)\, \delta(k, p-j)\, \delta(l, 0)
\end{aligned}$$

proving the first formula.

Next, we have

$$\begin{aligned}
&\hat{\Delta}(\omega_{p,1})(a^r b^s \otimes a^k b^l) \\
&\quad = \omega_{p,1}(a^r b^s a^k b^l) \\
&\quad = \delta(p, r+k)(\delta(s, 0)\, \delta(l, 1) + \lambda^{-k} \delta(s, 1)\, \delta(l, 0)) \\
&\quad = \sum_j \delta(r, j)\, \delta(p-j, k)(\delta(s, 0)\, \delta(l, 1) + \lambda^{-k} \delta(s, 1)\, \delta(l, 0)).
\end{aligned}$$

This means

$$\hat{\Delta}(\omega_{p,1}) = \sum_j (\omega_{j,0} \otimes \omega_{p-j,1} + \lambda^{-(p-j)} \omega_{j,1} \otimes \omega_{p-j,0}).$$

In other words

$$\hat{\Delta}(e_p d) = \sum_j (e_j \otimes e_{p-j} d + \lambda^{-(p-j)} e_j d \otimes e_{p-j})$$

and it follows that

$$\hat{\Delta}(d) = 1 \otimes d + d \otimes c$$

where $c = \sum \lambda^{-j} e_j$.

It is not so hard to verify that $\hat{\Delta}$ is indeed a comultiplication on $\hat{A}$ and that $(\hat{A}, \hat{\Delta})$ is a regular multiplier Hopf algebra. It is of discrete type and non-counimodular.

In the multiplier algebra $M(\hat{A})$ sits the Hopf algebra generated by c and d where c is invertible and $d^n = 0$ and $cd = \lambda^{-2} dc$ with the comultiplication $\hat{\Delta}(c) = c \otimes c$ and $\hat{\Delta}(d) = 1 \otimes d + d \otimes c$. Remark that this is very similar to the Hopf algebra that we started with.

If we take e.g. the quotient of A by imposing the extra relation $a^m = 1$ for some m satisfying $\lambda^m = 1$ (cf. the example in [14]), this will result in taking the subalgebra of $M(\hat{A})$ containing the element d and the elements $\sum_j e_{p+mj}$ for all p. It is not so hard to see that we recover here the duality found for this example in [14].

It is easy to find the counit and the antipode for $(\hat{A}, \hat{\Delta})$. We must have $\hat{\varepsilon}(e_p) = \delta(p, 0)$ and $\hat{\varepsilon}(d) = 0$ and for the antipode, we get $\hat{S}(e_p) = e_{-p}$ and $\hat{S}(d) = -dc^{-1}$.

We can also easily find the elements h and k. They are $h = e_0 d^{n-1}$ and $k = d^{n-1} e_0$. Indeed, $e_p h = e_p e_0 d^{n-1} = \delta(p, 0)\, e_0 d^{n-1}$ while $dh = 0$. Similarly for k.

We could now use these elements to calculate the dual invariant functionals. However, it turns out to be simpler to use the formulas for $\hat{\varphi}$ and $\hat{\psi}$ given in Proposition 4.8. We get the following result.

5.16. Proposition. *We have that* $\hat{\varphi}(e_r d^s) = 0$ *for all* r *and* s *except for* $s = n-1$. *Then we get* $\hat{\varphi}(e_r d^{n-1}) = 1$ *for all* r. *Similarly,* $\hat{\psi}(d^s e_r) = 0$ *except for* $s = n-1$ *and then* $\hat{\psi}(d^{n-1} e_r) = \lambda^{r(n-1)}$ *for all* r.

Proof. We know that $\hat{\varphi}(\omega) = \varepsilon(x)$ when $\omega = \psi(x\,\cdot)$. Take $x = a^p b^q$. Then

$$\begin{aligned}\omega(a^k b^l) &= \psi(a^p b^q a^k b^l) = \lambda^{-kq} \psi(a^{p+k} b^{q+l})\\ &= \lambda^{-kq} \delta(p+k, n-1)\, \delta(q+l, n-1)\\ &= \lambda^{-kq} \delta(k, n-p-1)\, \delta(l, n-q-1).\end{aligned}$$

So,

$$\begin{aligned}\omega &= \lambda^{-q(n-p-1)} \omega_{n-p-1,\, n-q-1}\\ &= \lambda^{-q(n-p-1)} r_1 \cdots r_{n-q-1} e_{2n-p-q-2} d^{n-q-1}.\end{aligned}$$

Now, $\hat{\varphi}(\omega) = \varepsilon(a^p b^q) = \delta(q, 0)$. So $\hat{\varphi}(e_r d^s) = 0$ except when $s = n-1$.

Let further $q = 0$. Then we find

$$r_1 r_2 \cdots r_{n-1} \hat{\varphi}(e_{2n-p-2} d^{n-1}) = 1$$

for all p. This gives, after rescaling, the formula for $\hat{\varphi}$. If we apply S, we get, again after rescaling,

$$\hat{\psi}(d^{n-1}c^{-n+1}e_{-r}) = 1$$

and so $\hat{\psi}(d^{n-1}e_{-r}) = \lambda^{r(n-1)}$ for all r and this gives the result.

Remark that also $\hat{\varphi}$ and $\hat{\psi}$ are different from each other so that this example is not only non-counimodular but also non-unimodular.

Let us finish this last section with a few concluding remarks.

5.17. *Remarks.* (i) We have mentioned already that it is possible to construct the quantum double within our category (see [3]). If we do this with an algebra of discrete type or with an algebra of compact type, we will, except in the finite-dimensional case, get an algebra which is neither of discrete nor of compact type. This will give interesting examples of such more general multiplier Hopf algebras. The example of this section can be used for this purpose.

(ii) It is known that quantum groups are used in the theory of q-special functions. It would also be interesting to find out if the duality theory that we developed in this paper, will give some new applications there.

ACKNOWLEDGMENTS

A large part of this work was done while I was visiting the University of Oslo. Therefore, I thank my colleagues and friends of the Department of Mathematics there: E. Alfsen, E. Bedos, O. Bratteli, and E. Størmer. Their warm hospitality created a good atmosphere. I especially thank E. Alfsen for many discussions on this subject (now and on the occasion of earlier visits). I am also grateful to M. Landstad for many discussions with him and for his hospitality during my visit at the University of Trondheim. I also thank my colleagues at the University of Warsaw, especially S. Woronowicz and S. Zakrzewski for organizing the Workshop on Quantum Groups and Quantum Spaces and for inviting me and giving me the opportunity to announce my results at this meeting. Finally, I am grateful to my co-workers at the University of Leuven, J. Kustermans and J. Quaegebeur, for discussions on this subject.

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