Isometric embedding of finite ultrametric spaces in Banach spaces

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Abstract

We prove that for any finite ultrametric space \( M \) and any infinite-dimensional Banach space \( B \) there exists an isometric embedding of \( M \) into \( B \).

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1. Introduction

A metric space \((M, d)\) is called ultrametric (or isosceles) if \( d(x, z) \leq \max\{d(x, y), d(y, z)\} \) for any \( x, y, z \in M \). Metrics which have this property are also called non-Archimedean. These spaces were intensively studied during last 50 years. They have applications in number theory, in \( p \)-adic analysis and in computer science [1–4].

We briefly review only the results related to isometric embedding of ultrametric spaces. Giving a partial solution of a Nikol’skii problem [18], Timan [6] in 1975 proved that elements of a certain (rather restricted) subclass of countable ultrametric spaces admit isometric embedding in the Banach space \( L_p[0, 1] \) for any \( p \in [1, +\infty] \). In 1979 Vestfrid and Timan [7] (see also [8]) proved that any separable ultrametric space is isometric to a subspace of \( l_2 \). Lemin [10] (1985) proved independently (and using a different method) the last result and generalized it for ultrametric spaces of arbitrary weight. Vestfrid (1984) [9] constructed a universal ultrametric space for the class of separable ultrametric spaces (i.e., such an ultrametric space \( M \) that any separable ultrametric space is isometric to a
subspace of \( M \). Using specific properties of the constructed space he showed that any separable ultrametric space admits an isometric embedding into Banach spaces \( l_1 \) and \( c_0 \).

Lemin [5] constructed universal spaces for the classes of ultrametric spaces of the weight \( \leq \tau \) for any infinite cardinal \( \tau \). As for the further development of the topic see [11]. In 1988 Fichet [12] proved that any finite ultrametric space admits an isometric embedding in the Banach space \( l_p \) for any \( p \in [1, \infty) \). Vestfrid [13] in 2001 proved the same theorem for a wider class of at most countable metric spaces including a certain subclass of countable ultrametric spaces. Note also that any countable ultrametric space is isometric to a subspace of \( L(\mathbb{R}) \), where \( L(\mathbb{R}) \) is the set of (equivalence classes of) Lebesgue measurable subsets of \( \mathbb{R} \) with the metric \( d(A, B) = \mu(A \Delta B) \) and \( \mu \) is the Lebesgue measure (see [4]). In [4] Lemin formulates the following three problems on embedding of ultrametric spaces in Banach spaces:

**Problem 1.** Given \( n \in \mathbb{N} \) does there exist \( N = N(n) \in \mathbb{N} \) such that any \( n \)-point ultrametric space admits an isometric embedding in any \( N \)-dimensional Banach space?

**Problem 2.** Given a cardinal \( \tau \) does there exist a cardinal \( \psi = \psi(\tau) \) such that any ultrametric space of weight \( \tau \) admits an isometric embedding in any Banach space of weight \( \psi \)?

**Problem 3.** Is it possible to embed isometrically any separable ultrametric space in any infinite dimensional Banach space?

In the present paper we answer in affirmative the following weaker version of the Problem 1:

**Problem 1’.** Given a finite ultrametric space \( M \) does there exist \( n \in \mathbb{N} \) such that \( M \) admits an isometric embedding in any \( n \)-dimensional Banach space?

The key result allowing to construct embeddings of separable ultrametric spaces into \( l_2 \) [7,8,10] is the following.

**Theorem 0** (See [7,8,10,4]). Let \( (M, d) \), \( M = \{x_0, x_1, \ldots, x_n\} \) be a finite ultrametric space. Then there exists an isometric embedding \( \varphi: M \to l_2 \). Moreover, for any such an embedding the vectors \( \{\varphi(x_j) - \varphi(x_0): 1 \leq j \leq n\} \) are linearly independent.

Let \( \mathcal{M} \) be the class of finite metric spaces \( (M, d) \), \( M = \{x_0, x_1, \ldots, x_n\} \), which admit an isometric embedding \( \varphi: M \to l_2 \) such that the vectors \( \{\varphi(x_j) - \varphi(x_0): 1 \leq j \leq n\} \) are linearly independent. By Theorem 0, \( \mathcal{M} \) contains finite ultrametric spaces. The main result of the present paper is the following.

**Theorem 1.** For any \( (M, d) \in \mathcal{M} \), there exists \( m = m(M, d) \in \mathbb{N} \) such that for any Banach space \( B \) with \( \dim B \geq m \) there exists an isometric embedding of \( M \) into \( B \).

Theorems 0 and 1 imply the following Corollary and give the affirmative answer to Problem 1’.
Corollary 1. Any finite ultrametric space admits an isometric embedding into any infinite-dimensional Banach space.

2. Proof of Theorem 1

The following lemma is known as the Dvoretzky theorem [15] (see also [14]).

Lemma 1. Let \( \varepsilon > 0 \) and \( n \in \mathbb{N} \). Then there exists \( m = m(\varepsilon, n) \) such that for any Banach space \( B \) with \( \dim B \geq m \) there exists a linear subspace \( L \subset B \) with \( \dim L = n \) and an inner product \((\cdot, \cdot)\) on \( L \) such that
\[
\|x\|_B \leq \sqrt{(x, x)} \leq (1 + \varepsilon)\|x\|_B \quad \text{for any } x \in L.
\]

Lemma 2. Let \( B_1 \) and \( B_2 \) be finite-dimensional Banach spaces, \( U \) be an open subset of \( B_1 \), \( x_0 \in U \) and \( f : U \to B_2 \) be a \( C^1 \)-mapping (in the Fréchet sense) such that the rank of \( f'(x_0) \) is equal to \( \dim B_2 \). Then there exists \( \varepsilon > 0 \) such that \( f(x_0) \in f(U) \) for any continuous mapping \( f : U \to B_2 \) satisfying the condition
\[
\|f(x) - f(x_0)\| \leq \varepsilon \quad \text{for any } x \in U.
\]

Proof. Without loss of generality we can assume that \( x_0 = 0 \) and \( f(x_0) = 0 \). Let \( n = \dim B_1, m = \dim B_2 \) and \( T = f'(x_0) \). Since \( \text{rk} T = m \), there exists an \( m \)-dimensional linear subspace \( L \subset B_1 \) such that the restriction \( A = T|_L : L \to B_2 \) is one-to-one. By the inverse mapping theorem [16], there exists a neighborhood \( V \) of 0 in \( L \) such that \( V \subset U \), \( f(V) \) is open in \( B_2 \) and the restriction \( f|_V : V \to f(V) \) is a diffeomorphism and hence a homeomorphism. Pick \( r > 0 \) such that \( D = \{x \in L : \|x\| \leq r\} \subset V \) and let \( S = \{x \in L : \|x\| = r\} \). Then \( S \) is homeomorphic to the \( m - 1 \)-dimensional sphere \( S^{m-1} \) and \( f_1 = f|_S : S \to B_2 \setminus \{0\} \) is a homeomorphic embedding. The \((m - 1)\)-dimensional homotopy group of \( B_2 \setminus \{0\} \) is \( \mathbb{Z} \) and the homotopy class of \( f_1 \) is 1. By the well-known theorem [17] there is \( \varepsilon > 0 \) such that maps \( \varepsilon \)-close to \( f_1 \) are homotopically equivalent to \( f_1 \). Thus, there exists \( \varepsilon > 0 \) such that for any \( \tilde{f} : U \to B_2 \) satisfying (2), we have that the homotopy class of \( \tilde{f}|_S \) is 1. Suppose that \( 0 \notin \tilde{f}(U) \). Then \( \tilde{f}|_S \) is homotopic to a constant mapping \( \varphi_{k}(x) = \tilde{f}(ix) \) is such a homotopy) and therefore the homotopy class of \( \tilde{f}|_S \) is 0 and we arrived to a contradiction. \( \square \)

Lemma 3. Let \( H \) be a finite-dimensional Hilbert space, \( n \in \mathbb{N} \) and \( x = (x_0, \ldots, x_n) \in H^{n+1} \) be such that vectors \( \{x_j - x_0 : 1 \leq j \leq n\} \) are linearly independent. Then there exists \( \varepsilon = \varepsilon(x) > 0 \) such that for any norm \( p \) on \( H \) satisfying the condition
\[
p(u) \leq \|u\|_H \leq (1 + \varepsilon)p(u) \quad \text{for any } u \in H,
\]
there exists \( y = (y_0, \ldots, y_n) \in H^{n+1} \) for which \( p(y_i - y_j) = \|x_i - x_j\|_H \) for any \( i, j \in \{0, 1, \ldots, n\} \).

Proof. Let \( J = \{(j, k) \in \mathbb{Z}^2 : 0 \leq j < k \leq n\} \) and \( F : H^{n+1} \to \mathbb{R}^J, F(z)_{j,k} = \|z_k - z_j\|^2 \).
Using induction with respect to \( n \) we shall verify that
\[
\text{rk} F'(x) = |J| = n(n + 1)/2.
\]
For $n = 1$ (4) is trivial. Suppose that $n > 1$ and for smaller $n$’s (4) holds. Without loss of generality we can assume that $x_0 = 0$. Let $J_0 = \{(j,k) \in \mathbb{Z}^2 : 1 \leq j < k \leq n\}$, $J_1 = \{(0,k) : 1 \leq k \leq n\}$, $X_0 = \{0\} \times \mathbb{H}^n$, $X_1 = \mathbb{H} \times \{0\}^n$, $Y_0 = \{w \in \mathbb{R}^J : u_{j,k} = 0 \text{ for } (j,k) \in J_1\}$ and $Y_1 = \{w \in \mathbb{R}^J : u_{j,k} = 0 \text{ for } (j,k) \in J_0\}$. Clearly $H^{n+1} = X_0 \oplus X_1$, $\mathbb{R}^J = Y_0 \oplus Y_1$ and $F(z) = (F_0(z^0), F_1(z^0, z^1))$, where $z^0 \in X_0$, $z^1 \in X_1$. $F_0 : X_0 \rightarrow Y_0$, $(F_0(z^0))_{j,k} = \|z^0_k - z^0_j\|^2$ if $(j,k) \in J_0$, $(F_0(z^0))_{j,k} = 0$ if $(j,k) \in J_1$ and $F_1 : X_0 \oplus X_1 \rightarrow Y_1$, $(F_1(z^0, z^1))_{0,k} = \|z_k\|^2$ if $1 \leq k \leq n$, $(F_1(z^0, z^1))_{j,k} = 0$ if $(j,k) \in J_0$. Therefore

$$F'(x) = \begin{pmatrix} F_0'(x^0) & (F_1'(x^0)(x))_0 \\ 0 & (F_1'(x^0)(x))_{1} \end{pmatrix}. \tag{5}$$

By the induction hypothesis $\text{rk } F_0'(x^0) = |J_1| = n(n - 1)/2$. The square matrix $[(F_1')_{j,i}(x)(x_i)]_{0,k}$ $(1 \leq j, k \leq n)$ has the form $2[(z_k, x_j)]$ and is non-degenerate since the vectors $x_1, \ldots, x_n$ are linearly independent. Therefore $\text{rk } F_1'(x^0)(x) = n$. Formula (5) implies $\text{rk } F'(x) = \text{rk } F_1'(x^0)(x) + \text{rk } F_0'(x^0) = n + n(n - 1)/2 = n + 1/2$ and proves (4).

Fix a bounded neighborhood $V$ of $x$ in $H^{n+1}$ and a norm in $\mathbb{R}^J$. By Lemma 2, there exists $\delta > 0$ such that for any continuous mapping $G : V \rightarrow \mathbb{R}^J$ with $\|F(z) - G(z)\| \leq \delta$ for all $z \in V$ we have $F(x) \in G(V)$. Since $V$ is bounded we can pick $\varepsilon > 0$ such that $\|F(z) - G_p(z)\| \leq \delta$ for all $z \in V$ for any norm $p$ on $H$ satisfying (3), where $[G_p(z)]_{j,k} = p(z_j - z_k)$. Therefore for any norm $p$ satisfying (3), there exists $y = (y_0, \ldots, y_n) \in V$ such that $p(y_i - y_j) = \|x_i - x_j\|$ for any $i, j \in \{0, 1, \ldots, n\}$. \hfill $\Box$

Now we can prove Theorem 1. Let $(M, d) \in \mathcal{M}$, $M = \{x_0, x_1, \ldots, x_n\}$ and $H$ be an $n$-dimensional Hilbert space. By definition of $\mathcal{M}$ there exists an isometric embedding $\varphi : M \rightarrow H$ such that $\varphi(x_0) = y_0 = 0$ and the vectors $y_j = \varphi(x_j)$ are linearly independent. By Lemma 3 there exists $\varepsilon > 0$ such that for any norm $p$ on $H$ satisfying (3) there exists $z = (z_0, \ldots, z_n) \in H^{n+1}$ for which $p(z_j - z_i) = \|y_j - y_i\|_H$ for any $0 \leq i, j \leq n$. Take this $\varepsilon$ and apply Lemma 1. Let $B$ be a Banach space with the dimension $\geq m(\varepsilon)$. Then there exists a linear subspace $L \subset B$ of dimension $n$ and an inner product $(\cdot, \cdot)$ on $L$ such that (1) holds. Pick a linear isometry $T : H \rightarrow L$, where $L$ is endowed with this inner product. Formula (1) implies (3) for the norm $p(x) = \|Tx\|_B$. Using Lemma 3 we obtain $z = (z_0, \ldots, z_n) \in H^{n+1}$ for which $p(z_j - z_i) = \|y_j - y_i\|_H$ for $0 \leq i, j \leq n$. Consider the mapping $\psi : M \rightarrow B$, $\psi(x_j) = T z_j$. Then for $i \neq j$ we have

$$\|\psi(x_i) - \psi(x_j)\|_B = \|T z_i - T z_j\|_B = p(z_i - z_j) = \|y_i - y_j\|_H = d(x_i, x_j).$$

Thus, $\psi$ is an isometry and Theorem 1 is proved.

3. Concluding remarks

Lemin’s Problem 1 (as well as Problem 2) remains open. Problem 1’, which we solve affirmatively in the present paper, is a weaker version of Problem 1.

It worth noting the difference between embedding into Hilbert spaces and into Banach spaces. As it was mentioned in the introduction Theorem 0 is the key result for proving the Vestfrid–Timan–Lemin theorem on isometric embedding of countable ultrametric spaces.
Moreover the last theorem follows from Theorem 0 immediately since a metric space admits an isometric embedding in a Hilbert space if and only if it is finite subspaces admit such embeddings. Of course this fails for general Banach spaces. Therefore Theorem 1 does not imply any analog of the Vestfrid–Timan–Lemin theorem. Thus, Problem 3 also remains open.

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References