On the *LU* Decomposition of *V*-Matrices

Volker Mehrmann
Fakultät für Mathematik
Universität Bielefeld
Postfach 8640
4800 Bielefeld 1, Federal Republic of Germany

Submitted by Hans Schneider

ABSTRACT

We show that the class of V-matrices, introduced by Mehrmann [6], which contains the M-matrices and the Hermitian positive semidefinite matrices, is invariant under Gaussian elimination.

1. INTRODUCTION

Recently there has been revived interest in the question, under which conditions an M-matrix A has an LU decomposition, where L is a nonsingular, lower triangular M-matrix and U is an upper triangular M-matrix (e.g. Funderlic and Plemmons [4], Kuo [5], Rothblum [8], Varga and Cai [10, 11]). It is natural to ask whether the results obtained for M-matrices still hold in more general classes. In this paper we will be primarily concerned with the V-matrices (e.g. Mehrmann [6]), which are defined as follows: An $n \times n$ complex matrix A is called a V-matrix if all principal minors of A are nonnegative and for all subsets $\mu, \nu \subseteq \{1,2,\dots,n\}$ and all real numbers t such that all principal minors of $(A-tI)[\mu \cup \nu]$ are nonnegative, $\det(A-tI)[\mu \cap \nu]\det(A-tI)[\mu \cup \nu] \leqslant \det(A-tI)[\mu]\det(A-tI)[\nu]$, where for $\alpha \subseteq \{1,\dots,n\}$, $A[\alpha]$ denotes the submatrix of A given by the rows and columns indexed in α.

It was shown in [6] that the set of V-matrices includes M-matrices and Hermitian positive semidefinite matrices and that the class of V-matrices is invariant under multiplication (addition) by positive (nonnegative) diagonal matrices. We will show that for a nonsingular V-matrix A, Gaussian elimination without pivoting is possible and the matrices created by the elimination process are again V-matrices, i.e., the class of nonsingular V-matrices is invariant under Gaussian elimination. For a singular V-matrix A we will show

LINEAR ALGEBRA AND ITS APPLICATIONS 61:175–186 (1984)

52 Vanderbilt Ave., New York, NY 10017 0024-3795/84/$3.00

that there always exists a permutation matrix P such that PAP^T is decomposable by Gaussian elimination into a product LU, where L is lower triangular, unit diagonal, and nonsingular and U is upper triangular and singular. Again, any of the matrices produced by the elimination is a V-matrix.

2. PRELIMINARIES

By $\mathbb{R}$ ($\mathbb{C}$) we denote the real (complex) field, and by $\mathbb{R}^{n,n}$ ($\mathbb{C}^{n,n}$) the real (complex) $n \times n$ matrices. For a positive integer n, we set $\langle n \rangle := \{1,\dots,n\}$. For $A \in \mathbb{C}^{n,n}$ and $\mu \subseteq \langle n \rangle$, we denote by $A[\mu]$ the matrix $[a_{ij}] \in \mathbb{C}^{|\mu|,|\mu|}$ with $i, j \in \mu$. (Here $|\mu|$ denotes the cardinality of the set μ.) For $t \in \mathbb{R}$ we set $A_t[\mu] := (A - tI)[\mu]$, where I is the identity matrix. By $\sigma(A)$ we denote the spectrum of A, by $\rho(A)$ the spectral radius of A; and furthermore we set

$$l(A) := \begin{cases} \min(\sigma(A) \cap \mathbb{R}) & \text{if } \sigma(A) \cap \mathbb{R} \neq \varnothing, \\ \infty & \text{otherwise.} \end{cases}$$

DEFINITION 1. A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is called

(i) a *V-matrix* ($A \in V_{\langle n \rangle}$) if all principal minors of A are nonnegative and if for all $\mu, \nu \subseteq \langle n \rangle$ ($\mu, \nu \neq \varnothing$) and all $t \in \mathbb{R}$ such that all principal minors of $A_t[\mu \cup \nu]$ are nonnegative,

$$\det A_t[\mu] \det A_t[\nu] \geqslant \det A_t[\mu \cup \nu] \det A_t[\mu \cap \nu]; \tag{1}$$

(ii) a *τ-matrix* ($A \in \tau_{\langle n \rangle}$) if

(a) $l(A[\mu]) < \infty$ $\forall \mu \subseteq \langle n \rangle$ ($\mu \neq \varnothing$),

(b) $l(A[\mu]) \leqslant l(A[\nu])$ $\forall \nu \subseteq \mu \subseteq \langle n \rangle$ ($\nu, \mu \neq \varnothing$);

(iii) an *M-matrix* ($A \in M_{\langle n \rangle}$) if $A = \alpha I - B$ with B a nonnegative matrix and $\alpha \geqslant \rho(B)$.

In order to see how a matrix in $V_{\langle n \rangle}$ behaves under Gaussian elimination we consider the following: Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geqslant 2$, and $a_{11} \neq 0$. Define the matrix $C = [c_{ij}] \in \mathbb{C}^{n-1,n-1}$ by

$$c_{ij} = \frac{a_{ij}a_{11} - a_{i1}a_{1j}}{a_{11}} \qquad \text{for} \quad i, j \in \{2,\dots,n\}. \tag{2}$$

It is clear that C is exactly the $(n-1)\times(n-1)$ submatrix that occurs in the lower right corner of the matrix A after one step of Gaussian elimination without pivoting. In the following lemma we list some properties of C.

LEMMA 1. *For* $A \in \mathbb{C}^{n,n}$, *with* $n \geqslant 2$ *and* $a_{11} \neq 0$, *let the matrix* C *be defined via* (2). *Then*:

(i) $\det C[\mu] = \det A[\mu \cup \{1\}]/a_{11}$ $\forall \mu \subseteq \{2,\dots,n\}$.

(ii) $\det C_t[\mu] = (1/a_{11})(\det A_t[\mu \cup \{1\}] + t \det A_t[\mu])$ $\forall \mu \subseteq \{2,\dots,n\}$, $\forall t \in \mathbb{R}$.

(iii)

$$\det C_t[\mu] = \frac{1}{a_{11}} \lim_{d_1 \to \infty} \left(\frac{1}{d_1} \det(D_1 A)_t [\mu \cup \{1\}] \right)$$

$\forall \mu \subseteq \{2,\dots,n\}$, $\forall t \in \mathbb{R}$, *where*

$$D_1 = \begin{bmatrix} d_1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}.$$

Proof. (i) follows directly by the Sylvester determinant identity.
(ii): For all $\mu \subseteq \{2,\dots,n\}$ and all $t \in \mathbb{R}$ we have

$$\begin{aligned}
\det A_t[\mu \cup \{1\}] &= \sum_{\alpha \subseteq \mu \cup \{1\}} (-t)^{|\alpha|} \det A[(\mu \cup \{1\}) \setminus \alpha] \\
&= \sum_{\alpha \subseteq \mu} (-t)^{|\alpha|} \det A[(\mu \cup \{1\}) \setminus \alpha] \\
&\quad + \sum_{\substack{\alpha \subseteq \mu \cup \{1\} \\ 1 \in \alpha}} (-t)^{|\alpha|} \det A_t[(\mu \cup \{1\}) \setminus \alpha] \\
&= \sum_{\alpha \subseteq \mu} (-t)^{|\alpha|} (a_{11} \det C[\mu \setminus \alpha]) \\
&\quad + (-t) \sum_{\alpha \subseteq \mu} (-t)^{|\alpha|} \det A[\mu \setminus \alpha] \qquad \text{[by (i)]} \\
&= a_{11} \det C_t[\mu] - t \det A_t[\mu].
\end{aligned}$$

Therefore

$$\det C_t[\mu] = \frac{1}{a_{11}}(\det A_t[\mu \cup \{1\}] + t\det A_t[\mu]).$$

(iii): $\det(D_1A)_t[\mu \cup \{1\}] = d_1 \det A_t[\mu \cup \{1\}] + (d_1 - 1)t\det A_t[\mu]$. This follows by expanding the determinant with respect to the first row. Thus

$$\frac{1}{d_1}\det(D_1A)_t[\mu \cup \{1\}] = \det A_t[\mu \cup \{1\}] + \left(\frac{d_1 - 1}{d_1}\right)t\det A_t[\mu],$$

and since $\lim_{d_1 \to \infty}(d_1 - 1)/d_1 = 1$, (iii) follows by (ii). ■

If we now consider the matrix C for a matrix $A \in V_{\langle n \rangle}$ we get the following:

Lemma 2. *Let $A \in V_{\langle n \rangle}$, $n \geqslant 2$, and let C be the matrix defined by A via* (2). *Then:*

(i) *$l(C[\mu]) < \infty$ and $l(C[\mu]) \in [l(A[\mu \cup \{1\}]), l(A[\mu])]$ $\forall \mu \subseteq \{2,\dots,n\}$.*
(ii) *Let $t \in \mathbb{R}$ and $\mu \subseteq \{2,\dots,n\}$. Then all principal minors of $C_t[\mu]$ are nonnegative iff $t \leqslant l(C[\mu])$.*

Proof. Let $\mu \subseteq \{2,\dots,n\}$. For any $d_1 > 0$ we have that the matrix D_1A defined as in Lemma 1 is in $V_{\langle n \rangle}$ (this follows by Theorem 2 of Mehrmann [6]). Thus for all $d_1 > 1$, $l(D_1A)$ exists and $l(D_1A[\mu \cup \{1\}]) \in [l(A[\mu \cup \{1\}]), l(A[\mu])]$. The latter follows because $\det(D_1A)_t[\mu \cup \{1\}] = d_1\det A_t[\mu \cup \{1\}] + (d_1 - 1)t\det A_t[\mu]$. For $t < 0$ this is positive, since $D_1A \in V_{\langle n \rangle}$. If $l(A[\mu \cup \{1\}]) = l(A[\mu])$, then $l(D_1A[\mu \cup \{1\}]) = l(A[\mu \cup \{1\}])$, which is in the given interval. If $l(A[\mu]) > l(A[\mu \cup \{1\}])$, then for all $0 \leqslant t \leqslant l(A[\mu \cup \{1\}])$ we have $\det(D_1A)_t[\mu \cup \{1\}] > 0$. But for $\tilde{t} = l(A[\mu])$, applying Theorem 5 of Mehrmann [6], we obtain that $\det A_{\tilde{t}}[\mu \cup \{1\}] \leqslant 0$. Thus

$$l(D_1A[\mu \cup \{1\}]) \in [l(A[\mu \cup \{n\}]), l(A[\mu])].$$

The limit $\lim_{d_1 \to \infty}(1/d_1)\det((D_1A)_t[\mu \cup \{1\}])$ exists, and $l(D_1A[\mu \cup \{1\}])$ is a continuous function in d_1. Thus, for $l(C[\mu])$ we also have $l(C[\mu]) \in [l(A[\mu \cup \{n\}]), l(A[\mu])]$.

(ii): By Theorem 2(i) of Mehrmann [6], for any $A \in V_{\langle n \rangle}$ and D_1 positive diagonal, $D_1 A \in V_{\langle n \rangle}$. Applying now Lemma 1 of Mehrmann [6], we have that for $A \in V_{\langle n \rangle}$, D_1 positive diagonal, $t \in \mathbb{R}$, and $\mu \subseteq \langle n \rangle$ the following are equivalent:

(a) all principal minors of $(D_1 A)_t[\mu]$ are nonnegative;
(b) $t \leqslant l((D_1 A)[\mu])$.

Again, since the eigenvalues and the characteristic polynomials of submatrices are continuous functions in d_1, it follows that this result also holds for the limit, i.e. for C. ■

It was shown by Fan [2] that for an M-matrix A the above constructed matrix C again is an M-matrix. Now we get a similar result for $V_{\langle n \rangle}$.

LEMMA 3. *Let $A \in V_{\langle n \rangle}$, $n \geqslant 2$, and let C be defined by the matrix A via (2). Then, $C \in V_{\langle n-1 \rangle}$.*

Proof. We have to show that for all $\mu, \nu \subseteq \{2, \dots, n\}$ and all $t \in \mathbb{R}$ such that all principal minors of $C_t[\mu \cup \nu]$ are nonnegative,

$$\det C_t[\mu] \det C_t[\nu] \geqslant \det C_t[\mu \cap \nu] \det C_t[\mu \cup \nu]. \tag{3}$$

Applying Lemma 2, it suffices to show this for $t \leqslant l(C[\mu \cup \nu])$. For all $\mu, \nu \subseteq \langle n \rangle$, for any

$$D_1 = \begin{bmatrix} d_1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \quad \text{with} \quad d_1 > 0,$$

and for all $t \leqslant l((D_1 A)_t[\mu \cup \nu])$,

$$\frac{1}{d_1^2}\big(\det(D_1 A)_t[\mu \cup \{1\}] \det(D_1 A)_t[\nu \cup \{1\}]$$

$$- \det(D_1 A)_t[(\mu \cap \nu) \cup \{1\}] \det(D_1 A)_t[\mu \cup \nu \cup \{1\}]\big) \geqslant 0, \quad (4)$$

since $D_1 A \in V_{\langle n \rangle}$. Since for $d_1 > 0$ the left side of (4) is a continuous function in d_1, it follows for all $t \leqslant \lim_{d_1 \to \infty} l(D_1 A[\mu \cup \nu])$ that

$$\lim_{d_1 \to \infty} \frac{1}{d_1^2} \big(\det(D_1 A)_t[\mu \cup \{1\}] \det(D_1 A)_t[\nu \cup \{1\}]$$

$$- \det(D_1 A)_t[(\mu \cap \nu) \cup \{1\}] \det(D_1 A)_t[\mu \cup \nu \cup \{1\}]\big) \geqslant 0.$$

By Lemma 1 we see that $\lim_{d_1 \to \infty} l((D_1 A)[\mu \cup \nu]) = l(C[\mu \cup \nu])$, and this finishes the proof, because in the limit (4) becomes (3). ■

We now consider Gaussian elimination applied to matrices in $V_{\langle n \rangle}$. We describe the elimination process (if it is possible) as follows: Let $A^{(1)} := [a_{ij}^{(1)}] := A$. In the first elimination step we multiply $A^{(1)}$ from the left by the matrix

$$L_1 = \begin{bmatrix} 1 & & & \\ -l_{21} & 1 & & 0 \\ \vdots & 0 & \ddots & \\ -l_{n1} & & & 1 \end{bmatrix},$$

where $l_{j1} = a_{j1}/a_{11}$ for $j = 2, \ldots, n$. Then

$$L_1 A := \left[\begin{array}{c|ccc} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ \hline 0 & & & \\ \vdots & & A^{(2)} & \\ 0 & & & \end{array}\right],$$

where $A^{(2)} = [a_{ij}^{(2)}]$. In the kth step $(k \geqslant 2)$ of the elimination process, we multiply

$$L_{k-1} \cdots L_1 A = \left[\begin{array}{cccc|c} a_{11}^{(1)} & \cdot & \cdot & \cdot & a_{1,k}^{(1)} \\ & \ddots & & & \vdots \\ & & \ddots & & \vdots \\ 0 & & & a_{k-1,k-1}^{(k-1)} & a_{k-1,n}^{(k-1)} \\ \hline & & 0 & & A^{(k)} \end{array}\right]$$

from the left by

$$L_k = \begin{bmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ 0 & & -l_{k+1,k} & & & & \\ & & \cdot & 1 & & & \\ & & \vdots & & \ddots & & \\ & & -l_{n,k} & & & & 1 \end{bmatrix},$$

where $l_{j,k} = a_{jk}^{(k)}/a_{k,k}^{(k)}$, and obtain

$$L_k \cdots L_1 A = \left[\begin{array}{cccc|ccc} a_{11}^{(1)} & \cdot & \cdots & & & \cdots & a_{1n}^{(1)} \\ & \ddots & & & & & \vdots \\ 0 & & & a_{k,k}^{(k)} & \cdot & \cdots & a_{k,n}^{(k)} \\ \hline & & 0 & & & A^{(k+1)} & \end{array}\right].$$

We now have the following first main result:

THEOREM 1. *Let* $A \in V_{\langle n \rangle}$, $A[\langle n-1 \rangle]$ *nonsingular. Then Gaussian elimination without pivoting applied to* A *is possible, and all matrices* $L_k L_{k-1} \cdots L_1 A$ *are in* $V_{\langle n \rangle}$ *too for* $k = 1, \dots, n$.

Proof. Since $A \in V_{\langle n \rangle} \subset \tau_{\langle n \rangle}$ and $A[\langle n-1 \rangle]$ is nonsingular, it follows that $l(A[\langle n-1 \rangle]) > 0$. Hence, all principal minors $A[\langle i \rangle]$, $i = 1, \dots, n-1$, are positive. (See, e.g., Remark 3.7 in Engel and Schneider [1, p. 161].) Thus by a well-known result (e.g., Theorem 2.5 in Stewart [8, p. 120]), Gaussian elimination without pivoting is possible. Thus it remains to show that $L_k \cdots L_1 A \in V_{\langle n \rangle}$ for all $k = 1, \dots, n-1$. We do this by induction on k.

$k = 1$: Looking at the above-described algorithm, we see that by construction, the matrix A_2 is equal to the matrix constructed from A via (2). Thus by Lemma 3, we have that $A^{(2)} \in V_{\langle n-1 \rangle}$, and then it follows immediately that $L_1 A \in V_{\langle n \rangle}$, since for any $\mu \subseteq \langle n \rangle$ with $1 \in \mu$, $\det(L_1 A)_t[\mu] = (a_{11}^{(1)} - t)\det A_t^{(2)}[\mu \setminus \{1\}]$ and $l(L_1 A[\mu]) = \min\{a_{11}^{(1)}, l(A[\mu \setminus \{1\}])\}$.

Assume now that for all k with $1 \leqslant k \leqslant i < n$ we have $L_k \cdots L_1 A \in V_{\langle n \rangle}$. This implies that $A^{(k+1)} \in V_{\langle n-k \rangle}$. Applying the same arguments as in the first step of the elimination on $A^{(k+1)}$, we see that

$$\begin{bmatrix} 1 & & & & 0 \\ -l_{k+2,k+1} & & & & \\ \cdot & 1 & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ -l_{n,k+1} & 0 & & & 1 \end{bmatrix} A^{(k+1)} \in V_{\langle n-k \rangle}.$$

But

$$L_{k+1} = I \oplus \begin{bmatrix} 1 & & & & 0 \\ -l_{k+2,k+1} & \cdot & & & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & 0 & & & \cdot \\ -l_{n,k+1} & & & & 1 \end{bmatrix},$$

and thus $B := L_{k+1} L_k \cdots L_1 A \in V_{\langle n \rangle}$. This follows now immediately since

$$\det B_t[\alpha] = \prod_{j \in \langle k+1 \rangle \cap \alpha} \left(a_{jj}^{(j)} - t \right) \det A^{(k+2)}[(\langle n \rangle \setminus \langle k+1 \rangle) \cap \alpha].$$

Thus in the inequality

$$\det B_t[\mu \cup \nu] \det B_t[\mu \cap \nu] \leqslant \det B_t[\mu] \det B_t[\nu],$$

the diagonal elements cancel. Hence, it remains only an inequality for subsets of $(\langle n \rangle \setminus \langle k+1 \rangle)$, and this inequality holds, since $A^{(k+2)} \in V_{\langle n-k-1 \rangle}$ for all $t \leqslant l(A[(\mu \cup \nu) \cap \langle n \rangle \setminus \langle k+1 \rangle])$. But

$$l(B[\mu \cup \nu]) = \min\left(\min_{j \in (\mu \cup \nu) \cap \langle k+1 \rangle} \left\{ a_{jj}^{(j)} \right\}, \right.$$
$$\left. l(A[(\mu \cup \nu) \cap (\langle n \rangle \setminus \langle k+1 \rangle)]) \right). \quad \blacksquare$$

This shows that $V_{\langle n \rangle}$ is invariant under Gaussian elimination without pivoting, but this doesn't hold for $\tau_{\langle n \rangle}$, as the following example shows: Let

$$A = \begin{bmatrix} 9 & 2 & 2 \\ 2 & 9 & \frac{1}{2} \\ 2 & \frac{1}{3} & 9 \end{bmatrix} \in \tau_{\langle n \rangle}.$$

After one step of elimination,

$$L_1 A = \begin{bmatrix} 9 & 2 & 2 \\ 0 & \frac{77}{9} & \frac{1}{18} \\ 0 & -\frac{1}{9} & \frac{77}{9} \end{bmatrix},$$

which is obviously not in $\tau_{\langle n \rangle}$, since $(L_1 A)_{23}(L_1 A)_{32} < 0$.

REMARK 1. Obviously, by looking at the determinantal inequality (1), any lower or upper triangular matrix with nonnegative diagonal elements is in $V_{\langle n \rangle}$. At this point one sees the great usefulness of a determinantal characterization for classes of matrices, like those for τ-matrices introduced in [1] or for V-matrices introduced in [6], although in many places it looks a lot more complicated than other characterizations.

COROLLARY 1. *Let $A \in V_{\langle n \rangle}$ and $\exists j \in \langle n \rangle$ such that $A[\langle n \rangle \setminus \{j\}]$ is nonsingular. Then there exists a permutation matrix P such that any matrix $L_k \cdots L_1 APAP^T$ in the Gaussian elimination process applied to PAP^T is again in $V_{\langle n \rangle}$.*

Proof. Choose P such that $\det(PAP^T)[\langle n-1 \rangle] \neq 0$. Since P is a permutation matrix, PAP^T is again in $V_{\langle n \rangle}$. Thus we can apply Theorem 1. ■

COROLLARY 2. *Let $A \in V_{\langle n \rangle}$ be nonsingular. Then, for any permutation matrix P, any matrix $L_k \cdots L_1 APAP^T$ in the Gaussian elimination process applied to PAP^T is again in $V_{\langle n \rangle}$.*

Proof. Since $A \in V_{\langle n \rangle}$ is nonsingular, then $A[\langle n \rangle \setminus \{j\}]$ is nonsingular for all $j \in \langle n \rangle$. Hence we can apply Theorem 1 to PAP^T for any P. ■

REMARK 2. For the M-matrices it was shown by Kuo [5], based on a result in Fiedler and Pták [3], that the irreducible M-matrices are closed with respect to Gaussian elimination. This doesn't hold for $V_{\langle n \rangle}$. For M-matrices one can use the Perron-Frobenius theorem to get that if $A \in M_{\langle n \rangle}$ is irreducible and singular, then still all principal minors of order $n-1$ are nonzero.

But this does not work for $V_{\langle n \rangle}$, as the following example shows. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in V_{\langle n \rangle}.$$

A is singular and irreducible, but $\det A = 0$ and $\det A[\{1,2\}] = 0$.

Now we consider the case of a singular $A \in V_{\langle n \rangle}$ more deeply.

THEOREM 2. *Let $A \in V_{\langle n \rangle}$, A singular. Then there exist a permutation P and $k \in \langle n \rangle \cup \{0\}$ such that PAP^T has an LU decomposition with*

$$L = \left[\begin{array}{c|c} L_{11} & 0 \\ \hline L_{21} & 1 \end{array}\right] \begin{array}{l} \}\, k \\ \}\, n-k \end{array}$$

nonsingular, unit diagonal, lower triangular, and with

$$U = \left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline 0 & U_{22} \end{array}\right] \begin{array}{l} \}\, k \\ \}\, n-k \end{array}$$

upper triangular and such that U_{11} is nonsingular and U_{22} has only zeros on the diagonal. Furthermore, the first k elimination steps suffice to produce this decomposition, and $L_j L_{j-1} \cdots L_1 A \in V_{\langle n \rangle}$ for all $j = 1, \dots, k$.

Proof. Let $\mu \subseteq \langle n \rangle$ be a subset of maximal cardinality such that $\det A[\mu] \neq 0$. Let $|\mu| = k$. If such a set does not exist, then set $k = 0$.

(1) $k \neq 0$. Then there exists a permutation matrix P_1 such that $\det P_1 A P_1^T[\langle k \rangle] \neq 0$ and $\det P_1 A P_1^T[\langle k \rangle \cup \{j\}] = 0$ for all $j \in \langle n \rangle \setminus \langle k \rangle$. Applying Gaussian elimination to $P_1 P A_1^T$, we can do this up to step k, and in any step the matrix is again in $V_{\langle n \rangle}$ by Theorem 1. After the kth step, the matrix $A^{(k+1)}$ has the form

$$\left[\begin{array}{cccccc} a_{11}^{(1)} & \cdot & \cdot & \cdot & \cdot & a_{1n}^{(k)} \\ & \ddots & & & & \vdots \\ 0 & & a_{kk}^{(k)} & \cdot & \cdot & a_{kn}^{(k)} \\ \hline & 0 & & & A^{(k+1)} & \end{array}\right].$$

By the fact that $P_1AP_1^T[\langle k\rangle \cup \{j\}] = 0 \ \forall j \in \langle n\rangle \setminus \langle k\rangle$ it follows that all diagonal elements of $A^{(k+1)}$ are zero. If this were not the case, one could say there exists a $j \in \langle n\rangle \setminus \langle k\rangle$ such that the (j, j) diagonal element of $A^{(k+1)}$ is nonzero, and then take a permutation matrix P_2 interchanging the jth row with the $(k+1)$st row; this would mean that in applying Gaussian elimination to $P_2P_1AP_1^TP_2^T$ the elimination would be possible up to the $(k+1)$st step. But the abovementioned result (e.g. Stewart [8, p. 120]) would imply that $\det(P_2P_1AP_1^TP_2^T)[\langle k+1\rangle] \neq 0$, which contradicts the maximal cardinality of the above-chosen set μ.

Thus for $A^{(k+1)} = [b_{ij}] \in V_{\langle n-k\rangle}$, we have that all diagonal elements are zero. But since $A^{(k+1)} \in V_{\langle n-k\rangle}$, it then follows that $\det A^{(k+1)}[\nu] = 0 \ \forall \nu \subseteq \langle n - k\rangle$. This implies that all cyclic products $b_{i_1,i_2} \cdots b_{i_p,i_1}$ of all orders p in $A^{(k+1)}$ are zero. But this implies that the index set $\{k+1,\dots,n\}$ can be ordered by a total order $\lesseqqgtr$ in such a way that $\forall i, j \in \langle n\rangle \setminus \langle k\rangle$, $a_{ij} \neq 0$ implies $i \lesseqqgtr j$. Thus there exists a permutation P_2 that permutes $A^{(k+1)}$ to upper triangular form with zeros in the diagonal.

The permutation matrix

$$\tilde{P}_2 = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & P_2 \end{array}\right] \begin{array}{l} \}\, k \\ \}\, n-k \end{array}$$

does not affect $(P_1AP_1^T)[\langle k\rangle]$. Therefore we can apply Gaussian elimination up to step k to $\tilde{P}_2P_1AP_1^T\tilde{P}_2$ and obtain the required form.

(2) $k = 0$. In this case we cannot do any elimination step, and applying the second part of the argument in (1), we just have to take the permutation that brings A into upper triangular form with zeros in the diagonal. The proof is completed by taking $L = I$. ∎

We wish to thank Professor H. Schneider for proposing the work on the above subject.

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Received 28 March 1983; revised 13 June 1983