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Partial cubes: Structures, characterizations, and constructions

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Abstract

Partial cubes are isometric subgraphs of hypercubes. Structures on a graph defined by means of semicubes, and Djoković's and Winkler's relations play an important role in the theory of partial cubes. These structures are employed in the paper to characterize bipartite graphs and partial cubes of arbitrary dimension. New characterizations are established and new proofs of some known results are given.

The operations of Cartesian product and pasting, and expansion and contraction processes are utilized in the paper to construct new partial cubes from old ones. In particular, the isometric and lattice dimensions of finite partial cubes obtained by means of these operations are calculated.

Keywords: Hypercube; Partial cube; Semicube

1. Introduction

A hypercube $\mathcal{H}(X)$ on a set X is a graph whose vertices are the finite subsets of X; two vertices are joined by an edge if they differ by a singleton. A partial cube is a graph that can be isometrically embedded into a hypercube.

There are three general graph-theoretical structures that play a prominent role in the theory of partial cubes; namely, semicubes, Djoković's relation θ, and Winkler's relation Θ. We use these structures, in particular, to characterize bipartite graphs and partial cubes. The characterization problem for partial cubes was considered as an important one and many characterizations are known. We list contributions in the chronological order: Djoković [9] (1973), Avis [2] (1981), Winkler [21] (1984), Roth and Winkler [18] (1986), Chepoi [6,7] (1988 and 1994). In the paper, we present new proofs for the results of Djoković [9], Winkler [21], and Chepoi [6], and obtain two more characterizations of partial cubes.

The paper is also concerned with some ways of constructing new partial cubes from old ones. Properties of subcubes, the Cartesian product of partial cubes, and expansion and contraction of a partial cube are investigated. We introduce a construction based on pasting two graphs together and show how new partial cubes can be obtained from old ones by pasting them together.

The paper is organized as follows.

Hypercubes and partial cubes are introduced in Section 2 together with two basic examples of infinite partial cubes. Vertex sets of partial cubes are described in terms of well-graded families of finite sets.

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In Section 3 we introduce the concepts of a semicube, Djoković's θ and Winkler's Θ relations, and establish some of their properties. Bipartite graphs and partial cubes are characterized by means of these structures. One more characterization of partial cubes is obtained in Section 4, where so-called fundamental sets in a graph are introduced.

The rest of the paper is devoted to constructions: subcubes and the Cartesian product (Section 6), pasting (Section 7), and expansions and contractions (Section 8). We show that these constructions produce new partial cubes from old ones. Isometric and lattice dimensions of new partial cubes are calculated. These dimensions are introduced in Section 5.

Few details about the conventions used in the paper are given as follows. The *sum (disjoint union)* $A + B$ of two sets A and B is the union

$$(\{1\} \times A) \cup (\{2\} \times B).$$

All graphs in the paper are simple undirected graphs. In the notation $G = (V, E)$, the symbol V stands for the set of vertices of the graph G and E stands for its set of edges. By abuse of language, we often write ab for an edge in a graph; if this is the case, ab is an unordered pair of distinct vertices. We denote by $\langle U \rangle$ the graph induced by the set of vertices $U \subseteq V$. If G is a connected graph, then $d_G(a, b)$ stands for the distance between two vertices a and b of the graph G. Wherever it is clear from the context which graph is under consideration, we drop the subscript G in $d_G(a, b)$. A subgraph $H \subseteq G$ is an *isometric subgraph* if $d_H(a, b) = d_G(a, b)$ for all vertices a and b of H; it is *convex* if any shortest path in G between vertices of H belongs to H.

2. Hypercubes and partial cubes

Let X be a set. We denote by $\mathcal{P}_f(X)$ the set of all finite subsets of X.

Definition 2.1. A graph $\mathcal{H}(X)$ has the set $\mathcal{P}_f(X)$ as the set of its vertices; a pair of vertices PQ are an edge of $\mathcal{H}(X)$ if the symmetric difference $P \Delta Q$ is a singleton. The graph $\mathcal{H}(X)$ is called the *hypercube on* X [9]. If X is a finite set of cardinality n, then the graph $\mathcal{H}(X)$ is the *n-cube* Q_n. The *dimension* of the hypercube $\mathcal{H}(X)$ is the cardinality of the set X.

The shortest path distance $d(P, Q)$ on the hypercube $\mathcal{H}(X)$ is the *Hamming distance* between sets P and Q:

$$d(P, Q) = |P \Delta Q| \quad \text{for } P, Q \in \mathcal{P}_f. \tag{2.1}$$

The set $\mathcal{P}_f(X)$ is a metric space with the metric d.

Definition 2.2. A graph G is a *partial cube* if it can be isometrically embedded into a hypercube $\mathcal{H}(X)$ for some set X. We often identify G with its isometric image in the hypercube $\mathcal{H}(X)$, and say that G is a *partial cube on the set* X.

Clearly, a family $\mathcal{F}$ of finite subsets of X induces a partial cube on X if and only if for any two distinct subsets $P, Q \in \mathcal{F}$ there is a sequence

$$R_0 = P, R_1, \ldots, R_n = Q$$

of sets in $\mathcal{F}$ such that

$$d(R_i, R_{i+1}) = 1 \quad \text{for all } 0 \leq i < n, \quad \text{and} \quad d(P, Q) = n. \tag{2.2}$$

The families of sets satisfying condition (2.2) are known as well-graded families of sets [10]. Note that a sequence (R_i) satisfying (2.2) is a shortest path from P to Q in $\mathcal{H}(X)$ (and in the subgraph induced by $\mathcal{F}$).

Definition 2.3. A family $\mathcal{F}$ of arbitrary subsets of X is a *wg-family* (*well-graded family of sets*) if, for any two distinct subsets $P, Q \in \mathcal{F}$, the set $P \Delta Q$ is finite and there is a sequence

$$R_0 = P, R_1, \ldots, R_n = Q$$

of sets in $\mathcal{F}$ such that $|R_i \Delta R_{i+1}| = 1$ for all $0 \leq i < n$ and $|P \Delta Q| = n$.

Any family $\mathcal{F}$ of subsets of X defines a graph $G_{\mathcal{F}} = (\mathcal{F}, E_{\mathcal{F}})$, where

$$E_{\mathcal{F}} = \{\{P, Q\} \subseteq \mathcal{F} : |P \Delta Q| = 1\}.$$

Theorem 2.1. *The graph $G_{\mathcal{F}}$ defined by a family $\mathcal{F}$ of subsets of a set X is isomorphic to a partial cube on X if and only if the family $\mathcal{F}$ is well-graded.*

Proof. We need to prove sufficiency only. Let S be a fixed set in $\mathcal{F}$. We define a mapping $f : \mathcal{F} \to \mathcal{P}_f(X)$ by $f(R) = R \Delta S$ for $R \in \mathcal{F}$. Then

$$d(f(R), f(T)) = |(R \Delta S) \Delta (T \Delta S)| = |R \Delta T|.$$

Thus f is an isometric embedding of $\mathcal{F}$ into $\mathcal{P}_f(X)$. Let (R_i) be a sequence of sets in $\mathcal{F}$ such that $R_0 = P$, $R_n = Q$, $|P \Delta Q| = n$, and $|R_i \Delta R_{i+1}| = 1$ for all $0 \leq i < n$. Then the sequence $(f(R_i))$ satisfies conditions (2.2). The result follows. □

A set $R \in \mathcal{P}_f(X)$ is said to be *lattice between* sets $P, Q \in \mathcal{P}_f(X)$ if

$$P \cap Q \subseteq R \subseteq P \cup Q.$$

It is *metrically between* P and Q if

$$d(P, R) + d(R, Q) = d(P, Q).$$

The following theorem is a well-known result about these two betweenness relations on $\mathcal{P}_f(X)$ (see, for instance, [3]).

Theorem 2.2. *Lattice and metric betweenness relations coincide on $\mathcal{P}_f(X)$.*

Let $\mathcal{F}$ be a family of finite subsets of X. The set of all $R \in \mathcal{F}$ that are between $P, Q \in \mathcal{F}$ is the *interval* $\mathcal{I}(P, Q)$ *between P and Q in* $\mathcal{F}$. Thus,

$$\mathcal{I}(P, Q) = \mathcal{F} \cap [P \cap Q, P \cup Q],$$

where $[P \cap Q, P \cup Q]$ is the usual interval in the lattice $\mathcal{P}_f$.

Two distinct sets $P, Q \in \mathcal{F}$ are *adjacent in* $\mathcal{F}$ if $\mathcal{J}(P, Q) = \{P, Q\}$. If sets P and Q form an edge in the graph induced by $\mathcal{F}$, then P and Q are adjacent in $\mathcal{F}$, but, generally speaking, not vice versa.

The following theorem is a 'local' characterization of wg-families of sets.

Theorem 2.3. *A family $\mathcal{F} \subseteq \mathcal{P}_f(X)$ is well-graded if and only if $d(P, Q) = 1$ for any two sets P and Q that are adjacent in $\mathcal{F}$.*

Proof. (Necessity) Let $\mathcal{F}$ be a wg-family of sets. Suppose that P and Q are adjacent in $\mathcal{F}$. There is a sequence $R_0 = P, R_1, \ldots, R_n = Q$ that satisfies conditions (2.2). Since the sequence (R_i) is a shortest path in $\mathcal{F}$, we have

$$d(P, P_i) + d(P_i, Q) = d(P, Q) \quad \text{for all } 0 \leq i \leq n.$$

Thus, $P_i \in \mathcal{I}(P, Q) = \{P, Q\}$. It follows that $d(P, Q) = n = 1$.

(Sufficiency) Let P and Q be two distinct sets in $\mathcal{F}$. We prove by induction on $n = d(P, Q)$ that there is a sequence $(R_i) \in \mathcal{F}$ satisfying conditions (2.2).

The statement is trivial for $n = 1$. Suppose that $n > 1$ and that the statement is true for all $k < n$. Let P and Q be two sets in $\mathcal{F}$ such that $d(P, Q) = n$. Since $d(P, Q) > 1$, the sets P and Q are not adjacent in $\mathcal{F}$. Therefore there exists $R \in \mathcal{F}$ that lies between P and Q and is distinct from these two sets. Then $d(P, R) + d(R, Q) = d(P, Q)$ and both distances $d(P, R)$ and $d(R, Q)$ are less than n. By the induction hypothesis, there is a sequence $(R_i) \in \mathcal{F}$ such that

$$P = R_0, \quad R = R_j, \qquad Q = R_n \quad \text{for some } 0 < j < n,$$

satisfying conditions (2.2) for $0 \leq i < j$ and $j \leq i < n$. It follows that $\mathcal{F}$ is a wg-family of sets. □

We conclude this section with an example of an infinite partial cube (more examples are found in [17]).

Example 2.1. Let us consider $\mathbb{Z}^n$ as a metric space with respect to the ℓ_1-metric. The graph $\mathcal{Z}^n$ has $\mathbb{Z}^n$ as the vertex set; two vertices in $\mathcal{Z}^n$ are connected if they are at unit distance from each other. We will show in Section 6 (Corollary 6.1) that $\mathcal{Z}^n$ is a partial cube.

3. Characterizations

Only connected graphs are considered in this section.

Definition 3.1. Let $G = (V, E)$ be a graph and d be its distance function. For any two adjacent vertices $a, b \in V$ let W_{ab} be the set of vertices that are closer to a than to b:

$$W_{ab} = \{w \in V : d(w, a) < d(w, b)\}.$$

Following [11], we call the sets W_{ab} and induced subgraphs $\langle W_{ab} \rangle$ *semicubes* of the graph G. The semicubes W_{ab} and W_{ba} are called *opposite semicubes*.

Remark 3.1. The subscript ab in W_{ab} stands for an ordered pair of vertices, not for an edge of G. In his original paper [9], Djoković uses notation $G(a, b)$ (cf. [8]). We use the notation from [15].

Clearly, two opposite semicubes are disjoint. They can be used to characterize bipartite graphs as follows.

Theorem 3.1. *A graph $G = (V, E)$ is bipartite if and only if the semicubes W_{ab} and W_{ba} form a partition of V for any edge $ab \in E$.*

Proof. Let us recall that a connected graph G is bipartite if and only if for every vertex x there is no edge ab with $d(x, a) = d(x, b)$ (see, for instance, [1]). For any edge $ab \in E$ and vertex $x \in V$ we clearly have

$$d(x, a) = d(x, b) \Leftrightarrow x \notin W_{ab} \cup W_{ba}.$$

The result follows. □

The following lemma is instrumental and will be used frequently in the rest of the paper.

Lemma 3.1. *Let $G = (V, E)$ be a graph and $w \in W_{ab}$ for some edge $ab \in E$. Then*

$$d(w, b) = d(w, a) + 1.$$

Accordingly,

$$W_{ab} = \{w \in V : d(w, b) = d(w, a) + 1\}.$$

Proof. By the triangle inequality, we have

$$d(w, a) < d(w, b) \leq d(w, a) + d(a, b) = d(w, a) + 1.$$

The result follows, since d takes values in $\mathbb{N}$. □

There are two binary relations on the set of edges of a graph that play a central role in characterizing partial cubes.

Definition 3.2. Let $G = (V, E)$ be a graph and $e = xy$ and $f = uv$ be two edges of G.

(i) (Djoković [9]) The relation θ on E is defined by

$$e \theta f \Leftrightarrow f \quad \text{joins a vertex in } W_{xy} \text{ with a vertex in } W_{yx}.$$

The notation can be chosen such that $u \in W_{xy}$ and $v \in W_{yx}$.

(ii) (Winkler [21]) The relation Θ on E is defined by

$$e \Theta f \Leftrightarrow d(x, u) + d(y, v) \neq d(x, v) + d(y, u).$$

It is clear that both relations θ and Θ are reflexive and Θ is symmetric. The results of the next two lemmas are straightforward consequences of Lemma 3.1 (see [20]).

Lemma 3.2. *The relation θ is a symmetric relation on E.*

Lemma 3.3. $\theta \subseteq \Theta$.

Example 3.1. It is easy to verify that θ is the identity relation on the set of edges of the cycle C_3. On the other hand, any two edges of C_3 stand in the relation Θ. Thus, $\theta \neq \Theta$ in this case.

Bipartite graphs can be characterized in terms of relations θ and Θ as follows.

Theorem 3.2. *A graph $G = (V, E)$ is bipartite if and only if $\theta = \Theta$.*

Proof. (Necessity) Suppose that G is a bipartite graph, two edges xy and uv stand in the relation Θ, that is,

$$d(x, u) + d(y, v) \neq d(x, v) + d(y, u),$$

and that edges xy and uv do not stand in the relation θ. By Theorem 3.1, we may assume that $u, v \in W_{xy}$. By Lemma 3.1, we have

$$d(x, u) + d(y, v) = d(y, u) - 1 + d(x, v) + 1 = d(x, v) + d(y, u),$$

a contradiction. It follows that $\Theta \subseteq \theta$. By Lemma 3.3, $\theta = \Theta$.

(Sufficiency) Suppose that G is not bipartite. By Theorem 3.1, there is an edge xy such that $W_{xy} \cup W_{yx}$ is a proper subset of V. Since G is connected, there is an edge uv with $u \notin W_{xy} \cup W_{yx}$ and $v \in W_{xy} \cup W_{yx}$. Clearly, uv does not stand in the relation θ to xy. On the other hand,

$$d(x, u) + d(y, v) \neq d(x, v) + d(y, u),$$

since $u \notin W_{xy} \cup W_{yx}$ and $v \in W_{xy} \cup W_{yx}$. Thus, $xy\, \Theta\, uv$, a contradiction, since we assumed that $\theta = \Theta$. □

By Theorem 3.2, the relations θ and Θ coincide on bipartite graphs. For this reason we use the relation θ in the rest of the paper.

Lemma 3.4. *Let $G = (V, E)$ be a bipartite graph such that all its semicubes are convex sets. Then two edges xy and uv stand in the relation θ if and only if the corresponding pairs of mutually opposite semicubes form equal partitions of V:*

$$xy\theta uv \Leftrightarrow \{W_{xy}, W_{yx}\} = \{W_{uv}, W_{vu}\}.$$

Proof. (Necessity) We assume that the notation is chosen such that $u \in W_{xy}$ and $v \in W_{yx}$. Let $z \in W_{xy} \cap W_{vu}$. By Lemma 3.1, $d(z, u) = d(z, v) + d(v, u)$. Since $z, u \in W_{xy}$ and W_{xy} is convex, we have $v \in W_{xy}$, a contradiction to the assumption that $v \in W_{yx}$. Thus $W_{xy} \cap W_{vu} = \varnothing$. Since two opposite semicubes in a bipartite graph form a partition of V, we have $W_{uv} = W_{xy}$ and $W_{vu} = W_{yx}$.

A similar argument shows that $W_{uv} = W_{yx}$ and $W_{vu} = W_{xy}$, if $u \in W_{yx}$ and $v \in W_{xy}$.

(Sufficiency) Follows from the definition of the relation θ. □

We need another general property of the relation θ (cf. Lemma 2.2 in [15]).

Lemma 3.5. *Let P be a shortest path in a graph G. Then no two distinct edges of P stand in the relation θ.*

Proof. Let $i < j$ and $x_i x_{i+1}$ and $x_j x_{j+1}$ be two edges in a shortest path P from x_0 to x_n. Then

$$d(x_i, x_j) < d(x_i, x_{j+1}) \quad \text{and} \quad d(x_{i+1}, x_j) < d(x_{i+1}, x_{j+1}),$$

so $x_i, x_{i+1} \in W_{x_j x_{j+1}}$. It follows that edges $x_i x_{i+1}$ and $x_j x_{j+1}$ do not stand in the relation θ. □

Lemma 3.6. *Let $G = (V, E)$ be a bipartite graph. The following statements are equivalent*

(i) *All semicubes of G are convex.*
(ii) *The relation θ is an equivalence relation on E.*

Proof. (i) $\Rightarrow$ (ii). Follows from Lemma 3.4.

(ii) $\Rightarrow$ (i). Suppose that θ is transitive and there is a nonconvex semicube W_{ab}. Then there are two vertices $u, v \in W_{ab}$ and a shortest path P from u to v that intersects W_{ba}. This path contains two distinct edges e and f joining vertices of semicubes W_{ab} and W_{ba}. The edges e and f stand in the relation θ to the edge ab. By transitivity of θ, we have $e\theta f$. This contradicts the result of Lemma 3.5. Thus all semicubes of G are convex. □

We now establish some basic properties of partial cubes (cf. [9]).

Theorem 3.3. *Let $G = (V, E)$ be a partial cube. Then*

(i) *G is a bipartite graph.*
(ii) *Each pair of opposite semicubes forms a partition of V.*
(iii) *All semicubes are convex subsets of V.*
(iv) *θ is an equivalence relation on E.*

Proof. We may assume that G is an isometric subgraph of some hypercube $\mathcal{H}(X)$, that is, $G = (\mathcal{F}, E_{\mathcal{F}})$ for a wg-family $\mathcal{F}$ of finite subsets of X. Clearly, G is a bipartite graph.

(ii) Follows from (i) and Theorem 3.1.
(iii) Let W_{AB} be a semicube of G. By Lemma 3.1 and Theorem 2.2, we have

$$W_{AB} = \{S \in \mathcal{F} : S \cap B \subseteq A \subseteq S \cup B\}.$$

Let $Q, R \in W_{AB}$ and P be a vertex of G such that

$$d(Q, P) + d(P, R) = d(Q, R).$$

By Theorem 2.2,

$$Q \cap R \subseteq P \subseteq Q \cup R.$$

Since $Q, R \in W_{AB}$, we have

$$Q \cap B \subseteq A \subseteq Q \cup B \quad \text{and} \quad R \cap B \subseteq A \subseteq R \cup B,$$

which implies

$$P \cap B \subseteq (Q \cup R) \cap B \subseteq A \subseteq (Q \cap R) \cup B \subseteq S \cup B.$$

Hence, $P \in W_{AB}$, and the result follows.
(iv) Follows from (iii) and Lemma 3.6. □

Remark 3.2. Since semicubes of a partial cube $G = (V, E)$ are convex subsets of the metric space V, they are *half-spaces* in V [19]. This terminology is used in [6,7].

The following theorem presents four characterizations of partial cubes. Characterizations (ii) and (iii) are due to Djoković [9] and Winkler [21] (cf. Theorem 2.10 in [15]), and (iv) is due to Wilkeit [20].

Theorem 3.4. *Let $G = (V, E)$ be a connected graph. The following statements are equivalent:*

(i) *G is a partial cube.*
(ii) *G is bipartite and all semicubes of G are convex.*
(iii) *G is bipartite and θ is an equivalence relation.*
(iv) *G is bipartite and, for all $xy, uv \in E$,*

$$xy\theta uv \Rightarrow \{W_{xy}, W_{yx}\} = \{W_{uv}, W_{vu}\}. \tag{3.1}$$

(v) *G is bipartite and, for any pair of adjacent vertices of G, there is a unique pair of opposite semicubes separating these two vertices.*

Proof. By Lemma 3.6, the statements (ii) and (iii) are equivalent and, by Theorem 3.3, (i) implies both (ii) and (iii).

(iii) $\Rightarrow$ (i). By Theorem 3.1, each pair $\{W_{ab}, W_{ba}\}$ of opposite semicubes of G forms a partition of V. We orient these partitions by calling, in an arbitrary way, one of the two opposite semicubes in each partition a *positive semicube*. Let us assign to each $x \in V$ the set $W^+(x)$ of all positive semicubes containing x. In the next paragraph we prove that the family $\mathcal{F} = \{W^+(x)\}_{x \in V}$ is well-graded and that the assignment $x \mapsto W^+(x)$ is an isometry between V and $\mathcal{F}$.

Let x and y be two distinct vertices of G. We say that a positive semicube W_{ab} *separates* x and y if either $x \in W_{ab},\ y \in W_{ba}$ or $x \in W_{ba},\ y \in W_{ab}$. It is clear that W_{ab} separates x and y if and only if $W_{ab} \in W^+(x) \Delta W^+(y)$. Let P be a shortest path $x_0 = x, x_1, \ldots, x_n = y$ from x to y. By Lemma 3.5, no two distinct edges of P stand in the relation θ. By Lemma 3.4, distinct edges of P define distinct positive semicubes; clearly, these semicubes separate x and y. Let W_{ab} be a positive semicube separating x and y, and, say, $x \in W_{ab}$ and $y \in W_{ba}$. There is an edge $f \in P$ that joins vertices in W_{ab} and W_{ba}. Hence, f stands in the relation θ to ab and, by Lemma 3.4, W_{ab} is defined by f. It follows that any semicube in $W^+(x) \Delta W^+(y)$ is defined by a unique edge in P and any edge in P defines a semicube in $W^+(x) \Delta W^+(y)$. Therefore, $d(W^+(x), W^+(y)) = d(x, y)$, that is $x \mapsto W^+(x)$ is an isometry. Clearly, $\mathcal{F}$ is a wg-family of sets.

By Theorem 2.1, the family $\mathcal{F}$ is isometric to a wg-family of finite sets. Hence, G is a partial cube.

(iv) $\Rightarrow$ (ii). Suppose that there exists an edge ab such that semicube W_{ba} is not convex. Let p and q be two vertices in W_{ba} such that there is a shortest path P from p to q that intersects W_{ab}. There are two distinct edges xy and uv in P such that $x, u \in W_{ab}$ and $y, v \in W_{ba}$. Since $ab\theta xy$ and $ab\theta uv$, we have, by (3.1),

$$W_{ab} = W_{xy} = W_{uv}.$$

Hence, $u \in W_{xy}$ and $v \in W_{yx}$. By Lemma 3.1,

$$d(x, u) = d(x, v) - 1 = 1 + d(v, y) - 1 = d(v, y),$$

a contradiction, since P is a shortest path from p to q.

(ii) $\Rightarrow$ (iv). Follows from Lemma 3.4.

It is clear that (iv) and (v) are equivalent. $\square$

4. Fundamental sets in partial cubes

Semicubes played an important role in the previous section. In this section we introduce three more classes of useful subsets of graphs. We also establish one more characterization of partial cubes.

Let $G = (V, E)$ be a connected graph. For a given edge $e = ab \in E$, we define the following sets (cf. [15,16]):

$$F_{ab} = \{f \in E : e\theta f\} = \{uv \in E : u \in W_{ab}, v \in W_{ba}\},$$
$$U_{ab} = \{w \in W_{ab} : w \text{ is adjacent to a vertex in } W_{ba}\},$$
$$U_{ba} = \{w \in W_{ba} : w \text{ is adjacent to a vertex in } W_{ab}\}.$$

The five sets are schematically shown in Fig. 4.1.

Remark 4.1. In the case of a partial cube $G = (V, E)$, the semicubes W_{ab} and W_{ba} are complementary half-spaces in the metric space V (cf. Remark 3.2). Then the set F_{ab} can be regarded as a 'hyperplane' separating these half-spaces (see [17] where this analogy is formalized in the context of hyperplane arrangements).

The following theorem generalizes the result obtained in [16] for median graphs (see also [15]).

Theorem 4.1. *Let ab be an edge of a connected bipartite graph G. If the semicubes W_{ab} and W_{ba} are convex, then the set F_{ab} is a matching and induces an isomorphism between the graphs $\langle U_{ab}\rangle$ and $\langle U_{ba}\rangle$.*

Proof. Suppose that F_{ab} is not a matching. Then there are distinct edges xu and xv with, say, $x \in U_{ab}$ and $u, v \in U_{ba}$. By the triangle inequality, $d(u, v) \leq 2$. Since G does not have triangles, $d(u, v) \neq 1$. Hence, $d(u, v) = 2$, which implies that x lies between u and v. This contradicts the convexity of W_{ba}, since $x \in W_{ab}$. Therefore F_{ab} is a matching.

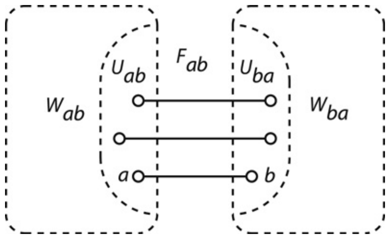

Fig. 4.1. Fundamental sets in a partial cube.

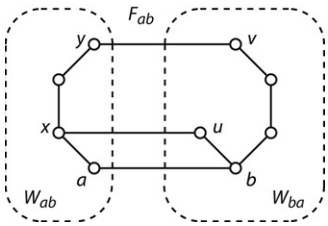

Fig. 4.2. Graph G.

To show that F_{ab} induces an isomorphism, let $xy, uv \in F_{ab}$ and $xu \in E$, where $x, u \in U_{ab}$ and $y, v \in U_{ba}$. Since G does not have odd cycles, $d(v, y) \neq 2$. By the triangle inequality,

$$d(v, y) \leq d(v, u) + d(u, x) + d(x, y) = 3.$$

Since W_{ba} is convex, $d(v, y) \neq 3$. Thus $d(v, y) = 1$, that is, vy is an edge. The result follows by symmetry. □

By Theorem 3.4(ii), we have the following corollary.

Corollary 4.1. *Let $G = (V, E)$ be a partial cube. For any edge ab the set F_{ab} is a matching and induces an isomorphism between induced graphs $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$.*

Example 4.1. Let G be the graph depicted in Fig. 4.2. The set

$$F_{ab} = \{ab, xu, yv\}$$

is a matching and defines an isomorphism between the graphs induced by subsets $U_{ab} = \{a, x, y\}$ and $U_{ba} = \{b, u, v\}$. The set W_{ba} is not convex, so G is not a partial cube. Thus the converse of Corollary 4.1 does not hold.

We now establish another characterization of partial cubes that utilizes a geometric property of families F_{ab}.

Theorem 4.2. *For a connected graph G the following statements are equivalent:*

(i) *G is a partial cube.*
(ii) *G is bipartite and*

$$d(x, u) = d(y, v) \quad \text{and} \quad d(x, v) = d(y, u), \tag{4.1}$$

for any $ab \in E$ and $xy, uv \in F_{ab}$.

Proof. (i) ⇒ (ii). We may assume that $x, u \in W_{ab}$ and $y, v \in W_{ba}$. Since θ is an equivalence relation, we have $xy\theta uv\theta ab$. By Lemma 3.4, $W_{uv} = W_{xy} = W_{ab}$. By Lemma 3.1,

$$d(x, u) = d(x, v) - 1 = d(v, y) + 1 - 1 = d(y, v).$$

We also have

$$d(x, v) = d(y, v) + 1 = d(y, u),$$

by the same lemma.

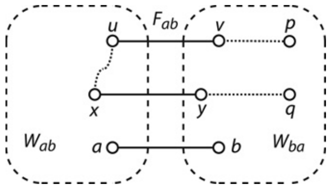

Fig. 4.3. An illustration of the proof of Theorem 4.2.

(ii) ⇒ (i). Suppose that G is not a partial cube. Then, by Theorem 3.4, there exists an edge ab such that, say, semicube W_{ba} is not convex. Let p and q be two vertices in W_{ba} such that there is a shortest path P from p to q that intersects W_{ab}. Let uv be the first edge in P which belongs to F_{ab} and xy be the last edge in P with the same property (see Fig. 4.3).

Since P is a shortest path, we have

$$d(v, y) = d(v, u) + d(u, x) + d(x, y) \neq d(x, u),$$

which contradicts condition (4.1). Thus all semicubes of G are convex. By Theorem 3.4, G is a partial cube. □

5. Dimensions of partial cubes

There are many different ways in which a given partial cube can be isometrically embedded into a hypercube. For instance, the graph K_2 can be isometrically embedded in different ways into any hypercube $\mathcal{H}(X)$ with $|X| > 2$.

Following Djoković [9] (see also [8]), we define the *isometric dimension*, $\dim_I(G)$, of a partial cube G as the minimum possible dimension of a hypercube $\mathcal{H}(X)$ in which G is isometrically embeddable. Recall (see Section 2) that the dimension of $\mathcal{H}(X)$ is the cardinality of the set X.

Theorem 5.1 (*Theorem 2 in [9]*). *Let $G = (V, E)$ be a partial cube. Then*

$$\dim_I(G) = |E/\theta|, \tag{5.1}$$

where θ is Djoković's equivalence relation on E and E/θ is the set of its equivalence classes (the quotient-set).

The quotient-set E/θ can be identified with the family of all distinct sets F_{ab} (see Section 4). If G is a finite partial cube, we may consider it as an isometric subgraph of some hypercube Q_n. Then the edges in each family F_{ab} are parallel edges in Q_n (cf. Theorem 4.2). This observation essentially proves (5.1) in the finite case.

Let G be a partial cube on a set X. The vertex set of G is a wg-family $\mathcal{F}$ of finite subsets of X (see Section 2). We define the *retraction* of $\mathcal{F}$ as a family $\mathcal{F}'$ of subsets of $X' = \cup\mathcal{F} \setminus \cap\mathcal{F}$ consisting of the intersections of sets in $\mathcal{F}$ with X'. It is clear that $\mathcal{F}'$ satisfies conditions

$$\cap\,\mathcal{F}' = \varnothing \quad \text{and} \quad \cup\,\mathcal{F}' = X'. \tag{5.2}$$

Proposition 5.1. *The partial cubes induced by a wg-family $\mathcal{F}$ and its retraction $\mathcal{F}'$ are isomorphic.*

Proof. It suffices to prove that metric spaces $\mathcal{F}$ and $\mathcal{F}'$ are isometric. Clearly, $\alpha : P \mapsto P \cap X'$ is a mapping from $\mathcal{F}$ onto $\mathcal{F}'$. For $P, Q \in \mathcal{F}$, we have

$$(P \cap X')\Delta(Q \cap X') = (P\Delta Q) \cap X' = (P\Delta Q) \cap (\cup\mathcal{F} \setminus \cap\mathcal{F}) = P\Delta Q.$$

Thus, $d(\alpha(P), \alpha(Q)) = d(P, Q)$. Consequently, α is an isometry. □

Let G be a partial cube on some set X induced by a wg-family $\mathcal{F}$ satisfying conditions (5.2), and let PQ be an edge of G. By definition, there is $x \in X$ such that $P\Delta Q = \{x\}$. The following two lemmas are instrumental.

Lemma 5.1. *Let PQ be an edge of a partial cube G on X and let $P\Delta Q = \{x\}$. The two sets*

$$\{R \in \mathcal{F} : x \in R\} \quad \textit{and} \quad \{R \in \mathcal{F} : x \notin R\}$$

form the same bipartition of the family $\mathcal{F}$ as semicubes W_{PQ} and W_{QP}.

Proof. We may assume that $Q = P + \{x\}$. Then, for any $R \in \mathcal{F}$,

$$R\Delta Q = R\Delta(P + \{x\}) = \begin{cases} (R\Delta P) \setminus \{x\}, & \text{if } x \in R, \\ (R\Delta P) + \{x\}, & \text{if } x \notin R. \end{cases}$$

Hence, $|R\Delta P| < |R\Delta Q|$ if and only if $x \notin R$. It follows that

$$W_{PQ} = \{R \in \mathcal{F} : x \notin R\}.$$

A similar argument shows that $W_{QP} = \{R \in \mathcal{F} : x \in R\}$. $\square$

Lemma 5.2. *If $\mathcal{F}$ is a wg-family of sets satisfying conditions (5.2), then for any $x \in X$ there are sets $P, Q \in \mathcal{F}$ such that $P\Delta Q = \{x\}$.*

Proof. By conditions (5.2), for a given $x \in X$ there are sets S and T in $\mathcal{F}$ such that $x \in S$ and $x \notin T$. Let $R_0 = S, R_1, \ldots, R_n = T$ be a sequence of sets in $\mathcal{F}$ satisfying conditions (2.2). It is clear that there is i such that $x \in R_i$ and $x \notin R_{i+1}$. Hence, $R_i \Delta R_{i+1} = \{x\}$, so we can choose $P = R_i$ and $Q = R_{i+1}$. $\square$

By Lemmas 5.1 and 5.2, there is one-to-one correspondence between the set X and the quotient-set E/θ. From Theorem 5.1 we obtain the following result.

Theorem 5.2. *Let $\mathcal{F}$ be a wg-family of finite subsets of a set X such that $\cap\mathcal{F} = \varnothing$ and $\cup\mathcal{F} = X$, and let G be a partial cube on X induced by $\mathcal{F}$. Then*

$$\dim_I(G) = |X|.$$

Clearly, a graph which is isometrically embeddable into a partial cube is a partial cube itself. We will show in Section 6 (Corollary 6.1) that the integer lattice $\mathbb{Z}^n$ is a partial cube. Thus a graph which is isometrically embeddable into an integer lattice is a partial cube. It follows that a finite graph is a partial cube if and only if it is embeddable in some integer lattice. Examples of infinite partial cubes isometrically embeddable into a finite dimensional integer lattice are found in [17].

We call the minimum possible dimension n of an integer lattice $\mathbb{Z}^n$, in which a given graph G is isometrically embeddable, its *lattice dimension* and denote it by $\dim_Z(G)$. The lattice dimension of a partial cube can be expressed in terms of maximum matchings in so-called semicube graphs [11].

Definition 5.1. The *semicube graph* $\mathrm{Sc}(G)$ has all semicubes in G as the set of its vertices. Two vertices W_{ab} and W_{cd} are connected in $\mathrm{Sc}(G)$ if

$$W_{ab} \cup W_{cd} = V \quad \text{and} \quad W_{ab} \cap W_{cd} \neq \emptyset. \tag{5.3}$$

If G is a partial cube, then condition (5.3) is equivalent to each of the two equivalent conditions:

$$W_{ba} \subset W_{cd} \Leftrightarrow W_{dc} \subset W_{ab}, \tag{5.4}$$

where $\subset$ stands for the proper inclusion.

Theorem 5.3 (*Theorem 1 in [11]*). *Let G be a finite partial cube. Then*

$$\dim_Z(G) = \dim_I(G) - |M|,$$

where M is a maximum matching in the semicube graph $\mathrm{Sc}(G)$.

Example 5.1. Let T be a tree with n edges and m leaves. Then

$$\dim_I(T) = n \quad \text{and} \quad \dim_Z(T) = \lceil m/2 \rceil$$

(cf. [8,14], respectively).

6. Subcubes and Cartesian products

Let G be a partial cube. We say that G' is a *subcube* of G if it is an isometric subgraph of G.

Clearly, a subcube is itself a partial cube. The converse does not hold; a subgraph of a graph G can be a partial cube but not an isometric subgraph of G.

If G' is a subcube of a partial cube G, then $\dim_I(G') \leq \dim_I(G)$ and $\dim_Z(G') \leq \dim_Z(G)$. In general, the two inequalities are not strict. For instance, the cycle C_6 is an isometric subgraph of the cube Q_3 (see Fig. 8.2) and

$$\dim_I(C_6) = \dim_Z(C_6) = \dim_I(Q_3) = \dim_Z(Q_3) = 3.$$

Semicubes of a partial cube are examples of subcubes. Indeed, by Theorem 3.4, semicubes are convex subgraphs and therefore isometric. In general, the converse is not true; a path connecting two opposite vertices in C_6 is an isometric subgraph but not a convex one.

Another common way of constructing new partial cubes from old ones is by forming their Cartesian products (see [15] for details and proofs).

Definition 6.1. Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, their *Cartesian product*

$$G = G_1 \square G_2$$

has vertex set $V = V_1 \times V_2$; a vertex $u = (u_1, u_2)$ is adjacent to a vertex $v = (v_1, v_2)$ if and only if $u_1 v_1 \in E_1$ and $u_2 = v_2$, or $u_1 = v_1$ and $u_2 v_2 \in E_2$.

The operation $\square$ is associative, so we can write

$$G = G_1 \square \cdots \square G_n = \prod_{i=1}^{n} G_i$$

for the Cartesian product of graphs $G_1, \ldots, G_n$. A Cartesian product $\prod_{i=1}^{n} G_i$ is connected if and only if the factors are connected. Then we have

$$d_G(u, v) = \sum_{i=1}^{n} d_{G_i}(u_i, v_i). \tag{6.1}$$

Example 6.1. Let $\{X_i\}_{i=1}^{n}$ be a family of sets and $Y = \sum_{i=1}^{n} X_i$ be their sum. Then the Cartesian product of the hypercubes $\mathcal{H}(X_i)$ is isomorphic to the hypercube $\mathcal{H}(Y)$. The isomorphism is established by the mapping

$$f : (P_1, \ldots, P_n) \mapsto \sum_{i=1}^{n} P_i.$$

Formula (6.1) yields immediately the following results.

Proposition 6.1. *Let H_i be isometric subgraphs of graphs G_i for all $1 \leq i \leq n$. Then the Cartesian product $\prod_{i=1}^{n} H_i$ is an isometric subgraph of the Cartesian product $\prod_{i=1}^{n} G_i$.*

Corollary 6.1. *The Cartesian product of a finite family of partial cubes is a partial cube. In particular, the graph $\mathcal{Z}^n$ (Example 2.1) is a partial cube.*

The results of the next two theorems can be easily extended to arbitrary finite products of finite partial cubes.

Theorem 6.1. *Let $G = G_1 \square G_2$ be the Cartesian product of two finite partial cubes. Then*

$$\dim_I(G) = \dim_I(G_1) + \dim_I(G_2).$$

Proof. We may assume that G_1 (resp. G_2) is induced by a wg-family $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) of subsets of a finite set X_1 (resp. X_2) such that $\cap\mathcal{F}_1 = \varnothing$ and $\cup\mathcal{F}_1 = X_1$ (resp. $\cap\mathcal{F}_2 = \varnothing$ and $\cup\mathcal{F}_2 = X_1$) (see Section 5). By Theorem 5.2,

$$\dim_I(G_1) = |X_1| \quad \text{and} \quad \dim_I(G_2) = |X_2|.$$

It is clear that the graph G is induced by the wg-family $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ of subsets of the set $X = X_1 + X_2$ (cf. Example 6.1) with $\cap\mathcal{F} = \varnothing,\ \cup\mathcal{F} = X$. By Theorem 5.2,

$$\dim_I(G) = |X| = |X_1| + |X_2| = \dim_I(G_1) + \dim_I(G_2). \quad \square$$

Theorem 6.2. *Let $G = (V, E)$ be the Cartesian product of two finite partial cubes $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then*

$$\dim_Z(G) = \dim_Z(G_1) + \dim_Z(G_2).$$

Proof. Let $W_{(a,b)(c,d)}$ be a semicube of the graph G. There are two possible cases:

(i) $c = a, bd \in E_2$. Let (x, y) be a vertex of G. Then, by (6.1),

$$d_G((x, y), (a, b)) = d_{G_1}(x, a) + d_{G_2}(y, b)$$

and

$$d_G((x, y), (c, d)) = d_{G_1}(x, c) + d_{G_2}(y, d).$$

Hence,

$$d_G((x, y), (a, b)) < d_G((x, y), (c, d)) \Leftrightarrow d_{G_2}(y, b) < d_{G_2}(y, d).$$

It follows that

$$W_{(a,b)(c,d)} = V_1 \times W_{bd}. \tag{6.2}$$

(ii) $d = b,\ ac \in E_1$. Like in (i), we have

$$W_{(a,b)(c,d)} = W_{ac} \times V_2. \tag{6.3}$$

Clearly, two semicubes given by (6.2) form an edge in the semicube graph $\mathrm{Sc}(G)$ if and only if their second factors form an edge in the semicube graph $\mathrm{Sc}(G_2)$. The same is true for semicubes in the form (6.3) with respect to their first factors. It is also clear that semicubes in the form (6.2) and in the form (6.3) are not connected by an edge in $\mathrm{Sc}(G)$. Therefore the semicube graph $\mathrm{Sc}(G)$ is isomorphic to the disjoint union of semicube graphs $\mathrm{Sc}(G_1)$ and $\mathrm{Sc}(G_2)$. If M_1 is a maximum matching in $\mathrm{Sc}(G_1)$ and M_2 is a maximum matching in $\mathrm{Sc}(G_2)$, then $M = M_1 \cup M_2$ is a maximum matching in $\mathrm{Sc}(G)$. The result follows from Theorems 5.3 and 6.1. $\square$

Remark 6.1. The result of Corollary 6.1 does not hold for infinite Cartesian products of partial cubes, as these products are disconnected. On the other hand, it can be shown that arbitrary *weak* Cartesian products (connected components of Cartesian products [15]) of partial cubes are partial cubes.

7. Pasting partial cubes

In this section we use the set pasting technique [5, Ch. I, Section 2.5] to build new partial cubes from old ones.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, $H_1 = (U_1, F_1)$ and $H_2 = (U_2, F_2)$ be two isomorphic subgraphs of G_1 and G_2, respectively, and $\psi : U_1 \to U_2$ be a bijection defining an isomorphism between H_1 and H_2. The bijection ψ defines an equivalence relation R on the sum $V_1 + V_2$ as follows: any element in $(V_1 \setminus U_1) \cup (V_2 \setminus U_2)$ is equivalent to itself only and elements $u_1 \in U_1$ and $u_2 \in U_2$ are equivalent if and only if $u_2 = \psi(u_1)$. We say that the quotient-set $V = (V_1 + V_2)/R$ is obtained by *pasting together the sets V_1 and V_2 along the subsets U_1 and U_2*. Since the graphs H_1 and H_2 are isomorphic, the pasting of the sets V_1 and V_2 can be naturally extended to a pasting of sets of edges E_1 and E_2 resulting in the set E of edges joining vertices in V. We say that the graph $G = (E, V)$ is obtained by *pasting together the graphs G_1 and G_2 along the isomorphic subgraphs H_1 and H_2*. The pasting construction

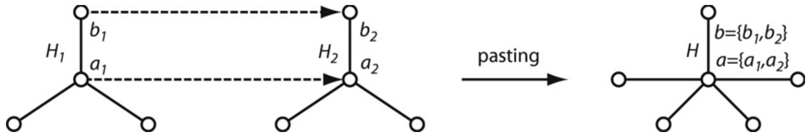

Fig. 7.1. Pasting of two trees.

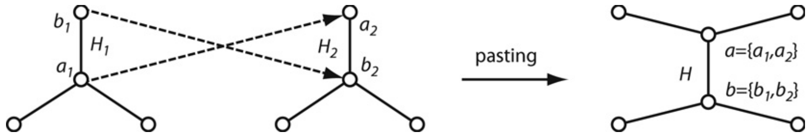

Fig. 7.2. Another pasting of the same trees.

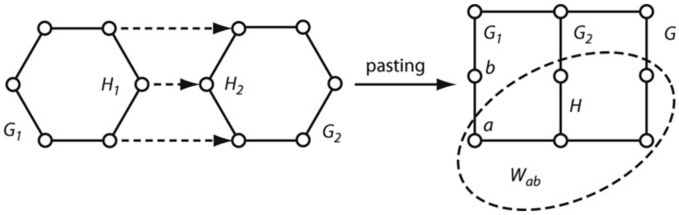

Fig. 7.3. Pasting partial cubes G_1 and G_2.

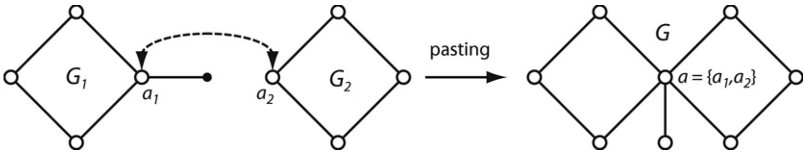

Fig. 7.4. An example of vertex-pasting.

allows for identifying in a natural way the graphs G_1 and G_2 with subgraphs of G, and the isomorphic graphs H_1 and H_2 with a common subgraph H of both graphs G_1 and G_2. We often follow this convention below.

Remark 7.1. Note that in the above construction the resulting graph G depends not only on graphs G_1 and G_2 and their isomorphic subgraphs H_1 and H_2 but also on the bijection ψ defining an isomorphism from H_1 onto H_2 (see the drawings in Figs. 7.1 and 7.2).

In general, pasting of two partial cubes G_1 and G_2 along two isomorphic subgraphs H_1 and H_2 does not produce a partial cube even under strong assumptions about these subgraphs as the next example illustrates.

Example 7.1. Pasting of two partial cubes $G_1 = C_6$ and $G_2 = C_6$ along subgraphs H_1 and H_2 is shown in Fig. 7.3. The resulting graph G is not a partial cube. Indeed, the semicube W_{ab} is not a convex set. Note that subgraphs H_1 and H_2 are convex subgraphs of the respective partial cubes.

In this section we study two simple pastings of connected graphs together, the vertex-pasting and the edge-pasting, and show that these pastings produce partial cubes from partial cubes. We also compute the isometric and lattice dimensions of the resulting graphs.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs, $a_1 \in V_1$, $a_2 \in V_2$, and $H_1 = (\{a_1\}, \varnothing)$, $H_2 = (\{a_2\}, \varnothing)$. Let G be the graph obtained by pasting G_1 and G_2 along subgraphs H_1 and H_2. In this case we say that the graph G is obtained from graphs G_1 and G_2 by *vertex-pasting*. We also say that G is obtained from G_1 and G_2 by *identifying* vertices a_1 and a_2. Fig. 7.4 illustrates this construction. Note that the vertex $a = \{a_1, a_2\}$ is a cut-vertex of G, since $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \{a\}$. (We follow our convention and identify graphs G_1 and G_2 with subgraphs of G.)

In what follows we use superscripts to distinguish subgraphs of the graphs G_1 and G_2. For instance, $W^{(2)}_{ab}$ stands for the semicube of G_2 defined by two adjacent vertices $a, b \in V_2$.

Theorem 7.1. *A graph $G = (V, E)$ obtained by vertex-pasting from partial cubes $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a partial cube.*

Proof. We denote by $a = \{a_1, a_2\}$ the vertex of G obtained by identifying vertices $a_1 \in V_1$ and $a_2 \in V_2$. Clearly, G is a bipartite graph. Let xy be an edge of G. Without loss of generality we may assume that $xy \in E_1$ and $a \in W_{xy}$. Note that any path between vertices in V_1 and V_2 must go through a. Since $a \in W_{xy}$, we have, for any $v \in V_2$,

$$d(v, x) = d(v, a) + d(a, x) < d(v, a) + d(a, y) = d(v, y),$$

which implies $V_2 \subseteq W_{xy}$ and $W_{yx} \subseteq V_1$. It follows that $W_{xy} = W^{(1)}_{xy} \cup V_2$ and $W_{yx} = W^{(1)}_{yx}$. The sets $W^{(1)}_{xy}$, $W^{(1)}_{yx}$ and V_2 are convex subsets of V. Since $W^{(1)}_{xy} \cap V_2 = \{a\}$, the set $W_{xy} = W^{(1)}_{xy} \cup V_2$ is also convex. By Theorem 3.4(ii), the graph G is a partial cube. □

The vertex-pasting construction introduced above can be generalized as follows. Let $\mathcal{G} = \{G_i = (V_i, E_i)\}_{i \in J}$ be a family of connected graphs and $\mathcal{A} = \{a_i \in G_i\}_{i \in J}$ be a family of distinguished vertices of these graphs. Let G be the graph obtained from the graphs G_i by identifying vertices in the set $\mathcal{A}$. We say that G is obtained by *vertex-pasting together the graphs G_i* (along the set $\mathcal{A}$).

Example 7.2. Let $J = \{1, \ldots, n\}$ with $n \geq 2$,

$$\mathcal{G} = \{G_i = (\{a_i, b_i\}, \{a_i b_i\})\}_{i \in J}, \quad \text{and} \quad \mathcal{A} = \{a_i\}_{i \in J}.$$

Clearly, each G_i is K_2. By vertex-pasting these graphs along $\mathcal{A}$, we obtain the n-star graph $K_{1,n}$.

Since the star $K_{1,n}$ is a tree it can be also obtained from K_1 by successive vertex-pasting as in Example 7.3.

Example 7.3. Let G_1 be a tree and $G_2 = K_2$. By vertex-pasting these graphs we obtain a new tree. Conversely, let G be a tree and v be its leaf. Let G_1 be a tree obtained from G by deleting the leaf v. Clearly, G can be obtained by vertex-pasting G_1 and K_2. It follows that any tree can obtained from the graph K_1 by successive vertex-pasting of copies of K_2 (cf. Theorem 2.3(e) in [12]).

Any connected graph G can be constructed by successive vertex-pasting of its blocks using its *block cut-vertex tree* [4] structure. Let G_1 be an endblock of G with a cut-vertex v and G_2 be the union of the remaining blocks of G. Then G can be obtained from G_1 and G_2 by vertex-pasting along the vertex v. It follows that any connected graph can be obtained from its blocks by successive vertex-pastings.

Let $G = (V, E)$ be a partial cube. We recall that the isometric dimension $\dim_I(G)$ of G is the cardinality of the quotient-set E/θ, where θ is Djoković's equivalence relation on the set E (cf. formula (5.1)).

Theorem 7.2. *Let $G = (V, E)$ be a partial cube obtained by vertex-pasting together partial cubes $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then*

$$\dim_I(G) = \dim_I(G_1) + \dim_I(G_2).$$

Proof. It suffices to prove that there are no edges $xy \in E_1$ and $uv \in E_2$ which are in Djoković's relation θ with each other. Suppose that G_1 and G_2 are vertex pasted along vertices $a_1 \in V_1$ and $a_2 \in V_2$ and let $a = \{a_1, a_2\} \in E$. Let $xy \in E_1$ and $uv \in E_2$ be two edges in E. We may assume that $u \in W_{xy}$. Since a is a cut-vertex of G and $u \in W_{xy}$, we have

$$d(u, a) + d(a, x) = d(u, x) < d(u, y) = d(u, a) + d(a, y).$$

Hence, $d(a, x) < d(a, y)$, which implies

$$d(v, x) = d(v, a) + d(a, x) < d(v, a) + d(a, y) = d(v, y).$$

It follows that $v \in W_{xy}$. Therefore the edge xy does not stand in the relation θ to the vertex uv. □

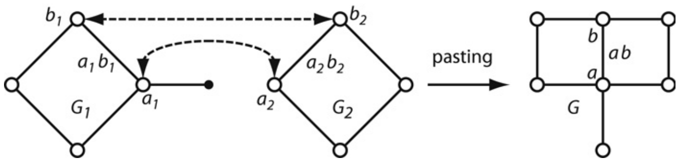

Fig. 7.5. An example of edge-pasting.

The next result follows immediately from the previous theorem. Note that blocks (maximal subgraphs without cut-vertices [4]) of a partial cube are partial cubes themselves.

Corollary 7.1. *Let G be a partial cube and $\{G_1, \dots, G_n\}$ be the family of its blocks. Then*

$$\dim_I(G) = \sum_{i=1}^{n} \dim_I(G_i).$$

In the case of the lattice dimension of a partial cube we can claim only a much weaker result than the one stated in Theorem 7.2 for the isometric dimension. We omit the proof.

Theorem 7.3. *Let G be a partial cube obtained by vertex-pasting together partial cubes G_1 and G_2. Then*

$$\max\{\dim_Z(G_1), \dim_Z(G_2)\} \le \dim_Z(G) \le \dim_Z(G_1) + \dim_Z(G_2).$$

The following example illustrates possible cases for inequalities in Theorem 7.3. Let us recall that the lattice dimension of a tree with m leaves is $\lceil m/2 \rceil$ (cf. [14]).

Example 7.4. The star $K_{1,6}$ can be obtained from the stars $K_{1,2}$ and $K_{1,4}$ by vertex-pasting these two stars along their centers. Clearly,

$$\max\{\dim_Z(K_{1,2}), \dim_Z(K_{1,4})\} < \dim_Z(K_{1,6}) = \dim_Z(K_{1,2}) + \dim_Z(K_{1,4}).$$

The same star $K_{1,6}$ is obtained from two copies of the star $K_{1,3}$ by vertex-pasting along their centers. We have $\dim_Z(K_{1,3}) = 2,\ \dim_Z(K_{1,6}) = 3$, so

$$\max\{\dim_Z(K_{1,3}), \dim_Z(K_{1,3})\} < \dim_Z(K_{1,6}) < \dim_Z(K_{1,3}) + \dim_Z(K_{1,3}).$$

Let us vertex paste two stars $K_{1,3}$ along two leaves. The resulting graph T is a tree with four leaves. Therefore,

$$\max\{\dim_Z(K_{1,3}), \dim_Z(K_{1,3})\} = \dim_Z(T) < \dim_Z(K_{1,3}) + \dim_Z(K_{1,3}).$$

We now consider another simple way of pasting two graphs together.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs, $a_1b_1 \in E_1$, $a_2b_2 \in E_2$, and $H_1 = (\{a_1, b_1\}, \{a_1b_1\})$, $H_2 = (\{a_2, b_2\}, \{a_2b_2\})$. Let G be the graph obtained by pasting G_1 and G_2 along subgraphs H_1 and H_2. In this case we say that the graph G is obtained from graphs G_1 and G_2 by *edge-pasting*. Figs. 7.1, 7.2 and 7.5 illustrate this construction.

As before, we identify the graphs G_1 and G_2 with subgraphs of the graph G and denote by $a = \{a_1, a_2\},\ b = \{b_1, b_2\}$ the two vertices obtained by pasting together vertices a_1 and a_2 and, respectively, b_1 and b_2. The edge $ab \in E$ is obtained by pasting together edges $a_1b_1 \in E_1$ and $a_2b_2 \in E_2$ (cf. Fig. 7.5). Then $G = G_1 \cup G_2,\ V_1 \cap V_2 = \{a, b\}$ and $E_1 \cap E_2 = \{ab\}$. We use these notations in the rest of this section.

Proposition 7.1. *A graph G obtained by edge-pasting together bipartite graphs G_1 and G_2 is bipartite.*

Proof. Let C be a cycle in G. If $C \subseteq G_1$ or $C \subseteq G_2$, then the length of C is even, since the graphs G_1 and G_2 are bipartite. Otherwise, the vertices a and b separate C into two paths each of odd length. Therefore C is a cycle of even length. The result follows. $\square$

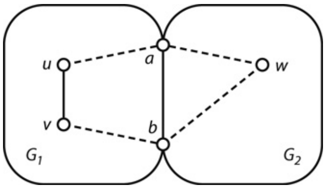

Fig. 7.6. Edge-pasting of graphs G_1 and G_2.

The following lemma is instrumental; it describes the semicubes of the graph G in terms of semicubes of graphs G_1 and G_2.

Lemma 7.1. *Let uv be an edge of G. Then*

(i) *For $uv \in E_1, a, b \in W_{uv} \Rightarrow W_{uv} = W_{uv}^{(1)} \cup V_2, W_{vu} = W_{vu}^{(1)}$.*
(ii) *For $uv \in E_2, a, b \in W_{uv} \Rightarrow W_{uv} = W_{uv}^{(2)} \cup V_1, W_{vu} = W_{vu}^{(2)}$.*
(iii) *$a \in W_{uv}, b \in W_{vu} \Rightarrow W_{uv} = W_{ab}$.*

Proof. We prove parts (i) and (iii) (see Fig. 7.6).

(i) Since any path from $w \in V_2$ to u or v contains a or b and $a, b \in W_{uv}$, we have $w \in W_{uv}$. Hence, $W_{uv} = W_{uv}^{(1)} \cup V_2$ and $W_{vu} = W_{vu}^{(1)}$.
(iii) Since $ab\theta uv$ in G_1, we have $W_{uv}^{(1)} = W_{ab}^{(1)}$, by Theorem 3.4(iv). Let w be a vertex in $W_{uv}^{(2)}$. Then, by the triangle inequality,

$$d(w, u) < d(w, v) \leq d(w, b) + d(b, v) < d(w, b) + d(b, u).$$

Since any shortest path from w to u contains a or b, we have

$$d(w, a) + d(a, u) = d(w, u).$$

Therefore,

$$d(w, a) + d(a, u) < d(w, b) + d(b, u).$$

Since $ab\theta uv$ in G_1, we have $d(a, u) = d(b, v)$, by Theorem 4.2. It follows that $d(w, a) < d(w, b)$, that is, $w \in W_{ab}^{(2)}$. We proved that $W_{uv}^{(2)} \subseteq W_{ab}^{(2)}$. By symmetry, $W_{vu}^{(2)} \subseteq W_{ba}^{(2)}$. Since two opposite semicubes form a partition of V_2, we have $W_{uv}^{(2)} = W_{ab}^{(2)}$. The result follows. $\square$

Theorem 7.4. *A graph G obtained by edge-pasting together partial cubes G_1 and G_2 is a partial cube.*

Proof. By Theorem 3.4(ii) and Proposition 7.1, we need to show that for any edge uv of G the semicube W_{uv} is a convex subset of V. There are two possible cases.

(i) $uv = ab$. The semicube W_{ab} is the union of semicubes $W_{ab}^{(1)}$ and $W_{ab}^{(2)}$ which are convex subsets of V_1 and V_2, respectively. It is clear that any shortest path connecting a vertex in $W_{ab}^{(1)}$ with a vertex in $W_{ab}^{(2)}$ contains vertex a and therefore is contained in W_{ab}. Hence, W_{ab} is a convex set. A similar argument proves that the set W_{ba} is convex.
(ii) $uv \neq ab$. We may assume that $uv \in E_1$. To prove that the semicube W_{uv} is a convex set, we consider two cases.

(a) $a, b \in W_{uv}$. (The case when $a, b \in W_{vu}$ is treated similarly.) By Lemma 7.1(i), the semicube W_{uv} is the union of the semicube $W_{uv}^{(1)}$ and the set V_2 which are both convex sets. Any shortest path P from a vertex in V_2 to a vertex in $W_{uv}^{(1)}$ contains either a or b. It follows that $P \subseteq W_{uv}^{(1)} \cup V_2 = W_{uv}$. Therefore the semicube W_{uv} is convex.
(b) $a \in W_{uv}, b \in W_{vu}$. (The case when $b \in W_{uv}, a \in W_{vu}$ are treated similarly.) By Lemma 7.1(ii), $W_{uv} = W_{ab}$. The result follows from part (i) of the proof. $\square$

Theorem 7.5. *Let G be a graph obtained by edge-pasting together finite partial cubes G_1 and G_2. Then*

$$\dim_I(G) = \dim_I(G_1) + \dim_I(G_2) - 1.$$

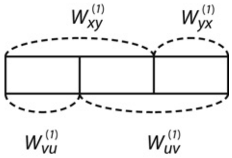

Fig. 7.7. Semicubes forming an edge in $\mathrm{Sc}(G_1)$.

Proof. Let θ, θ_1, and θ_2 be Djoković's relations on E, E_1, and E_2, respectively. By Lemma 7.1, for $uv, xy \in E_1$ (resp. $uv, xy \in E_2$) we have

$$uv\theta xy \Leftrightarrow uv\theta_1 xy \quad (\text{resp. } uv\theta xy \Leftrightarrow uv\theta_2 xy).$$

Let $uv \in E_1$, $xy \in E_2$, and $uv\theta xy$. Suppose that $(uv, ab) \notin \theta$. We may assume that $a, b \in W_{uv}$. By Lemma 7.1(i), $V_2 \subset W_{uv}$, a contradiction, since $xy \in E_2$. Hence, $uv\theta xy\theta ab$. It follows that each equivalence class of the relation θ is either an equivalence class of θ_1, an equivalence class of θ_2 or the class containing the edge ab. Therefore

$$|E/\theta| = |E_1/\theta_1| + |E_2/\theta_2| - 1.$$

The result follows, since the isometric dimension of a partial cube is equal to the cardinality of the set of equivalence classes of Djoković's relation (formula (5.1)). $\square$

We need some results about semicube graphs in order to prove an analog of Theorem 7.3 for a partial cube obtained by edge-pasting of two partial cubes.

Lemma 7.2. *Let G be a partial cube and $W_{pq}W_{uv}$, $W_{qp}W_{xy}$ be two edges in the graph $Sc(G)$. Then $W_{xy}W_{uv}$ is an edge in $Sc(G)$.*

Proof. By condition (5.4), $W_{qp} \subset W_{uv}$ and $W_{yx} \subset W_{qp}$. Hence, $W_{yx} \subset W_{uv}$. By the same condition, $W_{xy}W_{uv} \in \mathrm{Sc}(G)$. $\square$

If G is obtained by edge-pasting together graphs G_1 and G_2, we identify graphs G_1 and G_2 with subgraphs of the graph G. Then $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = (\{a, b\}, \{ab\}) = K_2$ (cf. Fig. 7.6).

Lemma 7.3. *Let G be a partial cube obtained by edge-pasting together partial cubes G_1 and G_2. Let $W_{uv}^{(1)}W_{xy}^{(1)}$ (resp. $W_{uv}^{(2)}W_{xy}^{(2)}$) be an edge in the semicube $Sc(G_1)$ (resp. $Sc(G_2)$). Then $W_{uv}W_{xy}$ is an edge in $Sc(G)$.*

Proof. It suffices to consider the case of $\mathrm{Sc}(G_1)$ (see Fig. 7.7). By condition (5.4), $W_{vu}^{(1)} \subset W_{xy}^{(1)}$ and $W_{yx}^{(1)} \subset W_{uv}^{(1)}$. Suppose that $a \in W_{vu}^{(1)}$ and $b \in W_{yx}^{(1)}$ (the case when $b \in W_{vu}^{(1)}$ and $a \in W_{yx}^{(1)}$ are treated similarly). Then $ab\theta_1 xy$ and $ab\theta_1 uv$. By transitivity of θ_1, we have $uv\theta_1 xy$, a contradiction, since semicubes $W_{uv}^{(1)}$ and $W_{xy}^{(1)}$ are distinct. Therefore we may assume that, say, $a, b \in W_{uv}^{(1)}$. Then, by Lemma 7.1, $W_{vu} = W_{vu}^{(1)} \subset V_1$. Since $W_{vu}^{(1)} \subset W_{xy}^{(1)} \subseteq W_{xy}$, we have $W_{vu} \subset W_{xy}$. By condition (5.4), $W_{uv}W_{xy}$ is an edge in $\mathrm{Sc}(G)$. $\square$

Lemma 7.4. *Let M_1 and M_2 be matchings in graphs $Sc(G_1)$ and $Sc(G_2)$. There is a matching M in $Sc(G)$ such that*

$$|M| \geq |M_1| + |M_2| - 1.$$

Proof. By Lemma 7.3, M_1 and M_2 induce matchings in $\mathrm{Sc}(G)$ which we denote by the same symbols. The intersection $M_1 \cap M_2$ is either empty or a subgraph of the empty graph with vertices W_{ab} and W_{ba}.

If $M_1 \cap M_2$ is empty, then $M = M_1 \cup M_2$ is a matching in $\mathrm{Sc}(G)$ and the result follows.

If $M_1 \cap M_2$ is an empty graph with a single vertex, say, in M_1, we remove from M_1 the edge that has this vertex as its end vertex, resulting in the matching M_1'. Clearly, $M = M_1' \cup M_2$ is a matching in $\mathrm{Sc}(G)$ and $|M| = |M_1| + |M_2| - 1$.

Suppose now that $M_1 \cap M_2$ is the empty graph with vertices W_{ab} and W_{ba}. Let $W_{ab}W_{uv}$, $W_{ba}W_{pq}$ (resp. $W_{ab}W_{xy}$, $W_{ba}W_{rs}$) be edges in M_1 (resp. M_2). By Lemma 7.2, $W_{xy}W_{rs}$ is an edge in $\mathrm{Sc}(G_2)$. Let us replace edges $W_{ab}W_{xy}$ and $W_{ba}W_{rs}$ in M_2 by a single edge $W_{xy}W_{rs}$, resulting in the matching M_2'. Then $M = M_1 \cup M_2'$ is a matching in $\mathrm{Sc}(G)$ and $|M| = |M_1| + |M_2| - 1$. $\square$

Corollary 7.2. *Let M_1 and M_2 be maximum matchings in $Sc(G_1)$ and $Sc(G_2)$, respectively, and M be a maximum matching in $Sc(G)$. Then*

$$|M| \geq |M_1| + |M_2| - 1. \tag{7.1}$$

By Theorem 5.3, we have

$$\dim_I(G_1) = \dim_Z(G_1) + |M_1|, \qquad \dim_I(G_2) = \dim_Z(G_2) + |M_2|,$$

and

$$\dim_I(G) = \dim_Z(G) + |M|,$$

where M_1 and M_2 are maximum matchings in $\mathrm{Sc}(G_1)$ and $\mathrm{Sc}(G_2)$, respectively, and M is a maximum matching in $\mathrm{Sc}(G)$. Therefore, by Theorem 7.5 and (7.1), we have the following result (cf. Theorem 7.3).

Theorem 7.6. *Let G be a partial cube obtained by edge-pasting from partial cubes G_1 and G_2. Then*

$$\max\{\dim_Z(G_1), \dim_Z(G_2)\} \leq \dim_Z(G) \leq \dim_Z(G_1) + \dim_Z(G_2).$$

Example 7.5. Let us consider two edge-pastings of the stars $G_1 = K_{1,3}$ and $G_2 = K_{1,3}$ of lattice dimension 2 shown in Figs. 7.1 and 7.2. In the first case the resulting graph is the star $G = K_{1,5}$ of lattice dimension 3. Then we have

$$\max\{\dim_Z(G_1), \dim_Z(G_2)\} < \dim_Z(G) < \dim_Z(G_1) + \dim_Z(G_2).$$

In the second case the resulting graph is a tree with 4 leaves. Therefore,

$$\max\{\dim_Z(G_1), \dim_Z(G_2)\} = \dim_Z(G) < \dim_Z(G_1) + \dim_Z(G_2).$$

Let c_1a_1 and c_2a_2 be edges of stars $G_1 = K_{1,4}$ and $G_2 = K_{1,4}$ (each of which has lattice dimension 2), where c_1 and c_2 are centers of the respective stars. Let us edge paste these two graphs by identifying c_1 with c_2 and a_1 with a_2, respectively. The resulting graph G is the star $K_{1,7}$ of lattice dimension 4. Thus,

$$\max\{\dim_Z(G_1), \dim_Z(G_2)\} \leq \dim_Z(G) = \dim_Z(G_1) + \dim_Z(G_2).$$

8. Expansions and contractions of partial cubes

The graph expansion procedure was introduced by Mulder in [16], where it is shown that a graph is a median graph if and only if it can be obtained from K_1 by a sequence of convex expansions (see also [15]). A similar result for partial cubes was established in [6] (see also [7]) as a corollary to a more general result concerning isometric embeddability into Hamming graphs; it was also established in [13] in the framework of oriented matroids theory.

In this section we investigate properties of (isometric) expansion and contraction operations and, in particular, prove in two different ways that a graph is a partial cube if and only if it can be obtained from the graph K_1 by a sequence of expansions.

A remark about notations is in order. In the product $\{1, 2\} \times (V_1 \cup V_2)$, we denote $V_i' = \{i\} \times V_i$ and $x^i = (i, x)$ for $x \in V_i$, where $i = 1, 2$.

Definition 8.1. Let $G = (V, E)$ be a connected graph, and let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two isometric subgraphs of G such that $G = G_1 \cup G_2$. The *expansion of G with respect to G_1 and G_2* is the graph $G' = (V', E')$ constructed as follows from G (see Fig. 8.1):

(i) $V' = V_1 + V_2 = V_1' \cup V_2'$;
(ii) $E' = E_1 + E_2 + M$, where M is the matching $\bigcup_{x \in V_1 \cap V_2} \{x^1x^2\}$.

In this case, we also say that G is a *contraction* of G'.

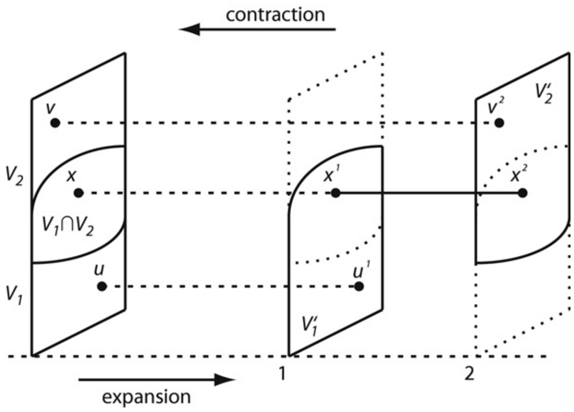

Fig. 8.1. Expansion/contraction processes.

It is clear that the graphs G_1 and $\langle V_1' \rangle$ are isomorphic, as well as the graphs G_2 and $\langle V_2' \rangle$.

We define a *projection* $p : V' \to V$ by $p(x^i) = x$ for $x \in V$. Clearly, the restriction of p to V_1' is a bijection $p_1 : V_1' \to V_1$ and its restriction to V_2' is a bijection $p_2 : V_2' \to V_2$. These bijections define isomorphisms $\langle V_1' \rangle \to G_1$ and $\langle V_2' \rangle \to G_2$.

Let P' be a path in G'. The vertices of G obtained from the vertices in P' under the projection p define a walk P in G; we call this walk P the *projection* of the path P'. It is clear that

$$\ell(P) = \ell(P'), \quad \text{if } P' \subseteq \langle V_1' \rangle \text{ or } P' \subseteq \langle V_2' \rangle. \tag{8.1}$$

In this case, P is a path in G and either $P = p_1(P')$ or $P = p_2(P')$. On the other hand,

$$\ell(P) < \ell(P'), \quad \text{if } P' \cap \langle V_1' \rangle \neq \varnothing \text{ and } P' \cap \langle V_2' \rangle \neq \varnothing, \tag{8.2}$$

and P is not necessarily a path.

We will frequently use the results of the following lemma in this section.

Lemma 8.1. (i) *For $u^1, v^1 \in V_1'$, any shortest path $P_{u^1v^1}$ in G' belongs to $\langle V_1' \rangle$ and its projection $P_{uv} = p_1(P_{u^1v^1})$ is a shortest path in G. Accordingly,*

$$d_{G'}(u^1, v^1) = d_G(u, v)$$

and $\langle V_1' \rangle$ is a convex subgraph of G'. A similar statement holds for $u^2, v^2 \in V_2'$.

(ii) *For $u^1 \in V_1'$ and $v^2 \in V_2'$,*

$$d_{G'}(u^1, v^2) = d_G(u, v) + 1.$$

Let $P_{u^1v^2}$ be a shortest path in G'. There is a unique edge $x^1x^2 \in M$ such that $x^1, x^2 \in P_{u^1v^2}$ and the sections $P_{u^1x^1}$ and $P_{x^2v^2}$ of the path $P_{u^1v^2}$ are shortest paths in $\langle V_1' \rangle$ and $\langle V_2' \rangle$, respectively. The projection P_{uv} of $P_{u^1v^2}$ in G' is a shortest path in G.

Proof. (i) Let $P_{u^1v^1}$ be a path in G' that intersects V_2'. Since $\langle V_1 \rangle$ is an isometric subgraph of G, there is a path P_{uv} in G that belongs to $\langle V_1 \rangle$. Then $p_1^{-1}(P_{uv})$ is a path in $\langle V_1' \rangle$ of the same length as P_{uv}. By (8.1) and (8.2),

$$\ell(p_1^{-1}(P_{uv})) < \ell(P_{u^1v^1}).$$

Therefore any shortest path $P_{u^1v^1}$ in G' belongs to $\langle V_1' \rangle$. The result follows.

(ii) Let $P_{u^1v^2}$ be a shortest path in G' and P_{uv} be its projection to V. By (8.2),

$$d_{G'}(u^1, v^2) = \ell(P_{u^1v^2}) > \ell(P_{uv}) \geq d_G(u, v).$$

Since there is no edge of G joining vertices in $V_1 \setminus V_2$ and $V_2 \setminus V_1$, a shortest path in G from u to v must contain a vertex $x \in V_1 \cap V_2$. Since G_1 and G_2 are isometric subgraphs, there are shortest paths P_{ux} in G_1 and P_{xv} in G_2

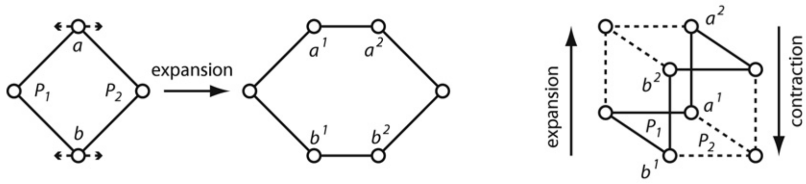

Fig. 8.2. An expansion of the cycle C_4.

such that their union is a shortest path from u to v. Then, by the triangle inequality and part (i) of the proof, we have (cf. Fig. 8.1)

$$d_{G'}(u^1, v^2) \le d_{G'}(u^1, x^1) + d_{G'}(x^1, x^2) + d_{G'}(x^2, v^2) = d_G(u, v) + 1.$$

The last two displayed formulas imply $d_{G'}(u^1, v^2) = d_G(u, v) + 1$.

Since $u^1 \in V_1'$ and $v^2 \in V_2'$ the path $P_{u^1v^2}$ must contain an edge, say x^1x^2, in M. Since this path is a shortest path in G', this edge is unique. Then the sections $P_{u^1x^1}$ and $P_{x^2v^2}$ of $P_{u^1v^2}$ are shortest paths in $\langle V_1' \rangle$ and $\langle V_2' \rangle$, respectively. Clearly, P_{uv} is a shortest path in G. $\square$

Let a^1a^2 be an edge in the matching $M = \cup_{x \in V_1 \cap V_2}\{x^1x^2\}$. This edge defines five fundamental sets (cf. Section 4): the semicubes $W_{a^1a^2}$ and $W_{a^2a^1}$, the sets of vertices $U_{a^1a^2}$ and $U_{a^2a^1}$, and the set of edges $F_{a^1a^2}$. The next theorem follows immediately from Lemma 8.1. It gives a hint to a connection between the expansion process and partial cubes.

Theorem 8.1. *Let G' be an expansion of a connected graph G and notations are chosen as above. Then*

(i) *$W_{a^1a^2} = V_1'$ and $W_{a^2a^1} = V_2'$ are convex semicubes of G'.*
(ii) *$F_{a^1a^2} = M$ defines an isomorphism between induced subgraphs $\langle U_{a^1a^2} \rangle$ and $\langle U_{a^2a^1} \rangle$, which are isomorphic to the subgraph $G_1 \cap G_2$.*

The result of Theorem 8.1 justifies the following constructive definition of the contraction process.

Definition 8.2. Let ab be an edge of a connected graph $G' = (V', E')$ such that

(i) semicubes W_{ab} and W_{ba} are convex and form a partition of V';
(ii) the set F_{ab} is a matching and defines an isomorphism between subgraphs $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$.

A graph G obtained from the graphs $\langle W_{ab} \rangle$ and $\langle W_{ba} \rangle$ by pasting them along subgraphs $\langle U_{ab} \rangle$ and $\langle U_{ba} \rangle$ is said to be a *contraction* of the graph G'.

Remark 8.1. If G' is bipartite, then semicubes W_{ab} and W_{ba} form a partition of its vertex set. Then, by Theorem 4.1, condition (i) implies condition (ii). Thus any pair of opposite convex semicubes in a connected bipartite graph defines a contraction of this graph.

By Theorem 8.1, a graph is a contraction of its expansion. It is not difficult to see that any connected graph is also an expansion of its contraction.

The following three examples give geometric illustrations for the expansion and contraction procedures.

Example 8.1. Let a and b be two opposite vertices in the graph $G = C_4$. Clearly, the two distinct paths P_1 and P_2 from a to b are isometric subgraphs of G defining an expansion $G' = C_6$ of G (see Fig. 8.2). Note that P_1 and P_2 are not convex subsets of V.

Example 8.2. Another isometric expansion of the graph $G = C_4$ is shown in Fig. 8.3. Here, the path P_1 is the same as in the previous example and $G_2 = G$.

Example 8.3. Lemma 8.1 claims, in particular, that the projection of a shortest path in an extension G' of a graph G is a shortest path in G. Generally speaking, the converse is not true. Consider the graph G shown in Fig. 8.4 and two paths in G:

$$V_1 = abcef \quad \text{and} \quad V_2 = bde.$$

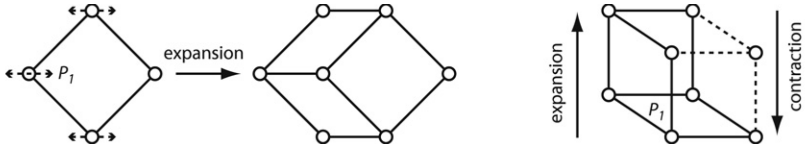

Fig. 8.3. Another isometric expansion of the cycle C_4.

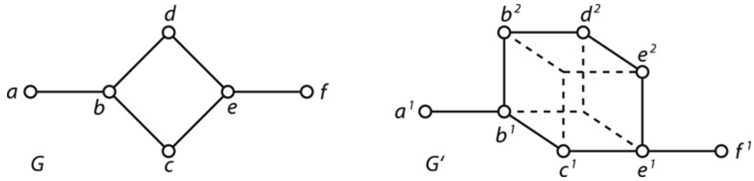

Fig. 8.4. A shortest path which is not a projection of a shortest path.

The graph G' in Fig. 8.4 is the convex expansion of G with respect to V_1 and V_2. The path $abdef$ is a shortest path in G; it is not a projection of a shortest path in G'.

One can say that, in the case of finite partial cubes, the contraction procedure is defined by an orthogonal projection of a hypercube onto one of its facets.

By Theorem 8.1, the sets V_1' and V_2' are opposite semicubes of the graph G' defined by edges in M. Their projections are the sets V_1 and V_2 which are not necessarily semicubes of G. For other semicubes in G' we have the following result.

Lemma 8.2. *For any two adjacent vertices $u, v \in V$,*

$$W_{u^i v^i} = p^{-1}(W_{uv}) \quad \textit{for } u, v \in V_i \textit{ and } i = 1, 2.$$

Proof. By Lemma 8.1,

$$d_{G'}(x^j, u^i) < d_{G'}(x^j, v^i) \Leftrightarrow d_G(x, u) < d_G(x, v)$$

for $x \in V$ and $i, j = 1, 2$. The result follows. □

Corollary 8.1. *If uv is an edge of $G_1 \cap G_2$, then $W_{u^1v^1} = W_{u^2v^2}$.*

The following lemma is an immediate consequence of Lemma 8.1. We shall use it implicitly in our arguments later.

Lemma 8.3. *Let $u, v \in V_1$ and $x \in V_1 \cap V_2$. Then*

$$x^1 \in W_{u^1v^1} \Leftrightarrow x^2 \in W_{u^1v^1}.$$

The same result holds for semicubes in the form $W_{u^2v^2}$.

Generally speaking, the projection of a convex subgraph of G' is not a convex subgraph of G. For instance, the projection of the convex path $b^2d^2e^2$ in Fig. 8.4 is the path bde which is not a convex subgraph of G. On the other hand, we have the following result.

Theorem 8.2. *Let $G' = (V', E')$ be an expansion of a graph $G = (V, E)$ with respect to subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The projection of a convex semicube of G' different from $\langle V_1' \rangle$ and $\langle V_2' \rangle$ is a convex semicube of G.*

Proof. It suffices to consider the case when $W_{uv} = p(W_{u^1v^1})$ for $u, v \in V_1$ (cf. Lemma 8.2). Let $x, y \in W_{uv}$ and $z \in V$ be a vertex such that

$$d_G(x, z) + d_G(z, y) = d_G(x, y).$$

We need to show that $z \in W_{uv}$.

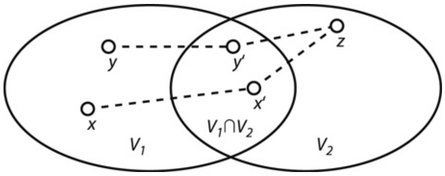

Fig. 8.5. A shortest path from x to y.

(i) $x, y \in V_1$ (the case when $x, y \in V_2$ is treated similarly). Suppose that $z \in V_1$. Then $x^1, y^1, z^1 \in V_1'$ and, by Lemma 8.1,

$$d_{G'}(x^1, z^1) + d_{G'}(z^1, y^1) = d_{G'}(z^1, y^1).$$

Since $x^1, y^1 \in W_{u^1v^1}$ and $W_{u^1v^1}$ is convex, $z^1 \in W_{u^1v^1}$. Hence, $z \in W_{uv}$.

Suppose now that $z \in V_2 \setminus V_1$. Consider a shortest path P_{xy} in G from x to y containing z. This path contains vertices $x', y' \in V_1 \cap V_2$ such that (see Fig. 8.5)

$$d_G(x, x') + d_G(x', z) = d_G(x, z) \quad \text{and} \quad d_G(y, y') + d_G(y', z) = d_G(y, z).$$

Since P_{xy} is a shortest path in G, we have

$$d_G(x, x') + d_G(x', y) = d_G(x, y), \qquad d_G(x, y') + d_G(y', y) = d_G(x, y),$$

and

$$d_G(x', z) + d_G(z, y') = d_G(x', y').$$

Since $x, x', y \in V_1$, we have $x^1, x'^1, y^1 \in V_1'$. Because $x^1, y^1 \in W_{u^1v^1}$ and $W_{u^1v^1}$ is convex, $x'^1 \in W_{u^1v^1}$. Hence, $x' \in W_{uv}$ and, similarly, $y' \in W_{uv}$. Since $x'^2, y'^2, z^2 \in V_2'$ and $W_{u^1v^1}$ is convex, $z^2 \in W_{u^1v^1}$. Hence, $z \in W_{uv}$.

(ii) $x \in V_1 \setminus V_2$ and $y \in V_2 \setminus V_1$. We may assume that $z \in V_1$. By Lemma 8.1,

$$\begin{aligned} d_{G'}(x^1, y^2) &= d_G(x, y) + 1 = d_G(x, z) + d_G(z, y) + 1 \\ &= d_{G'}(x^1, z^1) + d_{G'}(z^1, y^2). \end{aligned}$$

Since $x^1, y^2 \in W_{u^1v^1}$ and $W_{u^1v^1}$ is convex, $z^1 \in W_{u^1v^1}$. Hence, $z \in W_{uv}$. □

By using the results of Lemma 8.1, it is not difficult to show that the class of connected bipartite graphs is closed under the expansion and contraction operations. The next theorem establishes this result for the class of partial cubes.

Theorem 8.3. (i) *An expansion G' of a partial cube G is a partial cube.*
(ii) *A contraction G of a partial cube G' is a partial cube.*

Proof. (i) Let $G = (V, E)$ be a partial cube and $G' = (V', E')$ be its expansion with respect to isometric subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. By Theorem 3.4(ii), it suffices to show that the semicubes of G' are convex.

By Theorem 8.1, the semicubes $\langle V_1' \rangle$ and $\langle V_2' \rangle$ are convex, so we consider a semicube in the form $W_{u^1v^1}$ where $uv \in E_1$ (the other case is treated similarly). Let $P_{x'y'}$ be a shortest path connecting two vertices in $W_{u^1v^1}$ and P_{xy} be its projection to G. By Lemma 8.2, $x, y \in W_{uv}$ and, by Lemma 8.1, P_{xy} is a shortest path in G. Since W_{uv} is convex, P_{xy} belongs to W_{uv}. Let z' be a vertex in $P_{x'y'}$ and $z = p(z') \in P_{xy}$. By Lemma 8.1,

$$d_G(z, u) < d_G(z, v) \Rightarrow d_{G'}(z', u^1) \le d_{G'}(z', v^1).$$

Since G' is a bipartite graph, $d_{G'}(z', u^1) < d_{G'}(z', v^1)$. Hence, $P_{x'y'} \subseteq W_{u^1v^1}$, so $W_{u^1v^1}$ is convex.

(ii) Let $G = (V, E)$ be a contraction of a partial cube $G' = (V', E')$. By Theorem 3.4, we need to show that the semicubes of G are convex. By Lemma 8.2, all semicubes of G are projections of semicubes of G' distinct from $\langle V_1' \rangle$ and $\langle V_2' \rangle$. By Theorem 8.2, the semicubes of G are convex. □

Corollary 8.2. (i) *A finite connected graph is a partial cube if and only if it can be obtained from K_1 by a sequence of expansions.*
(ii) *The number of expansions needed to produce a partial cube G from K_1 is* $\dim_I(G)$.

Proof. (i) Follows immediately from Theorem 8.3.
(ii) Follows from Lemma 8.2 and Theorem 5.1 (see the discussion in Section 5 just before Theorem 5.2). □

The processes of expansion and contraction admit useful descriptions in the case of partial cubes on a set. Let $G = (V, E)$ be a partial cube on a set X, that is an isometric subgraph of the hypercube $\mathcal{H}(X)$. Then it is induced by some wg-family $\mathcal{F}$ of finite subsets of X (cf. Theorem 2.1). We may assume (see Section 5) that $\cap\mathcal{F} = \varnothing$ and $\cup\mathcal{F} = X$.

In what follows we present proofs of the results of Theorem 8.3 and Corollary 8.2 given in terms of wg-families of sets.

The expansion process for a partial cube G on X can be described as follows: Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be wg-families of finite subsets of X such that $\mathcal{F}_1 \cap \mathcal{F}_2 \neq \varnothing$, $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}$, and the distance between any two sets $P \in \mathcal{F}_1 \setminus \mathcal{F}_2$ and $Q \in \mathcal{F}_2 \setminus \mathcal{F}_1$ is greater than one. Note that $\langle\mathcal{F}_1\rangle$ and $\langle\mathcal{F}_2\rangle$ are partial cubes, $\langle\mathcal{F}_1\rangle \cap \langle\mathcal{F}_2\rangle \neq \varnothing$, and $\langle\mathcal{F}_1\rangle \cup \langle\mathcal{F}_2\rangle = \langle\mathcal{F}\rangle = G$. Let $X' = X + \{p\}$, where $p \notin X$, and

$$\mathcal{F}'_2 = \{Q + \{p\} : Q \in \mathcal{F}_2\}, \qquad \mathcal{F}' = \mathcal{F}_1 \cup \mathcal{F}'_2.$$

It is quite clear that the graphs $\langle\mathcal{F}'_2\rangle$ and $\langle\mathcal{F}_2\rangle$ are isomorphic and the graph $G' = \langle\mathcal{F}'\rangle$ is an isometric expansion of the graph G.

Theorem 8.4. *An expansion of a partial cube is a partial cube.*

Proof. We need to verify that $\mathcal{F}'$ is a wg-family of finite subsets of X'. By Theorem 2.3, it suffices to show that the distance between any two adjacent sets in $\mathcal{F}'$ is 1. It is obvious if each of these two sets belongs to one of the families $\mathcal{F}_1$ or $\mathcal{F}'_2$. Suppose that $P \in \mathcal{F}_1$ and $Q + \{p\} \in \mathcal{F}'_2$ are adjacent, that is, for any $S \in \mathcal{F}'$ we have

$$P \cap (Q + \{p\}) \subseteq S \subseteq P \cup (Q + \{p\}) \Rightarrow S = P \quad \text{or} \quad S = Q + \{p\}. \tag{8.3}$$

If $Q \in \mathcal{F}_1$, then

$$P \cap (Q + \{p\}) \subseteq Q \subseteq P \cup (Q + \{p\}),$$

since $p \notin P$. By (8.3), $Q = P$ implying $d(P, Q + \{p\}) = 1$.

If $Q \in \mathcal{F}_2 \setminus \mathcal{F}_1$, there is $R \in \mathcal{F}_1 \cap \mathcal{F}_2$ such that

$$d(P, R) + d(R, Q) = d(P, Q),$$

since $\mathcal{F}$ is well-graded. By Theorem 2.2,

$$P \cap Q \subseteq R \subseteq P \cup Q,$$

which implies

$$P \cap (Q + \{p\}) \subseteq R + \{p\} \subseteq P \cup (Q + \{p\}).$$

By (8.3), $R + \{p\} = Q + \{p\}$, a contradiction. □

It is easy to recognize the fundamental sets (cf. Section 4) in an isometric expansion G' of a partial cube $G = \langle\mathcal{F}\rangle$. Let $P \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $Q = P + \{p\} \in \mathcal{F}'_2$ be two vertices defining an edge in G' according to Definition 8.1(ii). Clearly, the families $\mathcal{F}_1$ and $\mathcal{F}'_2$ are the semicubes W_{PQ} and W_{QP} of the graph G' (cf. Lemma 5.1) and therefore are convex subsets of $\mathcal{F}'$. The set F_{PQ} is the set of edges defined by p as in Lemma 5.1. In addition, $U_{PQ} = \mathcal{F}_1 \cap \mathcal{F}_2$ and $U_{QP} = \{R + \{p\} : R \in \mathcal{F}_1 \cap \mathcal{F}_2\}$.

Let G be a partial cube induced by a wg-family $\mathcal{F}$ of finite subsets of a set X. As before, we assume that $\cap\mathcal{F} = \varnothing$ and $\cup\mathcal{F} = X$. Let PQ be an edge of G. We may assume that $Q = P + \{p\}$ for some $p \notin P$. Then (see Lemma 5.1)

$$W_{PQ} = \{R \in \mathcal{F} : p \notin R\} \quad \text{and} \quad W_{QP} = \{R \in \mathcal{F} : p \in R\}.$$

Let $X' = X \setminus \{p\}$ and $\mathcal{F}' = \{R \setminus \{p\} : R \in \mathcal{F}\}$. It is clear that the graph G' induced by the family $\mathcal{F}'$ is isomorphic to the contraction of G defined by the edge PQ. Geometrically, the graph G' is the orthogonal projection of the graph G along the edge PQ (cf. Figs. 8.2 and 8.3).

Theorem 8.5. (i) *A contraction G' of a partial cube G is a partial cube.*
(ii) *If G is finite, then* $\dim_I(G') = \dim_I(G) - 1$.

Proof. (i) For $p \in X$ we define $\mathcal{F}_1 = \{R \in \mathcal{F} : p \notin R\}$, $\mathcal{F}_2 = \{R \in \mathcal{F} : p \in R\}$, and $\mathcal{F}_2' = \{R \setminus \{p\} \in \mathcal{F} : p \in R\}$. Note that $\mathcal{F}_1$ and $\mathcal{F}_2$ are semicubes of G and $\mathcal{F}_2'$ is isometric to $\mathcal{F}_2$. Hence, $\mathcal{F}_1$ and $\mathcal{F}_2'$ are wg-families of finite subsets of X'. We need to prove that $\mathcal{F}' = \mathcal{F}_1 \cup \mathcal{F}_2'$ is a wg-family. By Theorem 2.3, it suffices to show that $d(P, Q) = 1$ for any two adjacent sets $P, Q \in \mathcal{F}'$. This is true if $P, Q \in \mathcal{F}_1$ or $P, Q \in \mathcal{F}_2'$, since these two families are well-graded. For $P \in \mathcal{F}_1 \setminus \mathcal{F}_2'$ and $Q \in \mathcal{F}_2' \setminus \mathcal{F}_1$, the sets P and $Q + \{p\}$ are not adjacent in $\mathcal{F}$, since $\mathcal{F}$ is well-graded and $Q \notin \mathcal{F}$. Hence there is $R \in \mathcal{F}_1$ such that

$$P \cap (Q + \{p\}) \subseteq R \subseteq P \cup (Q + \{p\})$$

and $R \neq P$. Since $p \notin R$, we have

$$P \cap Q \subseteq R \subseteq P \cup Q.$$

Since $R \neq P$ and $R \neq Q$, the sets P and Q are not adjacent in $\mathcal{F}'$. The result follows.

(ii) If G is a finite partial cube, then

$$\dim_I(G') = |X'| = |X| - 1 = \dim_I(G) - 1,$$

by Theorem 5.2. $\square$

9. Conclusion

The paper focuses on two themes of a rather general mathematical nature.

1. *The characterization problem.* It is a common practice in mathematics to characterize a particular class of object in different terms. We present new characterizations of the classes of bipartite graphs and partial cubes, and give new proofs for known characterization results.
2. *Constructions.* The problem of constructing new objects from old ones is a standard topic in many branches of mathematics. For the class of partial cubes, we discuss operations of forming the Cartesian product, expansion and contraction, and pasting. It is shown that the class of partial cubes is closed under these operations.

Because partial cubes are defined as graphs isometrically embeddable into hypercubes, the theory of partial cubes has a distinctive geometric flavor. The three main structures on a graph – semicubes and Djoković's and Winkler's relations – are defined in terms of the metric structure on a graph. One can say that this theory is a branch of discrete metric geometry. Not surprisingly, geometric structures play an important role in our treatment of the characterization and construction problems.

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