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# EMBEDDINGS OF STEINER TRIPLE SYSTEMS

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## 1. Introduction

If $X$ is a set whose elements are called points and $\mathcal{A}$ is a collection of subsets of $X$ (called lines) such that:

(i) any two distinct points of $X$ are contained in exactly one line,

(ii) every line contains at least two points,

we say that the pair $(X, \mathcal{A})$ is a *linear space*.

A *Steiner triple system* is defined as a finite non-empty linear space $(X, \mathcal{A})$ all of whose lines are of size 3, i.e., contain exactly 3 points. A Steiner triple system with $|X| = v$ is said to be of *order* $v$ and is denoted by $S(v)$. Kirkman [4] has proved that there exists an $S(v)$ if and only if $v \equiv 1$ or 3 (mod 6); any positive integer satisfying this congruence will be called *admissible*.

If $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are two Steiner triple systems such that $Y \subseteq X$ and $\mathcal{B} \subseteq \mathcal{A}$, we shall say that $(Y, \mathcal{B})$ is *embedded in* (or is a subsystem of) $(X, \mathcal{A})$ and that $(X, \mathcal{A})$ contains $(Y, \mathcal{B})$. If $(X, \mathcal{A})$ is of order $v$ and $(Y, \mathcal{B})$ is of order $u < v$, then $v \geqslant 2u + 1$. Indeed, let $p \in X - Y$. Any line containing $p$ has at most one point in $Y$. Therefore there are exactly $u$ lines

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joining $p$ to the $u$ points of $Y$ and each of these lines has two points distinct from $p$, so that $v \geqslant 2u + 1$. Our purpose is to prove that conversely,

**Theorem 1.1.** *Any $S(u)$ can be embedded in some $S(v)$ for every admissible $v \geqslant 2u + 1$.*

The proof of this result is greatly simplified by noticing that if there exists a single $S(v)$ containing a subsystem of order $u$, then any $S(u)$ can be embedded in a 'slightly modified' $S(v)$. Indeed, if $(X, A)$, $(Y, B_1)$, $(Y, B_2)$ are three Steiner triple systems and if $(Y, B_1)$ is a subsystem of $(X, A)$, then $(X, (A - B_1) \cup B_2)$ is a Steiner triple system containing $(Y, B_2)$ as a subsystem. Roughly speaking, subsystems can be unplugged and replaced.

We have still to introduce a few definitions and notations which will be used throughout this paper.

A *parallel class of lines* of a linear space $(X, A)$ is a subset $A'$ of $A$ such that every point of $X$ is contained in exactly one line of $A'$, i.e., such that $A'$ is a partition of $X$. Let $(X, A)$ be a finite linear space of cardinality $mn$, with a distinguished parallel class of lines $A' = \{A_1, A_2, ..., A_m\}$ such that

(i) every line of $A'$ has size $n$,

(ii) every line of $A - A'$ has size $m$.

Such a linear space will be called a *transversal system* $T(m, n)$; the lines of $A'$ will be called *groups*, the other lines *transversals*. Clearly, every transversal intersects each group in precisely one point. It is well-known (see for instance [2, Chapters 13 and 15]) that the existence of a $T(m, n)$ is equivalent to the existence of $m - 2$ mutually orthogonal latin squares of order $n$. Therefore, by a result of Bose, Parker and Shrikhande [1], there is a $T(4, n)$ for every $n \neq 2, 6$. Any transversal system $T(m, n)$ containing $s$ pairwise disjoint parallel classes of transversals will be denoted by $T_s(m, n)$.

A Steiner triple system $(X, A)$ is called a *Kirkman system* if the set $A$ admits a partition $A = A_1 \cup A_2 \cup ... \cup A_r$ into parallel classes. Ray-Chaudhuri and Wilson [6] have proved that there exists a Kirkman system of order $v$ if and only if $v \equiv 3 \pmod 6$. A Steiner triple system $(X, A)$ of order $v$ is called *cyclic* if its automorphism group contains a cyclic permutation $\alpha$ (i.e. a permutation consisting of a single cycle of length $v$). Peltesohn [5] has constructed a cyclic $S(v)$ for every admissible $v$, except $v = 9$.

In this paper, we shall sometimes consider cartesian products of the form $X \times \{1, 2, ..., t\}$; any element $(x, i)$ of such a product (where $x \in X$ and $1 \leq i \leq t$) will be denoted simply by $x_i$ and any subset $X \times \{i\}$ simply by $X_i$.

## 2. Main results

**Lemma 2.1.** *Let $u \geq u'$ be two admissible integers such that the congruences $u \equiv 1 \pmod 6$ and $u' \equiv 3 \pmod 6$ do not hold simultaneously. Then there exists a Steiner triple system $(Y, \mathcal{B})$ of order $u$ satisfying the following conditions*:

(i) *there is a subset $\mathcal{B}^*$ of $\mathcal{B}$ such that every point of $Y$ is contained in exactly $\frac{1}{2}(u' - 1)$ lines of $\mathcal{B}^*$,*

(ii) *there is a cyclic permutation $\alpha$ of $Y$ taking $\mathcal{B}^*$ onto itself, i.e., mapping any pair of points joined by a line of $\mathcal{B}^*$ onto a pair of points of the same type.*

**Proof.** Suppose $u \neq 9$ and let $(Y, \mathcal{B})$ be a cyclic Steiner triple system of order $u$, $\alpha$ one of its cyclic automorphisms. If $u = 6t + 1$, $\alpha$ partitions $\mathcal{B}$ into $t$ orbits $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_t$ of length $u$; write $u' = 6t' + 1$ and take $\mathcal{B}^* = \mathcal{B}_1 \cup \mathcal{B}_2 \cup ... \cup \mathcal{B}_{t'}$. If $u = 6t + 3$, $\alpha$ partitions $\mathcal{B}$ into $t$ orbits $\mathcal{B}_1, \mathcal{B}_2, ...,$ $\mathcal{B}_t$ of length $u$ and one orbit $\mathcal{B}_0$ of length $\frac{1}{3}u$; take $\mathcal{B}^* = \mathcal{B}_1 \cup \mathcal{B}_2 \cup ... \cup \mathcal{B}_{t'}$ when $u' = 6t' + 1$ and $\mathcal{B}^* = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup ... \cup \mathcal{B}_{t'}$ when $u' = 6t' + 3$.

Now let $u = 9$ and let $\mathcal{B}$ be the following collection of subsets of $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$:

$$\{1,2,3\},\quad \{1,5,9\},\quad \{1,6,8\},\quad \{1,4,7\},$$
$$\{4,5,6\},\quad \{2,6,7\},\quad \{2,4,9\},\quad \{2,5,8\},$$
$$\{7,8,9\},\quad \{3,4,8\},\quad \{3,5,7\},\quad \{3,6,9\}.$$

If $u' = 1$, $\mathcal{B}^* = \emptyset$ and $\alpha$ may be any cyclic permutation of $Y$.

If $u' = 3$, let $\mathcal{B}^*$ consist of the three lines in the last column and take $\alpha = (1, 2, 3, 4, 5, 6, 7, 8, 9)$.

If $u' = 7$, let $\mathcal{B}^*$ consist of the 9 lines in the first three columns and take $\alpha = (1, 2, 3, 4, 5, 6, 7, 8, 9)$.

If $u' = 9$, $\mathcal{B}^* = \mathcal{B}$ and $\alpha$ may be any cyclic permutation of $Y$.

**Proposition 2.2.** *Let $u \geqslant u'$ be two given admissible integers such that the congruences $u \equiv 1 \pmod 6$ and $u' \equiv 3 \pmod 6$ do not hold simultaneously. If $v = 2u + u'$, there exists an $S(v)$ containing an $S(u)$ and a disjoint $S(u')$.*[1]

**Proof.** Let $(Y, \mathcal{B})$ be an $S(u)$ satisfying the conditions of Lemma 2.1; select some point $p_1 \in Y$ and let $p_1, p_2, ..., p_{u'}$ denote the points of $Y$ lying on the lines of $\mathcal{B}^*$ containing $p_1$. Let $(Y', \mathcal{B}')$ be an $S(u')$ with $Y' = \{p'_1, p'_2, ..., p'_{u'}\}$.

Consider the set $X = (Y \times \{1, 2\}) \cup Y'$ of cardinality $v = 2u + u'$ and let $\mathcal{A}_\alpha$ denote the collection of all subsets of $X$ of the form $\{p'_j, (\alpha^i(p_1))_1, (\alpha^i(p_j))_2\}$, where $1 \leqslant i \leqslant u$ and $1 \leqslant j \leqslant u'$. For each line $B = \{x, y, z\} \in \mathcal{B}$, let

$$\mathcal{A}_B = \{\{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}\}$$

if $B \in \mathcal{B}^*$, and

$$\mathcal{A}_B = \{\{x_1, y_1, z_1\}, \{x_1, y_2, z_2\}, \{y_1, z_2, x_2\}, \{z_1, x_2, y_2\}\}$$

if $B \notin \mathcal{B}^*$.

Then, with $\mathcal{A} = \mathcal{A}_\alpha \cup (\bigcup_{B \in \mathcal{B}} \mathcal{A}_B) \cup \mathcal{B}'$, the pair $(X, \mathcal{A})$ is an $S(v)$ containing a subsystem of order $u$ on the subset $Y_1$ and a subsystem of order $u'$ on the subset $Y'$.

**Proposition 2.3.** *Let $u \equiv 3 \pmod 6$. If $v = 4u + 1$, there exists an $S(v)$ containing an $S(u)$.*

**Proof.** Let $(Y, \mathcal{B})$ be a Kirkman system of order $u$ and let $\mathcal{B}_1$ be a parallel class of lines of $\mathcal{B}$. Consider the set $X = (Y \times \{1, 2, 3, 4\}) \cup \{\infty\}$ of cardinality $v = 4u + 1$. Let $B = \{x, y, z\}$ be any line of $\mathcal{B}$. If $B \in \mathcal{B}_1$, let $\mathcal{A}_B$ denote the collection of lines of an $S(13)$ constructed on the subset $(B \times \{1, 2, 3, 4\}) \cup \{\infty\}$ of $X$ in such a way that $\{x_1, y_1, z_1\} \in \mathcal{A}_B$. If

[1] This has been noticed independently by Rosa [8] and later also by Robinson [7].

$B \notin \mathcal{B}_1$, let $\mathcal{A}_B$ denote the collection of transversals of a $T(3,4)$ whose groups are $\{x_1, x_2, x_3, x_4\}$, $\{y_1, y_2, y_3, y_4\}$, $\{z_1, z_2, z_3, z_4\}$ and such that $\{x_1, y_1, z_1\} \in \mathcal{A}_B$.

Then $(X, \bigcup_{B \in \mathcal{B}} \mathcal{A}_B)$ is an $S(v)$ containing a subsystem of order $u$ on the subset $Y_1$.

**Lemma 2.4.** *Let $(Y, \mathcal{B})$ be a finite linear space on $v$ points. If $|B| \equiv 0$ or 1 (mod 3) for every line $B \in \mathcal{B}$, there exists a Steiner triple system $(X, \mathcal{A})$ of order $2v+1$ containing a subsystem of order $2|B|+1$ for every $B \in \mathcal{B}$.*

**Proof.** Consider the set $X = (Y \times \{1,2\}) \cup \{\infty\}$ of cardinality $2v+1$. For each line $B \in \mathcal{B}$, the subset $X_B = (B \times \{1,2\}) \cup \{\infty\}$ has cardinality $2|B|+1 \equiv 1$ or 3 (mod 6); let $\mathcal{A}_B$ denote the collection of lines of a Steiner triple system constructed on $X_B$ in such a way that $\{x_1, x_2, \infty\} \in \mathcal{A}_B$ for every $x \in B$. Then $(X, \bigcup_{B \in \mathcal{B}} \mathcal{A}_B)$ is an $S(2v+1)$ with the required properties.

**Lemma 2.5.** *There exists a $T_n(3,n)$ for every positive integer $n \neq 2, 6$. There exists a $T_4(3,6)$ and no $T_1(3,2)$.*

**Proof.** If $n \neq 2$ and 6, there is a $T(4,n)$ with 4 groups $A_1, A_2, A_3, A_4$ of size $n$. The set $X = A_1 \cup A_2 \cup A_3$, provided with the groups $A_1, A_2, A_3$ and the restriction to $X$ of all transversals of the $T(4,n)$, is a $T_n(3,n)$; the $n$ parallel classes are obtained by taking the restriction to $X$ of all transversals containing one of the $n$ points of $A_4$.

Hanani has proved [3, Theorem 2.12] that the existence of a $T_s(m,n)$ and of a $T(m,s)$ implies the existence of a $T_{s^2}(m, ns)$. By applying this result in the particular case where $m = n = 3$ and $s = 2$, we get a $T_4(3,6)$.

Finally, it is easy to check that the $T(3,2)$ is unique (up to isomorphism) and contains no parallel class of transversals.

**Lemma 2.6.** *For every positive integer $t$, there exists a linear space $(X, \mathcal{A})$ on $24t+1$ points with one line of size $6t+1$, three lines of size $6t$, $6t$ lines of size 4 and all remaining lines of size 3.*

**Proof.** Let $(Y, \mathcal{B})$ be the linear space whose set of points is $Y = \{a, a', b, b', c, c', d, d'\}$ and whose set of lines is $\mathcal{B} = \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_3'$, where

$\mathcal{B}_2 = \{\{a, a'\}, \{b, b'\}, \{c, c'\}, \{d, d'\}\}$,

$\mathcal{B}_3 = \{\{a, b', d'\}, \{b, c', d'\}, \{c, a', d'\}, \{a', b, d\}, \{b', c, d\}, \{c', a, d\}\}$,

$\mathcal{B}'_3 = \{\{a, b, c\}, \{a', b', c'\}\}$.

Let $I_{3t} = \{1, 2, ..., 3t\}$ and consider the set $X = (Y \times I_{3t}) \cup \{\infty\}$ of cardinality $24t + 1$. By Lemma 2.5, there is a $T_1(3, 3t)$ for every $t$. For each line $B \in \mathcal{B}_3 \cup \mathcal{B}'_3$, let $\mathcal{A}_B$ denote the collection of transversals of a $T_1(3, 3t)$ constructed on $B \times I_{3t}$ and having as groups the 3 subsets $\{x\} \times I_{3t}$, where $x \in B$. Let $\mathcal{A}^*_B$ be a parallel class of transversals of $\mathcal{A}_B$ and, for each line $B \in \mathcal{B}'_3$, put $\mathcal{A}'_B = \mathcal{A}_B - \mathcal{A}^*_B$. Finally, let

$$\mathcal{A}_0 = \{\{a, a'\} \times I_{3t}, \{b, b'\} \times I_{3t}, \{c, c'\} \times I_{3t}\}$$

and let $\mathcal{A}_\infty$ denote the collection of the following $6t + 1$ subsets of $X$:

$$(\{d, d'\} \times I_{3t}) \cup \{\infty\}$$

and $A'_B \cup \{\infty\}$ for every $A'_B \in \mathcal{A}'_B$, where $B \in \mathcal{B}'_3$.

Then, with

$$\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_\infty \cup (\cup_{B \in \mathcal{B}_3} \mathcal{A}_B) \cup (\cup_{B \in \mathcal{B}'_3} \mathcal{A}'_B),$$

the pair $(X, \mathcal{A})$ is a linear space with the required properties.

**Proposition 2.7.** *Let $u \equiv 1 \pmod 6$. If $v = 4u - 1$, there exists an $S(v)$ containing an $S(u)$.*

**Proof.** (a) If $u = 12t + 1$ for some integer $t$, the existence of an $S(48t + 3)$ containing an $S(12t + 1)$ follows from Lemmas 2.4 and 2.6.

(b) Suppose now $u = 12t + 7$. By Proposition 2.3, there exists a Steiner triple system $(X, \mathcal{A})$ of order $24t + 13$ containing a subsystem $(Y, \mathcal{B})$ of order $6t + 3$. The pair $(X, \mathcal{A}')$, where $\mathcal{A}' = (\mathcal{A} - \mathcal{B}) \cup \{Y\}$, is a linear space on $24t + 13$ points with one line of size $6t + 3$ and all other lines of size 3. Therefore, by Lemma 2.4, there is an $S(48t + 27)$ containing an $S(12t + 7)$.

**Lemma 2.8.** *Let $s \leqslant n$ be two given positive integers* $\equiv 0$ *or* 1 (mod 3). *There exists a linear space $(X', \mathcal{A}')$ on $3n+s$ points such that every line has size* $\equiv 0$ *or* 1 (mod 3) *and at least one line has size $n$ if $n \geqslant 3$.*

**Proof.** For every such $n$ and $s$, except for $n = s = 6$, Lemma 2.5 implies the existence of a $T_s(3, n)$. Let $(Y, \mathcal{B})$ be such a transversal system and let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_s \cup \mathcal{B}'$, where $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_s$ are $s$ pairwise disjoint parallel classes of transversals. Take $X' = Y \cup Y'$, where the set $Y' = \{\infty_1, \infty_2, \ldots, \infty_s\}$ is disjoint from $Y$. Finally, for each $i = 1, \ldots, s$, let

$$\mathcal{B}'_i = \{B_i \cup \{\infty_i\} \mid B_i \in \mathcal{B}_i\}.$$

Then the pair $(X', \mathcal{A}')$, where

$$\mathcal{A}' = \mathcal{B}'_1 \cup \mathcal{B}'_2 \cup \ldots \cup \mathcal{B}'_s \cup \mathcal{B}' \cup \{Y'\},$$

is a linear space with the required properties.

If $n = s = 6$, let $(X, \mathcal{A})$ be the linear space on 25 points constructed in Lemma 2.6 (corresponding to the value $t = 1$); remember that $\mathcal{A}_\infty$ was the set of all lines of $\mathcal{A}$ containing the point $\infty$. Take $X' = X - \{\infty\}$ and $\mathcal{A}'_\infty = \{A - \{\infty\} \mid A \in \mathcal{A}_\infty\}$. The linear space $(X', \mathcal{A}')$ with $\mathcal{A}' = (\mathcal{A} - \mathcal{A}_\infty) \cup \mathcal{A}'_\infty$ satisfies the required properties.

**Proposition 2.9.** *Let $u, v \equiv 1$ or* 3 (mod 6). *If $3u \leqslant v \leqslant 4u - 3$, there exists an $S(v)$ containing an $S(u)$.*

**Proof.** Write $w = v - 3(u-1)$. Clearly, $w \equiv 1$ or 3 (mod 6) and the inequality $3u \leqslant v \leqslant 4u - 3$ implies $3 \leqslant w \leqslant u$.

Let $s = \frac{1}{2}(w-1)$ and $n = \frac{1}{2}(u-1)$, so that $s, n \equiv 0$ or 1 (mod 3) and $1 \leqslant s \leqslant n$. As $3n + s = \frac{1}{2}(v-1)$, the existence of an $S(v)$ containing an $S(u)$ follows immediately from Lemmas 2.4 and 2.8.

**Lemma 2.10.** *Let $k$ be an odd integer, $1 \leqslant k \leqslant 12t + 5$. There exists a linear space $(X, \mathcal{A})$ on $12t + 6$ points and a partition $\mathcal{A} = \mathcal{A}^* \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_k$ such that $\mathcal{A}^*$ consists of lines of size* 3 *and each $\mathcal{A}_i$, $1 \leqslant i \leqslant k$, is a parallel class of lines of size* 2.

**Proof.** Let $(Y, \mathcal{B})$ be a Kirkman system of order $6t+3$ and let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_r$ $(r = 3t+1)$ be a partition of $\mathcal{B}$ into parallel classes. Take $X = Y \times \{1, 2\}$ and write $k = 4l + m$, where $m = 1$ or $3$. Here $l \leqslant 3t+1$ if $m = 1$ and $l \leqslant 3t$ if $m = 3$.

For each line $B = \{x, y, z\} \in \mathcal{B}_i$, $1 \leqslant i \leqslant l$, let $\mathcal{A}_B$ denote the collection of the following subsets of $X$:

$$\{x_1, y_1\}, \{z_1, x_2\}, \{y_2, z_2\},$$

$$\{x_1, z_1\}, \{y_1, z_2\}, \{x_2, y_2\},$$

$$\{x_1, z_2\}, \{z_1, y_2\}, \{y_1, x_2\},$$

$$\{x_1, y_2\}, \{x_2, z_2\}, \{y_1, z_1\}.$$

Note that for each $i = 1, \ldots, l$, $\bigcup_{B \in \mathcal{B}_i} \mathcal{A}_B$ admits a partition into four parallel classes of lines of size 2. Let $\mathcal{A}_k = \{\{x_1, x_2\} \mid x \in Y\}$. We now have $4l + 1$ parallel classes on $X$.

Suppose $m = 1$, so that $k = 4l + 1$. For each $B = \{x, y, z\} \in \mathcal{B}_i$, $l < i \leqslant r$, let

$$\mathcal{A}_B = \{\{x_1, y_1, z_1\}, \{x_1, y_2, z_2\}, \{y_1, z_2, x_2\}, \{z_1, x_2, y_2\}\}.$$

Then with $\mathcal{A} = (\bigcup_{B \in \mathcal{B}} \mathcal{A}_B) \cup \mathcal{A}_k$, $(X, \mathcal{A})$ is a linear space with the required properties.

Consider now the case $m = 3$, so that $k = 4l + 3$. For each $B = \{x, y, z\} \in \mathcal{B}_{l+1}$, let $\mathcal{A}_B$ denote the collection of the following subsets of $X$:

$$\{x_1, y_1, z_1\}, \{x_2, y_2, z_2\},$$

$$\{x_1, y_2\}, \{y_1, z_2\}, \{z_1, x_2\},$$

$$\{x_1, z_2\}, \{y_1, x_2\}, \{z_1, y_2\},$$

and for each $B = \{x, y, z\} \in \mathcal{B}_i$, $l + 1 < i \leqslant r$, as before let

$$\mathcal{A}_B = \{\{x_1, y_1, z_1\}, \{x_1, y_2, z_2\}, \{y_1, z_2, x_2\}, \{z_1, x_2, y_2\}\}.$$

Note that the subsets of size 2 occurring in $\bigcup_{B \in \mathcal{B}_{l+1}} \mathcal{A}_B$ can be partitioned into two parallel classes and thus the linear space $(X, \mathcal{A})$, where $\mathcal{A} = (\bigcup_{B \in \mathcal{B}} \mathcal{A}_B) \cup \mathcal{A}_k$, has the required properties.

**Proposition 2.11.** *Let* $u \equiv 1$ *or* 3 (mod 6) *and* $v = u + (12t + 6)$ *for some integer* $t$. *If* $v \geq 2u + 1$, *there exists an* $S(v)$ *containing an* $S(u)$.

**Proof.** Let $(Y, \mathcal{B})$ be an $S(u)$ with $Y = \{\infty_1, \infty_2, \ldots, \infty_u\}$. Clearly, $u$ is odd and $1 \leq u \leq 12t + 5$ because of the hypothesis $v \geq 2u + 1$. Let $(X, \mathcal{A})$ be the linear space of Lemma 2.10, with $k = u$ and $\mathcal{A} = \mathcal{A}^* \cup \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_u$.

Consider the set $X' = X \cup Y$ of cardinality $v = u + (12t + 6)$. For each $i = 1, \ldots, u$, let

$$\mathcal{A}_i' = \{A \cup \{\infty_i\} \mid A \in \mathcal{A}_i\}.$$

Then, with $\mathcal{A}' = \mathcal{A}^* \cup \mathcal{A}_1' \cup \ldots \cup \mathcal{A}_u' \cup \mathcal{B}$, $(X', \mathcal{A}')$ is an $S(v)$ containing $(Y, \mathcal{B})$ as a subsystem of order $u$.

**Theorem 2.12.** *Let* $u$ *and* $v$ *be two admissible integers. Whenever* $v \geq 2u + 1$, *there is an* $S(v)$ *containing an* $S(u)$.

**Proof.** The theorem is clearly valid for $u = 1$ or 3, so we shall fix $u$ and assume $u > 3$ (and hence $u \geq 7$).

We remark that if there is an $S(v)$ containing an $S(w)$ and an $S(w)$ containing an $S(u)$, then surely there is an $S(v)$ containing an $S(u)$.

We first claim that there is an $S(v)$ containing an $S(u)$ for every $v \equiv 3$ (mod 6) with $2u + 1 \leq v \leq 3u$; for $u' = v - 2u$ is admissible and satisfies $1 \leq u' \leq u$, whence our claim follows from Proposition 2.2. Propositions 2.3, 2.7 and 2.9 show that the theorem is valid whenever $3u \leq v < 4u + 3$. The theorem also holds for $v = 4u + 3 = 2(2u + 1) + 1$, since we have an $S(2u + 1)$ containing an $S(u)$ and an $S(4u + 3)$ containing an $S(2u + 1)$, both by Proposition 2.2.

We now proceed by induction to prove the validity of the theorem for $v \geq 3u$. Let an admissible $v_0$ be given, $v_0 \geq 3u$, and assume the assertion of the theorem for all $v$, $3u \leq v < v_0$. If $v_0 \leq 4u + 3$, there is an $S(v_0)$ containing an $S(u)$ as we have observed above. If $v_0 > 4u + 3$, write (uniquely) $v_0 = 2w + u'$, where $w \equiv 3$ (mod 6) and $u' = 1, 3, 7$ or 9. The inequalities $v_0 > 4u + 3$, $u \geq 7$, $u' \leq 9$ and the congruence $w \equiv 3$ (mod 6) imply $u' < 2u + 1 \leq w < v_0$. Now by our claim of the previous paragraph (if $w \leq 3u$) or by our induction hypothesis (if $w \geq 3u$), there is an $S(w)$ containing an $S(u)$; and by Proposition 2.2, there is an $S(v_0)$

containing an $S(w)$. We conclude that the theorem holds for $v_0$, and, inductively, for every admissible $v \geqslant 3u$.

Suppose that $u \equiv 3 \pmod 6$, $v \equiv 1 \pmod 6$ and $2u + 1 \leqslant v \leqslant 3u$. Here $u' = v - 2u$ is admissible and $1 \leqslant u' \leqslant u$, so there is an $S(v)$ containing an $S(u)$ by Proposition 2.2. The proof is now complete in the case that $u \equiv 3 \pmod 6$.

So it remains only to prove that if $u, v \equiv 1 \pmod 6$, $2u + 1 \leqslant v \leqslant 3u$, there exists an $S(v)$ containing an $S(u)$. The proof given below uses the fact that the theorem is now known to be true for every admissible $v \geqslant 3u$.

**Proposition 2.13.** *Let* $u \equiv 1$ *or* $3 \pmod 6$ *be given. Then for every* $v \equiv u \pmod 6$ *with* $2u + 1 \leqslant v \leqslant 3u$, *there exists an* $S(v)$ *containing an* $S(u)$.

**Proof.** We proceed by induction on $u$. Let $u_0 \equiv 1$ or $3 \pmod 6$ and assume the validity of Proposition 2.13 for every $u \equiv 1$ or $3 \pmod 6$, $u < u_0$. (Proposition 2.13 is clearly valid for $u = 1$ or $3$.)

Now let $v$ be given, $2u_0 + 1 \leqslant v \leqslant 3u_0$. If $v - u_0 \equiv 6 \pmod{12}$, the existence of an $S(v)$ containing an $S(u)$ is asserted by Proposition 2.11, so we assume $v = u_0 + 12t$ for some integer $t$. Let $w = u_0 - 6t$, so that $w \equiv u_0 \pmod 6$. The inequality $2u_0 + 1 \leqslant v \leqslant 3u_0$ implies $u_0 \geqslant 2w + 1$ and thus, by the partial result of the above theorem (if $u_0 > 3w$) or our induction hypothesis (if $u_0 \leqslant 3w$), there exists an $S(u_0)$ containing an $S(w)$. Let $\mathcal{A}'$ be the collection of all transversals of a $T(3, 6t)$ with 3 groups $X_1', X_2', X_3'$ of size $6t$ and let $(Y, \mathcal{B})$ be an $S(w)$. Consider the set $X = X_1' \cup X_2' \cup X_3' \cup Y$ of cardinality $18t + w = v$. For each $i = 1, 2, 3$, let $(X_i' \cup Y, \mathcal{A}_i' \cup \mathcal{B})$ be an $S(u_0)$ containing $(Y, \mathcal{B})$ as a subsystem of order $w$. Then, with $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}_1' \cup \mathcal{A}_2' \cup \mathcal{A}_3' \cup \mathcal{B}$, the pair $(X, \mathcal{A})$ is an $S(v)$ containing an $S(u_0)$.

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