ELSEVIER
Discrete Mathematics 192 (1998) 187–204
DISCRETE MATHEMATICS

On one maximum multiflow problem and related metrics [1]

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Received 4 November 1996; revised 20 July 1997; accepted 14 August 1997

Abstract

We consider the undirected maximum multiflow (multicommodity flow) problem in the case when the commodity graph is the disjoint union of K_3 and K_2. We prove that if the supply graph satisfies a certain Eulerian-type condition, then the problem has an integer optimal solution. To obtain this result, we first study the corresponding dual problem on metrics and show that an optimal solution to the latter is achieved on some (2,3)-metric or some 3-cut metric. © 1998 Elsevier Science B.V. All rights reserved

Keywords: Multicommodity flow; Finite metric; Cut; Edge-disjoint paths

1. Introduction

The main problem that we deal with in this paper is defined by a *supply* graph $G=(V,E)$ and a *commodity* graph $H=(W,U)$. Both graphs are undirected, G is allowed to have parallel edges, and W is a subset of V called the set of *terminals*. An *s–t path* is a path P from a node s to a node t in G; if $\{s,t\}$ forms an edge of H, then P is called an *H-path*. In the *maximum edge-disjoint paths problem*, one wishes to

$$\text{Find a largest set of pairwise edge-disjoint } H\text{-paths in } G. \tag{1.1}$$

The maximum number of such paths is denoted by $v=v(G,H)$. Besides, we consider the fractional relaxation of (1.1). More precisely, by a *multiflow* (multicommodity flow) we mean a collection of H-paths $P_1,\dots,P_k$ along with nonnegative real weights $\lambda_1,\dots,\lambda_k$. A multiflow $f=(P_1,\dots,P_k;\lambda_1,\dots,\lambda_k)$ is *admissible* if the total multiflow through each edge is at most one, i.e.,

$$f^e := \sum(\lambda_i : e\in P_i)\leqslant 1 \quad \text{for each } e\in E. \tag{1.2}$$

[1] This research was done when the author was visiting LRI, University Paris-XI and supported by European Union grant INTAS-93-2530-ext.

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PII S0012-365X(98)00071-5

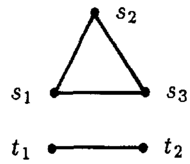

Fig. 1.

Unless otherwise is said, we assume that every multiflow in question is admissible. The *value* of f is $\lambda_1 + \cdots + \lambda_k$, denoted by val($f$). The *maximum multiflow problem* for (G,H) is:

$$\text{Find a multiflow } f \text{ whose value is as large as possible.} \tag{1.3}$$

This maximum value is denoted by v^*, and f is called *maximum* if $\mathrm{val}(f) = v^*$. Obviously, $v \leqslant v^*$ holds in general, and one is often interested in special cases when this inequality turns into equality, or, roughly, when (1.3) becomes equivalent to (1.1). The simplest case with $v = v^*$ arises if H consists of a single edge (while G is arbitrary), due to the classic result that the maximum single commodity flow problem has an integer optimal solution [1,12]. Subsequently, other interesting special cases of (G,H) with $v = v^*$ have been found (see [2,4,8,11] for a survey). In this paper we describe one more non-trivial class with such a property.

More precisely, we assume that the commodity graph H is formed by five nodes s_1,s_2,s_3,t_1,t_2 and four edges $s_1s_2,s_2s_3,s_1s_3,t_1t_2$. In other words, H represents the disjoint union of a triangle and an edge, which is abbreviated as $H = K_3 + K_2$; see Fig. 1. In addition, we assume that G is *pseudo-Eulerian*; this means that

$$|\delta(X)| \text{ is even if either } X \subseteq V - W \text{ or } X \cap W = \{t_1,t_2\}. \tag{1.4}$$

Hereinafter for $X \subseteq V$, $\delta(X) = \delta^G(X)$ is the set of edges of G with one end in X and the other in $V - X$, a *cut* in G.

We prove the following theorem.

Theorem 1. *If $H = K_3 + K_2$ and G is pseudo-Eulerian, then $v = v^*$.*

This fact immediately implies that for an arbitrary G and $H = K_3 + K_2$, problem (1.3) has a *half-integer* optimal solution $f = (P_1,\dots,P_k;\lambda_1,\dots,\lambda_k)$, i.e., with all λ_i's multiple of $\frac{1}{2}$.

Remark. A similar property of half-integrality has been established for a wide class of commodity graphs by Lomonosov and the author in [9] (proofs in details are given in [5,11]). Namely, it turned out that (1.3) always has a half-integer optimal solution if H is *anticlique-bipartite*, in the sense that the family of all (inclusion) maximal stable sets of H admits a partition into two subfamily, each consisting of pairwise disjoint sets (see also [7,11] for a stronger version and [3] for a shorter proof of this version). On the other hand, it was shown in [8, Section 5] that for each fixed H, (1.3) has no half-integer optimal solution for *some* G, unless H is anticlique-bipartite or $H = K_3 + K_2$. Also [8] announced that a half-integer optimal solution exists if $H = K_3 + K_2$. The present paper proves this result in a sharper form.

Since (1.3) is a linear program which has an integer optimal solution in our case by Theorem 1, it is reasonable to ask whether the dual program has an optimal solution of a special form. A 'nice' optimal dual solution does exist: we reveal a combinatorial minimax relation involving v^* and a value depending on certain metrics.

To state this, consider a metric m on V, i.e., a function $m : V \times V \to \mathbb{R}_+$ satisfying (i) $m(x,x) = 0$, (ii) $m(x,y) = m(y,x)$, and (iii) $m(x,y) + m(y,z) \geqslant m(x,z)$, for all $x,y,z \in V$. We allow $m(x,y) = 0$ for distinct x,y. Because of (i) and (ii) we may think that m is, in fact, defined on the edges of the complete undirected graph $K_V = (V,E_V)$ on V, and write $m(xy)$ instead of $m(x,y)$. The linear program dual of (1.3) can be viewed as follows (see [11]):

$$\text{Find a metric } m \text{ on } V \text{ such that } m(E) \text{ is as small as possible, provided that } m(uv) = 1 \text{ for all pairs } uv \in U. \tag{1.5}$$

(For a function $g : S \to \mathbb{R}$ and a subset $S' \subseteq S$, $g(S')$ denotes $\sum(g(e) : e \in S')$.) Let $\tau = \tau(G,H)$ be the minimum $m(E)$ in (1.5). Then $\tau = v^* \geqslant v$ holds for arbitrary G and H, and $\tau = v$ holds for our special case stated in Theorem 1.

One sort of metrics feasible to (1.5) with $H = K_3 + K_2$ is described as follows. For disjoint subsets $X, Y \subset V$, let (X,Y) be the set of edges of K_V with one end in X and the other in Y. Consider a partition $\Pi = (S_1,S_2,S_3,T_1,T_2)$ of V such that $s_i \in S_i$ for $i = 1,2,3$ and $t_j \in T_j$ for $j = 1,2$. Then Π induces the metric $m = m^{\Pi}$ defined by

$$\begin{aligned} m(e) &= 1/2 && \text{for } e \in (S_i,T_j),\ i = 1,2,3,\ j = 1,2, \\ &= 1 && \text{for } e \in (S_1,S_2) \cup (S_2,S_3) \cup (S_1,S_3) \cup (T_1,T_2), \\ &= 0 && \text{otherwise.} \end{aligned} \tag{1.6}$$

We refer to m as a (2,3)-*metric* (this slightly differs from the usual definition where by a (2,3)-metric one means the metric $2m$). Let $M^{2,3} = M^{2,3}(V,H)$ denote the set of such metrics.

Another sort of feasible metrics comes up from triples $\Xi = (A,B,C)$ of pairwise disjoint subsets of V such that

$$A \cap W = \{s_i,t_1\}, \quad B \cap W = \{s_j,t_2\} \quad \text{and} \quad C \cap W = \{s_k\}, \tag{1.7}$$

where $\{i,j,k\} = \{1,2,3\}$. Then Ξ induces the metric $m = m^{\Xi}$ to be the half-sum of the cut metrics corresponding to the cuts $\delta(A), \delta(B), \delta(C)$, i.e.,

$$\begin{aligned} m(e) &= 1 && \text{for } e \in (A,B) \cup (B,C) \cup (A,C), \\ &= 1/2 && \text{for } e = uv,\ u \in A \cup B \cup C,\ v \in V - (A \cup B \cup C), \\ &= 0 && \text{otherwise.} \end{aligned} \tag{1.8}$$

We call m a 3-*cut metric* and denote their set by $M^3 = M^3(V,H)$. Obviously, our (2,3)-metrics and 3-cut metrics are feasible to (1.5). Moreover, it turns out that, in fact, only such metrics are essential.

Theorem 2. *If $H = K_3 + K_2$, then $\tau = m(E)$ for some $m \in M^{2,3} \cup M^3$.*

This paper is organized as follows. Theorem 2 is proved in Section 2. Note that this theorem can be derived from a general result in [8]; however, we prefer to give an independent combinatorial proof. Using Theorem 2, we then prove Theorem 1. The proof is divided into two parts. First, in Section 3, we show the existence of a *half-integer* maximum multiflow in our case. This exploits a variant of splitting-off techniques similar to that elaborated in [6] for the multiflow demand problem in the five terminal case (and also applied for another multiflow problem in [10]). Second, in Section 4, we show that a maximum half-integer multiflow can be transformed into an integer multiflow, thus proving Theorem 1. However, such a transformation is more complicated than the corresponding transformation in [6].

When it is not confusing, a path $P = (x_0, e_1, x_1, \ldots, e_k, x_k)$ is denoted by $x_0x_1 \ldots x_k$. For a function g on the edges of a graph where P is defined, $g(P)$ denotes $\sum(g(e_i) : i = 1, \ldots, k)$.

2. Optimal metrics

Since the metrics feasible to (1.5) are described via linear constraints, they form a polyhedron $\mathscr{P} = \mathscr{P}(V,H)$ in the Euclidean space $\mathbb{R}^{E_V}$ whose coordinates are indexed by the edges of the complete graph K_V. Let $m, m' \in \mathscr{P}$. We say that m' *decomposes* m with respect to H if there exists $m'' \in \mathscr{P}$ such that $m \geqslant \lambda m' + (1-\lambda)m''$ for some $0 < \lambda \leqslant 1$. If m is optimal and m' decomposes m, then the obvious inequality $m(E) \geqslant \lambda m'(E) + (1-\lambda)m''(E)$ together with $m(E) \leqslant m'(E), m''(E)$ implies $m'(E) = m(E)$, i.e., m' is also optimal. Therefore, Theorem 2 will follow from the fact that any metric in $\mathscr{P}(V, K_3 + K_2)$ is decomposed by some (2,3)-metric or some 3-cut metric. We prove the latter in the rest of this section.

Consider $m \in \mathscr{P}(V,H)$ for $H = (W,U) = K_3 + K_2$. Let $\overline{U}$ be the set of pairs s_it_j for $i = 1,2,3$ and $j = 1,2$. Without loss of generality, one may assume that

each two $x, y \in V$ belong to a shortest H-path; in other words,
$$m(ux) + m(xy) + m(yv) = 1 \text{ holds for some } uv \in U. \tag{2.1}$$

For otherwise one can decrease m on some pairs so as to get a smaller metric m' on V still satisfying $m'(e) = 1$ for all $e \in U$, and then work with m' instead of m. (2.1) easily implies that

for each $e \in \overline{U}$, there is an adjacent edge e' in $\overline{U}$ such that
$$m(e) + m(e') = 1 \tag{2.2}$$

(edges are *adjacent* is they share a common node).

Claim 1. *At least one of the following is true*:
(i) $m(e) = \frac{1}{2}$ *for all* $e \in \overline{U}$;
(ii) *there are two non-adjacent edges* $e, e' \in \overline{U}$ *such that* $m(e), m(e') < \frac{1}{2}$, *and* $m(\overline{e}) > \frac{1}{2}$ *for the other edges* $\overline{e}$ *in* $\overline{U}$;
(iii) *there is an* $e \in \overline{U}$ *with* $m(e) < \frac{1}{2}$ *such that the two adjacent edges* $e', e'' \in \overline{U}$ *disjoint from* e *satisfy* $m(e') = m(e'') = \frac{1}{2}$, *and* $m(\overline{e}) > \frac{1}{2}$ *for the other edges* $\overline{e}$ *in* $\overline{U}$.

Proof. For any two adjacent edges e, e' in $\overline{U}$, their non-common ends form an edge in U, therefore, $m(e) + m(e') \geqslant 1$. Also, any triple of edges in $\overline{U}$ includes an adjacent pair. Hence, at most two edges $e \in \overline{U}$ with $m(e) < \frac{1}{2}$ are possible, and such edges cannot be adjacent. In view of (2.2), only cases (i)–(iii) are possible. □

In case (i) of this claim, the restriction of m to W is exactly $\frac{1}{2}$ times the distance function of the graph $(W, \overline{U}) = K_{2,3}$. As is shown in [6], such an m is decomposed w.r.t. H by a (2,3)-metric.

Consider case (ii) of Claim 1. Let for definiteness $m(s_1t_1) = a < \frac{1}{2}$ and $m(s_2t_2) = b < \frac{1}{2}$. We show that m is decomposed by the 3-cut metric induced by the triple $\Xi = (A, B, C)$, where

$$\begin{aligned} A &= \{x \in V : m(s_1x) + m(xt_1) = a\}, \\ B &= \{x \in V : m(s_2x) + m(xt_2) = b\}, \\ C &= \{x \in V : m(s_3x) = 0\}. \end{aligned} \tag{2.3}$$

Clearly, $s_1, t_1 \in A$, $s_2, t_2 \in B$ and $s_3 \in C$, whence Ξ satisfies (1.7).

Claim 2. *Let* $x \in A$ *and* $y \in B$. *Then* $m(xy) > \frac{1}{2}$.

Proof. By (2.3), $m(s_1x) + m(xt_1) < \frac{1}{2}$ and $m(s_2y) + m(yt_2) < \frac{1}{2}$. Since m is a metric,

$$m(s_1x) + m(xy) + m(ys_2) \geqslant m(s_1s_2) = 1$$

and

$$m(t_1x) + m(xy) + m(yt_2) \geqslant m(t_1t_2) = 1.$$

This implies $2m(xy) > 1$, or $m(xy) > \frac{1}{2}$, as required. □

Claim 3. *Let* $x \in A$ *and* $z \in C$. *Then* $m(xz) > \frac{1}{2}$. *Similarly*, $m(yz) > \frac{1}{2}$ *for* $y \in B$ *and* $z \in C$. □

Proof. We have $m(s_1x) < \frac{1}{2}$, $m(s_3z) = 0$ and $m(s_1x) + m(xz) + m(zs_3) \geqslant m(s_1s_3) = 1$. Therefore, $m(xz) > \frac{1}{2}$. □

These two claims show that A, B, C are pairwise disjoint.

Claim 4. *Every two nodes x, y in A belong to a shortest path from s_1 to t_1. Similarly, every pair of nodes in B belongs to a shortest path from s_2 to t_2.*

Proof. By (2.1), $m(ux)+m(xy)+m(yv) = 1$ for some $uv \in U$. Suppose that $\{s_1t_1\} \cap \{u,v\} \neq \emptyset$; say, $s_1 = u$. Since the path s_1xyv is shortest for m, $m(s_1x)+m(xy) = m(s_1y)$. Then $m(s_1y)+m(yt_1) = a$ (by (2.3)) implies $m(s_1x)+m(xy)+m(yt_1) = a$, i.e., the path s_1xyt_1 is shortest.

Now, suppose that $\{s_1,t_1\} \cap \{u,v\} = \emptyset$. This is possible only if $\{u,v\} = \{s_2,s_3\}$. Then $m(s_1x)+m(xu) \geqslant 1$ and $m(s_1y)+m(yv) \geqslant 1$. Adding these inequalities and subtracting $m(ux)+m(xy)+m(yv) = 1$ yields $m(s_1x)+m(s_1y)-m(xy) \geqslant 1$. But both $m(s_1x)$ and $m(s_1y)$ are at most $a < \frac{1}{2}$; a contradiction. □

Claim 5. $m(xy) = 0$ *for any* $x, y \in C$.

Proof. This follows from $m(s_3x) = m(s_3y) = 0$ and $m(xy) \leqslant m(xs_3)+m(s_3y)$. □

An immediate corollary from Claims 4 and 5 is that

if nodes x and y belong to the same set X among A, B, C, and
P is a shortest $x-y$ path, then all nodes of P belong to X as well. (2.4)

Claim 6. *Let $P = x_0x_1 \ldots x_k$ be a shortest H-path. For $X = A, B, C$, let n_X be the number of times P meets the cut $\delta(X)$ (in K_V). Let $n = n_A + n_B + n_C$. Then $n = 2$.*

Proof. Since x_0 and x_k belong to different sets among A, B and C, n is an even integer $\geqslant 2$. We call a part $x_ix_{i+1} \ldots x_j$ of P a *segment* if x_i and x_j occur in different sets among A, B, C, and $x_{i+1}, \ldots, x_{j-1} \notin A \cup B \cup C$.

Consider a segment $x_i \ldots x_j$. Suppose that one of x_i and x_j, x_i say, is in A and the other, x_j, in B. Then $m(s_1x_i)+m(x_it_1) = a$ and $m(s_2x_j)+m(x_jt_2) = b$. These together with $m(s_1x_i)+m(x_ix_j)+m(x_js_2) \geqslant 1$ and $m(t_1x_i)+m(x_ix_j)+m(x_jt_2) \geqslant 1$ imply $2m(x_ix_j) \geqslant 2-(a+b)$. Since $a+b < 1$, we obtain $m(x_ix_j) > \frac{1}{2}$. Now suppose that one of x_i and x_j is in $A \cup B$, $x_i \in A$ say, and the other, x_j, is in C. Then $m(s_1x_i) \leqslant a$, $m(s_3x_j) = 0$ and $m(s_1x_i)+m(x_ix_j)+m(x_js_3) \geqslant 1$ yield $m(x_ix_j) \geqslant 1-a > \frac{1}{2}$.

Now, since $m(P) = 1$, P has at most one segment, and the claim follows. □

Claim 6 enables us to prove that m^{Ξ} decomposes m. First of all we observe that $m(e) > 0$ for any $e \in \delta(X)$. Indeed, for $x \in A$ and $y \in V$, $m(xy) = 0$ would imply $m(s_1y) = m(s_1x)$ and $m(t_1y) = m(s_1y)$, whence $m(s_1y)+m(yt_1) = a$, i.e., $y \in X$. Similarly, $m(e) > 0$ for any $e \in \delta(B) \cup \delta(C)$. Therefore, there exists $\varepsilon_1 > 0$ such that the function $m^{\varepsilon} := m - \varepsilon m^{\Xi}$ is nonnegative for any $0 \leqslant \varepsilon \leqslant \varepsilon_1$ (note that m^{ε} is not necessarily a metric). Next, Claim 6 implies that any *shortest* H-path P satisfies

$$1 = m(P) = m^{\varepsilon}(P) + \varepsilon m^{\Xi}(P) = m^{\varepsilon}(P) + \varepsilon.$$

Therefore, there exists $0 < \varepsilon \leqslant \varepsilon_1$ such that *any* H-path satisfies $m^{\varepsilon}(P) \geqslant 1 - \varepsilon$. Let m' be the distance function in K_V whose edges are weighted by $m^{\varepsilon}/(1-\varepsilon)$. Then $m'(e) = 1$ for all $e \in U$, and $m \geqslant \varepsilon m^{\Xi} + (1-\varepsilon)m'$. Thus, m^{Ξ} decomposes m.

Finally, in case (iii) of Claim 1, the proof is similar. Let for definiteness $m(s_1t_1) = a < \frac{1}{2}$ and $m(s_2t_2) = b = \frac{1}{2}$. Define A, B, C by (2.3), and let $\Xi = (A, B, C)$. Observe that Claims 2–6 remain valid (except for the second part of Claim 3 where $m(yz) > \frac{1}{2}$ should be replaced by $m(yz) \geqslant \frac{1}{2}$). In the above proofs of these claims we only need a slight correction in the proof of Claim 6 because it is now possible that $m(yz) = \frac{1}{2}$ for some $y \in B$ and $z \in C$. The latter implies that if some segment $x_i \ldots x_j$ of P connects B and C, then $m(x_ix_j) \geqslant \frac{1}{2}$ (instead of $m(x_ix_j) > \frac{1}{2}$). Nevertheless, in view of (2.4), at most one segment connecting B and C is possible, and now the claim that P has at most one segment at all remains correct.

This completes the proof of Theorem 2. □

3. Existence of a half-integer maximum multiflow

As mentioned in the Introduction, the first part of the proof of Theorem 1 given in this section is based on splitting-off techniques. To describe this, we first associate with the supply graph $G = (V, E)$ the function $c = c^G$ on E_V which, for $x, y \in V$, indicates the number of edges between x and y in G. We call $c(e)$ the *capacity* of an edge e. Accordingly, a multiflow $f = (P_1, \ldots, P_k; \lambda_1, \ldots, \lambda_k)$ for K_V and H is called *c-admissible* if

$$f^e \leqslant c(e) \quad \text{for each } e \in E_V, \tag{3.1}$$

cf. (1.2). A c-admissible multiflow determines in a natural way an admissible multiflow in G, and vice versa. The corresponding dual problem (cf. (1.5)) consists in finding a metric $m \in \mathscr{P}(V, H)$ with cm minimum (where gh denotes the inner product $\sum(g(e)h(e) : e \in D)$ of functions g and h within the common part D of their domains). Since we will vary the function c during the proof, the corresponding numbers v, v^* and τ are specified as $v(c), v^*(c)$ and $\tau(c)$, respectively.

As above $H = (W, U)$ is $K_3 + K_2$; let $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2\}$. A metric $m \in M^{2,3} \cup M^3$ is called *tight* if $cm = \tau(c)$; let $\mathscr{T}(c)$ denote the set of tight metrics for c. Let $\|c\|$ denote $\sum(c(e) : e \in E_V)$. We apply induction, assuming that the equality $v(c') = \tau(c')$ holds for every pseudo-Eulerian function c' on E_V such that either $|\mathscr{T}(c')| > |\mathscr{T}(c)|$, or $|\mathscr{T}(c')| = |\mathscr{T}(c)|$ and $\|c'\| < \|c\|$ (preserving V and H throughout the proof). Here $c'' : E_V \to \mathbb{Z}_+$ is called *pseudo-Eulerian* if $c''(\delta(X))$ is even for every $X \subset V$ such that either $X \subseteq V - W$ or $X \cap W = T$. The base case $c = 0$ (implying $\mathscr{T}(c) = M^{2,3} \cup M^3$) is obvious.

Fix an inner node $x \in V - W$ such that the set $\Phi = \Phi(x, c)$ of incident edges xy with $c(xy) > 0$ is non-nempty. Note that if Φ consists of a single edge xy, then reducing c to zero on xy we obtain the function c' which is also pseudo-Eulerian and obviously

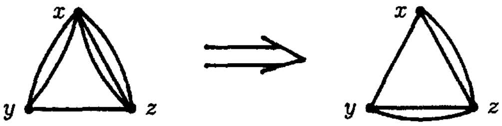

Fig. 2.

satisfies $\nu(c') = \nu(c)$ and $\mathscr{T}(c') \supseteq \mathscr{T}(c)$. Since $\|c'\| < \|c\|$, the theorem follows by induction.

So we may assume that $|\Phi| \geqslant 2$. Consider a pair $\{xy, xz\}$ in Φ. The (integer) *splitting-off operation* applied to xy, xz transforms c into c' as follows:

$$\begin{aligned} c'(e) &= c(e) - 1 && \text{for } e = xy, xz, \\ &= c(e) + 1 && \text{for } e = yz, \\ &= c(e) && \text{otherwise.} \end{aligned} \tag{3.2}$$

(In the original graph G, this corresponds to deletion of one edge connecting x and y and one edge connecting x and z and addition of a new edge between y and z; see Fig. 2.)

Clearly, c' remains nonnegative and pseudo-Eulerian. Moreover, for any metric m on V, $cm - c'm = m(xy) + m(xz) - m(yz) \geqslant 0$. Therefore, $\tau(c') \leq \tau(c)$. We say that $\{xy, xz\}$ is *splittable* if $\tau(c') = \tau(c)$. In this case $cm \geqslant c'm \geqslant \tau(c') = \tau(c)$ for any $m \in \mathscr{P}$ implies $\mathscr{T}(c) \subseteq \mathscr{T}(c')$. Also $\|c'\| = \|c\| - 1$. Hence, by induction there exists a c'-admissible integer multiflow f' such that $\mathrm{val}(f') = \tau(c')$. One can transform f' into a c-admissible integer multiflow f with the same value; then $\nu(c) \geqslant \mathrm{val}(f) = \tau(c') = \tau(c)$ proves the theorem for c. (More precisely, assuming without loss of generality, that all paths in f' have unit weights, the desired f is constructed as follows:

if $(f')^{yz} \leqslant c(yz)$, put $f := f'$ (as f' is already c-admissible); otherwise choose a path $P' \in f$ going through yz and replace in P' the edge yz by the pair yx, xz.) (3.3)

Our goal is to show that at least one pair in $\Phi(x, c)$ is splittable. By induction this reduces the problem to the case $V = W$; this case will be considered in the end of Section 4.

For a contradiction, we suppose that all pairs in Φ are not splittable. In the rest of this section we show the following.

Lemma 3.1. *There exists a maximum multiflow which is half-integer.*

The proof of this lemma falls into several claims.

Claim 1. *cm is an integer for each $m \in M^{2,3} \cup M^3$.*

Proof. Let $m = m^{\Pi} \in M^{2,3}$, where $\Pi = (S_1, S_2, S_3, T_1, T_2)$ (cf. (1.6)). Let $X = T_1 \cup T_2$. Then m takes value $\frac{1}{2}$ on the edges of the cut $\delta(X)$ and integer values on the other edges in K_V. Since c is pseudo-Eulerian, $c(\delta(X))$ is even. This implies that $\sum(c(e)m(e)$: $e \in \delta(X))$ is an integer, whence cm is also an integer.

Next, let $m = m^{\Xi} \in M^3$, where $\Xi = (A, B, C)$ (cf. (1.8)). Let $Q = V - (A \cup B \cup C)$. Then m takes value $\frac{1}{2}$ on the edges of the cut $\delta(Q)$ and integer values on the other edges in K_V. The integrality of cm follows from the fact that $c(\delta(Q))$ is even. □

This claim together with Theorem 2 implies that

$$\tau(c) \text{ and } cm - \tau(c) \text{ are integers for any } m \in M^{2,3} \cup M^3. \tag{3.4}$$

Consider a pair $\{xy, xz\}$ in Φ, and let c' be obtained by splitting-off (3.2).

Claim 2. *Let $m \in M^{2,3} \cup M^3$, and let $\Delta = cm - c'm$. Then Δ is equal to* 0, 1 *or* 2. *Moreover, if $\Delta = 2$ then $m(xy) = m(xz) = 1$ and $m(yz) = 0$.*

Proof. Observe that the lenght $m(P)$ of any closed path P in K_V is an integer. Also $m(e) \in \{0, 1, \frac{1}{2}\}$ for all $e \in E_V$. Hence, $\Delta = m(xy) + m(xz) - m(yz)$ is an integer $\leqslant 2$, and the result follows. □

In view of Theorem 2, the fact that $\{xy, xz\}$ is non-splittable means the existence of $m \in M^{2,3} \cup M^3$ such that $c'm = \tau(c') < \tau(c)$. From (3.4) and Claim 2 we obtain that

$$\text{if } m \in M^{2,3} \cup M^3 \text{and } c'm < \tau(c), \text{ then either} \tag{3.5}$$

(i) m is tight for c, and $m(xy) + m(xz) - m(yz) > 0$, or
(ii) $cm = \tau(c) + 1$, $c'm = \tau(c) - 1$, $m(xy) = m(xz) = 1$ and $m(yz) = 0$.

A metric m satisfying (ii) in (3.5) is called *critical* for $\{xy, xz\}$. Let $f = (P_1, \ldots, P_k;$ $\lambda_1, \ldots, \lambda_k)$ be a multiflow which is an optimal solution for c. One may assume that all λ_i's are nonzero.

Claim 3. *Let m be tight for c. Then*:
(i) *for $e \in E_V$, if $m(e) > 0$ then e is saturated by f, i.e., $f^e = c(e)$;*
(ii) *each path P_i in f is shortest for m, i.e., $m(P_i) = 1$.*

Proof. (i) and (ii) are equivalent to the complementary slackness conditions for (1.3) and its dual. More precisely,

$$\begin{aligned} \mathrm{val}(f) &= \lambda_1 + \cdots + \lambda_k \leqslant \lambda_1 m(P_1) + \cdots + \lambda_k m(P_k) \\ &= \sum (f^e m(e) : e \in E_V) \leqslant cm. \end{aligned} \tag{3.6}$$

Since $\mathrm{val}(f) = \tau(c) = cm$, equality holds throughout, proving (i) and (ii). □

This claim shows that

if some of the edges xy, xz is not saturated by f or if both xy and xz belong to a path in f, then only alternative (ii) in (3.5) is possible. (3.7)

Note that at least one path in f meets x; otherwise $f^e = 0$ holds for all $e \in \Phi$, and therefore, each pair in Φ is splittable.

Now the proof of Lemma 3.1 is completed as follows. Choose a pair $\{xy, xz\}$ as in (3.7). Consider the capacity function $\widetilde{c} = 2c$. Then $\tau(\widetilde{c}) = 2\tau(c)$. Furthermore, the impossibility of (i) in (3.5) provides that any metric $m' \in M^{2,3} \cup M^3$ with $m'(xy) + m'(xz) > m'(yz)$ satisfies $\widetilde{c}m' \geqslant \tau(\widetilde{c}) + 2$. Then the function $\widetilde{c}'$ obtained by splitting-off (3.2) from $\widetilde{c}$ satisfies $\tau(\widetilde{c}') = \tau(\widetilde{c}) = 2\tau(c)$. Let m be a critical metric for c and $\{xy, xz\}$. Then

$$\widetilde{c}m = \tau(\widetilde{c}) + 2 \quad \text{and} \quad \widetilde{c}'m = \tau(\widetilde{c}) = \tau(\widetilde{c}').$$

This means that m becomes tight for $\widetilde{c}'$. Therefore, $\mathscr{T}(\widetilde{c}')$ strictly includes $\mathscr{T}(\widetilde{c}) = \mathscr{T}(c)$. Since $\widetilde{c}'$ is, obviously, pseudo-Eulerian, by induction there exists an integer $\widetilde{c}'$-admissible multiflow h with $\mathrm{val}(h) = \tau(\widetilde{c}')$. Then the problem for $\widetilde{c}$ also has an integer optimal solution, by (3.3) applied to $\widetilde{c}$ and h. Hence, the problem for c has a *half-integer* optimal solution, as required. □

The above proof reveals some additional properties of maximum multiflows and critical metrics; these will be used in the next section. Let $f = (P_1, \ldots, P_k; \lambda_1, \ldots, \lambda_k)$ be a half-integer maximum multiflow for c. Repeating, if needed, some paths in f, we may assume that $\lambda_1 = \cdots = \lambda_k = \frac{1}{2}$. Also, assume that f is chosen so that

$$\xi(f) := |P_1| + \cdots + |P_k| \quad \text{is as small as possible,} \tag{3.8}$$

where $|P_i|$ is the number of edges of P_i. In particular, all P_i's are simple. Consider a path $P = P_i$ which passes x using edges xy and xz say. By (3.7), there exists a critical metric m for c and $\{xy, xz\}$. Moreover, taking into account that $cm = \tau(c) + 1$, that $\lambda_i = \frac{1}{2}$ and that $m(P) \geqslant 3$ (by (3.5)(ii)), and considering an expression similar to (3.6), one can see that

if m is critical for $\{xy, xz\}$ and P is a path in f containing xy and xz, then all edges e with $m(e) > 0$ are saturated by f; also $m(P) = 3$ and $m(P') = 1$ for the remaining paths P' in f. (3.9)

In particular, xy belongs to a path in f different from P, and similarly for xz. Varying $\{xy, xz\}$, we conclude that

there is a sequence $e_0, \ldots, e_{r-1}, e_r = e_0$ $(r \geqslant 2)$ of different edges incident to x such that each pair e_i, e_{i+1} belongs to a path in f, P_i say, and these paths are different. (3.10)

Moreover, $r \geqslant 3$ because if $r = 2$ then both P_0 and P_1 use e_0 and e_1, so $\{e_0, e_1\}$ is splittable. Here and later on the indices are taken modulo r. We call $D = (P_0, \ldots, P_{r-1})$ a *paths cycle* and assume that, among all maximum half-integer multiflows satisfying (3.8), f and D are chosen so that $|D|$ is minimum.

4. Existence of an integer maximum multiflow

To show this, we will try to re-arrange some paths in the above paths cycle $D = (P_0, \ldots, P_{r-1})$ by breaking them up at the node x into pieces and then combining the pieces in another way in order to obtain a 'better configuration' of paths through x; for example, to get a smaller paths cycle. We keep terminology and notation from the previous section and will use the following fact:

$$\text{if } c' \text{is obtained from } c \text{ by splitting-off (3.2) for a pair } \{xy, xz\} \text{ and there is a } c'\text{-admissible multiflow } g \text{ such that } \operatorname{val}(g) > \tau(c) - 1, \text{ then } \{xy, xz\} \text{ is splittable.} \tag{4.1}$$

Indeed, since $\tau(c)$ and $\tau(c')$ are integers (cf. (3.4)) and $\tau(c') \geqslant \operatorname{val}(g)$, it must be $\tau(c') = \tau(c)$.

For a path $P = v_0 v_1 \ldots v_k$, P^{-1} denotes the reverse path $v_k v_{k-1} \ldots v_0$, and $P(v_i, v_j)$ the subpath of P from v_i to v_j. The concatenation $v_0 \ldots v_k u_1 \ldots u_q$ of P and a path $Q = u_0 \ldots u_q$ with $u_0 = v_k$ is denoted by $P \cdot Q$. For $i = 0, \ldots, r-1$, let y_i be the end of the edge e_i, defined in (3.10), that is different from x. We assume that each path $P_i \in D$ meets the nodes y_i, x, y_{i+1} in this order, and denote by a_i and b_i the first and last nodes in P_i, respectively. If a_i, b_i are in S (resp. T), P_i is called a path of *type* S, or an S*-path* (resp. a T*-path*).

Consider P_i and fix a metric m_i critical for $\{e_i, e_{i+1}\}$. If $m_i \in M^{2,3}$ ($m_i \in M^3$), the partition (resp. triple) that induces m_i is denoted by $\Pi^i = (S_1^i, S_2^i, S_3^i, T_1^i, T_2^i)$ (resp. $\Xi^i = (A^i, B^i, C^i)$). Let X_i and Y_i denote the sets in Π^i (resp. Ξ^i) that contain x and $\{y_i, y_{i+1}\}$, respectively. Note that if $m_i \in M^3$, then all x, y_i, y_{i+1} are in $A^i \cup B^i \cup C^i$, by (3.5)(ii). Since $m_i(y_i x) = 1 = m_i(P_{i-1})$ and $m_i(y_{i-1} x) = 1 = m(P_{i+1})$ (by (3.5)(ii) and (3.9)), we observe that

$$\text{the subpaths } P_{i-1}(a_{i-1}, x) \text{ and } P_{i+1}(x, b_{i+1}) \text{ are entirely contained in } X_i, \text{ while } P_{i-1}(y_i, b_{i-1}) \text{ and } P_{i+1}(a_{i+1}, y_{i+1}) \text{ in } Y_i; \text{ in particular, } a_{i-1} \text{ and } b_{i+1} \text{ are terminals in } X_i, \text{ while } b_{i-1} \text{ and } a_{i+1} \text{ are terminals in } Y_i \tag{4.2}$$

(considering a path as a set of nodes). Recall that each of X_i, Y_i contains exactly one terminal if m_i is a (2,3)-metric, and one or two terminals if m_i is a 3-cut metric. Also, none of X_i, Y_i has two terminals in S or in T. Therefore, (4.2) implies that

$$\text{either } a_{i-1} = b_{i+1} \text{ and } b_{i-1} = a_{i+1}, \text{ or } a_{i-1} \neq b_{i+1} \text{ and } b_{i-1} \neq a_{i+1}; \text{ in the latter case, } m_i \text{ is a 3-cut metric with } \{A^i, B^i\} = \{X_i, Y_i\}, \text{ and one of } P_{i-1} \text{ and } P_{i+1} \text{ is an } S\text{-path while the other is a } T\text{-path.} \tag{4.3}$$

Regarding the path P_i, the facts that $m_i(P_i) = 3$ (cf. (3.9)) and $m_i(x'y') = 1$ for any $x' \in X_i$ and $y' \in Y_i$ imply that

$$\text{either one of } a_i, b_i \text{ lies in } Y_i \text{ (and therefore, it coincides with some of } b_{i-1}, a_{i+1}\text{), or both } a_i, b_i \text{ are outside } Y_i\text{; the latter occurs only if } m_i \text{ is a (2,3)-metric and the paths } P_i \text{ and } P_{i+1} \text{ have different types.} \tag{4.4}$$

We now study possible combinations of types of consecutive paths in D.

$$\text{If both } P_j, P_{j+1} \text{ have the same type, then } b_j = a_{j+1}. \tag{4.5}$$

Indeed, suppose that $b_j \neq a_{j+1}$. Since P_j and P_{j+1} have the same type, we obtain from (4.4) that one of a_j, b_j is in Y_j. Also $a_{j+1} \in Y_j$, by (4.3). Then a_{j+1} coincides with some of a_j, b_j. Therefore, $b_j \neq a_{j+1}$ implies $a_j = a_{j+1}$, whence $a_j \neq b_{j+1}$. Now replace in f the paths P_j and P_{j+1} by the new paths $Q = P_j(a_j, x) \cdot P_{j+1}(x, b_{j+1})$ and $Q' = P_{j+1}(a_{j+1}, y_{j+1}) \cdot P_j(y_{j+1}, b_j)$. Since both Q, Q' are H-paths and $|Q| + |Q'| < |P_j| + |P_{j+1}|$ (as xy_{j+1} is not used in Q, Q'), the resulting multiflow is again maximum and $\xi(f') < \xi(f)$, contrary to (3.8).

Let L_i and R_i stand for $P_i(a_i, x)$ and $P_i(x, b_i)$, respectively.

$$\text{For } r \geqslant 4, \text{ there are no four consecutive } P_j, \dots, P_{j+3} \text{ such that } P_j \text{ and } P_{j+2} \text{ are } S\text{-paths while } P_{j+1} \text{ and } P_{j+3} \text{ are } T\text{-paths.} \tag{4.6}$$

For suppose such paths exist. By (4.3) for $i = j+1$, $a_j = b_{j+2}$ and $b_j = a_{j+2}$. Similarly, $a_{j+1} = b_{j+3}$ and $b_{j+1} = a_{j+3}$. Break these four paths at x and combine the resulting pieces by concatenating L_j with L_{j+2}^{-1}, L_{j+1} with L_{j+3}^{-1}, R_j^{-1} with R_{j+2}, and R_{j+1}^{-1} with R_{j+3}, forming paths $Q_1, \dots, Q_4$, respectively. Since all $Q_1, \dots, Q_4$ are H-paths, we can replace $P_j, \dots, P_{j+3}$ in f by these paths. But both Q_2 and Q_3 pass through e_{j+1} and e_{j+3}, therefore, this pair of edges is splittable; a contradiction.

$$\text{For } r \geqslant 4, \text{ there are no four consecutive } P_j, \dots, P_{j+3} \text{ such that } P_j \text{ and } P_{j+1} \text{ are } S\text{-paths while } P_{j+2} \text{ and } P_{j+3} \text{ are } T\text{-paths.} \tag{4.7}$$

For suppose the contrary. Since $b_j \in S$, $a_{j+2} \in T$ and both b_j, a_{j+2} belong to Y_{j+1} (by (4.2)), the set Y_{j+1} contains exactly two terminals, namely, b_j and a_{j+2}. Hence, m_{j+1} is a 3-cut metric. Similarly, $Y_{j+2} \cap W = \{b_{j+1}, a_{j+3}\}$ and $m_{j+2} \in M^3$. Notice also that $b_j = a_{j+1}$ and $b_{j+2} = a_{j+3}$ (by (4.5)). Therefore, $\{b_j, a_{j+2}\}$ and $\{b_{j+1}, a_{j+3}\}$ have no common terminal (in view of $a_{j+1} \neq b_{j+1}$ and $a_{j+2} \neq b_{j+2}$). Next we use submodularity. Let for definiteness $a_{j+2} = t_1$ and $a_{j+3} = t_2$; then $Y_{j+1} = A^{j+1}$ and $Y_{j+2} = B^{j+2}$. Let $A = A^{j+1} - B^{j+2}$ and $B = B^{j+2} - A^{j+1}$. Since the sets of terminals in A^{j+1} and in B^{j+2} are disjoint,

$$A \cap W = A^{j+1} \cap W \quad \text{and} \quad B \cap W = B^{j+2} \cap W.$$

Therefore, the triples $\Xi' = (A, B^{j+1}, C^{j+1})$ and $\Xi'' = (A^{j+2}, B, C^{j+2})$ satisfy (1.7). Also each triple consists of pairwise disjoint sets; so $m' = m^{\Xi'}$ and $m'' = m^{\Xi''}$ are 3-cut metrics. We observe that the obvious submodular inequality

$$c(\delta(A)) + c(\delta(B)) \leqslant c(\delta(A^{j+1})) + c(\delta(B^{j+2}))$$

is strict because the edge e_{j+2} is contained in both $\delta(A^{j+1})$ and $\delta(B^{j+2})$ but none of $\delta(A)$ and $\delta(B)$. Therefore, $cm' + cm'' < cm_{j+1} + cm_{j+2}$. Let for definiteness $cm' < cm_{j+1}$. Since $cm' - \tau(c)$ and $cm_{j+1} - \tau(c)$ are integers and $cm_{j+1} - \tau(c) = 1$, we have $cm' = \tau(c)$, i.e., m' is tight for c. But the set $X_{j+1} = B^{j+1}$ contains x and none of y_{j+1}, y_{j+2}; so P_{j+1} intersects the cut $\delta(B^{j+1})$ at least twice. This implies that P_{j+1} cannot be shortest for m', whence m' is not tight (cf. Claim 3 in Section 3); a contradiction.

$$\text{For } r \geqslant 3, \text{ there are no three consecutive } T\text{-paths } P_j, P_{j+1}, P_{j+2}. \tag{4.8}$$

Otherwise, letting for definiteness $b_j = t_1$, we have $a_{j+1} = t_1$ and $b_{j+1} = a_{j+2} = t_2$ (in view of (4.5)). Then $a_j = t_2$ and $b_{j+2} = t_1$. Hence, both $Q = L_j \cdot R_{j+2}$ and $Q' = L_{j+2} \cdot R_j$ are H-paths. Replace P_j, P_{j+2} by Q, Q'. Then the resulting multiflow f' is maximum. Since the paths P_{j+1} and Q' in f' go through e_{j+1} and e_{j+2}, the pair of these edges is splittable; a contradiction.

$$\text{For } r \geqslant 3, \text{there are no three consecutive } S\text{-paths } P_j, P_{j+1}, P_{j+2}. \tag{4.9}$$

Otherwise form Q, Q' as in the previous case. Then Q' is an H-path (as it connects $a_{j+2} = b_{j+1}$ with $b_j = a_{j+1}$), while Q needs not connect different terminals in S. Nevertheless, replacing P_j and P_{j+2} by the only path Q' results in a multiflow f' with $\mathrm{val}(f') = \mathrm{val}(f) - \frac{1}{2} > \tau(c) - 1$. Since f' contains two paths that use both e_{j+1} and e_{j+2}, f' can be transformed into a c'-admissible multiflow g with $\mathrm{val}(g) = \mathrm{val}(f')$, where c' is obtained from c by the splitting-off operation for e_{j+1}, e_{j+2}. Now (4.1) says that the latter pair is splittable; a contradiction.

$$\begin{aligned}&\text{For } r \geqslant 4, \text{ there are no } j, q \text{ such that } 2 \leqslant q - j \leqslant r - 2\\ &\quad\text{and all } P_j, P_{j+1}, P_q, P_{q+1} \text{ are } T\text{-paths.}\end{aligned} \tag{4.10}$$

Indeed, suppose such j, q exist. By (4.5), $b_j = a_{j+1}$ and $b_q = a_{q+1}$. Let for definiteness $b_j = t_1$. Then $a_j = b_{j+1} = t_2$. Two cases are possible. (i) Let $b_q = t_2$. Then $a_q = b_{q+1} = t_1$. Replace the above four paths by the H-paths $L_j \cdot R_{q+1}$, $L_{q+1} \cdot R_j$, $L_{j+1} \cdot R_q$ and $L_q \cdot R_{j+1}$. Since two of the latter paths contain the pair e_{j+1}, e_{q+1}, this pair is splittable. (ii) Let $b_q = t_1$. Then $a_q = t_2$. Hence, both $Q = L_j \cdot R_q$ and $Q' = L_q \cdot R_j$ are H-paths. Now replacing P_j, P_q by Q, Q' creates two paths cycles in place of D, namely, $(Q, P_{q+1}, \ldots, P_{j-1})$ and $(P_{j+1}, \ldots, P_{q-1}, Q')$, contradicting the minimality of D.

$$\begin{aligned}&\text{For } r \geqslant 4, \text{ there are no } j, q \text{ such that } 2 \leqslant q - j \leqslant r - 2\\ &\quad\text{and all } P_j, P_{j+1}, P_q, P_{q+1} \text{ are } S\text{-paths.}\end{aligned} \tag{4.11}$$

Again suppose such j, q exist. If $a_j \neq b_q$ and $b_j \neq a_q$ or, symmetrically, $a_{j+1} \neq b_{q+1}$ and $b_{j+1} \neq a_{q+1}$, we act similar to (ii) in the previous proof. So assume we are not in these cases. Then $b_j \neq b_q$; let $b_j = s_1$ and $b_q = s_2$ say. Then $a_j, b_{j+1} \in \{s_2, s_3\}$ and $a_q, b_{q+1} \in \{s_1, s_3\}$. Moreover, it is impossible that $a_j = a_q = s_3$ or $b_{j+1} = b_{q+1} = s_3$. Let for definiteness $a_j = s_2$. Consider two possible cases. (a) Let $b_{j+1} = s_2$. Then $a_j \neq b_{q+1}$ and $b_{j+1} \neq a_q$, and we act as in (i) from the previous proof. (b) Let $b_{j+1} = s_3$. Then $b_{q+1} = s_1$, and we have $a_j \neq a_q$ and $b_{j+1} \neq b_{q+1}$. Therefore, each of $L_j \cdot L_q^{-1}$ and $R_{j+1}^{-1} \cdot R_{q+1}$ is an H-path, which implies that $\{e_{j+1}, e_{q+1}\}$ is splittable.

Putting together the eliminations exhibited in (4.6)–(4.11) and considering D up to reversing (so the configurations reversed to those in (4.6) and (4.7) cannot occur either), we conclude that only two cases still remain possible, as follows.

Lemma 4.1. *$r = 3$ and, up to shifting D cyclically, either* (i) *P_1 and P_2 are S-paths and P_3 is a T-path, or* (ii) *P_1 and P_2 are T-paths and P_3 is an S-path.* □

We say that D in case (i) of this lemma has type S–S–T, and in case (ii) type T–T–S. First of all we describe D of type S–S–T in more details. Notice that

$$b_1 = a_2 \quad \text{and} \quad a_1 = b_2. \tag{4.12}$$

Indeed, $b_1 = a_2$ is true by (4.5). Suppose that $a_1 \neq b_2$. Then a_1 and b_2 are different terminals in S, so $Q = L_1 \cdot R_2$ is an H-path. Replacing P_1 and P_2 by the only Q gives a multiflow f' with $\mathrm{val}(f') = \mathrm{val}(f) - \frac{1}{2} > \tau(c) - 1$ in which two paths (P_3 and Q) use e_1, e_2. Hence, the latter pair is splittable, by (4.1); a contradiction.

Thus, without loss of generality, one may assume that $a_1 = b_2 = s_1$, $a_2 = b_1 = s_2$, $a_3 = t_1$ and $b_3 = t_2$; see Fig. 3(a).

Next, by (4.2), X_1 contains two different terminals $b_2 = s_1$ and $a_3 = t_1$, and Y_1 contains $a_2 = s_2$ and $b_3 = t_2$. Therefore, m_1 is a 3-cut metric with $A^1 = X_1$ and $B^1 = Y_1$. Similarly, m_2 is a 3-cut metric, $A^2 = Y_2$ and $B^2 = X_2$. On the other hand, the fact that each of $a_3 = t_1$ and $b_3 = t_2$ is different from $b_2 = a_1 = s_1$ implies that m_3 is a (2,3)-metric, by (4.3).

By similar reasons, if D is of type T–T–S, then (4.12) holds, m_1 and m_2 are 3-cut metrics, and m_3 is a (2,3)-metric; see Fig. 3(b). (where $a_1 = t_1$, $b_1 = t_2$, $a_3 = s_1$ and $b_3 = s_2$).

To finish the case $V \neq W$, we first make one important observation from the above proofs of properties (4.5)–(4.12). Every time we proved the impossibility of one or

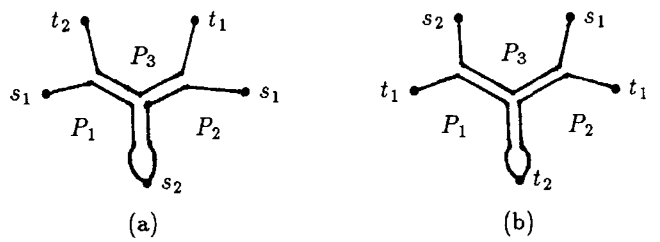

Fig. 3.

another situation, it turned out that its occurrence would imply that some pair of edges is splittable, or there exists a half-integer maximum multiflow f' with $\xi(f')$ smaller than $\xi(f)$, or some paths in the chosen cycle D can be rearranged so that D be transformed into two smaller paths cycles at x. It is essential to emphasize that the latter transformation changes no other paths cycle at this or any other node (assuming that some partitions into paths cycles are simultaneously fixed for the sets of paths at all nodes in $V - W$). This implies the following.

> There exists a half-integer maximum multiflow f such that $\xi(f)$ is minimum add, in addition, for each $y \in V - W$, the set of paths in f containing y is partitioned into paths cycles so that *each* paths cycle D consists of three paths and has type either S–S–T or T–T–S; moreover, the set $W(D)$ of end nodes of paths in D consists of four terminals. (4.13)

Assuming that (4.13) is satisfied, consider a paths cycle $D = (P_1, P_2, P_3)$ with $W(D) = \{s_1, s_2, t_1, t_2\}$ as above. For $v \in V$ let $Z(v)$ be the set of nodes z reachable from v by paths in K_V whose all edges are non-saturated by f. Let N be the smallest subset of V with the following properties:

(i) $x \in N$;

(ii) if $v \in N$ then $Z(v) \subseteq N$;

(iii) if uv is an edge with $u \in N$ which simultaneously belongs to an S-path and a T-path in f, then $v \in N$. (4.14)

In particular, all edges in $\delta(N)$ are saturated.

Lemma 4.2. *The following are true:*

(i) *N contains no terminal; and*

(ii) *each S-path in f meeting N connects s_1 and s_2.*

Proof. It falls into three claims. We prove these claims assuming that D (and paths cycles considered below) has type S–S–T, using the above properties of such cycles and critical metrics and keeping the above notation. For paths cycles of type T–T–S, the proofs below are similar.

Claim 1. *None of intermediate nodes of any path in f meeting $V - W$ is a terminal.*

Proof. It suffices to show this for the paths P_1, P_2, P_3. If P_i has type S (resp. T) and z is an intermediate node in P_i, then the fact that $\xi(f)$ is minimum that $z \notin S$ (resp. $z \notin T$).

Consider P_1 and m_3. By (4.2), the part of P_1 from $a_1 = s_1$ to y_1 is contained in Y_3, while its part from x to $b_1 = s_2$ is contained in X_3. Since m_3 is a (2,3)-metric, s_1 is a unique terminal in Y_3 and s_2 is a unique terminal in X_3. Therefore, no intermediate node of P_1 is a terminal. Similarly, no intermediate node of P_2 is a terminal. Finally, consider P_3, m_1 and m_2. The part P' of P_3 from $a_3 = t_1$ to y_3 lies simultaneously in X_1

and Y_2. Since $X_1 \cap W = \{s_1, t_1\}$ and $Y_2 \cap W = \{s_2, t_1\}$, P' does not meet S. Similarly, the part of P_3 from y_1 to $b_3 = t_2$ does not meet S. □

Claim 2. *For any $v \in N - W$, the set $Z(v)$ has no terminal.*

Proof. It suffices to show this for $v = x$ (because $\Phi(v)$ is non-empty for any node $v \in N - W$, and therefore, there is a paths cycle at v).

Consider X_1, X_2, X_3. Since the edges of each $\delta(X_i)$ are saturated and $x \in X_i$, $Z(x) \subseteq X_1 \cap X_2 \cap X_3$. Also one can see that X_1, X_2, X_3 have no terminal in common. □

Because of Claim 2, a terminal can appear in N only if some of its incident edges belongs to paths of different types (cf. (4.14)(iii)). The latter is impossible by Claim 1, therefore, (i) in the lemma is true.

Claim 3. *Let D' be a paths cycle at $v \in V - W$ such that every S-path in D' has ends s_1 and s_2. Let P be an S-path in f which meets $Z(v)$ at a node z. Then P connects s_1 and s_2.*

Proof. Again one may assume that $v = x$ and $D' = D$. Suppose that P connects s_3 and s_i, $i \in \{1,2\}$. Then P is different from P_1, P_2, P_3, so P is a shortest path for each of m_1, m_2, m_3. The facts that m_1 is a 3-cut metric, that P is shortest for m_1, and that $z \in Z(x) \subseteq X_1$ show that P meets $\delta(X_1)$ in exactly one edge. Thus, one end of P belongs to $X_1 \cap W = \{s_1, t_1\}$, whence $i = 1$. But $z \in X_3$ and $s_2 \in X_3$ imply that $m_3(s_3 z) + m_3(z s_1) = 2$. Therefore, P cannot be shortest for m_3; a contradiction. □

Now, part (ii) in the lemma follows from Claim 3 and the minimality of N subject to (4.14). □

Lemma 4.2 enables us to apply the following construction. Let C be the cut $\delta(N)$ from which the edges e with $c(e) = 0$ are deleted. Consider an edge $e = uv \in C$ with $u \in N$ and the set $\mathscr{L}(e)$ of paths in f going through e. Assume that, at the moment we deal with e, each path in $\mathscr{L}(e)$ is directed so that it passes the nodes u, v in this order. By (4.14)(iii), all paths in $\mathscr{L}(e)$ have the same type; moreover, by (ii) in Lemma 4.2, if they are S-paths, then they connect s_1 and s_2. Note that the last nodes of these paths must be the same (and the beginning nodes are the same). For if, e.g., $\mathscr{L}(e)$ contains a path P from t_1 to t_2 and a path Q from t_2 to t_1, then replacing them by the H-paths $P(t_1, u) \cdot (Q(t_2, u))^{-1}$ and $(P(v, t_2))^{-1} \cdot Q(v, t_1)$ decreases ξ in (3.8).

Using this, we construct an auxiliary graph $\Gamma = (V(\Gamma), E(\Gamma))$ as follows. Take the subgraph of K_V induced by N, add the terminals s_1, s_2, t_1, t_2, and for each edge $uv \in C$ with $u \in N$, connect u with $w \in \{s_1, s_2, t_1, t_2\}$ by an edge e_{uv}, where w is the last node of paths in $\mathscr{L}(uv)$. We endow the edges of Γ with the capacity function $\widetilde{c}$ that coincides with c within N and takes value $\widetilde{c}(e_{uv}) = c(uv)$ for each $uv \in C$. The fact that c is pseudo-Eulerian implies that $\widetilde{c}$ is *inner Eulerian*, which means that $\widetilde{c}(\delta^{\Gamma}(z))$ is even for each $z \in V(\Gamma) - W$.

Consider the maximum multiflow problem for Γ, $\widetilde{c}$ and the commodity graph forming by the edges s_1s_2 and t_1t_2 (*problem* $(*)$). The above property for f and C implies that the set $\mathscr{R}$ of subpaths $(v_0, e_1, v_1, \ldots, e_q, v_q)$ of the paths in f with $v_1, \ldots, v_{q-1} \in N$ and $e_0, e_q \in C$ determine an optimal solution to $(*)$. On the other hand, by a sharpened version of the two-commodity flow theorem due to Rothschild and Whinston [13], problem $(*)$ has an *integer* optimal solution. This gives a half-integer optimal solution $h = (Q_1, \ldots, Q_p; \mu_1, \ldots, \mu_p)$ to $(*)$ in which $\mu_1 = \cdots = \mu_p = \frac{1}{2}$ and each path Q_i is repeated ($Q_i = Q_j$ for some $j \neq i$). Sticking the paths from h in place of the corresponding paths from $\mathscr{R}$ transforms f into a half-integer maximum multiflow f' for the original problem. Observe that there are two paths in f' that share a pair e, e' of edges incident to a node in $V - W$. Hence, this pair is splittable. This final contradiction completes our study of the case when $V - W$ is non-empty.

It remains to consider case $V = W$. Let $f = (P_1, \ldots, P_k; \lambda_1, \ldots, \lambda_k)$ be a maximum multiflow. One may assume that all λ_i's are non-zero and f contains no repeated paths. Also, without loss of generality, one may assume that

(i) $|P_i| < 3$ for each i (otherwise P_i includes a smaller H-path);
(ii) each nonzero capacity edge connects some s_j and t_q; and
(iii) $\lambda_i < 1$ for all i (otherwise reduce by $\lfloor \lambda_i \rfloor$ the number λ_i and the capacities of the edges of P_i; this preserves the pseudo-Eulerianness).

Note that (i) and (ii) imply $|P_i| = 2$ for all i. Finally, we assume that f is chosen so that $\eta(f) = \sum(\lambda_i : P_i \text{ is an } S\text{-path})$ is as small as possible. This implies that:

(iv) no two S-paths in f have the same pair of end nodes; and
(v) for each S-path $s_it_ps_j$ in f, both edges s_it_q and s_jt_q are saturated, where $\{p, q\} = \{1, 2\}$

(otherwise one can decrease η). We show that f is an integer multiflow (and therefore, all λ_i's are zero, by (iii)). Let $\mathscr{L}$ be the set of S-paths in f. One can see that f is integer if $|\mathscr{L}| \leqslant 1$. So assume $|\mathscr{L}| \geqslant 2$. Property (iv) implies that, up to symmetry, only three cases for $\mathscr{L}$ are possible, namely: (a) $\mathscr{L}$ consists of $s_1t_1s_2$ and either $s_2t_1s_3$ or $s_2t_2s_3$; (b) $\mathscr{L}$ consists of $s_1t_1s_2$, $s_1t_1s_3$ and $s_2t_1s_3$; or (c) $\mathscr{L}$ consists of $P_1 = s_1t_1s_2$, $P_2 = s_2t_1s_3$ and $P_3 = s_1t_2s_3$. Note that (iii) and (v) provide that $c(e) = 0$ for all edges e not occurring in members of $\mathscr{L}$. Also $\sum(c(s_it_j) : i = 1, 2, 3,\ j = 1, 2)$ is even since c is pseudo-Eulerian. In case (a), the integrality of f is obvious. In case (b), the integrality of f follows from the fact that $c(s_1t_1) + c(s_2t_1) + c(s_3t_1)$ is even (since $c(s_it_2) = 0$ for $i = 1, 2, 3$). Finally, in case (c), we have $c(s_2t_2) = 0$, whence f contains at most two T-paths, namely, $P_4 = t_1s_1t_2$ and $P_5 = t_1s_3t_2$. In this case, the facts that each edge in $P_1 \cup \cdots \cup P_5$ is covered by these paths exactly twice and that the sum of their capacities is even imply that all λ_i's are integers.

This completes the proof of Theorem 1. $\square$

In conclusion notice that an integer optimal solution to our problem can be found in strongly polynomial time (provided that G is given by the corresponding capacity function c). This uses a weighted version of the splitting-off method similar to that developed in [6] (or in [10]). At iterations of such a method, one applies an algorithm to minimize $c'm$ over all (2,3)-metrics and 3-cut metrics, for a current $c' : E_V \to \mathbb{Z}_+$. The minimization problem over the set of (2,3)-metrics is known to be solvable in strongly polynomial time [6], while that over the set of 3-cut metrics is obviously reduced to $O(1)$ usual minimum cut problems.

Acknowledgements

I am indebted to the referee for useful suggestions.

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