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Longest Cycles in Regular Graphs

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The paper is concerned with the longest cycles in regular three- (or two-) connected graphs. In particular, the following results are proved: (i) every 3-connected k-regular graph on n vertices has a cycle of length at least $\min(3k, n)$; (ii) every 2-connected k-regular graph on n vertices, where $n < 3k+4$, has a cycle of length at least $\min(3k, n)$. © 1985 Academic Press, Inc.

1. Introduction

All graphs considered here are simple. Let $c(G)$ denote the length of a longest cycle in a graph G. The following is a well-known result by Dirac.

THEOREM A [5]. *If G is a 2-connected graph on n vertices with minimum degree k, then $c(G) \geqslant \min(2k, n)$.*

By adding a regularity condition Bollobas and Hobbs proved

THEOREM B [6]. *If G is a 2-connected, k-regular graph on at most $\frac{9}{4} \cdot k$ vertices, then G is hamiltonian.*

Jackson later obtained the following, much stronger, result.

THEOREM C [1]. *If G is a 2-connected, regular graph on at most $3k$ vertices, then G is hamiltonian.*

In this paper we shall prove

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THEOREM 1. *Let G be any graph with n vertices. If*

(i) *G is 2-connected and k-regular, and*

(ii) *for any two disjoint subsets of vertices A and B, $|A| \geqslant k$ and $|B| \geqslant 2k+4$ imply that G has 3 pairwise vertex-disjoint $A-B$ paths, then $c(G) \geqslant \min(3k, n)$.*

In Theorem 1, an $A-B$ path is a path which has one end-vertex in A and the other in B and has nothing else in common with $A \cup B$. Note that if G is 3-connected or $n < 3k+4$ then the condition (ii) is trivially satisfied and so we have the following two immediate consequences of Theorem 1.

COROLLARY 1.1. *Let G be a 3-connected, k-regular graph with n vertices. Then $c(G) \geqslant \min(3k, n)$.*

COROLLARY 1.2. *Let G be a 2-connected, k-regular graph with n vertices. If $n < 3k+4$, then $c(G) \geqslant \min(3k, n)$.*

It is clear that Theorem C is an immediate consequence of Corollary 1.2.

The results above are almost best possible, since one can construct, in the way described in [7], a 3-connected, k-regular graph G with $c(G) \leqslant 3k+5$, for k even, and $c(G) \leqslant 3k+6$ for k odd. The 2-connected, k-regular graph with $3k+4$ vertices described in [7], which has $c(G) \leqslant 2k+4$, shows that the condition (ii) of Theorem 1 cannot be improved when $k \geqslant 4$.

2. NOTATIONS

For H, a subgraph of a graph G, let $V(H)$ denote the set of vertices of H. For $v \in V(G)$, let $N_H(v)$ denote the set, and $d_H(v)$ the number, of neighbors of v in H. Further, if H is connected then, for each pair of vertices v and u in H, let $d_H(v, u)$ be the minimum length (the number of edges), and $L_H(v, u)$ the maximum length, of a $v-u$ path in H. In order to simplify notation we shall denote $V(G)$, $N_G(v)$, and $d_G(v)$ by V, $N(v)$, and $d(v)$, respectively. For $A \subseteq V(G)$, put

$$N(A) = \bigcup_{v \in A} N(v),$$

and let $e(A)$ denote the number of edges in G between the vertices of A. For H and F, subgraphs of G, let $E(H, F)$ denote the set, and $e(H, F)$ the number, of edges in G joining vertices of H to vertices of F. (Notice, the subgraphs H and F can be sets of vertices.)

Let $q_1 q_2 \cdots q_g$ be a path in G followed from q_1 to q_g. Define

$$N^*(q_1) = \{q_i \mid q_{i+1} \in N(q_1)\}, \ N^*(q_g) = \{q_i \mid q_{i-1} \in N(q_g)\}.$$

For any real number r, denote by $\lceil r \rceil$ the minimum integer not less than r.

3. The Proof of Theorem 1

If $k=2$ then G is a cycle and there is nothing to prove, and so we may assume that $k \geqslant 3$.

Let G be a graph satisfying the conditions of Theorem 1. Choose a longest cycle C so that the number of components in $R = G \backslash C$ is minimal. Let $c_1, c_2, \ldots, c_m, c_1$ be the vertices in order around C, the subscripts of the c_i will be reduced modulo m, where $m = |V(C)|$. For $A \subseteq V(C)$, put

$$A^+ = \{c_{i+1} \mid c_i \in A\} \qquad \text{and} \qquad A^- = \{c_{i-1} \mid c_i \in A\}.$$

The proof is by contradiction. Hence assume that $m \leqslant 3k-1$ and $R \neq \varnothing$. The proof is divided into two parts:

Part 1

R consists of isolated vertices: For $v \in R$, following Woodall [2] and Jackson [1] put $Y_0 = \varnothing$ and, for $j \geqslant 1$, put

$$X_j = N(Y_{j-1} \cup \{v\}) \qquad \text{and} \qquad Y_j = \{c_i \in V(C) \mid c_{i-1} \in X_j \text{ and } c_{i+1} \in X_j\}.$$

Put $X = \bigcup_{j=1}^{\infty} X_j$ and $Y = \bigcup_{j=1}^{\infty} Y_j$. Then (see [1 or 2])

(i) $X \subseteq V(C)$ and X does not contain two consecutive vertices of C.

(ii) $Y = X^+ \cap X^-$, $N(Y) \subseteq X$ and $X \cap Y = \varnothing$.

Let $Z^+ = X^+ \backslash Y$ and $Z^- = X^- \backslash Y$. We have the following results, due to Jackson.

Lemma 1 [1, Corollary 1]. (a) *Z^+ and Z^- are independent sets of vertices;*

(b) *given $c_i \in Z^+$ and $c_j \in Z^-$ there do not exist neighbors b_i of c_i and b_j of c_j which are consecutive on C and lie in the set $\{c_{i-2}, c_{i-1}, \ldots, c_{j+2}\}$,*

(c) *given c_i, $c_j \in Z^+$ or c_i, $c_j \in Z^-$ there does not exsit $c_p \in \{c_{i+2}, c_{i+3}, \ldots, c_{j-1}\}$ such that c_i is joined to c_p and c_j to c_{p-1}, and*

(d) *for any $u \in R$, $e(u, Z^+) \leqslant 1$ and $e(u, Z^-) \leqslant 1$.*

Put $y = |Y|$ and $x = |X|$. Then there are x open segments of C between vertices of X. Let $S_1, S_2, \ldots, S_b$ be the sets of vertices contained in the open segments which contain two or more vertices. Obviously,

$$b = x - y.$$

PROPOSITION 1. $1 < b < k$.

Proof. Note that $Y \cup \{v\}$ is an independent set of vertices, $N(Y \cup \{v\}) \subseteq X$ and, for all $i = 1, \ldots, b$, $e(X, S_i) \geqslant 2$. We see that

$$k \cdot x \geqslant e(X, Y \cup \{v\}) + \sum_{i=1}^{b} e(X, S_i) \geqslant k(y+1) + 2b,$$

implying

$$k \leqslant k \cdot x - ky - 2b = b(k-2).$$

Consequently,

$$b \geqslant \frac{k}{k-2} > 1.$$

On the other hand,

$$m = x + y + \sum_{i=1}^{b} |S_i| \geqslant x + y + 2b.$$

Note that $x \geqslant k$ and, by the assumption, $m < 3k$. The last inequality gives

$$b \leqslant \frac{m - x - y}{2} < \frac{3k - k}{2} = k. \quad \blacksquare$$

From the proof of Proposition 1 we have found that

$$\frac{k}{k-2} \leqslant b < k.$$

This means that

$$k \geqslant 4. \tag{1}$$

Put $S = \bigcup_{i=1}^{b} S_i$, $s_i = |S_i|$, and $s = |S| = \sum_{i=1}^{b} s_i$. Then

$$m = x + y + s. \tag{2}$$

For all $i=1,\dots,b$, let Z_i denote the set of vertices in S_i which are adjacent on C to vertices of X. It is clear that Z_i contains exactly two vertices, one belongs to Z^+, the other belongs to Z^-. Put

$$Z=\bigcup_{i=1}^{b} Z_i=Z^+\cup Z^-.$$

LEMMA 2. $e(Z, S\backslash Z)+2e(Z)\leqslant b(s-b+2)$.

Proof. Let

$$S_i=\{c_p, c_{p+1},\dots, c_q\} \qquad \text{and} \qquad S_j=\{c_t, c_{t+1},\dots, c_w\}$$

be distinct elements of $\{S_1, S_2,\dots, S_b\}$, where $\{c_p, c_q\}=Z_i$ and $\{c_t, c_w\}=Z_j$. Put

$$A=N(c_p)\cap S_j, \qquad B=N(c_q)\cap S_j,$$
$$D=N(c_t)\cap S_j \qquad \text{and} \qquad F=N(c_w)\cap S_j.$$

By Lemma 1(a), (b), c_q is not joined to any elements of $A^-\cup\{c_w\}$, and thus

$$B\subseteq S_j\backslash(A^-\cup\{c_w\}).$$

However

$$A^-\cup\{c_w\}\subseteq S_j \qquad \text{and} \qquad |A^-\cup\{c_w\}|=|A|+1.$$

Hence

$$|B|\leqslant s_j-|A|-1,$$

and so

$$e(Z_i, S_j)=|B|+|A|\leqslant s_j-1. \tag{3}$$

Furthermore, by Lemma 1(c), c_t is not joined to any elements of A^+ and c_w is not joined to any elements of B^-. Thus

$$D\subseteq(S_j\backslash\{c_t\})\backslash A^+ \qquad \text{and} \qquad F\subseteq(S_j\backslash\{c_w\})\backslash B^-.$$

However

$$|A^+|\geqslant|A|-1 \qquad \text{and} \qquad |B^-|\geqslant|B|-1.$$

Hence

$$|D| \leqslant (s_j - 1) - |A^+| \leqslant s_j - |A| \qquad \text{and} \qquad |F| \leqslant (s_j - 1) - |B^-| \leqslant s_j - |B|,$$

i.e.,

$$|D| + |A| \leqslant s_j \qquad \text{and} \qquad |F| + |B| \leqslant s_j,$$

and then

$$e(Z_i, S_j) + e(Z_j, S_j) = |A| + |B| + |D| + |F| \leqslant 2s_j. \tag{4}$$

By the definition of Z,

$$\begin{aligned} e(Z, S_j) &= \sum_{i=1}^{b} e(Z_i, S_j) \\ &= \sum_{h \neq i,j}^{b} (Z_h, S_j) + e(Z_i, S_j) + e(Z_j, S_j). \end{aligned}$$

Note that (3) holds for any i and j such that $i \neq j$. Using (3) and (4) we see that, for all $j = 1, ..., b$,

$$e(Z, S_j) \leqslant \sum_{h \neq i,j}^{b} (s_j - 1) + 2s_j = b(s_j - 1) + 2.$$

Summing these inequalities gives

$$\sum_{j=1}^{b} e(Z, S_j) \leqslant \sum_{j=1}^{b} (b(s_j - 1) + 2) = b(s - b) + 2b.$$

But

$$\sum_{j=1}^{b} e(Z, S_j) = e(Z, S \backslash Z) + 2e(Z),$$

and so the lemma is proved. ∎

LEMMA 3. *$m \geqslant 2x + k - 2 + (k + e(X, R^*) - e(Z, R^*))/b$, where $R^* = R \backslash \{v\}$.*

Proof. It is easily seen that

$$\begin{aligned} e(X, Z) &\leqslant k \cdot x - e(X, Y \cup \{v\}) - e(X, R^*) \\ &= k \cdot x - k(y + 1) - e(X, R^*) \\ &= k \cdot b - k - e(X, R^*). \end{aligned}$$

On the other hand,

$$e(X, Z)=k\cdot|Z|-e(Z, S\backslash Z)-2e(Z)-e(Z, R^*).$$

Applying $|Z|=2b$ and Lemma 2 we find that

$$k\cdot b-k-e(X, R^*)\geqslant e(X, Z)\geqslant k(2b)-b(s-b+2)-e(Z, R^*).$$

From this and $b\neq 0$, we obtain

$$s\geqslant k+b-2+\frac{k+e(X, R^*)-e(Z, R^*)}{b}.$$

Putting this into (2) and using $b=x-y$, Lemma 3 is proved.

We show now that

$$e(X, R^*)\geqslant e(Z, R^*). \tag{5}$$

If $R^*=\varnothing$ there is nothing to prove. Hence assume that $R^*\neq\varnothing$. If (5) is not true then there must exist a vertex u in R^* such that

$$e(X, u)<e(Z, u). \tag{6}$$

By Lemma 1(d), $e(Z, u)\leqslant 2$ and so (6) gives that $e(X, u)\leqslant 1$, i.e., $|N_C(u)\cap X|\leqslant 1$. Put

$$F=N_C(u)\cup X.$$

Then

$$\begin{aligned}|F|&=|N_C(u)|+|X|-|N_C(u)\cap X|\\&\geqslant|N_C(u)|+k-1.\end{aligned}$$

Since R consists of isolated vertices, u is an isolated vertex of R^* and $|N_C(u)|=k$. Thus

$$|F|\geqslant 2k-1.$$

The set F divides C into $|F|$ open segments. Since $N_C(u)$ or X does not contain two consecutive vertices of C, and by Lemma 1(d), there are only two ones containing no vertices in the segments above. Therefore,

$$\begin{aligned}m&\geqslant 2|F|-2\geqslant 4k-4\\&\geqslant 3k\qquad\text{by (1).}\end{aligned}$$

This contradicts the assumption $m \leqslant 3k-1$ and shows therefore that (5) is true. However, substituting (5) into the inequality of Lemma 3 yields

$$m \geqslant 2x + k - 2 + \frac{k}{b} \geqslant 3k - 2 + \frac{k}{b}$$

$$> 3k - 1 \qquad \text{by Proposition 1,}$$

which is a contradiction and completes the discussion of Part 1.

Part 2

R contains a component with two or more vertices. We first prove some general lemmas.

LEMMA 4. *Let H be any graph and let $Q = q_1 q_2 \cdots q_g$ be a path in H such that $N(q_1) \subseteq V(Q)$ and $N(q_g) \subseteq V(Q)$, and suppose that q_h is the last vertex which belongs to $N(q_1)$ and q_f the first vertex which belongs to $N(q_g)$. Denote by F the subgraph generated by $V(Q)$, we have*

(a) *If $h > f$ then, for any $q_i, q_j \in V(Q)$,*

$$L_F(q_i, q_j) \geqslant \min(d(q_1), d(q_g)).$$

(b) *If $h > f$ then, for any $q_i, q_j \in N^*(q_1)$,*

$$L_F(q_i, q_j) \geqslant d(q_g).$$

(c) *If H is 2-connected then, for any $q_i, q_j \in N^*(q_1)$,*

$$L_F(q_i, q_j) \geqslant \min(\lceil \tfrac{1}{2}(d(q_1) + d(q_g) + 1) \rceil, d(q_g)).$$

(d) *If H is 2-connected then, for any $q_i, q_j \in V(H)$,*

$$L_H(q_i, q_j) \geqslant \min(d(q_1), d(q_g)).$$

Remark. By symmetry, there are results similar to (b) and (c) for vertices of $N^*(q_g)$.

Proof. In the following we suppose, without loss of generality, that $i < j$:

(a) Let $q_i, q_j \in V(Q)$. We shall show that there is a $q_i - q_j$ path in F of length at least $\min(d(q_1), d(q_g))$.

(i) $j \leqslant f$ (or $i \geqslant h$). Let $n = \max\{t \mid t < h \text{ and } q_t \in N(q_g)\}$. The existence of n follows from the fact that $f < h$ and $q_f \in N(q_g)$. Then $P = q_i q_{i-1} \cdots q_1 q_h q_{h+1} \cdots q_g q_n q_{n-1} \cdots q_j$ is a required $q_i - q_j$ path since

$V(Q) \supseteq V(P) \supseteq N(q_g) \cup \{q_g\}$. For the case $i \geqslant h$, a similar argument gives a $q_i - q_j$ path in F which contains $N(q_1) \cup \{q_g\}$.

(ii) $j > f \geqslant i$ (or $j \geqslant h > i$). In this case there exists $n = \max\{t \mid t < j$ and $q_t \in N(q_g)\}$ such that $P = q_i q_{i+1} \cdots q_n q_g q_{g-1} \cdots q_j$ is a required $q_i - q_j$ path since $V(Q) \supseteq V(P) \supseteq N(q_g) \cup \{q_g\}$. For the case $j \geqslant h > i$, a similar discussion holds.

(iii) $f < i < j < h$. Put

$$B = \{q_{i+1}, q_{i+2}, \ldots, q_{j-1}\}.$$

$B = \varnothing$ if $j = i+1$. Note that $B \cap N(q_1) = \varnothing$ then $P = q_i q_{i-1} \cdots q_1 q_h q_{h-1} \cdots q_j$ is a required path since $V(Q) \supseteq V(P) \supseteq N(q_1) \cup \{q_1\}$. A similar discussion holds for the case when $B \cap N(q_g) = \varnothing$. Thus, we may assume that

$$B \cap N(q_1) \neq \varnothing \qquad \text{and} \qquad B \cap N(q_g) \neq \varnothing.$$

Let $r = \min\{t \mid q_t \in B \cap N(q_1)\}$ and $n = \max\{t \mid q_t \in B \cap N(q_1)\}$. If there is a p such that $n \leqslant p < j$ and $q_p \in N(q_g)$ then $P = q_i q_{i-1} \cdots q_1 q_r q_{r+1} \cdots q_p q_g q_{g-1} \cdots q_j$ is a required path since $V(Q) \supseteq V(P) \supseteq N(q_1) \cup \{q_1\}$, and thus suppose that no such p exists, i.e.,

$$\{q_n, q_{n+1}, \ldots, q_{j-1}\} \cap N(q_g) = \varnothing. \tag{8}$$

Choose $q_s \in B \cap N(q_g)$ so that s is as small as possible. By (8) $s < n$, and then $P = q_i q_{i-1} \cdots q_1 q_n q_{n-1} \cdots q_s q_g q_{g-1} \cdots q_j$ is a required path since $V(Q) \supseteq V(P) \supseteq N(q_g) \cup \{q_g\}$, and so the proof of (a) is completed.

(b) Let $q_i, q_j \in N^*(q_1)$. If $j \leqslant f$, this follows immediately from (i) of the proof of (a) since $N^*(q_1) \subseteq V(Q)$. So suppose that $j > f$, and let $n = \max\{t \mid t < j$ and $q_t \in N(q_g)\}$. If $n > i$ then $P = q_i q_{i-1} \cdots q_1 q_{i+1} \cdots q_n q_g q_{g-1} \cdots q_j$ is a $q_i - q_j$ path in F which contains $N(q_g) \cup \{q_g\}$, and is therefore of length at least $d(q_g)$. This means that $L_F(q_i, q_j) \geqslant d(q_g)$. If $n \leqslant i$ then, by the same arguments the path $P = q_i q_{i-1} \cdots q_n q_{n-1} \cdots q_f q_g q_{g-1} \cdots q_j$ implies that $L_F(q_i, q_j) \geqslant d(q_g)$.

(c) Let $q_i, q_j \in N^*(q_1)$. If $h > f$ then, by (b) $L_H(q_i, q_j) \geqslant d(q_g)$. If $h \leqslant f$ then, since H is 2-connected, and from the proof of Theorem 4.4 in [3], H contains a cycle of length at least $d(q_1) + d(q_g) + 1$. This implies, as H is 2-connected, that every two vertices of H are connected by a path of length at least $\lceil \frac{1}{2}(d(q_1) + d(q_g) + 1) \rceil$, and so does every two vertices of $N^*(q_1)$.

(d) Let $q', q'' \in V(H)$. If $h \leqslant f$ then, from the proof of (c), q' and q'' are connected by a path of length at least $\lceil \frac{1}{2}(d(q_1) + d(q_g) + 1) \rceil$, i.e.,

$$L_H(q', q'') \geqslant \lceil \tfrac{1}{2}(d(q_1) + d(q_g) + 1) \rceil \geqslant \min(d(q_1), d(q_g)).$$

If $h > f$ then, as H is 2-connected, there are two disjoint $\{q', q''\} - V(Q)$ paths in H. However, by (a), every two vertices of $V(Q)$ are connected by a path of length at least $\min(d(q_1), d(q_g))$ whose vertices all belong to $V(Q)$, and so it is not difficult to see that

$$L_H(q', q'') \geqslant \min(d(q_1), d(q_g)). \quad \blacksquare$$

LEMMA 5. *Suppose that H is any graph and C is a longest cycle in H. Let W be a component in $H \backslash C$* with two or more vertices. If there are $t \geqslant 2$ pairwise vertex-disjoint $V(W) - V(C)$ paths such that between every pair of endvertices in $V(W)$ of the paths there is a path in W of length at least d, then $|V(C)| \geqslant t(d+2)$.

Proof. For $i = 1, \ldots, t$, let v_i be the end vertices in $V(W)$, u_i the corresponding end vertices in $V(C)$, of the paths. By the given conditions, $L_W(v_i, v_j) \geqslant d$ for every pair of subscripts i and j. This means, by the maximality of C, that $d_C(u_i, u_j) \geqslant d + 2$, and hence $|V(C)| \geqslant t(d+2)$. $\blacksquare$

LEMMA 6. *Suppose that H is any graph and $C = c_1 c_2 \cdots c_m c_1$ is a longest cycle in H, the subscripts of the c_i will be reduced modulo m. Let W be a component in $H \backslash C$ with at least two vertices and d, k be any integers such that $1 \leqslant d \leqslant k - 2$. Suppose $T \subseteq V(W)$ such that*

(i) $|T| \geqslant \max(2, d)$,

(ii) *$e(v, C) \geqslant k - d$ for every $v \in T$, and*

(iii) *$L_W(v, u) \geqslant d$ for any $v, u \in T$,*

Assume fruther that $m < 3k$. Then, for every v in T,

$$N_C(v) = N_C(T) \qquad \textit{and} \qquad |N_C(T)| = k - d.$$

Proof. Let

$$T = \{q_1, q_2, \ldots, q_t\},$$

and

$$A_i = N_C(q_i), \qquad i = 1, \ldots, t.$$

Note that, for all $i = 1, \ldots, t$,

$$|A_i| = e(q_i, C) \geqslant k - d, \tag{9}$$

which means $|A_i| \geqslant 2$, since $d \leqslant k - 2$. Take into account $t \geqslant 2$ (where $t = |T|$), it is easy to see that there are at least two ordered pairs (p, n) such that c_p is joined to one, and c_n to another, of vertices of T and $e(T, \{c_{p+1},$

$c_{p+2},\ldots, c_{n-1}\})=0$. Furthermore, the given condition (iii) and the maximality of C give that

$$|\{c_{p+1}, c_{p+2},\ldots, c_{n-1}\}| \geqslant d+1,$$

and hence

$$m \geqslant \left|\bigcup_{i=1}^{t} (A_i \cup A_i^+)\right| + 2(d+1) - 2.$$

If $A_i \cap A_j = \varnothing$ for every pair of subscripts i and j, then

$$\begin{aligned} m &\geqslant \sum_{i=1}^{t} |A_i \cup A_i^+| + 2d \\ &\geqslant 2t(k-d) + 2d \qquad \text{using (9).} \end{aligned}$$

If $d=1$ then $t \geqslant 2$ and

$$m \geqslant 4(k-1) + 2 \geqslant 3k.$$

If $d \geqslant 2$ then $t \geqslant d$, and $k \geqslant 4$ since $d \leqslant k-2$. So

$$\begin{aligned} m &\geqslant 2d(k-d) + 2d \geqslant 2d(k-d) + 4 \\ &= 2(d-2)(k-d-2) + 4k - 4 \qquad \text{(rearranging)} \\ &\geqslant 4k - 4 \geqslant 3k. \end{aligned}$$

In each case $m \geqslant 3k$, which is contrary to hypothesis and shows that there are subscripts i and j such that $A_i \cap A_j \neq \varnothing$. Put

$$X = \{u \in V(C) \mid \exists i \neq j,\ A_i \cap A_j \ni u\}.$$

Then $X \neq \varnothing$. Let $x = |X|$ and $S_1, S_2,\ldots, S_x$ be the sets of vertices contained in the open segments of C between vertices of X (define $S_1 = V(C)\backslash X$ if $x=1$). Since C is a longest cycle, the given condition (iii) and the definition of X give that

$$|S_i| \geqslant d+1 \qquad \text{for all} \quad i = 1,\ldots, x. \tag{10}$$

Since $m = \sum_{i=1}^{x} |S_i| + x$ we find that

$$m \geqslant \sum_{i=1}^{x} (d+1) + x = x(d+2),$$

giving

$$x \leqslant \frac{m}{d+2} < \frac{3k}{d+2} = k - d + 1 - \frac{(d-1)(k-d-2)}{d+2}.$$

Using $1 \leqslant d \leqslant k-2$, we obtain

$$x < k - d + 1. \tag{11}$$

We now prove that

$$x = k - d. \tag{12}$$

If (12) is false then, by (11),

$$x \leqslant k - d - 1. \tag{13}$$

Put

$$\bar{X} = N_C(T) \backslash X,$$

and then

$$\begin{aligned} |\bar{X}| &= \sum_{i=1}^{t} |A_i \backslash X| \geqslant \sum_{i=1}^{t} |A_i \backslash X| \geqslant \sum_{i=1}^{t} (k-d-x) \qquad \text{using (9)}, \\ &= t(k-d-x). \end{aligned} \tag{14}$$

In fact, $\bar{X}$ is the set of vertices of C which are joined exactly to one vertex of T, and X is the set of vertices of C which are joined to two or more vertices of T. Put

$$\mathscr{F} = \{S_i \mid S_i \cap \bar{X} \neq \varnothing,\ 1 \leqslant i \leqslant t\}.$$

Since $\bar{X} \neq \varnothing$ by (13) and (14), $|\mathscr{F}| \geqslant 1$. We now show that

$$m \geqslant x(d+2) + 2d + 2\,|\bar{X}|. \tag{15}$$

(a) Assume first $|\mathscr{F}| = 1$. For convenience, let $\mathscr{F} = \{S_1\}$. Then all vertices of $\bar{X}$ lie in S_1. From (9) and (13), $x < |A_i|$, and therefore $A_i \cap \bar{X} \neq \varnothing$, for all $i = 1, ..., t$. On the other hand, from the maximality of C and the given condition (iii), if $c_i \in A_i \cap \bar{X}$ and $c_j \in A_j \cap \bar{X}$, $i \neq j$, then $d_C(c_i, c_j) \geqslant d + 2$. Hence it is not difficult to see that

$$|S_1| \geqslant (t+1)(d+1) + |\bar{X} \cup \bar{X}^+| - t \geqslant 3d + 1 + 2\,|\bar{X}|,$$

and so

$$\begin{aligned} m = \sum_{i=2}^{x} |S_i| + |S_1| + x \geqslant \sum_{i=2}^{x} (d+1) + |S_1| + x \qquad \text{using (10)}, \\ \geqslant x(d+2) + 2d + 2\,|\bar{X}|. \end{aligned}$$

(b) Assume that $|\mathscr{F}| \geqslant 2$. By reasoning similar to that used in (a), for each $S_i \in \mathscr{F}$,

$$|S_i| \geqslant 2(d+1) + 2\,|\bar{X} \cap S_i| - 1 = 2d + 1 + 2\,|\bar{X} \cap S_i|, \tag{16}$$

and so

$$\begin{aligned} m &= \sum_{S_i \bar{\in} \mathscr{F}} |S_i| + \sum_{S_i \in \mathscr{F}} |S_i| + x \\ &\geqslant \sum_{S_i \bar{\in} \mathscr{F}} (d+1) + \sum_{S_i \in \mathscr{F}} (2d + 1 + 2\,|\bar{X} \cap S_i|) + x \qquad \text{by (10) and (16),} \\ &= (x - |\mathscr{F}|)(d+1) + |\mathscr{F}|\,(2d+1) + 2\,|\bar{X}| + x \qquad (17) \\ &= x(d+2) + |\mathscr{F}|\,d + 2\,|\bar{X}| \\ &\geqslant x(d+2) + 2d + 2\,|\bar{X}|. \end{aligned}$$

In each case (15) holds. Combining (15) with (14) we have

$$m \geqslant x(d+2) + 2d + 2t(k-d-x).$$

If $d = 1$ then $t \geqslant 2$, and thus

$$\begin{aligned} m &\geqslant 3x + 2 + 4(k-1-x) = 3k + (k-x-2) \\ &\geqslant 3k \qquad \text{by (13).} \end{aligned}$$

If $d \geqslant 2$ then $t \geqslant d$, and so $k \geqslant 4$ since $d \leqslant k-2$. Thus

$$\begin{aligned} m &\geqslant x(d+2) + 2d + 2d(k-d-x) = 2d(k-d+1) - x(d-2) \\ &\geqslant 2d(k-d+1) - (k-d-1)(d-2) \qquad \text{by (13),} \\ &= (d+2)(k-d-1) + 4d \qquad \text{(rearranging)} \\ &\geqslant 4(k-d-1) + 4d = 4k-4 \\ &\geqslant 3k. \end{aligned}$$

Each case gives that $m \geqslant 3k$, contrary to hypothesis. Hence (12) is proved. Also, we have

$$\bar{X} = \varnothing.$$

If $\bar{X} \neq \varnothing$ then $|\mathscr{F}| \geqslant 1$. Note that (17) also holds for the case $|\mathscr{F}| \geqslant 1$ we have

$$\begin{aligned} m \geqslant x(d+2) + d + 2 &= (k-d)(d+2) + d + 2 \qquad \text{by (12),} \\ &= 3k + (d-1)(k-d-2) \\ &\geqslant 3k. \end{aligned}$$

This contradiction shows that $\bar{X}=\varnothing$, which means that $A_i \subseteq X$ for all $i=1,\dots, t$, but (9) and (12) imply that $|A_i| \geqslant |X|$, and so $A_1 = A_2 = \cdots = A_t = X = N_C(T)$. Hence Lemma 6 is proved.

Let us return to the proof of Part 2. Let W be a component in R with two or more vertices. We consider two cases:

Case 1. W is a block. Let $Q=q_1q_2\cdots q_g$ be a longest path in W followed from q_1 to q_g, chosen so that $d_W(q_1)+d_W(q_g)$ is maximal. Suppose without loss of generality that

$$d_W(q_g) \geqslant d_W(q_1) = d. \tag{18}$$

Let q_h be the last vertex which belongs to $N(q_1)$ and q_f the first vertex which belongs to $N(q_g)$. Note that, as Q is a longest path, $N_W(q_1) \subseteq V(Q)$ and $N_W(q_g) \subseteq V(Q)$. Replacing H by W in Lemma 4, we see that the path Q satisfies the requirement of Lemma 4.

Case 1a. $d \leqslant k-3$. In this case we have

PROPOSITION 2. *Q is a hamilton path in W with $d+1$ vertices, and $N_C(q_i)=N_C(q_j)$ for any q_i and q_j in $V(Q)$.*

Proof. Note that, as W is a block, if $d=1$ then $W=Q=q_1q_2$ and so applying Lemma 6 to the set $\{q_1, q_2\}$ we complete the proof. Thus suppose that $d \geqslant 2$, and so $N^*(q_1)=d \geqslant 2$. Note that for each $q_i \in N^*(q_1)$ the path $q_iq_{i-1}\cdots q_1q_{i+1}\cdots q_g$ has the same length as Q so $d_W(q_i) \leqslant d_W(q_1)=d$, implying $e(q_i, C)=k-d_W(q_i) \geqslant k-d$. Furthermore, using Lemma 4(d), and taking (18) into account we obtain that $L_W(q_i, q_j) \geqslant d$ for every pair of vertices q_i and q_j in $N^*(q_1)$. From the above discussion, the set $N^*(q_1)$ satisfies the requirement of Lemma 6 and thus, putting $X=N_C(N^*(q_1))$, and using Lemma 6 we have that, for each $q_i \in N^*(q_1)$,

$$N_C(q_i)=X \tag{19}$$

and $|X|=k-d$. Let $S_1, S_2,\dots, S_{k-d}$ be the sets of vertices contained in the open egments of C between vertices of X. Then

$$m=\sum_{i=1}^{k-d} |S_i| + (k-d).$$

If $|S_i| \geqslant d+2$ for all $i=1,\dots, k-d$, then

$$\begin{aligned} m &\geqslant \sum_{i=1}^{k-d} (d+2)+(k-d) \\ &= 3k+d(k-d-3) \\ &\geqslant 3k, \end{aligned}$$

contrary to hypothesis, and thus there must be some j, such that

$$|S_j| < d+2. \tag{20}$$

If $d_W(q_g) \geqslant d+1$ then, by Lemma 4(c), $L_W(q_i, q_j) \geqslant d+1$ for any q_i, $q_j \in N^*(q_1)$, which implies that $|S_i| \geqslant d+2$ for all $i = 1,..., k-d$, contrary to (20). Hence $d_W(q_g) \leqslant d$, giving $e(q_g, C) \geqslant k-d$. Furthermore, applying Lemma 6 to the set $N^*(q_1) \cup \{q_g\}$ we obtain

$$N_C(q_g) = X. \tag{21}$$

This means $|S_i| \geqslant g$ for all $i = 1,..., k-d$. It follows from (20) that $g < d+2$, but on the other hand, $g \geqslant d_W(q_1) + 1 = d+1$. Consequently

$$g = d+1,$$

implying

$$V(Q) = N^*(q_1) \cup \{q_g\}. \tag{22}$$

As Q is a longest path in W, no vertex of $N^*(q_1) \cup \{q_g\}$ can be joined to vertices of $V(W) \setminus V(Q)$, and so the connectness of W implies that Q is a hamiltonian path in W. Also, combining (19), (21) with (22) we find that $N_C(q_i) = X = N_C(q_j)$ for any q_i, $q_j \in V(Q)$. Hence the proposition is proved. ∎

Put $X = N_C(V(Q))$. By Proposition 2, $|X| = k-d$ and

$$e(X, Q) = |X| \cdot |V(Q)| = (k-d)(d+1).$$

As defined above, let $S_1, S_2,..., S_{k-d}$ be the sets of vertices contained in the open segments of C between vertices of X, and then

$$\begin{aligned} \sum_{i=1}^{k-d} e(X, S_i) &\leqslant k\,|X| - e(X, Q) = k(k-d) - (k-d)(d+1) \\ &= (k-d)(k-d-1). \end{aligned} \tag{23}$$

Put $s_i = |S_i|$, $i = 1,..., k-d$. Note that, for all $i = 1,..., k-d$,

$$s_i \geqslant g = d+1.$$

Suppose that there is some j such that

$$s_j = d+1.$$

Replacing by Q the open segment of C which contains S_j, we obtain another cycle C' of the same length as C. From the choice of C,

$$e(S_j, R)=0, \tag{24}$$

since otherwise $G\backslash C'$ has less components. As C is a longest cycle, it is easily checked that, for any $n \neq j$, if $s_n \leqslant d+2$ then

$$e(S_j, S_n)=0; \tag{25}$$

if $s_n = d+3$ then

$$e(S_j, S_n) \leqslant s_j, \tag{26}$$

since in this case each vertex of S_j can be joined to at most one vertex of S_n.

If every s_i takes its minimal value $d+1$, then

$$\begin{aligned} m = \sum_{i=1}^{k-d} s_i + (k-d) &\geqslant \sum_{i=1}^{k-d} (d+1) + (k-d) \\ &= 3k-3+(d-1)(k-d-3) \\ &\geqslant 3k-3. \end{aligned}$$

However, by the assumption, $m \leqslant 3k-1$, and so there are only the following two cases for values of $\{s_1, s_2, ..., s_{k-d}\}$:

(i) There is exactly one element $s_n = d+3$, and $s_j = d+1$ for every $j \neq n$. Then for all $j \neq n$

$$\begin{aligned} e(S_j, X) &= k \cdot s_j - 2e(S_j) - e\left(S_j, \bigcup_{i \neq j}^{k-d} S_i\right) - e(S_j, R) \\ &= k \cdot s_j - 2e(S_j) - e(S_j, S_n) \qquad \text{by (25) and (24),} \\ &\geqslant ks_j - s_j(s_j-1) - s_j \qquad \text{since } e(S_j) \leqslant \tbinom{s_j}{2}, \text{ and by (26),} \\ &= (d+1)(k-d-1) \qquad \text{since } s_j = d+1, \\ &\geqslant 2(k-d-1). \end{aligned}$$

Summing these inequalities yields

$$\begin{aligned} \sum_{j \neq n}^{k-d} e(S_j, X) \geqslant \sum_{j \neq n}^{k-d} 2(k-d-1) &= 2(k-d-1)(k-d-1) \\ &> (k-d)(k-d-1). \end{aligned}$$

This is contrary to (23).

(ii) There are two elements s_n and s_t such that $s_n \leqslant d+2$ and $s_t \leqslant d+2$, and for every $j \neq n$, $s_j = d+1$ and so

$$\begin{aligned} e(S_j, X) &= k \cdot s_j - 2e(S_j) - e\left(S_j, \bigcup_{i \neq j}^{k-d} S_i\right) - e(S_j, R) \\ &= ks_j - 2e(S_j) \qquad \text{by (25) and (24).} \\ &\geqslant (d+1)(k-d) \quad \text{since } e(S_j) \leqslant \tbinom{s_j}{2} \text{ and } s_j = d+1, \\ &\geqslant 2(k-d). \end{aligned}$$

Summing these inequalities gives

$$\sum_{j \neq n,t}^{k-d} e(S_j, X) \geqslant \sum_{j \neq n,t}^{k-d} 2(k-d) = 2(k-d)(k-d-2),$$

and so

$$\begin{aligned} \sum_{j=1}^{k-d} e(S_j, X) &= \sum_{j \neq n,t}^{k-d} e(S_j, X) + e(S_n, X) + e(S_t, X) \\ &\geqslant 2(k-d-2)(k-d) + 2 + 2 \\ &> (k-d)(k-d-1), \end{aligned}$$

which contradicts (23).

In each case we obtain a contradiction, and hence the discussion of Case 1a is completed.

Case 1b. $d \geqslant k-2$. If $d=1$, this is the case discussed in Case 1a, where we did not use the condition $d \leqslant k-3$ when $d=1$. Therefore, in the following, we consider only the case when $d \geqslant 2$ and $k \geqslant 4$. We shall prove

$$m \geqslant 2k+4. \tag{27}$$

By contradiction, suppose that $m < 2k+4$. Consider the following three subcases:

(i) $d=k$. Since G is 2-connected, there are two disjoint $V(W)-V(C)$ paths in G. Using Lemma 4(d) and Lemma 5 we obtain that

$$m \geqslant 2(d+2) = 2(k+2) \qquad \text{a contradiction.}$$

(ii) $d=k-1$. From the choice of Q, for every $q_i \in N^*(q_1)$, $d_W(q_i) \leqslant d = k-1$, implying $|N_C(q_i)| \geqslant 1$. Note that $|N^*(q_1)| = k-1$ and $d(u) \leqslant k$ for $u \in C$, we find that there are at least two distinct vertices of C, say c_i and c_j, which are respectively joined to two distinct vertices of

$N^*(q_1)$, say v_i and v_j. If $d_W(q_g)=k$ then, by Lemma 4(c), $L_W(v_i, v_j) \geqslant k$, giving $m \geqslant 2(k+2)$, which is a contradiction and shows that $d_W(q_g)<k$. Therefore we may assume that q_g is joined to a vertex, say c_p, of C. If $c_p \bar{\in} \{c_i, c_j\}$, then there are three independent edges in $E(W, C)$, and then using Lemma 4(d) and Lemma 5 we have that $m \geqslant 3(d+2)=3(k+1)$. This contradiction shows that $c_p \in \{c_i, c_j\}$. Put

$$S_1=\{c_{i+1}, c_{i+2}, ..., c_{j-1}\} \qquad \text{and} \qquad S_2=\{c_{j+1}, c_{j+2}, ..., c_{i-1}\}.$$

Clearly

$$|S_1| \geqslant g \geqslant d+1=k \tag{28}$$

and

$$|S_2| \geqslant g \geqslant d+1=k.$$

However, by the supposition,

$$|S_1|+|S_2|=m-2<2k+2,$$

and so we may assume, without loss of generality, that

$$|S_1|=k \qquad \text{and} \qquad |S_2| \leqslant k+1.$$

Then (28) gives $g=k$, and by reasoning similar to that used in the proof of Proposition 2, Q is a hamilton path in W. Therefore, analogously to the proof of (24) and (25),

$$e(S_1, R)=0 \qquad \text{and} \qquad e(S_2, S_1)=0,$$

giving

$$e(S_1, \{c_i, c_j\})=k\,|S_1|-2e(S_1) \geqslant k\,|S_1|-|S_1| \cdot (|S_1|-1)=k.$$

On the other hand, $|S_1|=k$ and $|S_2| \leqslant k+1$ imply $e(N^*(q_1), S_1 \cup S_2)=0$, and therefore

$$e(N^*(q_1), \{c_i, c_j\})=e(N^*(q_1), V(C)) \geqslant |N^*(q_1)| \geqslant k-1.$$

Then

$$e(\{c_i, c_j\}, S_2) \leqslant 2k-e(S_1, \{c_i, c_j\})-e(N^*(q_1), \{c_i, c_j\}) \leqslant 1,$$

which is contrary to the fact that $e(\{c_i, c_j\}, S_2) \geqslant 2$ and completes the discussion of (ii).

(iii) $d=k-2$. Applying Lemma 6 to the set $N^*(q_1)$ we deduce that there are exactly two vertices on C, say c_i and c_j, which are joined to every vertex of $N^*(q_1)$. Note that $d(c_i)$, $d(c_j) \leqslant k$ and $|N^*(q_1)|=k-2$, it is easy to see that q_g cannot be joined to either c_i nor c_j. Therefore if q_g is joined to some vertex, say c_p, of C then $c_p \bar{\in} \{c_i, c_j\}$, and so there are three independent edges in $E(W, C)$. It follows from Lemma 4(d) and Lemma 5 that $m \geqslant 3(d+2)=3k$. This is impossible since $m<3k$ is assumed throughout the proof of the theorem. Hence q_g is joined to no vertex on C. This means

$$d_W(q_g)=k. \tag{29}$$

Applying Lemma 4(c) and the maximality of C we have that $d_C(c_i, c_j) \geqslant k+2$, giving $m \geqslant 2(k+2)$, a contradiction.

In each case we arrive at a contradiction, and hence (27) is proved.

We shall now complete the discussion of Case 1b by using the given condition (ii) of the theorem. Noting that if $d \geqslant k-1$ then $|V(W)| \geqslant d+1 \geqslant k$ and if $d=k-2$ then by (29) $|V(W)| \geqslant k+1$, and taking (27) into account we find, by the given condition (ii) of the theorem, that G contains 3 pairwise disjoint $V(W)-V(C)$ paths. It follows from Lemma 4(d) and Lemma 5 that $m \geqslant 3(d+2) \geqslant 3k$. This is contrary to the assumption and completes the discussion of Case 1b.

Case 2. W is not a block. Let H_1 and H_2 be two end-blocks of W (blocks containing only one cut-vertex), and let

$$Q_1=a_1a_2\cdots a_g \qquad \text{and} \qquad Q_2=b_1b_2\cdots b_{g'}$$

be longest paths in H_1 and H_2, respectively, such that

$$d_{H_1}(a_1)+d_{H_1}(a_g) \qquad \text{and} \qquad d_{H_2}(b_1)+d_{H_2}(b_{g'})$$

as large as possible. Without loss of generality, suppose that

$$d_{H_1}(a_g) \geqslant d_{H_1}(a_1)=d_1, \qquad d_{H_2}(b_{g'}) \geqslant d_{H_2}(b_1)=d_2$$

and

$$d_1 \geqslant d_2.$$

Let h_1 be the unique cutvertex in H_1 and h_2 in H_2. Choose a vertex b',

$$b' \in N^*(b_1)\backslash\{h_2\} \qquad \text{if} \quad N^*(b_1) \neq \{h_2\},$$

$$b' \in V(H_2)\backslash\{h_2\} \qquad \text{if} \quad N^*(b_1)=\{h_2\}.$$

As H_2 is a edge when $N^*(b_1)=\{h_2\}$, it is quite easy to see that

$$d_W(b')=d_{H_2}(b')\leqslant d_{H_2}(b_1)=d_2\leqslant d_1,$$

implying

$$e(b', C)\geqslant k-d_1.$$

Let $W_1=W\backslash(H_1\backslash\{h_1\})$ and contract W_1 to a vertex, denoted by h^*, in other words, add a new vertex h^* to the graph $G\backslash W_1$ and join it to every vertex $u\in G\backslash W_1$ for which G contains a edge from u to W_1. The resulting graph is denoted by $\tilde{G}$. It is not difficult to see that C remains a longest cycle in $\tilde{G}$ and H_1 is a component in $\tilde{G}\backslash C$. However, in the new graph $\tilde{G}$, h_1 has been replaced by h^* and

$$e_{\tilde{G}}(h^*, C)\geqslant e(b', C)\geqslant k-d_1,$$

and so

$$e_{\tilde{G}}(a_i, C)\geqslant k-d_1 \qquad \text{for any} \quad a_i\in N^*(a_1),$$

where the suffix $\tilde{G}$ indicates that the underlying graph is $\tilde{G}$, for example, $e_{\tilde{G}}(a_i, C)$ denotes the number of edges in $\tilde{G}$ joining a_i to vertices of C. We replace G, C, W, Q, and d by $\tilde{G}$, C, H_1, Q_1, and d_1, respectively. Note that $d_{\tilde{G}}(v)\leqslant k$ for all $v\in N_C(h^*)$ and $d_G(v)=k$ for all $v\in V(G)\backslash N_C(h^*)$, $v\neq h^*$, we see that all arguments in Case 1 can be repeated. This completes the discussion of Case 2, and with it the proof of Theorem 1.

In the above proof, we used condition (ii) of Theorem 1 only in Case 1b of Part 2. If condition (ii) is deleted, then Case 1b gives that $m\geqslant 2k+4$. This means

THEOREM 2 (Chen [4]). *Every 2-connected, k-regular graph on n vertices has a cycle of length at least* $\min(3k, 2k+4, n)$.

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