Longest Cycles in Regular Graphs

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The paper is concerned with the longest cycles in regular three- (or two-) connected graphs. In particular, the following results are proved: (i) every 3-connected $k$-regular graph on $n$ vertices has a cycle of length at least $\min(3k, n)$; (ii) every 2-connected $k$-regular graph on $n$ vertices, where $n < 3k + 4$, has a cycle of length at least $\min(3k, n)$. © 1985 Academic Press, Inc.

1. Introduction

All graphs considered here are simple. Let $c(G)$ denote the length of a longest cycle in a graph $G$. The following is a well-known result by Dirac.

**Theorem A [5].** If $G$ is a 2-connected graph on $n$ vertices with minimum degree $k$, then $c(G) \geq \min(2k, n)$.

By adding a regularity condition Bollobas and Hobbs proved

**Theorem B [6].** If $G$ is a 2-connected, $k$-regular graph on at most $\frac{9}{2} \cdot k$ vertices, then $G$ is hamiltonian.

Jackson later obtained the following, much stronger, result.

**Theorem C [1].** If $G$ is a 2-connected, regular graph on at most $3k$ vertices, then $G$ is hamiltonian.

In this paper we shall prove

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THEOREM 1. Let $G$ be any graph with $n$ vertices. If

(i) $G$ is 2-connected and $k$-regular, and

(ii) for any two disjoint subsets of vertices $A$ and $B$, $|A| \geq k$ and $|B| \geq 2k + 4$ imply that $G$ has 3 pairwise vertex-disjoint $A - B$ paths, then $c(G) \geq \min(3k, n)$.

In Theorem 1, an $A - B$ path is a path which has one end-vertex in $A$ and the other in $B$ and has nothing else in common with $A \cup B$. Note that if $G$ is 3-connected or $n < 3k + 4$ then the condition (ii) is trivially satisfied and so we have the following two immediate consequences of Theorem 1.

**COROLLARY 1.1.** Let $G$ be a 3-connected, $k$-regular graph with $n$ vertices. Then $c(G) \geq \min(3k, n)$.

**COROLLARY 1.2.** Let $G$ be a 2-connected, $k$-regular graph with $n$ vertices. If $n < 3k + 4$, then $c(G) \geq \min(3k, n)$.

It is clear that Theorem C is an immediate consequence of Corollary 1.2.

The results above are almost best possible, since one can construct, in the way described in [7], a 3-connected, $k$-regular graph $G$ with $c(G) \leq 3k + 5$, for $k$ even, and $c(G) \leq 3k + 6$ for $k$ odd. The 2-connected, $k$-regular graph with $3k + 4$ vertices described in [7], which has $c(G) \leq 2k + 4$, shows that the condition (ii) of Theorem 1 cannot be improved when $k \geq 4$.

2. Notations

For $H$, a subgraph of a graph $G$, let $V(H)$ denote the set of vertices of $H$. For $v \in V(G)$, let $N_H(v)$ denote the set, and $d_H(v)$ the number, of neighbors of $v$ in $H$. Further, if $H$ is connected then, for each pair of vertices $v$ and $u$ in $H$, let $d_H(v, u)$ be the minimum length (the number of edges), and $L_H(v, u)$ the maximum length, of a $v - u$ path in $H$. In order to simplify notation we shall denote $V(G)$, $N_G(v)$, and $d_G(v)$ by $V$, $N(v)$, and $d(v)$, respectively. For $A \subseteq V(G)$, put

$$N(A) = \bigcup_{v \in A} N(v),$$

and let $e(A)$ denote the number of edges in $G$ between the vertices of $A$. For $H$ and $F$, subgraphs of $G$, let $E(H, F)$ denote the set, and $e(H, F)$ the number, of edges in $G$ joining vertices of $H$ to vertices of $F$. (Notice, the subgraphs $H$ and $F$ can be sets of vertices.)
Let \( q_1 q_2 \cdots q_g \) be a path in \( G \) followed from \( q_1 \) to \( q_g \). Define
\[
N^*(q_1) = \{ q_i \mid q_{i+1} \in N(q_1) \}, \quad N^*(q_g) = \{ q_i \mid q_{i-1} \in N(q_g) \}.
\]
For any real number \( r \), denote by \( \lceil r \rceil \) the minimum integer not less than \( r \).

3. The Proof of Theorem 1

If \( k = 2 \) then \( G \) is a cycle and there is nothing to prove, and so we may assume that \( k \geq 3 \).

Let \( G \) be a graph satisfying the conditions of Theorem 1. Choose a longest cycle \( C \) so that the number of components in \( R = G \setminus C \) is minimal. Let \( c_1, c_2, \ldots, c_m, c_1 \) be the vertices in order around \( C \), the subscripts of the \( c_i \) will be reduced modulo \( m \), where \( m = |V(C)| \). For \( A \subseteq V(C) \), put
\[
A^+ = \{ c_{i+1} \mid c_i \in A \} \quad \text{and} \quad A^- = \{ c_{i-1} \mid c_i \in A \}.
\]
The proof is by contradiction. Hence assume that \( m < 3k - 1 \) and \( R \neq \emptyset \). The proof is divided into two parts:

Part 1

\( R \) consists of isolated vertices: For \( v \in R \), following Woodall [2] and Jackson [1] put \( Y_0 = \emptyset \) and, for \( j \geq 1 \), put
\[
X_j = N(Y_{j-1} \cup \{ v \}) \quad \text{and} \quad Y_j = \{ c_i \in V(C) \mid c_{i-1} \in X_j \text{ and } c_{i+1} \in X_j \}.
\]
Put \( X = \bigcup_{j=1}^{\infty} X_j \) and \( Y = \bigcup_{j=1}^{\infty} Y_j \). Then (see [1 or 2])
(i) \( X \subseteq V(C) \) and \( X \) does not contain two consecutive vertices of \( C \).
(ii) \( Y = X^+ \cap X^- \), \( N(Y) \subseteq X \) and \( X \cap Y = \emptyset \).

Let \( Z^+ = X^+ \setminus Y \) and \( Z^- = X^- \setminus Y \). We have the following results, due to Jackson.

Lemma 1 [1, Corollary 1]. (a) \( Z^+ \) and \( Z^- \) are independent sets of vertices;
(b) given \( c_i \in Z^+ \) and \( c_j \in Z^- \) there do not exist neighbors \( b_i \) of \( c_i \) and \( b_j \) of \( c_j \) which are consecutive on \( C \) and lie in the set \( \{ c_{i-2}, c_{i-1}, \ldots, c_{i+2} \} \),
(c) given \( c_i, c_j \in Z^+ \) or \( c_i, c_j \in Z^- \) there does not exist \( c_p \in \{ c_{i+2}, c_{i+3}, \ldots, c_{j-1} \} \) such that \( c_i \) is joined to \( c_p \) and \( c_j \) to \( c_{p-1} \), and
(d) for any \( u \in R \), \( e(u, Z^+) \leq 1 \) and \( e(u, Z^-) \leq 1 \).
Put \( y = |Y| \) and \( x = |X| \). Then there are \( x \) open segments of \( C \) between vertices of \( X \). Let \( S_1, S_2, \ldots, S_b \) be the sets of vertices contained in the open segments which contain two or more vertices. Obviously,

\[
b = x - y.
\]

**Proposition 1.** \( 1 < b < k \).

**Proof.** Note that \( Y \cup \{v\} \) is an independent set of vertices, \( N(Y \cup \{v\}) \subseteq X \) and, for all \( i = 1, \ldots, b \), \( e(X, S_i) \geq 2 \). We see that

\[
k \cdot x \geq e(X, Y \cup \{v\}) + \sum_{i=1}^{b} e(X, S_i) \geq k(y + 1) + 2b,
\]

implying

\[
k \leq k \cdot x - ky - 2b = b(k - 2).
\]

Consequently,

\[
b \geq \frac{k}{k - 2} > 1.
\]

On the other hand,

\[
m = x + y + \sum_{i=1}^{b} |S_i| \geq x + y + 2b.
\]

Note that \( x \geq k \) and, by the assumption, \( m < 3k \). The last inequality gives

\[
b \leq \frac{m - x - y}{2} < \frac{3k - k}{2} = k.
\]

From the proof of Proposition 1 we have found that

\[
\frac{k}{k - 2} \leq b < k.
\]

This means that

\[
k \geq 4.
\]  \hspace{1cm} (1)

Put \( S = \bigcup_{i=1}^{b} S_i, s_i = |S_i| \), and \( s = |S| = \sum_{i=1}^{b} s_i \). Then

\[
m = x + y + s.
\]  \hspace{1cm} (2)
For all $i = 1, \ldots, b$, let $Z_i$ denote the set of vertices in $S_i$ which are adjacent on $C$ to vertices of $X$. It is clear that $Z_i$ contains exactly two vertices, one belongs to $Z^+$, the other belongs to $Z^-$. Put

$$Z = \bigcup_{i=1}^{b} Z_i = Z^+ \cup Z^-.$$

**Lemma 2.** $e(Z, S \setminus Z) + 2e(Z) \leq b(s - b + 2)$.

**Proof.** Let

$$S_i = \{c_p, c_{p+1}, \ldots, c_q\} \quad \text{and} \quad S_j = \{c_r, c_{r+1}, \ldots, c_w\}$$

be distinct elements of $\{S_1, S_2, \ldots, S_b\}$, where $\{c_p, c_q\} = Z_i$ and $\{c_r, c_w\} = Z_j$. Put

$$A = N(c_p) \cap S_j, \quad B = N(c_q) \cap S_j, \quad D = N(c_r) \cap S_j \quad \text{and} \quad F = N(c_w) \cap S_j.$$

By Lemma 1(a), (b), $c_q$ is not joined to any elements of $A^- \cup \{c_w\}$, and thus

$$B \subseteq S_j \setminus (A^- \cup \{c_w\}).$$

However

$$A^- \cup \{c_w\} \subseteq S_j \quad \text{and} \quad |A^- \cup \{c_w\}| = |A| + 1.$$

Hence

$$|B| \leq s_j - |A| - 1,$$

and so

$$e(Z_i, S_j) = |B| + |A| \leq s_j - 1. \quad (3)$$

Furthermore, by Lemma 1(c), $c_r$ is not joined to any elements of $A^+$ and $c_w$ is not joined to any elements of $B^-$. Thus

$$D \subseteq (S_j \setminus \{c_r\}) \setminus A^+ \quad \text{and} \quad F \subseteq (S_j \setminus \{c_w\}) \setminus B^-.$$

However

$$|A^+| \geq |A| - 1 \quad \text{and} \quad |B^-| \geq |B| - 1.$$
Hence

\[ |D| \leq (s_j - 1) - |A^+| \leq s_j - |A| \quad \text{and} \quad |F| \leq (s_j - 1) - |B^-| \leq s_j - |B|, \]

i.e.,

\[ |D| + |A| \leq s_j \quad \text{and} \quad |F| + |B| \leq s_j, \]

and then

\[ e(Z_i, S_j) + e(Z_j, S_j) = |A| + |B| + |D| + |F| \leq 2s_j. \]  \hspace{1cm} (4)

By the definition of \( Z \),

\[ e(Z, S_j) = \sum_{i=1}^{b} e(Z_i, S_j) - \sum_{h \neq i,j}^{b} (Z_h, S_j) + e(Z_i, S_j) + e(Z_j, S_j). \]

Note that (3) holds for any \( i \) and \( j \) such that \( i \neq j \). Using (3) and (4) we see that, for all \( j = 1, \ldots, b \),

\[ e(Z, S_j) \leq \sum_{h \neq i,j}^{b} (s_j - 1) + 2s_j = b(s_j - 1) + 2. \]

Summing these inequalities gives

\[ \sum_{j=1}^{b} e(Z, S_j) \leq \sum_{j=1}^{b} (b(s_j - 1) + 2) - b(s - b) + 2b. \]

But

\[ \sum_{j=1}^{b} e(Z, S_j) = e(Z, S \setminus Z) + 2e(Z), \]

and so the lemma is proved. \( \square \)

**Lemma 3.** \( m \geq 2x + k - 2 + (k + e(X, R^*) - e(Z, R^*))/b \), where \( R^* = R \setminus \{v\} \).

*Proof.* It is easily seen that

\[ e(X, Z) \leq k \cdot x - e(X, Y \cup \{v\}) - e(X, R^*) \]

\[ = k \cdot x - k(y + 1) - e(X, R^*) \]

\[ = k \cdot b - k - e(X, R^*). \]
On the other hand,
\[ e(X, Z) = k \cdot |Z| - e(Z, S \setminus Z) - 2e(Z) - e(Z, R^*). \]

Applying \( |Z| = 2b \) and Lemma 2 we find that
\[ k \cdot b - k - e(X, R^*) \geq e(X, Z) \geq k(2b) - b(s - b + 2) - e(Z, R^*). \]

From this and \( b \neq 0 \), we obtain
\[ s \geq k + b - 2 + \frac{k + e(X, R^*) - e(Z, R^*)}{b}. \]

Putting this into (2) and using \( b = x - y \), Lemma 3 is proved.

We show now that
\[ e(X, R^*) \geq e(Z, R^*). \tag{5} \]

If \( R^* = \emptyset \) there is nothing to prove. Hence assume that \( R^* \neq \emptyset \). If (5) is not true then there must exist a vertex \( u \) in \( R^* \) such that
\[ e(X, u) < e(Z, u). \tag{6} \]

By Lemma 1(d), \( e(Z, u) \leq 2 \) and so (6) gives that \( e(X, u) < 1 \), i.e., \( |N_C(u) \cap X| \leq 1 \). Put
\[ F = N_C(u) \cup X. \]

Then
\[ |F| = |N_C(u)| + |X| - |N_C(u) \cap X| \geq |N_C(u)| + k - 1. \]

Since \( R \) consists of isolated vertices, \( u \) is an isolated vertex of \( R^* \) and \( |N_C(u)| = k \). Thus
\[ |F| \geq 2k - 1. \]

The set \( F \) divides \( C \) into \( |F| \) open segments. Since \( N_C(u) \) or \( X \) does not contain two consecutive vertices of \( C \), and by Lemma 1(d), there are only two ones containing no vertices in the segments above. Therefore,
\[ m \geq 2|F| - 2 \geq 4k - 4 \geq 3k \quad \text{by (1)}. \]
This contradicts the assumption $m \leq 3k - 1$ and shows therefore that (5) is true. However, substituting (5) into the inequality of Lemma 3 yields

$$m \geq 2x + k - 2 + \frac{k}{b} \geq 3k - 2 + \frac{k}{b}$$

$$> 3k - 1 \quad \text{by Proposition 1},$$

which is a contradiction and completes the discussion of Part 1.

**Part 2**

$R$ contains a component with two or more vertices. We first prove some general lemmas.

**Lemma 4.** Let $H$ be any graph and let $Q = q_1q_2 \cdots q_s$ be a path in $H$ such that $N(q_1) \subseteq V(Q)$ and $N(q_s) \subseteq V(Q)$, and suppose that $q_h$ is the last vertex which belongs to $N(q_1)$ and $q_f$ the first vertex which belongs to $N(q_s)$. Denote by $F$ the subgraph generated by $V(Q)$, we have

(a) If $h > f$, then, for any $q_i, q_j \in V(Q)$,

$$L_F(q_i, q_j) \geq \min(d(q_1), d(q_g)).$$

(b) If $h > f$, then, for any $q_i, q_j \in N^*(q_1)$,

$$L_F(q_i, q_j) \geq d(q_g).$$

(c) If $H$ is 2-connected then, for any $q_i, q_j \in N^*(q_1)$,

$$L_F(q_i, q_j) \geq \min\left(\left\lfloor \frac{1}{2}(d(q_1) + d(q_g) + 1)\right\rfloor, d(q_g)\right).$$

(d) If $H$ is 2-connected then, for any $q_i, q_j \in V(H)$,

$$L_H(q_i, q_j) \geq \min(d(q_1), d(q_g)).$$

**Remark.** By symmetry, there are results similar to (b) and (c) for vertices of $N^*(q_g)$.

**Proof.** In the following we suppose, without loss of generality, that $i < j$:

(a) Let $q_i, q_j \in V(Q)$. We shall show that there is a $q_i-q_j$ path in $F$ of length at least $\min(d(q_1), d(q_g))$.

(i) $j \geq f$ (or $i \geq h$). Let $n = \max\{t \mid t < h \text{ and } q_t \in N(q_g)\}$. The existence of $n$ follows from the fact that $f < h$ and $q_f \in N(q_g)$. Then $P = q_iq_{i-1} \cdots q_1q_hq_{h+1} \cdots q_gq_nq_{n-1} \cdots q_j$ is a required $q_i-q_j$ path since
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V(Q) ⊇ V(P) ⊇ N(q_1) ∪ \{q_g\}. For the case i ≥ h, a similar argument gives a q_i - q_j path in F which contains N(q_1) ∪ \{q_g\}.

(ii) j ≥ f ≥ i (or j ≥ h > i). In this case there exists n = max\{t | t < j and q_t ∈ N(q_g)\} such that P = q_i q_{i+1} \cdots q_n q_g q_{g-1} \cdots q_j is a required q_i - q_j path since V(Q) ⊇ V(P) ⊇ N(q_g) ∪ \{q_g\}. For the case j ≥ h > i, a similar discussion holds.

(iii) f < i < j < h. Put

B = \{q_{i+1}, q_{i+2}, \ldots, q_{j-1}\}.

B = \emptyset if j = i + 1. Note that B ∩ N(q_1) = \emptyset then P = q_i q_{i-1} \cdots q_1 q_h q_{h-1} \cdots q_j is a required path since V(Q) ⊇ V(P) ⊇ N(q_1) ∪ \{q_1\}. A similar discussion holds for the case when B ∩ N(q_g) = \emptyset. Thus, we may assume that

B ∩ N(q_1) ≠ \emptyset and B ∩ N(q_g) ≠ \emptyset.

Let r = min\{t | q_t ∈ B ∩ N(q_1)\} and n = max\{t | q_t ∈ B ∩ N(q_1)\}. If there is a p such that n < p < j and q_p ∈ N(q_g) then P = q_i q_{i-1} \cdots q_1 q_r q_{r+1} \cdots q_p q_g q_{g-1} \cdots q_j is a required path since V(Q) ⊇ V(P) ⊇ N(q_1) ∪ \{q_1\}, and thus suppose that no such p exists, i.e.,

\{q_n, q_{n+1}, \ldots, q_{j-1}\} ∩ N(q_g) = \emptyset. (8)

Choose q_s ∈ B ∩ N(q_g) so that s is as small as possible. By (8) s < n, and then P = q_i q_{i-1} \cdots q_1 q_n q_{n-1} \cdots q_s q_g q_{g-1} \cdots q_j is a required path since V(Q) ⊇ V(P) ⊇ N(q_g) ∪ \{q_g\}, and so the proof of (a) is completed.

(b) Let q_i, q_j ∈ N^*(q_1). If j ≤ f, this follows immediately from (i) of the proof of (a) since N^*(q_1) ⊇ V(Q). So suppose that j > f, and let n = max\{t | t < j and q_t ∈ N(q_g)\}. If n > i then P = q_i q_{i+1} \cdots q_1 q_{i+1} \cdots q_n q_g q_{g-1} \cdots q_j is a q_i - q_j path in F which contains N(q_g) ∪ \{q_g\}, and is therefore of length at least d(q_g). This means that L_r(q_i, q_j) ≥ d(q_g). If n < i then, by the same arguments the path P = q_i q_{i-1} \cdots q_n q_{n-1} \cdots q_f q_g q_{g-1} \cdots q_j implies that L_r(q_i, q_j) ≥ d(q_g).

(c) Let q_i, q_j ∈ N^*(q_1). If h > f then, by (b) L_H(q_i, q_j) ≥ d(q_g). If h ≤ f then, since H is 2-connected, and from the proof of Theorem 4.4 in [3], H contains a cycle of length at least d(q_1) + d(q_g) + 1. This implies, as H is 2-connected, that every two vertices of H are connected by a path of length at least \left\lceil \frac{1}{2}(d(q_1) + d(q_g) + 1) \right\rceil, and so does every two vertices of N^*(q_1).

(d) Let q', q" ∈ V(H). If h ≤ f then, from the proof of (c), q' and q" are connected by a path of length at least \left\lceil \frac{1}{2}(d(q_1) + d(q_g) + 1) \right\rceil, i.e.,

L_H(q', q") ≥ \left\lceil \frac{1}{2}(d(q_1) + d(q_g) + 1) \right\rceil ≥ \min(d(q_1), d(q_g)).
If \( h > f \) then, as \( H \) is 2-connected, there are two disjoint \( \{ q', q'' \} - V(Q) \) paths in \( H \). However, by (a), every two vertices of \( V(Q) \) are connected by a path of length at least \( \min(d(q_1), d(q_g)) \) whose vertices all belong to \( V(Q) \), and so it is not difficult to see that

\[
L_H(q', q'') \geq \min(d(q_1), d(q_g)).
\]

**Lemma 5.** Suppose that \( H \) is any graph and \( C \) is a longest cycle in \( H \). Let \( W \) be a component in \( H \setminus C \) with two or more vertices. If there are \( t \geq 2 \) pairwise vertex-disjoint \( V(W) - V(C) \) paths such that between every pair of endvertices in \( V(W) \) of the paths there is a path in \( W \) of length at least \( d \), then \( |V(C)| \geq t(d+2) \).

**Proof.** For \( i = 1, \ldots, t \), let \( v_i \) be the end vertices in \( V(W) \), \( u_i \) the corresponding end vertices in \( V(C) \), of the paths. By the given conditions, \( L_{W}(v_i, v_j) \geq d \) for every pair of subscripts \( i \) and \( j \). This means, by the maximality of \( C \), that \( d_{C}(u_i, u_j) \geq d + 2 \), and hence \( |V(C)| \geq t(d+2) \).

**Lemma 6.** Suppose that \( H \) is any graph and \( C = c_1c_2 \cdots c_mC_1 \) is a longest cycle in \( H \), the subscripts of the \( c_i \)'s will be reduced modulo \( m \). Let \( W \) be a component in \( H \setminus C \) with at least two vertices and \( d, k \) be any integers such that \( 1 \leq d \leq k-2 \). Suppose \( T \subseteq V(W) \) such that

(i) \( |T| \geq \max(2, d) \),
(ii) \( e(v, C) \geq k - d \) for every \( v \in T \), and
(iii) \( L_W(v, u) \geq d \) for any \( v, u \in T \).

Assume further that \( m < 3k \). Then, for every \( v \) in \( T \),

\[
N_C(v) = N_C(T) \quad \text{and} \quad |N_C(T)| = k - d.
\]

**Proof.** Let

\[
T = \{ q_1, q_2, \ldots, q_i \},
\]

and

\[
A_i = N_C(q_i), \quad i = 1, \ldots, t.
\]

Note that, for all \( i = 1, \ldots, t \),

\[
|A_i| = e(q_i, C) \geq k - d,
\]

which means \( |A_i| \geq 2 \), since \( d \leq k - 2 \). Take into account \( t \geq 2 \) (where \( t = |T| \)), it is easy to see that there are at least two ordered pairs \( (p, n) \) such that \( c_p \) is joined to one, and \( c_n \) to another, of vertices of \( T \) and \( e(T, \{ c_{p+1}, \)}
$c_{p+2, \ldots, c_{n-1}} = 0$. Furthermore, the given condition (iii) and the maximality of $C$ give that

$$|\{c_{p+1}, c_{p+2}, \ldots, c_{n-1}\}| \geq d + 1.$$  

and hence

$$m \geq \left| \bigcup_{i=1}^{t} (A_i \cup A_i^+) \right| + 2(d + 1) - 2.$$  

If $A_i \cap A_j = \emptyset$ for every pair of subscripts $i$ and $j$, then

$$m \geq \sum_{i=1}^{t} |A_i \cup A_i^+| + 2d$$

$$\geq 2t(k - d) + 2d \quad \text{using (9).}$$  

If $d = 1$ then $t \geq 2$ and

$$m \geq 4(k - 1) + 2 \geq 3k.$$  

If $d \geq 2$ then $t \geq d$, and $k \geq 4$ since $d \leq k - 2$. So

$$m \geq 2d(k - d) + 2d \geq 2d(k - d) + 4$$

$$= 2(d - 2)(k - d - 2) + 4k - 4 \quad \text{(rearranging)}$$

$$\geq 4k - 4 \geq 3k.$$  

In each case $m \geq 3k$, which is contrary to hypothesis and shows that there are subscripts $i$ and $j$ such that $A_i \cap A_j \neq \emptyset$. Put

$$X = \{u \in V(C) \mid \exists i \neq j, A_i \cap A_j \ni u\}.$$  

Then $X \neq \emptyset$. Let $x = |X|$ and $S_1, S_2, \ldots, S_x$ be the sets of vertices contained in the open segments of $C$ between vertices of $X$ (define $S_1 = V(C) \setminus X$ if $x = 1$). Since $C$ is a longest cycle, the given condition (iii) and the definition of $X$ give that

$$|S_i| \geq d + 1 \quad \text{for all} \quad i = 1, \ldots, x.$$  

(10)

Since $m = \sum_{i=1}^{x} |S_i| + x$ we find that

$$m \geq \sum_{i=1}^{x} (d + 1) + x = x(d + 2),$$  

giving

$$x \leq \frac{m}{d + 2} \leq \frac{3k}{d + 2} = k - d + 1 - \frac{(d - 1)(k - d - 2)}{d + 2}.$$  


Using $1 \leq d \leq k - 2$, we obtain

$$x < k - d + 1.$$

(11)

We now prove that

$$x = k - d.$$

(12)

If (12) is false then, by (11),

$$x \leq k - d - 1.$$

(13)

Put

$$\bar{X} = N_c(T) \setminus X,$$

and then

$$|\bar{X}| = \sum_{i=1}^{t} |A_i \setminus X| \geq \sum_{i=1}^{t} |A_i \setminus X| \geq \sum_{i=1}^{t} (k - d - x) \quad \text{using (9),}$$

$$= t(k - d - x).$$

(14)

In fact, $X$ is the set of vertices of $C$ which are joined exactly to one vertex of $T$, and $\bar{X}$ is the set of vertices of $C$ which are joined to two or more vertices of $T$. Put

$$s_i = (S_i \setminus \bar{X}) \setminus \bigcup_{j \neq i} S_j \setminus \bar{X}, \quad 1 \leq i \leq t.$$}

(15)

(a) Assume first $|s_i| = 1$. For convenience, let $s_i = \{S_i\}$. Then all vertices of $\bar{X}$ lie in $S_i$. From (9) and (13), $x < |A_i|$, and therefore $A_i \cap \bar{X} \neq \emptyset$, for all $i = 1, \ldots, t$. On the other hand, from the maximality of $C$ and the given condition (iii), if $c_i \in A_i \cap \bar{X}$ and $c_j \in A_j \cap \bar{X}$, $i \neq j$, then $d_C(c_i, c_j) \geq d + 2$. Hence it is not difficult to see that

$$|S_i| \geq (t + 1)(d + 1) + |\bar{X} \cup \bar{X}^+| - t \geq 3d + 1 + 2 |\bar{X}|,$$

and so

$$m = \sum_{i=2}^{x} |S_i| + |S_i| + x \geq \sum_{i=2}^{x} (d + 1) + |S_i| + x \quad \text{using (10),}$$

$$\geq x(d + 2) + 2d + 2 |\bar{X}|.$$
(b) Assume that $|\mathcal{F}| \geq 2$. By reasoning similar to that used in (a), for each $S_i \in \mathcal{F}$,

$$|S_i| \geq 2(d + 1) + 2 |\overline{X} \cap S_i| - 1 = 2d + 1 + 2 |\overline{X} \cap S_i|,$$  

and so

$$m = \sum_{S_i \in \mathcal{F}} |S_i| + \sum_{S_i \in \mathcal{F}} |S_i| + x$$
$$\geq \sum_{S_i \in \mathcal{F}} (d + 1) + \sum_{S_i \in \mathcal{F}} (2d + 1 + 2 |\overline{X} \cap S_i|) + x$$
$$= (x - |\mathcal{F}|)(d + 1) + |\mathcal{F}| (2d + 1) + 2 |\overline{X}| + x$$
$$= x(d + 2) + |\mathcal{F}| d + 2 |\overline{X}|$$
$$\geq x(d + 2) + 2d + 2 |\overline{X}|.  

In each case (15) holds. Combining (15) with (14) we have

$$m \geq x(d + 2) + 2d + 2t(k - d - x).$$

If $d = 1$ then $t \geq 2$, and thus

$$m \geq 3x + 2 + 4(k - 1 - x) = 3k + (k - x - 2)$$
$$\geq 3k \quad \text{by (13)}.$$

If $d \geq 2$ then $t \geq d$, and so $k \geq 4$ since $d \leq k - 2$. Thus

$$m \geq x(d + 2) + 2d + 2d(k - d - x) = 2d(k - d + 1) - x(d - 2)$$
$$\geq 2d(k - d + 1) - (k - d - 1)(d - 2) \quad \text{by (13)},$$
$$= (d + 2)(k - d - 1) + 4d \quad \text{(rearranging)}$$
$$\geq 4(k - d - 1) + 4d = 4k - 4$$
$$\geq 3k.$$

Each case gives that $m \geq 3k$, contrary to hypothesis. Hence (12) is proved. Also, we have

$$\overline{X} = \emptyset.$$

If $\overline{X} \neq \emptyset$ then $|\mathcal{F}| \geq 1$. Note that (17) also holds for the case $|\mathcal{F}| \geq 1$ we have

$$m \geq x(d + 2) + d + 2 = (k - d)(d + 2) + d + 2 \quad \text{by (12)},$$
$$= 3k + (d - 1)(k - d - 2)$$
$$\geq 3k.$$
This contradiction shows that \( \bar{x} = \emptyset \), which means that \( A_i \subseteq X \) for all \( i = 1, \ldots, t \), but (9) and (12) imply that \( |A_i| \geq |X| \), and so \( A_1 = A_2 = \cdots = A_t = X = N_C(T) \). Hence Lemma 6 is proved.

Let us return to the proof of Part 2. Let \( W \) be a component in \( R \) with two or more vertices. We consider two cases:

Case 1. \( W \) is a block. Let \( Q = q_1, q_2, \ldots, q_g \) be a longest path in \( W \) followed from \( q_1 \) to \( q_g \), chosen so that \( d_w(q_1) + d_w(q_g) \) is maximal. Suppose without loss of generality that

\[
d_w(q_g) \geq d_w(q_1) = d.
\]

Let \( q_h \) be the last vertex which belongs to \( N(q_1) \) and \( q_f \) the first vertex which belongs to \( N(q_g) \). Note that, as \( Q \) is a longest path, \( N_w(q_1) \subseteq V(Q) \) and \( N_w(q_g) \subseteq V(Q) \). Replacing \( H \) by \( W \) in Lemma 4, we see that the path \( Q \) satisfies the requirement of Lemma 4.

Case 1a. \( d \leq k - 3 \). In this case we have

**Proposition 2.** \( Q \) is a hamilton path in \( W \) with \( d + 1 \) vertices, and \( N_C(q_i) = N_C(q_j) \) for any \( q_i \) and \( q_j \) in \( V(Q) \).

**Proof.** Note that, as \( W \) is a block, if \( d = 1 \) then \( W = Q = q_1 q_2 \) and so applying Lemma 6 to the set \( \{ q_1, q_2 \} \) we complete the proof. Thus suppose that \( d \geq 2 \), and so \( N^*(q_1) = d \geq 2 \). Note that for each \( q_i \in N^*(q_1) \) the path \( q_i q_{i-1} \cdots q_1 q_{i+1} \cdots q_g \) has the same length as \( Q \) so \( d_w(q_i) \leq d_w(q_1) = d \), implying \( e(q_i, C) = k - d_w(q_i) \geq k - d \). Furthermore, using Lemma 4(d), and taking (18) into account we obtain that \( L_w(q_i, q_j) \geq d \) for every pair of vertices \( q_i \) and \( q_j \) in \( N^*(q_1) \). From the above discussion, the set \( N^*(q_1) \) satisfies the requirement of Lemma 6 and thus, putting \( X = N_C(N^*(q_1)) \), and using Lemma 6 we have that, for each \( q_i \in N^*(q_1) \),

\[
N_C(q_i) = X
\]

and \( |X| = k - d \). Let \( S_1, S_2, \ldots, S_{k-d} \) be the sets of vertices contained in the open segments of \( C \) between vertices of \( X \). Then

\[
m = \sum_{i=1}^{k-d} |S_i| + (k - d).
\]

If \( |S_i| \geq d + 2 \) for all \( i = 1, \ldots, k-d \), then

\[
m \geq \sum_{i=1}^{k-d} (d + 2) + (k - d)
\]

\[
= 3k + d(k - d - 3)
\]

\[
\geq 3k,
\]
contrary to hypothesis, and thus there must be some \( j \), such that

\[ |S_j| < d + 2. \tag{20} \]

If \( d_{w}(q_{g}) \geq d + 1 \) then, by Lemma 4(c), \( L_{w}(q_{i}, q_{j}) \geq d + 1 \) for any \( q_{i}, q_{j} \in N^{*}(q_{1}) \), which implies that \( |S_{i}| \geq d + 2 \) for all \( i = 1, \ldots, k - d \), contrary to (20). Hence \( d_{w}(q_{g}) \leq d \), giving \( e(q_{g}, C) \geq k - d \). Furthermore, applying Lemma 6 to the set \( N^{*}(q_{1}) \cup \{ q_{g} \} \) we obtain

\[ N_{c}(q_{g}) = X. \tag{21} \]

This means \( |S_{i}| \geq g \) for all \( i = 1, \ldots, k - d \). It follows from (20) that \( g < d + 2 \), but on the other hand, \( g \geq d_{w}(q_{1}) + 1 = d + 1 \). Consequently

\[ g = d + 1, \]

implying

\[ V(Q) = N^{*}(q_{1}) \cup \{ q_{g} \}. \tag{22} \]

As \( Q \) is a longest path in \( W \), no vertex of \( N^{*}(q_{1}) \cup \{ q_{g} \} \) can be joined to vertices of \( V(W) \setminus V(Q) \), and so the connectness of \( W \) implies that \( Q \) is a hamiltonian path in \( W \). Also, combining (19), (21) with (22) we find that \( N_{c}(q_{i}) = X = N_{c}(q_{g}) \) for any \( q_{i}, q_{g} \in V(Q) \). Hence the proposition is proved. \( \blacksquare \)

Put \( X = N_{c}(V(Q)) \). By Proposition 2, \( |X| = k - d \) and

\[ e(X, Q) = |X| \cdot |V(Q)| = (k - d)(d + 1). \]

As defined above, let \( S_{1}, S_{2}, \ldots, S_{k - d} \) be the sets of vertices contained in the open segments of \( C \) between vertices of \( X \), and then

\[
\sum_{i=1}^{k-d} e(X, S_{i}) \leq k |X| - e(X, Q) = k(k - d) - (k - d)(d + 1) = (k - d)(k - d - 1). \tag{23}
\]

Put \( s_{i} = |S_{i}|, \ i = 1, \ldots, k - d \). Note that, for all \( i = 1, \ldots, k - d \),

\[ s_{i} \geq g = d + 1. \]

Suppose that there is some \( j \) such that

\[ s_{j} = d + 1. \]
Replacing by $Q$ the open segment of $C$ which contains $S_j$, we obtain another cycle $C'$ of the same length as $C$. From the choice of $C$,

$$e(S_j, R) = 0,$$  

(24)

since otherwise $G \setminus C'$ has less components. As $C$ is a longest cycle, it is easily checked that, for any $n \neq j$, if $s_n \leq d + 2$ then

$$e(S_j, S_n) = 0;$$  

(25)

if $s_n = d + 3$ then

$$e(S_j, S_n) \leq s_j,$$  

(26)

since in this case each vertex of $S_j$ can be joined to at most one vertex of $S_n$.

If every $s_i$ takes its minimal value $d + 1$, then

$$m = \sum_{i=1}^{k-d} s_i + (k-d) \geq \sum_{i=1}^{k-d} (d+1) + (k-d)$$

$$= 3k - 3 + (d-1)(k-d-3)$$

$$\geq 3k - 3.$$

However, by the assumption, $m \leq 3k - 1$, and so there are only the following two cases for values of \{s_1, s_2, ..., s_{k-d}\}:

(i) There is exactly one element $s_n = d + 3$, and $s_j = d + 1$ for every $j \neq n$. Then for all $j \neq n$

$$e(S_j, X) = k \cdot s_j - 2e(S_j) - e\left( S_j, \bigcup_{i \neq j} S_i \right) - e(S_j, R)$$

$$= k \cdot s_j - 2e(S_j) - e(S_j, S_n)$$

by (25) and (24),

$$\geq ks_j - s_j(s_j - 1) - s_j$$

since $e(S_j) \leq (\frac{s_j}{2})$, and by (26),

$$= (d+1)(k-d-1)$$

since $s_j = d + 1$,

$$\geq 2(k-d-1).$$

Summing these inequalities yields

$$\sum_{j \neq n}^{k-d} e(S_j, X) \geq \sum_{j \neq n}^{k-d} 2(k-d-1) = 2(k-d-1)(k-d-1)$$

$$> (k-d)(k-d-1).$$

This is contrary to (23).
There are two elements $s_n$ and $s_1$ such that $s_n \leq d + 2$ and $s_1 \leq d + 2$, and for every $j \neq n$, $s_j = d + 1$ and so

$\sum_{j \neq n, t} e(S_j, X) = k \cdot s_j - 2e(S_j) \cdot e \left( S_j, \bigcup_{i \neq j} S_i \right) - e(S_j, R)$

$= ks_j - 2e(S_j)$ by (25) and (24).

$(d + 1)(k - d)$ since $e(S_j) \leq \left( \frac{k}{2} \right)$ and $s_j = d + 1$,

$\geq 2(k - d)$.

Summing these inequalities gives

$\sum_{j \neq n, t} e(S_j, X) \geq 2(k - d) = 2(k - d)(k - d - 2),$

and so

$\sum_{j = 1}^{k - d} e(S_j, X) = \sum_{j \neq n, t} e(S_j, X) + e(S_n, X) + e(S_t, X)$

$\geq 2(k - d - 2)(k - d) + 2 + 2$

$> (k - d)(k - d - 1),$

which contradicts (23).

In each case we obtain a contradiction, and hence the discussion of Case 1a is completed.

**Case 1b.** $d \geq k - 2$. If $d = 1$, this is the case discussed in Case 1a, where we did not use the condition $d \leq k - 3$ when $d = 1$. Therefore, in the following, we consider only the case when $d \geq 2$ and $k \geq 4$. We shall prove

$m \geq 2k + 4.$

By contradiction, suppose that $m < 2k + 4$. Consider the following three subcases:

(i) $d = k$. Since $G$ is 2-connected, there are two disjoint $V(W) - V(C)$ paths in $G$. Using Lemma 4(d) and Lemma 5 we obtain that

$m \geq 2(d + 2) = 2(k + 2)$ a contradiction.

(ii) $d = k - 1$. From the choice of $Q$, for every $q_i \in N^*(q_1)$, $d_{W}(q_i) \leq d = k - 1$, implying $|N_C(q_i)| \geq 1$. Note that $|N^*(q_1)| = k - 1$ and $d(u) \leq k$ for $u \in C$, we find that there are at least two distinct vertices of $C$, say $c_i$ and $c_j$, which are respectively joined to two distinct vertices of
If $d_W(q_g) - k$ then, by Lemma 4(c), $L_W(v_i, v_j) \geq k$, giving $m \geq 2(k + 2)$, which is a contradiction and shows that $d_W(q_g) < k$. Therefore we may assume that $q_g$ is joined to a vertex, say $c_p$, of $C$. If $c_p \in \{c_i, c_j\}$, then there are three independent edges in $E(W, C)$, and then using Lemma 4(d) and Lemma 5 we have that $m \geq 3(d + 2) = 3(k + 1)$. This contradiction shows that $c_p \in \{c_i, c_j\}$. Put

\[ S_1 = \{c_{i+1}, c_{i+2}, \ldots, c_{j-1}\} \quad \text{and} \quad S_2 = \{c_{j+1}, c_{j+2}, \ldots, c_{i-1}\}. \]

Clearly

\[ |S_1| \geq g \geq d + 1 = k \quad \text{(28)} \]

and

\[ |S_2| \geq g \geq d + 1 = k. \]

However, by the supposition,

\[ |S_1| + |S_2| = m - 2 < 2k + 2, \]

and so we may assume, without loss of generality, that

\[ |S_1| = k \quad \text{and} \quad |S_2| \leq k + 1. \]

Then (28) gives $g = k$, and by reasoning similar to that used in the proof of Proposition 2, $Q$ is a hamilton path in $W$. Therefore, analogously to the proof of (24) and (25),

\[ e(S_1, R) = 0 \quad \text{and} \quad e(S_2, S_1) = 0, \]

giving

\[ e(S_1, \{c_i, c_j\}) = k |S_1| - 2e(S_1) \geq k |S_1| - |S_1| \cdot (|S_1| - 1) = k. \]

On the other hand, $|S_1| = k$ and $|S_2| \leq k + 1$ imply $e(N^*(q_1), S_1 \cup S_2) = 0$, and therefore

\[ e(N^*(q_1), \{c_i, c_j\}) = e(N^*(q_1), V(C)) \geq |N^*(q_1)| \geq k - 1. \]

Then

\[ e(\{c_i, c_j\}, S_2) \leq 2k - e(S_1, \{c_i, c_j\}) - e(N^*(q_1), \{c_i, c_j\}) \leq 1, \]

which is contrary to the fact that $e(\{c_i, c_j\}, S_2) \geq 2$ and completes the discussion of (ii).
(iii) \( d = k - 2 \). Applying Lemma 6 to the set \( N^*(q_i) \) we deduce that there are exactly two vertices on \( C \), say \( c_i \) and \( c_j \), which are joined to every vertex of \( N^*(q_i) \). Note that \( d(c_i), d(c_j) \leq k \) and \( |N^*(q_i)| = k - 2 \), it is easy to see that \( q_g \) cannot be joined to either \( c_i \) nor \( c_j \). Therefore if \( q_g \) is joined to some vertex, say \( c_p \), of \( C \) then \( c_p \notin \{c_i, c_j\} \), and so there are three independent edges in \( E(W, C) \). It follows from Lemma 4(d) and Lemma 5 that \( m \geq 3(d + 2) = 3k \). This is impossible since \( m < 3k \) is assumed throughout the proof of the theorem. Hence \( q_p \) is joined to no vertex on \( C \). This means

\[
d_w(q_g) = k.
\] (29)

Applying Lemma 4(c) and the maximality of \( C \) we have that \( d_C(c_i, c_j) \geq k + 2 \), giving \( m \geq 2(k + 2) \), a contradiction.

In each case we arrive at a contradiction, and hence (27) is proved.

We shall now complete the discussion of Case 1b by using the given condition (ii) of the theorem. Noting that if \( d \geq k - 1 \) then \( |V(W)| \geq d + 1 \geq k \) and if \( d = k - 2 \) then by (29) \( |V(W)| \geq k + 1 \), and taking (27) into account we find, by the given condition (ii) of the theorem, that \( G \) contains 3 pairwise disjoint \( V(W) - V(C) \) paths. It follows from Lemma 4(d) and Lemma 5 that \( m \geq 3(d + 2) = 3k \). This is contrary to the assumption and completes the discussion of Case 1b.

Case 2. \( W \) is not a block. Let \( H_1 \) and \( H_2 \) be two end-blocks of \( W \) (blocks containing only one cut-vertex), and let

\[
Q_1 = a_1a_2\cdots a_g \quad \text{and} \quad Q_2 = b_1b_2\cdots b_g.
\]

be longest paths in \( H_1 \) and \( H_2 \), respectively, such that

\[
d_{H_1}(a_1) + d_{H_2}(a_g) \quad \text{and} \quad d_{H_1}(b_1) + d_{H_2}(b_g)
\]

as large as possible. Without loss of generality, suppose that

\[
d_{H_1}(a_g) \geq d_{H_1}(a_1) = d_1, \quad d_{H_2}(b_g) \geq d_{H_2}(b_1) = d_2
\]

and

\[
d_1 \geq d_2.
\]

Let \( h_1 \) be the unique cutvertex in \( H_1 \) and \( h_2 \) in \( H_2 \). Choose a vertex \( b' \),

\[
b' \in N^*(b_1) \setminus \{h_2\} \quad \text{if} \quad N^*(b_1) \neq \{h_2\},
\]

\[
b' \in V(H_2) \setminus \{h_2\} \quad \text{if} \quad N^*(b_1) = \{h_2\}.
\]
As $H_2$ is an edge when $N^*(b_1) = \{h_2\}$, it is quite easy to see that

$$d_w(b') = d_{H_2}(b') = d_{H_2}(b_1) = d_2 < d_1,$$

implying

$$e(b', C) \geq k - d_1.$$

Let $W_1 = W \setminus (H_1 \setminus \{h_1\})$ and contract $W_1$ to a vertex, denoted by $h^*$, in other words, add a new vertex $h^*$ to the graph $G \setminus W_1$ and join it to every vertex $u \in G \setminus W_1$ for which $G$ contains an edge from $u$ to $W_1$. The resulting graph is denoted by $\bar{G}$. It is not difficult to see that $C$ remains a longest cycle in $\bar{G}$ and $H_1$ is a component in $\bar{G} \setminus C$. However, in the new graph $\bar{G}$, $h_1$ has been replaced by $h^*$ and

$$e_{\bar{G}}(h^*, C) \geq e(b', C) \geq k - d_1,$$

and so

$$e_{\bar{G}}(a_i, C) \geq k - d_1$$

for any $a_i \in N^*(a_1)$,

where the suffix $\bar{G}$ indicates that the underlying graph is $\bar{G}$, for example, $e_{\bar{G}}(a_i, C)$ denotes the number of edges in $\bar{G}$ joining $a_i$ to vertices of $C$. We replace $G$, $C$, $W$, $Q$, and $d$ by $\bar{G}$, $C$, $H_1$, $Q_1$, and $d_1$, respectively. Note that $d_{\bar{G}}(v) \leq k$ for all $v \in N_C(h^*)$ and $d_{\bar{G}}(v) = k$ for all $v \in V(G) \setminus N_C(h^*)$, $v \neq h^*$, we see that all arguments in Case 1 can be repeated. This completes the discussion of Case 2, and with it the proof of Theorem 1.

In the above proof, we used condition (ii) of Theorem 1 only in Case 1b of Part 2. If condition (ii) is deleted, then Case 1b gives that $m \geq 2k + 4$. This means

**Theorem 2 (Chen [4]).** Every 2-connected, $k$-regular graph on $n$ vertices has a cycle of length at least $\min(3k, 2k + 4, n)$.

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**References**


