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A Topology for Operator Modules over W^*-Algebras

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Given a von Neumann algebra R on a Hilbert space $\mathscr{H}$, the so-called R-topology is introduced into $B(\mathscr{H})$, which is weaker than the norm and stronger than the

such that $ba \in X$, is weak* closed in R. Equivalently, X is closed in the R-topology if and only if for each $b \in B(\mathscr{H})$ and each orthogonal family of projections e_i in R with the sum 1 the condition $be_i \in X$ for all i implies that $b \in X$.

1. INTRODUCTION

Given a C*-algebra R on a Hilbert space $\mathscr{H}$, a concrete *operator right R-module* is a subspace X of $B(\mathscr{H})$ (the algebra of all bounded linear operators on $\mathscr{H}$) such that $XR \subseteq X$. Such modules can be characterized abstractly as L_∞-matricially normed spaces in the sense of Ruan [21], [11] which are equipped with a completely contractive R-module multiplication (see [6] and [9]). If R is a von Neumann algebra, it is natural to study the R-submodules of $B(\mathscr{H})$ which are closed in the weak* topology. It turns out, however, that many properties of weak* closed modules are valid in fact for the larger class of so-called strong modules. A right R-submodule X of $B(\mathscr{H})$ is called *strong* if $\sum_{i \in \mathbb{I}} x_i a_i \in X$ for any two sets $\{x_i\}_{i \in \mathbb{I}} \subseteq X$ and $\{a_i\}_{i \in \mathbb{I}} \subseteq R$ such that the sums $\sum_{i \in \mathbb{I}} x_i x_i^*$ and $\sum_{i \in \mathbb{I}} a_i^* a_i$ are convergent to bounded operators in the strong operator topology. All weak* closed modules are strong of course, and it turns out that all strong R-submodules of R (that is, right ideals) are necessarily weak* closed, but in general the class of all strong R-modules is much larger than the class of all weak* closed R-modules. For example, in the case $R = \mathbb{C}$ all norm closed R-submodules are strong.

Strong modules appeared first in a remark at the end of the paper [9] by Effros and Ruan under the name $\mathbb{M}_\infty$-submodules. Later, in [15], a few

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basic properties of strong modules were developed and used to study a module version of the weak* Haagerup tensor product of Blecher and Smith [5]. Recently this concept has been used by Blecher [2] to study tensor products of selfdual Hilbert modules. In [15] it has been observed in particular that for each strong right R-submodule X of $\mathrm{B}(\mathscr{H})$ and each element $b \in \mathrm{B}(\mathscr{H})$ the right ideal

$$R(b, X) := \{a \in R : ba \in X\}$$

is weak* closed. Here we shall show (in Section 2) that this property characterizes strong R-modules. Another algebraic characterization will be that X is strong if and only if for each $b \in \mathrm{B}(\mathscr{H})$ and each orthogonal set of projections e_i in R with the sum 1 the condition $be_i \in X$ for all i implies that $b \in X$. We shall try to give an intrinsic characterization of strong modules as a part of an abstract characterization of normal operator modules in Section 6.

In Section 3 we shall study for each von Neumann algebra $R \subseteq \mathrm{B}(\mathscr{H})$ the so-called R-topology, defined on $\mathrm{B}(\mathscr{H})$ by the family of seminorms

$$s_\omega(b) = \inf\{\|y\|\, \omega(a^*a)^{1/2} : b = ya,\ y \in \mathrm{B}(\mathscr{H}),\ a \in R\},$$

where ω is a normal positive functional on R. It will be shown in Section 5 that strong R-submodules of $\mathrm{B}(\mathscr{H})$ are precisely the submodules which are closed in the R-topology. In general, the R-topology on $\mathrm{B}(\mathscr{H})$ is strictly weaker than the norm topology and strictly stronger than the ultrastrong topology, but its restriction to R agrees with the ultrastrong operator topology. A linear functional ρ turns out to be continuous in the R-topology if and only if the functional $a \mapsto \rho(ba)$ on R is normal for each $b \in \mathrm{B}(\mathscr{H})$ (Section 4). Although the R-topology is in general different from (the ultrastrong and) the weak* topology, a variant of the Krein–Smulian theorem can be proved (Theorem 5.1).

Throughout the paper R will be a von Neumann algebra and X an operator R-module such that $x1 = x$ for all $x \in X$, where 1 is the unit in R. By an R-module we shall always mean a right module. X is called *normal* if there exists a Hilbert space $\mathscr{H}$, a normal representation $\varphi: R \to \mathrm{B}(\mathscr{H})$ and a complete isometry $\Phi: X \to \mathrm{B}(\mathscr{H})$ such that $\Phi(xa) = \Phi(x)\,\varphi(a)$ for all $x \in X$ and $a \in R$. (We refer to [16] for the definition of a complete isometry and for basic theory of completely bounded maps.) Most of the time X will be a normal R-module.

The set of all $1 \times n$ matrices (rows) with entries in a set Y will be denoted by $\mathscr{R}_n(Y)$, and the set of all $1 \times \infty$ matrices with only finitely many non-zero entries in Y by $\mathscr{R}_{\mathrm{fin}}(Y)$. The corresponding sets of columns will be denoted by $\mathscr{C}_n(Y)$ and $\mathscr{C}_{\text{fin}}(Y)$. Given $x \in \mathscr{R}_{\text{fin}}(X)$ and $a \in \mathscr{C}_{\text{fin}}(R)$ the product xa is defined by

$$xa = \sum_i x_i a_i,$$

where x_i and a_i are the components of x and a (respectively). If Y is a left R-module, $y \in \mathscr{C}_{\text{fin}}(Y)$ and $x \in \mathscr{R}_{\text{fin}}(X)$, then $x \odot_R y$ denotes the element $\sum_i x_i \otimes_R y_i$ in the (algebraic) tensor product $X \otimes_R Y$.

2. ALGEBRAIC CHARACTERIZATION OF STRONG MODULES

THEOREM 2.1. *The following properties are equivalent for a norm closed R-submodule $X \subseteq B(\mathscr{H})$, where R is a von Neumann algebra represented normally on $\mathscr{H}$:*

(i) *X is strong;*

(ii) *for each $x \in B(\mathscr{H})$ and each orthogonal family $\{e_i : i \in \mathbb{I}\}$ of projections in R with the sum 1 the condition $xe_i \in X$ for all $i \in \mathbb{I}$ implies that $x \in X$;*

(iii) *for each $x \in B(\mathscr{H})$ the right ideal $R(x, X) = \{a \in R : xa \in X\}$ in R is weak* closed.*

Proof. (i) $\Rightarrow$ (ii). Let $x \in \mathrm{B}(\mathscr{H})$ and let $\{e_i : i \in \mathbb{I}\}$ be an orthogonal family of projections in R with the sum 1 such that $xe_i \in X$ for all $i \in \mathbb{I}$. Since $x = \sum_{i \in \mathbb{I}} (xe_i)\, e_i$, it follows directly from the definition of a strong module (see Section 1) that $x \in X$.

(ii) $\Rightarrow$ (i). Let $\{x_i : i \in \mathbb{I}\} \subseteq X$ and $\{a_i : i \in \mathbb{I}\} \subseteq R$ be such that the sums

$$\sum_{i \in \mathbb{I}} x_i x_i^* \qquad \text{and} \qquad \sum_{i \in \mathbb{I}} a_i^* a_i =: c$$

converge (to bounded operators) in the strong operator topology. Then by the non-commutative Egoroff theorem (see [23, p. 85]) for each nonzero projection $f \in R$ there exists a nonzero subprojection $e \leqslant f$ in R and a sequence of finite subsets $\mathbb{F}_k$ of $\mathbb{I}$ such that the sequence of finite sums $\sum_{i \in \mathbb{F}_k} a_i^* a_i e$ converges in norm to ce. Since the finite sums $\sum_{i \in \mathbb{F}_k} e a_i^* a_i e$ form an increasing net, it follows that the sum $\sum_{i \in \mathbb{I}} e a_i^* a_i e$ converges in norm to ece. This implies that the sum $\sum_{i \in \mathbb{I}} x_i a_i e$ is also norm convergent. Namely, the finite subsums $\sum_{i \in \mathbb{F}} x_i a_i e$ form a Cauchy net since

$$\left\| \sum_{i \in \mathbb{F}} x_i a_i e \right\| \leqslant \left\| \sum_{i \in \mathbb{F}} x_i x_i^* \right\|^{(1/2)} \left\| \sum_{i \in \mathbb{F}} e a_i^* a_i e \right\|^{(1/2)}$$
$$\leqslant \left\| \sum_{i \in \mathbb{I}} x_i x_i^* \right\|^{(1/2)} \left\| \sum_{i \in \mathbb{F}} e a_i^* a_i e \right\|^{(1/2)}.$$

Since X is norm closed, it follows that $xe \in X$, where $x = \sum_{i \in \mathbb{I}} x_i a_i$. Let now $\{e_j : j \in \mathbb{J}\}$ be a maximal orthogonal set of nonzero projections in R such that $xe_j \in X$ for all j. Since the projection f is arbitrary, the above argument shows that $\sum_{j \in \mathbb{J}} e_j = 1$, hence (ii) implies that $x \in X$.

(i) $\Rightarrow$ (iii). This implication is proved in the same way as [15, Proposition 2.3].

(iii) $\Rightarrow$ (ii). Let $x \in \mathrm{B}(\mathscr{H})$ and let $\{e_i : i \in \mathbb{I}\}$ be an orthogonal family of projections in R with the sum 1 such that $xe_i \in X$ for all i. By hypothesis the ideal $R(x, X)$ is weak* closed, hence of the form eR for some projection $e \in R$. Thus $xe \in X$ (since $e \in R(x, X)$) and it suffices now to prove that $e = 1$. Suppose the contrary that $e^{\perp} \neq 0$. Then $e^{\perp} e_i e^{\perp} \neq 0$ for some $i \in \mathbb{I}$. From $xe_i e^{\perp} \in X$ we have $e_i e^{\perp} \in R(x, X)$ and therefore $e_i e^{\perp} = ee_i e^{\perp}$. But this is in contradiction with $e^{\perp} e_i e^{\perp} \neq 0$. ∎

The following proposition shows that the smallest strong R-submodule of $\mathrm{B}(\mathscr{H})$ containing a given norm closed module X can be obtained in one step.

Proposition 2.2. *For any norm closed R-submodule X of $B(\mathscr{H})$ (where R is a von Neumann algebra with a normal representation on $B(\mathscr{H})$) the set $\tilde{X}$, which consists of all $b \in B(\mathscr{H})$ such that there exists an orthogonal family $\{e_i : i \in \mathbb{I}\}$ of projections in R with the sum* 1 *satisfying $be_i \in X$ for all i (the family may depend on b), is a strong R-module and it is then clearly the smallest strong R-submodule of $B(\mathscr{H})$ containing X.*

To prove this proposition, it will be convenient to isolate a part of the argument as a lemma.

Lemma 2.3. *Let X, R and $B(\mathscr{H})$ be as in Proposition 2.2, $b \in B(\mathscr{H})$ and let $\{e_i : i \in \mathbb{I}\}$ be an orthogonal set of projections in R with the sum* 1 *such that $be_i \in X$ for all i. Then for each projection $p \in R$ there exists an orthogonal family of projections $\{p_j : j \in \mathbb{J}\}$ in R with the sum p such that $bp_j \in X$ for all $j \in \mathbb{J}$.*

Proof. For each finite subset $\mathbb{F} \subseteq \mathbb{I}$ let $e_{\mathbb{F}} = \sum_{i \in \mathbb{F}} e_i$. Then

$$\{e_{\mathbb{F}} : \mathbb{F} \text{ a finite subset of } \mathbb{I}\}$$

is a net of projections in R converging strongly to 1, hence by the noncommutative Egoroff theorem for each nonzero projection $q \in R$ there exists a nonzero subprojection $q_0 \in R$ and a sequence $\{\mathbb{F}_n\}_{n \in \mathbb{N}}$ of finite subsets of $\mathbb{I}$ such that the sequence $\{be_{\mathbb{F}_n} q_0\}_{n \in \mathbb{N}}$ converges to bq_0 in norm, hence $bq_0 \in X$. Thus, if $\{p_j : j \in \mathbb{J}\}$ is a maximal orthogonal set of subprojections of p in R such that $bp_j \in X$, then $\sum_{j \in \mathbb{J}} p_j = p$. ∎

Proof of Proposition 2.2. First we shall show that $\tilde{X}$ is a linear subspace of $B(\mathscr{H})$. Let $b, c \in \tilde{X}$ and let $\{e_i : i \in \mathbb{I}\}$ and $\{f_j : j \in \mathbb{J}\}$ be the sets of orthogonal projections in R with the sum 1 such that $be_i \in X$ and $cf_j \in X$ for all i and j. By Lemma 2.3 there exists for each i an orthogonal set $\{g_{ik} : k \in \mathbb{K}_i\}$ of projections in R with the sum e_i such that $cg_{ik} \in X$ for all $k \in \mathbb{K}_i$. Since $g_{ik} \leqslant e_i$, we also have $bg_{ik} = (be_i)\, g_{ik} \in X$. Thus, $\{g_{ik} : k \in \mathbb{K}_i, i \in \mathbb{I}\}$ is an orthogonal set of projections in R with the sum 1 such $(b+c)\, g_{ik} \in X$. This proves that $b + c \in \tilde{X}$. Moreover, Lemma 2.3 implies that $bp \in \tilde{X}$ for each $b \in \tilde{X}$ and each projection $p \in R$. Since each element of R can be approximated in norm by a linear combination of projections and $\tilde{X}$ will be shown to be norm closed, this will prove that $\tilde{X}$ is an R-submodule of $B(\mathscr{H})$.

Let $\{b_n : n \in \mathbb{N}\}$ be a sequence in $\tilde{X}$ converging in norm to some element $b \in B(\mathscr{H})$. We shall show that $b \in \tilde{X}$. It suffices to prove that each nonzero projection $e \in R$ contains a nonzero subprojection $f \in R$ such that $bf \in X$, for then a standard maximality argument gives a required orthogonal family of projections f_j with the sum 1 such that $bf_j \in X$ for all j. Let ω be a normal state on R with the support projection dominated by e. Applying the noncommutative Egoroff theorem (as in the proof of Lemma 2.3, but this time the whole theorem), we find a projection $f_1 \leqslant e$ in R such that $\omega(e - f_1) < 1/2$ and $b_1 f_1 \in X$. Proceeding inductively, we find a decreasing sequence of projections $e =: f_0 \geqslant f_1 \geqslant f_2 \ldots$ in R such that

$$\omega(f_{n-1} - f_n) < \frac{1}{2^n} \qquad \text{and} \qquad b_n f_n \in X$$

for all n. Then the projection $f := \bigwedge_{n \in \mathbb{N}} f_n$ is nonzero (since $\omega(f^{\perp}) = \sum_{n=1}^{\infty} \omega(f_{n-1} - f_n) < 1$) and $b_n f \in X$ for all n. Since the sequence $\{b_n f : n \in \mathbb{N}\}$ converges to bf in norm and X is norm closed, it follows that $bf \in X$.

Finally, using Lemma 2.3 it is easy to show that $\tilde{X}$ satisfies the requirement (ii) of Theorem 2.1, hence $\tilde{X}$ is a strong R-submodule of $B(\mathscr{H})$. ∎

3. A TOPOLOGY FOR OPERATOR MODULES

In this section we shall define a locally convex topology, the so-called R-topology, on each operator module over a von Neumann algebra R. The initial definition will be slightly indirect, but we shall soon see that this topology has a very simple description.

Recall that the Haagerup tensor product $X \otimes_R^h Y$, where X is a right and Y a left operator R-module, is obtained from the algebraic tensor product $X \otimes_R Y$ by dividing with the zero space of the seminorm

$$\|w\| = \inf \{ \|x\| \, \|y\| : w = x \odot_R y, \ x \in \mathscr{R}_{\text{fin}}(X), \ y \in \mathscr{C}_{\text{fin}}(Y) \}.$$

(We refer to [8], [17], [3] or [22] for the Haagerup tensor product of operator spaces and to [4] and [14] for the Haagerup tensor product of modules. If X and Y are normal operator modules over a von Neumann algebra R the above seminorm is indeed a norm, which does does not change if X or Y is embedded completely isometrically into a larger module. In the present paper we shall use the Haagerup tensor product more ore less only for motivation. The reader who is not acquainted with this tensor product may regard the first displayed formula in the proof of Lemma 3.1 below as the definition of the seminorms determining the R-topology; to see that these are indeed seminorms, he can use the proof that the Haagerup norm satisfies the triangle inequality, which can be found in [22, p. 31] or in [8].)

We denote by $R_\sharp$ the space of all normal linear functionals on R and by $R_{\sharp+}$ the subset of all positive functionals. Given $\omega \in R_{\sharp+}$, let $\mathscr{H}_\omega$ be the Hilbert space obtained from ω and R by the GNS construction and let ξ_ω be the corresponding cyclic vector in $\mathscr{H}_\omega$ for the action of R. Let $\mathscr{H}_\omega$ be equipped with the operator space structure of a column Hilbert space (see [10]). Given a (not necessarily normal) operator R-module X, we can consider the Haagerup tensor product $X \otimes_R^h \mathscr{H}_\omega$. There is a natural bounded linear map

$$Q_\omega : X \to X \otimes_R^h \mathscr{H}_\omega, \qquad Q_\omega x = x \otimes_R^h \xi_\omega.$$

The *R-topology* on X is defined by the family of seminorms

$$s_\omega(x) = \|Q_\omega x\| \qquad (\omega \in R_{\sharp+}).$$

Let $\mathscr{H}_\omega^0$ be the pre-Hilbert space obtained from ω by the GNS construction; thus $\mathscr{H}_\omega$ is the completion of $\mathscr{H}_\omega^0$. We note that the natural map $X \otimes_R^h \mathscr{H}_\omega^0 \to X \otimes_R^h \mathscr{H}_\omega$ is completely isometric, so that in the definition of the R-topology the Hilbert space $\mathscr{H}_\omega$ can be replaced by $\mathscr{H}_\omega^0$. To see this, first recall that the spaces $X \otimes^h \mathscr{H}_\omega$ and $X \otimes^h \mathscr{H}_\omega^0$ have the same operator dual $(X \otimes^h \mathscr{H}_\omega)^\sharp$ since completely bounded bilinear matrix valued maps on $X \times \mathscr{H}_\omega^0$ can be uniquely extended to $X \times \mathscr{H}_\omega$ without incresing the norm. (A precise description of the dual of the Haagerup tensor product can be found in [10] and [5].) Since $X \otimes_R^h \mathscr{H}_\omega$ is a quotient operator space of

$X\otimes^h \mathcal{H}_\omega$, $(X\otimes^h_R \mathcal{H}_\omega)^\sharp$ is a subspace of $(X\otimes^h \mathcal{H}_\omega)^\sharp$, consisting of R-balance completely bounded bilinear mappings (in the sense that $\theta(xa, \eta)=\theta(x, a\eta)$ for $x\in X$, $\eta\in\mathcal{H}$ and $a\in R$). The operator dual of $X\otimes^h_R \mathcal{H}^0_\omega$ can be described in a similar way, and we conclude that $(X\otimes^h_R \mathcal{H}^0_\omega)^\sharp = (X\otimes^h_R \mathcal{H}_\omega)^\sharp$. This implies that the natural map $X\otimes^h_R \mathcal{H}^0_\omega \to X\otimes^h_R \mathcal{H}_\omega$ is completely isometric.

Note also that the left R-module $\mathcal{H}^0_\omega$ is (algebraically) isomorphic to Rp_ω (by the map $ap_\omega \mapsto ap_\omega\xi_\omega$), where $p_\omega\in R$ is the support projection of ω. Since Rp_ω is obviously a direct summand in the free R-module R, this easily implies that the kernel of the map $x\mapsto x\otimes_R \xi_\omega$ from X to the algebraic tensor product $X\otimes_R \mathcal{H}^0_\omega$ is $Xp^\perp_\omega$. Thus, the relation $x\otimes_R \xi_\omega=0$ is equivalent to $xp_\omega=0$ for $x\in X$. (If X is normal over R, then the natural map $X\otimes_R \mathcal{H}^0_\omega \to X\otimes^h_R \mathcal{H}^0_\omega$ turns out to be injective, hence the identity $x\otimes^h_R \xi_\omega=0$ holds in $X\otimes^h_R \mathcal{H}^0_\omega$ if and only if $xp_\omega=0$, but if X is not normal, this is not necessarily true any more.)

By the above comments $s_\omega(x)$ is the norm of $x\otimes^h_R \xi_\omega$ in $X\otimes^h_R \mathcal{H}^0_\omega$, which is by the definition equal to the Haagerup seminorm of $x\otimes_R \xi_\omega$ in the algebraic tensor product $X\otimes_R \mathcal{H}^0_\omega$. Thus, the R-topology on X is determined by the family of seminorms

$$s_\omega(x)=\inf\{\|y\|\ \|a\xi_\omega\| : x\otimes_R \xi_\omega = ya\otimes_R \xi_\omega,\ y\in\mathcal{R}_{\text{fin}}(X),\ a\in\mathcal{C}_{\text{fin}}(R)\}, \tag{3.1}$$

where $\omega\in R_{\sharp+}$. Note that $\|a\xi_\omega\|=\omega(a^*a)^{1/2}$ and that by the previous paragraph the condition $x\otimes_R \xi_\omega = ya\otimes_R \xi_\omega$ is equivalent to $xp_\omega = yap_\omega$, where $p_\omega\in R$ is the support projection of ω.

LEMMA 3.1. *The seminorms s_ω $(\omega\in R_{\sharp+})$ can be expressed as*

$$s_\omega(x)=\inf\{\|y\|\ \omega(a^2)^{1/2} : x=ya,\ y\in X,\ a\in R,\ 0\leqslant a\leqslant 1\}. \tag{3.2}$$

Moreover, if for each ω we define the function $t_\omega: X\to\mathbb{R}$ by

$$t_\omega(x)=\inf\{\|y\|\ \omega(a^2)^{1/2} : x=ya,\ y\in X,\ \|y\|\leqslant\sqrt{2}\,\|x\|,\ a\in R,\ 0\leqslant a\leqslant 1\}, \tag{3.3}$$

then

$$s_\omega(x)\leqslant t_\omega(x)\leqslant\sqrt{2}\,s_\omega(x).$$

If Y is an R-submodule of X, then the R-topology on Y is just the restriction to Y of the R-topology on X.

Proof. We show first that in the definition of $s_\omega(x)$ the condition $x \otimes_R \xi_\omega = ya \otimes_R \xi_\omega$ can be replaced by $x = ya$. Denote temporarily

$$\sigma_\omega(x) = \inf\{\|y\|\,\omega(a^*a)^{1/2} : x = ya,\ y \in \mathscr{R}_{\mathrm{fin}}(X),\ a \in \mathscr{C}_{\mathrm{fin}}(R)\}.$$

Clearly $s_\omega(x) \leqslant \sigma_\omega(x)$. To prove the reverse inequality, let $\varepsilon > 0$ and choose $y \in \mathscr{R}_{\mathrm{fin}}(X)$ and $a \in \mathscr{C}_{\mathrm{fin}}(R)$ such that $x \otimes_R \xi_\omega = ya \otimes_R \xi_\omega$ and

$$\|y\|\,\omega(a^*a)^{1/2} < s_\omega(x) + \varepsilon. \tag{3.4}$$

Then $xp_\omega = yap_\omega$, hence

$$x = y(ap_\omega) + \lambda^{-1}x(\lambda p_\omega^\perp)$$

for each real $\lambda > 0$. This implies that

$$\begin{aligned}\sigma_\omega(x) &\leqslant \|yy^* + \lambda^{-2}xx^*\|^{1/2}\,\omega(p_\omega a^*ap_\omega + \lambda^2 p_\omega^\perp)^{1/2}\\ &= \|yy^* + \lambda^{-2}xx^*\|^{1/2}\,\omega(a^*a)^{1/2}.\end{aligned}$$

Letting $\lambda \to \infty$ and using (3.4), it follows that $\sigma_\omega(x) \leqslant s_\omega(x)$ since $\varepsilon > 0$ is arbitrary.

Choose now $y \in \mathscr{R}_{\mathrm{fin}}(X)$ and $a \in \mathscr{C}_{\mathrm{fin}}(R)$ so that $x = ya$ and (3.4) holds. Replacing y and a by δy and $\delta^{-1}a$, respectively, where δ is a suitable positive number, we can achieve that $\|y\| = \eta\,\|x\|$, where η is any pre-assigned positive real number. Choose $n \in \mathbb{N}$ so that all nonzero components of y and a are on the first n positions, hence $y = [\tilde{y}, 0]$ and $a = \binom{\tilde{a}}{0}$ for some $\tilde{y} \in \mathscr{R}_n(X)$ and $\tilde{a} \in \mathscr{C}_n(R)$. Let $|a| = \sqrt{a^*a}$ and let e be the spectral projection of $|a|$ corresponding to the interval $[0, 1]$. Define $z \in \mathscr{R}_{\mathrm{fin}}(X)$ and $b \in \mathscr{C}_{\mathrm{fin}}(R)$ by

$$z = [\tilde{y}, xe^\perp, 0] \qquad \text{and} \qquad b = \begin{bmatrix} \tilde{a}e \\ e^\perp \\ 0 \end{bmatrix}.$$

Then

$$x = zb, \qquad \|b\| = \|ea^*ae + e^\perp\|^{1/2} \leqslant 1,$$

$$\|z\| \leqslant \sqrt{\|y\|^2 + \|x\|^2} = \frac{\sqrt{1+\eta^2}}{\eta}\,\|y\|,$$

and $\omega(b^*b) \leqslant \omega(a^*a)$ since $b^*b \leqslant a^*a$ by the functional calculus. Let $b = uc$ be the polar decomposition of b, where $c = \sqrt{b^*b} \in R$ and $u \in \mathscr{C}_{\text{fin}}(R)$ is a partial isometry. Put $v = zu \in X$. Then

$$x = vc, \qquad 0 \leqslant c \leqslant 1, \qquad \|v\| \leqslant \frac{\sqrt{1+\eta^2}}{\eta}\, \|y\| \qquad \text{and}$$

$$\omega(c^2) \leqslant \omega(a^*a). \tag{3.5}$$

This implies that the right hand side of (3.2) is dominated by

$$\|v\|\, \omega(c^2)^{1/2} \leqslant \frac{\sqrt{1+\eta^2}}{\eta}\, \|y\|\, \omega(a^*a)^{1/2} \leqslant \frac{\sqrt{1+\eta^2}}{\eta}\, (s_\omega(x) + \varepsilon). \tag{3.6}$$

Letting $\eta \to \infty$ and then $\varepsilon \to 0$ it follows that the right hand side of (3.2) is less than or equal to $s_\omega(x)$. Since the reverse inequality is obvious, this proves (3.2).

By (3.2) it is clear that $s_\omega(x) \leqslant t_\omega(x)$. Moreover, if we choose $\eta = 1$ in the above computation (hence $\|y\| = \|x\|$), then by (3.5) we have $x = vc$, where $v \in X$ and $c \in R$ satisfy

$$0 \leqslant c \leqslant 1, \qquad \|v\| \leqslant \sqrt{2}\, \|x\| \qquad \text{and} \qquad \omega(c^2) \leqslant \omega(a^*a).$$

This implies (together with (3.4)) that

$$t_\omega(x) \leqslant \|v\|\, \omega(c^2)^{1/2} \leqslant \sqrt{2}\|y\|\, \omega(a^*a)^{1/2} \leqslant \sqrt{2}(s_\omega(x) + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, this proves that $t_\omega(x) \leqslant \sqrt{2}\, s_\omega(x)$.

Let now Y be an R-submodule of X and for clarity let us denote by s_ω^X and s_ω^Y the seminorms in X and Y, respectively, corresponding to $\omega \in R_{\sharp+}$. For each $x \in Y$ we clearly have $s_\omega^X(x) \leqslant s_\omega^Y(x)$. To prove the reverse inequality, recall from the above that, given $\varepsilon > 0$ and $\eta > 0$, we can choose $v \in X$ and $c \in R$ such that $0 \leqslant c \leqslant 1$, $\|v\| \leqslant \sqrt{1+\eta^2}\, \|x\|$, $x = vc$ and (3.5) holds. Let f be the spectral projection of c corresponding to the interval $[\varepsilon, 1]$. By the spectral theorem there exists $d \in R$ such that $cd = f = dc$, hence $vf = vcd = xd \in Y$ and therefore

$$s_\omega^Y(x) \leqslant s_\omega^Y(vfc) + s_\omega^Y(xf^\perp) \leqslant \|vf\|\, \omega(c^2)^{1/2} + \|xf^\perp\|\, \omega(1)^{1/2}. \tag{3.7}$$

Since

$$\|vf\| \leqslant \|v\| \qquad \text{and} \qquad \|xf^\perp\| \leqslant \|v\|\, \|cf^\perp\| \leqslant \sqrt{1+\eta^2}\, \|x\|\, \varepsilon,$$

it follows from (3.7) and (3.6) that

$$s_\omega^Y(x) \leqslant \frac{\sqrt{1+\eta^2}}{\eta}(s_\omega^X(x)+\varepsilon)+\varepsilon\sqrt{1+\eta^2}\,\|x\|\,\omega(1)^{1/2}.$$

Letting first $\varepsilon \to 0$ and then $\eta \to \infty$, we conclude that $s_\omega^Y(x) \leqslant s_\omega^X(x)$. This proves that the R-topology on Y is just the restriction to Y of the R-topology on X. ∎

Proposition 3.2. *The R-topology on X is weaker than the norm topology and, if X is normal over R, than the R-topology on X is stronger than the ultrastrong operator topology.*

Proof. By definition we have $\|Q_\omega x\| \leqslant \|x\|\,\|\xi_\omega\| = \|x\|\,\omega(1)^{1/2}$ for each $\omega \in R_{\sharp+}$, which implies that the R-topology is weaker than the norm topology. The ultrastrong topology on $X \subseteq \mathrm{B}(\mathscr{H})$ is defined by the family of seminorms $x \mapsto \omega(x^*x)^{1/2}$, where $\omega \in \mathrm{B}(\mathscr{H})_{\sharp+}$. If $x = ya$, with $y \in X$ and $a \in \mathrm{B}(\mathscr{H})$, then $\omega(x^*x)^{1/2} \leqslant \|y\|\,\omega(\varphi(a^*a))^{1/2}$, where $\varphi\colon R \to \mathrm{B}(\mathscr{H})$ is the normal representation defining the normal R-module structure on X. Since $\omega\varphi \in R_{\sharp+}$, this implies that $\omega(x^*x)^{1/2} \leqslant s_{\omega\varphi}(x)$ and the R-topology is therefore stronger than the ultrastrong topology.

Proposition 3.3. *The R-topology on an operator R-module X is the strongest locally convex topology on X such that the module multiplication $\mu\colon X \times R \to X$ is continuous, where $X \times R$ carries the product of the norm topology on X and the ultrastrong topology on R.*

Proof. It is easy to verify that the module multiplication is continuous if the target space X is equipped with the R-topology. Suppose that σ is any seminorm on X such that the composition $\sigma\mu\colon X \times R \to \mathbb{R}$ is continuous. Then it follows from the continuity at $(0, 0)$ (and using Lemma 3.1) that there exist $\delta > 0$ and $\omega \in R_{\sharp+}$ such that the conditions $\|y\| < \delta$ and $\omega(a^*a)^{1/2} < 1$ imply that $\sigma(ya) < 1$ ($y \in X$, $a \in R$). By homogeneity it follows that $\sigma(ya) \leqslant \delta^{-1}\,\|y\|\,\omega(a^*a)^{1/2}$ for all $y \in X$ and $a \in R$, hence $\sigma(x) \leqslant \delta^{-1} s_\omega(x)$ for all $x \in X$ by Lemma 3.1. This implies that the R-topology on X is the strongest locally convex topology on X with the required continuity of the module multiplication.

Example 3.4. If R is finite dimensional then the R-topology on each operator R-module X is just the norm topology. Indeed, let ω be a faithful state on R. There exists a constant $\kappa \in \mathbb{R}$ such that $\|a\| \leqslant \kappa\omega(a^*a)^{1/2}$ for each $a \in R$. Given $x \in X$, $y \in X$ and $a \in R$ such that $x = ya$, we have

$$\|x\| \leqslant \|y\|\,\|a\| \leqslant \kappa\,\|y\|\,\omega(a^*a)^{1/2},$$

hence $\|x\| \leqslant \kappa s_\omega(x)$. This shows that the norm topology is weaker than the R-topology, hence by Proposition 3.2 the two topologies are the same.

EXAMPLE 3.5. Let $X = R^n$ for some $n \in \mathbb{N}$. (So X is a von Neumann algebra, the usual direct sum of n copies of a von Neumann algebra R.) Then the R-topology on X is just the ultrastrong topology. To see this, let $e_i = (0, ..., 1, ..., 0)$ $(i = 1, ..., n)$ be the standard basic elements of R^n over R. Given $\omega \in R_{\sharp+}$ let $\tilde{\omega} \in (R^n)_{\sharp+}$ be defined by $\tilde{\omega}(x) = \sum \omega(x_i)$, where $x_i \in R$ are the components of x. By definition of s_ω we have

$$s_\omega(x) \leqslant \left\| \sum_{i=1}^{n} e_i e_i^* \right\|^{1/2} \omega\left(\sum_{i=1}^{n} x_i^* x_i \right)^{1/2} = \tilde{\omega}(x^* x)^{1/2}.$$

This implies that the R-topology on X is weaker than the ultrastrong topology, hence the two topologies are the same by Proposition 3.2. This example can be generalized as follows.

Suppose that S is a von Neumann algebra containing R such that there exist a normal conditional expectation $E: S \to R$ and a finite set

$$\{u_1, ..., u_n\} \subseteq S$$

such that

$$x = \sum_{i=1}^{n} u_i E(u_i^* x) \qquad \text{for each} \quad x \in S \qquad \text{and} \qquad E(u_i^* u_j) = \delta_{i,j} f_i, \tag{3.8}$$

where $f_i \in R$ is a projection for each i. (For example, S may be the algebra $\mathscr{M}_m(R)$ of all $m \times m$ matrices with entries in R. In this case $n = m^2$, $E([a_{ij}]) = (1/m) \sum_{i=1}^{m} a_{ii}$ and the elements u_k are the standard matrix units e_{ij} multiplied by $\sqrt{m}$. As another example, R and S may be factors of type II_1 such that the Jones index of R in S is finite (see [12, p. 165]).) On the submodule $M := \bigoplus_{i=1}^{n} f_i R$ of R^n the R-topology coincides with the ultrastrong topology (by the previous paragraph and the last sentence in Lemma 3.1). Observe that (3.8) implies that $f_i E(u_i^* x) = E(u_i^* x)$ for each $x \in S$ and all $i = 1, ..., n$ (it suffices to check this for $x = u_j$). We can define two homomorphisms of right R-modules

$$\psi: M \to S, \qquad \psi(a_1, ..., a_n) = \sum_{i=1}^{n} u_i a_i$$

and

$$\theta: S \to M, \qquad \theta(x) = (E(u_1^* x), ..., E(u_n^* x)).$$

It is easy to verify that $\psi\theta = 1_S$, $\theta\psi = 1_M$, ψ and θ are continuous in the ultrastrong topology and ψ is continuous in the R-topology. (To see the ultrastrong continuity of θ, recall that normal completely positive mappings on von Neumann algebras are continuous in the ultrastrong operator topology; this applies in particular to the conditional expectation E.) Since the R-topology on S is stronger than the ultrastrong topology by Proposition 3.2 and the two topologies coincide on M, it follows that θ is continuous also when S and M carry the R-topology. Thus θ must be a homeomorphism in both topologies and the two topologies on S must coincide.

EXAMPLE 3.6. Let X be a selfdual Hilbert C^*-module over R (see [19]). Then X is a normal operator R-module (as can be seen, for example, by using induced representations [19]). The ultrastrong topology on X can be defined (intrinsically) by the family of seminorms $x \mapsto \omega(\langle x, x \rangle)^{1/2}$, where $\omega \in R_{\sharp +}$ and $\langle \cdot, \cdot \rangle$ denotes the R-valued inner product on X. Since each $x \in X$ has a polar decomposition $x = u\langle x, x \rangle^{1/2}$, where $u \in X$ and $\|u\| \leqslant 1$, we have $s_\omega(x) \leqslant \omega(\langle x, x \rangle)^{1/2}$. Together with Proposition 3.2 this implies that the R-topology on X coincides with the ultrastrong topology. Note that the hypothesis that X is selfdual is not essential since each Hilbert C*-module over R can be embedded into a selfdual Hilbert R-module completely isometrically. (The hypothesis was needed only to justify the use of the polar decomposition.)

In general, the R-topology is different from the ultrastrong and the norm topology. For example, using Proposition 2.2 and Theorem 5.3 one can see that if $R \subseteq \mathrm{B}(\mathscr{H})$ has trivial intersection with the ideal $\mathrm{K}(\mathscr{H})$ of compact operators, then the closure of $\mathrm{K}(\mathscr{H})$ in the R-topology is different from $\mathrm{K}(\mathscr{H})$ and $\mathrm{B}(\mathscr{H})$.

4. *R*-CONTINUOUS LINEAR FUNCTIONALS

If X is an operator R-module, we denote by $X^\sharp$ the space of all bounded linear functionals on X and by $X^{\sharp R}$ the subspace of all functionals which are continuous in the R-topology (recall that the R-topology is weaker than the norm topology). Mappings which are continuous in the R-topology will be called simply R-continuous. Each $\rho \in X^\sharp$ induces a linear map

$$T_\rho\colon R \to X^\sharp, \qquad T_\rho(a)(x) = \rho(xa) \qquad (a \in R,\ x \in X)$$

and $\|T_\rho\| \leqslant \|\rho\|$.

LEMMA 4.1. *A functional $\rho \in X^\sharp$ is R-continuous if and only if the map T_ρ is continuous when R carries the ultrastrong and $X^\sharp$ the norm topology.*

Proof. If ρ is R-continuous, then there exists an $\omega \in R_{\sharp +}$ such that the condition $s_\omega(x) < 1$ implies that $|\rho(x)| < 1$, where $x \in X$. Since $s_\omega(ya) < 1$ for each $y \in X$ with $\|y\| \leqslant 1$ and $a \in R$ with $\omega(a^*a) < 1$, it follows that $\|T_\rho(a)\| \leqslant 1$ for each $a \in R$ satisfying $\omega(a^*a) < 1$. This proves that T_ρ is ultrastrong to norm continuous at 0, hence (by linearity) continuous.

Conversely, if T_ρ is ultrastrong to norm continuous, then there exists an $\omega \in R_{\sharp +}$ such that $\|T_\rho(a)\| < 1$ for all $a \in R$ satisfying $\omega(a^*a)^{1/2} < 1$. This means that $|\rho(ya)| = |T_\rho(a)(y)| < \|y\|$ for all $y \in X$ if $\omega(a^*a)^{1/2} < 1$, hence

$$|\rho(ya)| \leqslant \|y\| \, \omega(a^*a)^{1/2} \tag{4.1}$$

for all $y \in X$ and $a \in R$. Using Lemma 3.1, (4.1) implies that $|\rho(x)| \leqslant s_\omega(x)$ for each $x \in X$, which proves that ρ is R-continuous. ∎

For each $\rho \in X^\sharp$ and $x \in X$ let ρ_x be the functional on R defined by $\rho_x(a) = \rho(xa)$.

THEOREM 4.2. *A bounded linear functional ρ on an operator R-module X is R-continuous if and only if for each $x \in X$ the functional ρ_x on R is normal.*

Proof. Observe that ρ_x is normal for all $x \in X$ if and only if the map T_ρ (defined above) is weak* to weak* continuous.

If ρ is R-continuous, then by Lemma 4.1 T_ρ is ultrastrong to norm continuous, hence also ultrastrong to weak* continuous, and thus (since the ultrastrong and the weak* topology on R have the same continuous linear functionals) weak* to weak* continuous. This last continuity is equivalent to the fact that all functionals ρ_x $(x \in X)$ on R are normal.

Suppose now conversely, that ρ_x is normal for all x, hence T_ρ is weak* continuous. Consider the completely bounded bilinear map

$$\tilde{\rho}\colon X \times R \to \mathbb{C}, \qquad \tilde{\rho}(x, a) = \rho(xa).$$

By the representation theorem for such maps (see [7], [1] or [22]) there exists a representation π of some C*-algebra containing X on some Hilbert space $\mathscr{K}$, a representation σ of R on a Hilbert space $\mathscr{L}$, a bounded operator $T\colon \mathscr{L} \to \mathscr{K}$ and two vectors $\xi \in \mathscr{K}$, $\eta \in \mathscr{L}$ such that

$$\rho(xa) = \langle \pi(x) \, T\sigma(a) \, \eta, \xi \rangle$$

for all $x \in X$ and $a \in R$. Since the left side of this identity is normal in a, we can replace σ by its normal part without violating the identity, hence we may assume that σ is normal. From $|\rho(xa)| \leqslant \|\pi(x)\| \, \|T\| \, \|\xi\| \, \|\sigma(a)\,\eta\|$ it follows that

$$\|T_\rho(a)\| \leqslant \|T\| \, \|\xi\| \, \|\sigma(a)\,\eta\|. \tag{4.2}$$

Since σ is a normal representation, σ is ultrastrongly continuous, hence the map $a \mapsto \|\sigma(a)\,\eta\|$ is continuous on R in the ultrastrong topology. The relation (4.2) then implies that T_ρ is ultrastrong to norm continuous (at 0, hence everywhere), which is equivalent to the R-continuity of ρ by Lemma 4.1. ▮

The following corollary is an immediate consequence of Theorem 4.2 and the well-known fact that $R_\sharp$ is norm closed in $R^\sharp$.

Corollary 4.3. *$X^{\sharp R}$ is a norm closed subspace of $X^\sharp$, hence a Banach space.*

To each $\rho \in X^\sharp$ we can associate in a natural way a functional $\rho_R \in X^{\sharp R}$ as follows. For each $x \in X$ let $\rho_x = (\rho_x)_{\text{nor}} + (\rho_x)_{\text{sing}}$ be the decomposition of ρ_x into the normal and the singular part. Then define ρ_R by

$$\rho_R(x) = (\rho_x)_{\text{nor}}(1) \qquad (x \in X).$$

To see that ρ_R is R-continuous, first recall that for each $\theta \in R^\sharp$ and each $a \in R$ the identity $(\theta a)_{\text{nor}} = \theta_{\text{nor}} a$ holds (see [13, p. 723]), where θa is defined by $(\theta a)(b) = \theta(ab)$. Since $\rho_{xa} = \rho_x a$, it follows that $(\rho_{xa})_{\text{nor}} = (\rho_x)_{\text{nor}}\, a$, hence

$$\rho_R(xa) = (\rho_{xa})_{\text{nor}}(1) = ((\rho_x)_{\text{nor}} a)(1) = (\rho_x)_{\text{nor}}(a).$$

This implies that the map $a \mapsto \rho_R(xa)$ is normal on R for each $x \in X$, hence $\rho_R \in X^{\sharp R}$ by Theorem 4.2.

From the well known properties of the decomposition of $R^\sharp$ into the normal and the singular part we immediately deduce the following proposition.

Proposition 4.4. *If X is an operator R-module, then the map $P\colon X^\sharp \to X^\sharp$, $P\rho = \rho_R$, is a linear contractive homomorphism of (left) R-modules with the range $X^{\sharp R}$ and $P^2 = P$. (Moreover, the complementary idempotent $I - P$ is also contractive.)*

Proposition 4.5. *If Y is a submodule of an operator R-module X, then each $\rho \in Y^{\sharp R}$ can be extended to a $\tilde{\rho} \in X^{\sharp R}$ such that $\|\tilde{\rho}\| = \|\rho\|$.*

Proof. Choose any extension $\psi \in X^\sharp$ of ρ with $\|\psi\| = \|\rho\|$ and put $\tilde{\rho} = \psi_R$. If $y \in Y$, then $\psi_y = \rho_y$ is normal, hence

$$\tilde{\rho}(y) = \psi_R(y) = (\psi_y)_{\text{nor}}(1) = (\rho_y)_{\text{nor}}(1) = \rho_y(1) = \rho(y)$$

and $\tilde{\rho}$ is an extension of ρ. Moreover, $\|\tilde{\rho}\| = \|\psi_R\| \leqslant \|\psi\| = \|\rho\|$. ▮

If $\Phi: X \to Y$ is an R-continuous homomorphism of operator R-modules, then clearly $\rho\Phi \in X^{\sharp^R}$ for each $\rho \in Y^{\sharp^R}$. Conversely, if $\Phi: X \to Y$ is a homomorphism of norm complete normal operator R-modules such that $\rho\Phi \in X^{\sharp^R}$ for each $\rho \in Y^{\sharp^R}$, then a standard application of the closed graph theorem shows that Φ is norm bounded and by the following proposition Φ is then R-continuous.

PROPOSITION 4.6. *If $\Phi: X \to Y$ is a bounded homomorphism of operator R-modules, then Φ is continuous in the R-topology.*

Proof. It follows directly from the description of the seminorms s_ω ($\omega \in R_{\sharp+}$) in Lemma 3.1 that

$$s_\omega(\Phi(x)) \leqslant \|\Phi\| \, s_\omega(x)$$

for each $x \in X$, which implies the R-continuity of Φ. ▮

5. A CHARACTERIZATION OF CLOSED SUBMODULES

The main result of this section is that the R-closed R-submodules of $\mathrm{B}(\mathscr{H})$ are precisely the strong submodules. For the proof of this we shall need an analogue of the Krein–Smulian theorem (see [18, p. 73] for the Krein–Smulian theorem).

Throughout this section, $\mathscr{H}$ will be a Hilbert space, R a von Neumann algebra acting normally on $\mathscr{H}$, and $X \subseteq \mathrm{B}(\mathscr{H})$ a right R-submodule.

A subset C of X is called *R-balanced* if $xa \in C$ for all $x \in C$ and all $a \in R$ with $\|a\| \leqslant 1$.

By C° we denote the polar of C in $X^\sharp$ and by $C^\diamond$ the polar of C in $X^{\sharp^R}$, thus

$$C^\circ = \{\rho \in X^\sharp : |\rho(x)| \leqslant 1 \ \forall x \in C\} \qquad \text{and} \qquad C^\diamond = C^\circ \cap X^{\sharp^R}.$$

The polar in X of a subset $Z \subseteq X^{\sharp^R}$ will be denoted by $Z^\diamond$. The polars in $X^{\sharp^R}$ (or in $X^\sharp$) of closed balls in X with center 0 are just the closed balls in $X^{\sharp^R}$ (or in $X^\sharp$) with the center 0 and vice versa (note that $X^{\sharp^R}$ includes all restrictions to X of normal functionals on $\mathrm{B}(\mathscr{H})$). The weak* closure of a subset $Z \subseteq X^\sharp$ is denoted by $\bar{Z}$ and for a subset $Z \subseteq X^{\sharp^R}$ we denote $\bar{Z} \cap X^{\sharp^R}$ by $\bar{Z}^R$.

THEOREM 5.1. *Let C be a convex R-balanced subset of X. Then C is R-closed in X if (and only if) $C \cap U$ is R-closed in X for each closed ball U in X with the center* 0.

To prove this theorem, we need a lemma.

LEMMA 5.2. *If C is an R-balanced subset of X, then for each ball U in X with the center* 0 *we have*

$$\overline{C^\diamond + U^\diamond}^R = C^\diamond + U^\diamond.$$

Proof. Since U° is weak* compact and C° is weak* closed, $C^\circ + U^\circ$ is weak* closed, hence

$$\overline{C^\diamond + U^\diamond}^R \subseteq \overline{C^\circ + U^\circ} = C^\circ + U^\circ.$$

Thus, each $\rho \in \overline{C^\diamond + U^\diamond}^R$ can be written as $\rho = \theta + \tau$, where $\theta \in C^\circ$ and $\tau \in U^\circ$. Using Proposition 4.4, it follows that

$$\rho = \theta_R + \tau_R,$$

where $\theta_R, \tau_R \in X^{\sharp^R}$ and $\|\tau_R\| \leqslant \|\tau\|$. Thus, $\tau_R \in U^\diamond$ and it remains to be shown that $\theta_R \in C^\circ$.

For each $x \in C$ and a in the unit ball of R we have $xa \in C$, hence $|\theta_x(a)| = |\theta(xa)| \leqslant 1$, hence $\|\theta_x\| \leqslant 1$. This implies that $\|(\theta_x)_{\mathrm{nor}}\| \leqslant 1$, therefore $|\theta_R(x)| = |(\theta_x)_{\mathrm{nor}}(1)| \leqslant 1$ and $\theta_R \in C^\circ$. ▮

It seems that the well known proof of the Krein–Smulian theorem (see, e.g., [18]) can not easily be extended to the situation considered here; however, using Lemma 5.2 the arguments from [20, pp. 112 and 125] suffice to prove Theorem 5.1. For convenience of the reader we shall now recall these arguments. Assume that $C \subseteq X$ is convex and R-balanced and $C \cap U$ is R-closed for each closed ball U in X with the center 0. To prove that C is R-closed, it suffices to prove that

$$(C \cap U)^\diamond \subseteq C^\diamond + 2U^\diamond \tag{5.1}$$

for each U. Indeed, replacing in (5.1) U by $\varepsilon^{-1}U$, where $\varepsilon > 0$, we get

$$(C \cap \varepsilon^{-1}U)^\diamond \subseteq C^\diamond + 2\varepsilon U^\diamond.$$

For each $x \in C^{\diamond\diamond}$ (= the R-closure of C by the bipolar theorem) we can choose a closed ball U with the center 0 so that $x \in U$. For each

$\rho \in (C \cap \varepsilon^{-1}U)^{\diamond}$ we than have (since $\rho = \theta + 2\varepsilon\tau$ for some $\theta \in C^{\diamond}$ and $\tau \in U^{\diamond}$) that $|\rho(x)| \leqslant 1 + 2\varepsilon$, hence

$$x \in (1+2\varepsilon)(C \cap \varepsilon^{-1}U)^{\diamond\diamond} = (1+2\varepsilon)(C \cap \varepsilon^{-1}U) \subseteq (1+2\varepsilon)\, C,$$

where the equality uses the assumption of the theorem that $C \cap \varepsilon^{-1}U$ is R-closed. Since $x \in U$, we have now $x \in (1+2\varepsilon)\, C \cap U \subseteq (1+2\varepsilon)(C \cap U)$ for each $\varepsilon > 0$. By taking the intersection over all $\varepsilon > 0$, it follows that $x \in C \cap U$ since $C \cap U$ is closed. This proves (assuming (5.1)) that $C^{\diamond\diamond} \subseteq C$, hence C is R-closed.

To prove (5.1), put $C_n = C \cap 2^n U$ for each $n \in \mathbb{N}$. Then $C_n = C_{n+1} \cap 2^n U$, hence by the bipolar theorem (applied to the dual pair $(X, X^{\sharp^R})$) we have

$$C_n^{\diamond} = \overline{\mathrm{co}}^R(C_{n+1}^{\diamond} \cup 2^{-n}U^{\diamond}) \subseteq \overline{C_{n+1}^{\diamond} + 2^{-n}U^{\diamond}}^R,$$

where "co" denotes the convex hull (which in our case is balanced). Since the sets C_n are R-balanced, Lemma 5.2 now implies that

$$C_n^{\diamond} \subseteq C_{n+1}^{\diamond} + 2^{-n}U^{\diamond}. \tag{5.2}$$

Let $\rho \in (C \cap U)^{\diamond} = C_0^{\diamond}$. Using (5.2) with $n = 0$, we find an element $\theta_1 \in U^{\diamond}$ such that $\rho - \theta_1 \in C_1^{\diamond}$. Inductively, we can find a sequence of elements $\theta_n \in U^{\diamond}$ such that $\tau_n := \rho - \sum_{k=1}^{n} 2^{-k+1}\theta_k \in C_n^{\diamond}$. The series $\sum_{k=1}^{\infty} 2^{1-k}\theta_k$ converges in norm to an element $\theta \in 2U^{\circ}$, hence $\theta \in 2U^{\diamond}$ by Corollary 4.3. If $x \in C$, then $x \in C_n$ for all sufficiently large n, hence $|(\rho - \theta)(x)| = \lim_{n \to \infty} |\tau_n(x)| \leqslant 1$. This shows that $\rho - \theta \in C^{\diamond}$, hence $\rho \in 2U^{\diamond} + C^{\diamond}$. This proves (5.1), hence also Theorem 5.1.

THEOREM 5.3. *An R-submodule $X \subseteq B(\mathscr{H})$ is R-closed if and only if X is strong.*

Proof. Suppose that X is R-closed. Let $\{y_i : i \in \mathbb{I}\} \subseteq X$ and $\{a_i : i \in \mathbb{I}\} \subseteq R$ be such that the sums $\sum_{i \in \mathbb{I}} y_i y_i^*$ and $\sum_{i \in \mathbb{I}} a_i^* a_i$ converge in the strong operator topology. We must show that $x := \sum_{i \in \mathbb{I}} y_i a_i \in X$. For each finite subset $\mathbb{F}$ of $\mathbb{I}$ put $x_{\mathbb{F}} = \sum_{i \in \mathbb{F}} y_i a_i$. Then

$$s_{\omega}(x - x_{\mathbb{F}}) \leqslant \left\| \sum_{i \in \mathbb{I} \setminus \mathbb{F}} y_i y_i^* \right\|^{1/2} \omega\left(\sum_{i \in \mathbb{I} \setminus \mathbb{F}} a_i^* a_i \right)^{1/2} \tag{5.3}$$

for each $\omega \in R_{\sharp+}$. (This is not completely obvious since the sums involved may have infinitely many terms. But, using the polar decomposition of the column with the components a_i $(i \in \mathbb{I} \setminus \mathbb{F})$, the sum $x - x_{\mathbb{F}} = \sum_{i \in \mathbb{I} \setminus \mathbb{F}} y_i a_i$ can be reduced to one term of the form vc, where $v \in \mathrm{B}(\mathscr{H})$ satisfies $\|v\| \leqslant \|\sum_{i \in \mathbb{I} \setminus \mathbb{F}} y_i y_i^*\|^{1/2}$ and $c = (\sum_{i \in \mathbb{I} \setminus \mathbb{F}} a_i^* a_i)^{1/2} \in R$.) Since the first factor on the right side of (5.3) is bounded (independently of $\mathbb{F}$) and the second factor tends to 0 as $\mathbb{F} \to \mathbb{I}$, it follows that the net $\{x_{\mathbb{F}} : \mathbb{F} \subseteq \mathbb{I}, \mathbb{F} \text{ finite}\}$ converges to x in the R-topology. Since X is R-closed, this proves that $x \in X$.

Suppose now conversely, that X is strong. It is easy to see that X is norm closed (this has been observed already in [15, p. 202]). We must prove that X is closed in the R-topology of $\mathrm{B}(\mathscr{H})$. By Theorem 5.1 it suffices to show that the ball with the center 0 and radius 1/4 (or any other radius) is closed in $\mathrm{B}(\mathscr{H})$ in the R-topology. So, let $x \in \mathrm{B}(\mathscr{H})$ be an element in the closure of this ball; then $\|x\| \leqslant 1/4$ since norm closed balls in $\mathrm{B}(\mathscr{H})$ are R-closed (they are even weak* closed). Then for each $\varepsilon > 0$ and each $\omega \in R_{\sharp +}$ there exists an $x_{\omega, \varepsilon} \in X$ with $\|x_{\omega, \varepsilon}\| \leqslant 1/4$ such that $s_\omega(x - x_{\omega, \varepsilon}) < \varepsilon/\sqrt{2}$. By Lemma 3.1 this implies that there exist $y_{\omega, \varepsilon} \in \mathrm{B}(\mathscr{H})$ and $a_{\omega, \varepsilon} \in R$ such that

$$x - x_{\omega, \varepsilon} = y_{\omega, \varepsilon} a_{\omega, \varepsilon}, \qquad \|y_{\omega, \varepsilon}\| \leqslant \sqrt{2}\, \|x - x_{\omega, \varepsilon}\| \leqslant 1, \qquad 0 \leqslant a_{\omega, \varepsilon} \leqslant 1$$

and $\|y_{\omega, \varepsilon}\|\, \omega(a_{\omega, \varepsilon}^2)^{1/2} < \varepsilon$. Replacing $a_{\omega, \varepsilon}$ by $\|y_{\omega, \varepsilon}\|\, a_{\omega, \varepsilon}$ and $y_{\omega, \varepsilon}$ by $y_{\omega, \varepsilon}/\|y_{\omega, \varepsilon}\|$ (where the quotient is defined to be 0 if $y_{\omega, \varepsilon} = 0$), we can achieve that

$$\omega(a_{\omega, \varepsilon}^2)^{1/2} < \varepsilon.$$

This implies that the bounded set $\{a_{\omega, \varepsilon} : \omega \in R_{\sharp +},\ \varepsilon > 0\}$ contains 0 in its strong operator closure, hence by the noncommutative Egoroff theorem for each nonzero projection $f \in R$ there exists a nonzero subprojection $e \in R$ and a sequence $\{a_{\omega_k, \varepsilon_k}\}_{k \in \mathbb{N}}$ such that the sequence $\{a_{\omega_k, \varepsilon_k} e\}_{k \in \mathbb{N}}$ is norm convergent to 0. From

$$\|(x - x_{\omega_k, \varepsilon_k})\, e\| \leqslant \|y_{\omega_k, \varepsilon_k} a_{\omega_k, \varepsilon_k} e\| \leqslant \|a_{\omega_k, \varepsilon_k} e\|$$

it follows then that the sequence $\{x_{\omega_k, \varepsilon_k} e\}_{k \in \mathbb{N}}$ converges to xe in norm, hence $xe \in X$ since X is norm closed. Since the projection f in this argument is arbitrary, the argument shows that if $\{e_j : j \in \mathbb{J}\}$ is a maximal orthogonal family of projections in R such that $xe_j \in X$ for all j, then $\sum_{j \in \mathbb{J}} e_j = 1$. Since X is strong, this implies that $x \in X$ by Theorem 2.1. ∎

PROPOSITION 5.4. *If R is countably decomposable* (=*σ-finite*) *then the* (*closed*) *unit ball U of X is metrizable in the R-topology with the metric* $\mathrm{d}(x, y) = s_\omega(x - y)$ $(x, y \in U)$, *where ω is a faithful normal state on R.*

Proof. To show that the topology defined by d is equivalent to the R-topology on U, it suffices to prove that if $\{x_i\}_{i \in \mathbb{I}}$ is a net in U such that $\{s_\omega(x_i)\}_{i \in \mathbb{I}}$ converges to 0, then $\{s_\rho(x_i)\}_{i \in \mathbb{I}}$ converges to 0 for each $\rho \in R_{\sharp +}$. By Lemma 3.1 there exist $y_i \in X$ and $a_i \in R$ such that

$$x_i = y_i a_i, \qquad \|y\|_i \leqslant 2\, \|x_i\| \leqslant 2, \qquad 0 \leqslant a_i \leqslant 1$$

and

$$\|y_i\| \, \omega(a_i^2)^{1/2} \to 0.$$

Multiplying y_i and a_i by suitable scalars, we can achieve in addition that $\omega(a_i^2) \to 0$. Since ω is faithful and $\{a_i\}_{i \in \mathbb{I}}$ is a bounded net, this implies that $\rho(a_i^2) \to 0$ for each $\rho \in R_{\sharp+}$. (This is well known and for the proof one may assume that ω is given by $\omega(a) = \langle a\xi, \xi \rangle$, where ξ is a cyclic and separating vector for R. Then $a_i a' \xi \to 0$ for each $a' \in R'$, where the subspace $R'\xi$ is dense in the Hilbert space of R since ξ is separating for R. This implies that the bounded net $\{a_i\}_{i \in \mathbb{I}}$ converges to 0 strongly, hence $\rho(a_i^2) \to 0$ for each $\rho \in R_{\sharp+}$.) It follows that $s_\rho(x_i) \to 0$. ∎

THEOREM 5.5. *The closed unit ball U of* $\mathrm{B}(\mathscr{H})$ *is complete in the R-topology, hence so is the closed unit ball of each R-closed submodule of* $\mathrm{B}(\mathscr{H})$.

Proof. We may assume that $R \subseteq \mathrm{B}(\mathscr{H})$. Suppose first that R is countably decomposable and let ω be a faithful normal state on R, hence

$$\omega(a) = \sum_{k=1}^{\infty} \langle a\xi_k, \xi_k \rangle$$

for some vectors $\xi_k \in \mathscr{H}$ satisfying $\sum_{k=1}^{\infty} \|\xi_k\|^2 = 1$. Then by Proposition 5.4 the R-topology on U is determined by the metric $\mathrm{d}(x, y) = s_\omega(x - y)$, and to prove completeness, it suffices to show that each Cauchy sequence $\{x_n\}_{n \in \mathbb{N}}$ in this metric converges to some $x \in U$. Choose a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ such that $s_\omega(x_{n_{k+1}} - x_{n_k}) < 8^{-k}$ for all $k \in \mathbb{N}$. It suffices to show that this subsequence converges, for then standard argument shows that the whole sequence must converge. Thus, to simplify the notation, we may assume that $s_\omega(x_{n+1} - x_n) < 8^{-n}$ for all n. Further, we may assume that $\|x_n\| \leqslant 1/4$ for all n. Then by Lemma 3.1 there exist $y_n \in U$ and $a_n \in R$ such that $0 \leqslant a_n \leqslant 1$, $x_{n+1} - x_n = y_n a_n$ and $\|y_n\| \, \omega(a_n^2)^{1/2} < 4^{-n}$. Replacing y_n by $y_n / \|y\|_n$ (which is interpreted as 0 if $y_n = 0$) and a_n by $\|y_n\| \, a_n$, we may further assume that $\omega(a_n^2)^{1/2} < 4^{-n}$. Put $z_n = 2^{-n} y_n$ and $b_n = 2^n a_n$. Then

$$x_{n+1} - x_n = z_n b_n, \qquad \|z_n\| \leqslant 2^{-n}, \qquad 0 \leqslant b_n \leqslant 2^n, \tag{5.4}$$

and

$$\omega(b_n^2)^{1/2} < 2^{-n}.$$

For each n consider now the (possibly unbounded) operator v_n with the domain $\mathscr{D}_n = \{\eta \in \mathscr{H} : \sum_{j=n}^{\infty} \|b_j \eta\|^2 < \infty\}$ and the range contained in $\mathscr{H}^{\infty}$, defined by

$$v_n \eta = (b_n \eta, b_{n+1} \eta, ...) \qquad (\eta \in \mathscr{D}_n).$$

For each $a' \in R'$ and each ξ_k we have

$$\sum_{j=n}^{\infty} \|b_j a' \xi_k\|^2 \leqslant \|a'\|^2 \sum_{j=n}^{\infty} \|b_j \xi_k\|^2 \leqslant \|a'\|^2 \sum_{j=n}^{\infty} \omega(b_j^2) < \infty,$$

hence $\mathscr{D}_n$ contains the linear span $\mathscr{L}$ of the set $\{a'\xi_k : a' \in R', k = 1, 2, ...\}$. Since ω is faithful, $\mathscr{L}$ is dense in $\mathscr{H}$, hence v_n is densely defined. Further, it is easy to verify that v_n is closed. Let $v_n = u_n |v_n|$ be the polar decomposition of v_n. Since $v_n u' = u'^{(\infty)} v_n$ for each unitary element $u' \in R'$, it follows that the self-adjoint operator $|v_n|$ is affiliated with R and the partial isometry u_n satisfies $u_n u' = u'^{(\infty)} u_n$ for each unitary element $u' \in R'$ (see [13, 6.1.11]). In particular, the components of u_n, denoted by $u_{n,j}$, are in R. Note that

$$b_{n+j-1} = u_{n,j} |v_n| \qquad (j = 1, 2, ...). \tag{5.5}$$

Since the sequence $\{x_n\}_{n \in \mathbb{N}}$ in U is Cauchy also in the strong operator topology, it converges strongly to some $x \in U$; we must show that the convergence is in fact in the R-topology. For each n we have

$$x - x_n = \sum_{j=n}^{\infty} (x_{j+1} - x_j) = \sum_{j=n}^{\infty} z_j b_j = \sum_{j=n}^{\infty} z_j u_{n,j-n+1} |v_n| \tag{5.6}$$

by (5.4) and (5.5), where the series converge in the strong operator topology. Let e_n be the spectral projection of $|v_n|$ corresponding to the interval $[0, 1]$. Then (5.6) implies that

$$x - x_n = \left[\sum_{j=n}^{\infty} z_j u_{n,j-n+1} e_n, (x - x_n) e_n^{\perp} \right] \left[\begin{matrix} |v_n| e_n \\ e_n^{\perp} \end{matrix} \right]. \tag{5.7}$$

Note that the series $\sum_{j=n}^{\infty} z_j u_{n,j+n-1}$ is norm convergent since

$$\|z_j u_{n,j-n+1}\| \leqslant \|z_j\| < 2^{-j}$$

by (5.4); moreover, the sum of this series has norm less than 2^{1-n}, hence the norm of the first factor on the right hand side of (5.7) is dominated by

2 (since $\|x - x_n\| \leqslant 1/2$). The norm of the second factor on the right hand side of (5.7) is bounded by $\sqrt{2}$ and we shall show that

$$\omega(|v_n|^2 e_n + e_n^\perp) \leqslant 2 \sum_{j=n}^{\infty} \omega(b_j^2). \tag{5.8}$$

By the estimate following (5.4) this will imply that $\omega(|v_n|^2 e_n + e_n^\perp)^{1/2}$ converges to 0 and consequently $x - x_n$ converges to 0 in the R-topology by (5.7).

To prove (5.8), recall that the vectors ξ_k are in the domain of v_n, which is the same as the domain of $|v_n|$. Moreover, for each spectral projection e of $|v_n|$ we have

$$\langle |v_n|^2 e\xi_k, \xi_k \rangle = \|e\,|v_n|\,\xi_k\|^2 \leqslant \|v_n \xi_k\|^2 = \sum_{j=n}^{\infty} \langle b_j^2 \xi_k, \xi_k \rangle.$$

If e corresponds to a bounded subset of $\mathbb{R}$ (so that $|v_n|\, e \in R$), then adding the above relations over all k we get

$$\omega(|v_n|^2 e) \leqslant \sum_{j=n}^{\infty} \omega(b_j^2). \tag{5.9}$$

If $e_{[1,l]}$ is the spectral projection of $|v_n|$ corresponding to the interval $[1, l]$, where $l > 1$, then $e_{[1,l]} \leqslant |v_n|^2 e_{[1,l]}$, hence $\omega(e_{[1,l]}) \leqslant \sum_{j=n}^{\infty} \omega(b_j^2)$ by (5.9). Letting $l \to \infty$, we see that

$$\omega(e_n^\perp) \leqslant \sum_{j=n}^{\infty} \omega(b_j^2). \tag{5.10}$$

Finally, putting $e = e_n$ in (5.9) and then adding to (5.10), we get (5.8). This proves the theorem in the case R is countably decomposable. In this case the unit ball of any normal operator module over R is a complete metric space for the R-topology.

In general, let $\{x_i\}_{i \in \mathbb{I}}$ be any Cauchy net in the R-topology in U and let x be the limit of this net in the strong operator topology. We must prove that the net $\{s_\omega(x - x_i)\}_{i \in \mathbb{I}}$ converges to 0 for each normal state ω on R. Let $p \in R$ be the support projection of ω. We may regard $\mathrm{B}(\mathscr{H})\,p$ as a normal module over the countably decomposable algebra pRp in the obvious way. It is easy to verify (using a polar decomposition in R) that $\{x_i p\}_{i \in \mathbb{I}}$ is a Cauchy net in the pRp-topology of Up. Since ω is a faithful normal state on pRp, the net $\{x_i p\}_{i \in \mathbb{I}}$ must converge to xp in the pRp-topology, which means that $\{s_\omega((x - x_i)\,p)\}_{i \in \mathbb{I}}$ converges to 0. Since $s_\omega((x - x_i)\,p^\perp) \leqslant \|x - x_i\|\, \omega(p^\perp)^{1/2} = 0$, it follows that the net $\{s_\omega(x - x_i)\}_{i \in \mathbb{I}}$ converges to 0. ∎

6. A CHARACTERIZATION OF NORMAL MODULES

A norm complete abstract operator R-module X is called *strong* if there exists a completely isometric R-module isomorphism from X onto a concrete strong (hence normal by definition) operator R-module. By Proposition 4.6 such an isomorphism is necessarily a homeomorphism in the R-topology. A *dual* operator R-module is an operator R-module which is completely isometrically isomorphic to a weak* closed R-submodule of $\mathrm{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. Such operator modules can be characterized abstractly as duals of L^1-matricially normed (left) R-modules with completely contractive module multiplication. (See in particular Theorem 3.3 and the proof of Theorem 3.4 in [9]. We do not assume that the action of R on such a module is normal.)

We can use the R-topology to give a characterization of normal modules among the dual operator R-modules.

Proposition 6.1. *A dual operator R-module Y is normal if and only if the R-topology on Y is Hausdorff.*

Proof. We already know that the R-topology of a normal operator R-module is Hausdorff (since it is stronger than the ultrastrong topology).

Suppose now that Y is a dual operator R-module such that the R-topology on Y is Hausdorff. Then there is a Hilbert space $\mathscr{H}$ and a representation $\pi\colon R \to \mathrm{B}(\mathscr{H})$ such that Y is a weak* closed subspace of $\mathrm{B}(\mathscr{H})$ and $ya = y\pi(a)$ for all $y \in Y$ and $a \in R$. Let $e' \in \mathrm{B}(\mathscr{H})$ be the projection with the range $[Y^*\mathscr{H}]$. Then $Ye'^{\perp} = 0$ and $Yp \neq 0$ for each nonzero subprojection p of e' in $\mathrm{B}(\mathscr{H})$. Since the range of e' is invariant under $\pi(R)$, $e' \in \pi(R)'$ and we can define a subrepresentation π_0 of π by

$$\pi_0(a) = \pi(a) \mid e'\mathscr{H} \qquad (a \in R).$$

To prove that π_0 is normal, it suffices to show that for any net of projections $e_i \in R$ decreasing to 0 (strongly) the limit $f \in \mathrm{B}(e'\mathscr{H})$ of the decreasing net $\{\pi_0(e_i)\}_{i \in \mathbb{I}}$ must be 0. (This is based on the fact that completely additive states on a von Neumann algebra are normal.) Since Y is weak* closed, $Yf \subseteq Y$. For each $y \in Y$ and $\omega \in R_{\sharp +}$ we have (with the notation introduced in the beginning of Section 3)

$$\|Q_\omega(yf)\| = \|yf \otimes_R \xi_\omega\| = \|yf \otimes_R e_i \xi_\omega\| \leqslant \|yf\|\, \omega(e_i)^{1/2}$$

for each $i \in \mathbb{I}$ since $f = f\pi_0(e_i)$. Since the net $\{e_i\}_{i \in \mathbb{I}}$ converges to 0, this implies that $Q_\omega(yf) = 0$. Since the R-topology on Y is Hausdorff, it follows that $yf = 0$. Thus $Yf = 0$ and therefore $f = 0$ (since $f \leqslant e'$). This proves that π_0 is normal.

Since $Ye'^{\perp}=0$, we may regard Y as a submodule of $B(e'\mathcal{H}, \mathcal{H})$, and Y is normal since π_0 is normal. (It is now routine (by using 2×2 operator matrices) to represent Y as a normal weak* closed operator R-submodule of $B(\mathcal{K})$ for some Hilbert space of the form $\mathcal{K}=e'\mathcal{H}\oplus\mathcal{L}$, where $\mathcal{L}\supseteq\mathcal{H}$ is a Hilbert space admitting a faithful normal representation of R.) ▮

We would like to give an abstract characterization of normal (not necessarily dual) operator modules, but for this we need a preliminary result.

If V is a normed left (right, respectively) R-module, we denote by $V^{\sharp^R}$ the subspace of the dual $V^\sharp$ of V consisting of all $\theta\in V^\sharp$ such that for each $v\in V$ the functional

$$a\mapsto\langle av, \theta\rangle \qquad (a\mapsto\langle va, \theta\rangle, \text{ respectively})$$

is normal on R. If V is an operator R-module, then $V^{\sharp^R}$ is just the space of all R-continuous linear functionals on V by Theorem 4.2, hence the present notation extends the one introduced in Section 4. If, on the other hand, V is an L^1-matricially normed left R-module such that the module multiplication is completely contractive (see [9]), then $V^\sharp$ and $V^{\sharp^R}$ are L^∞-matricially normed completely contractive right R-modules, hence operator modules. In any case $V^{\sharp^R}$ is a norm closed R-submodule of $V^\sharp$.

Suppose that V is an L^1-matricially normed left R-module with a completely contractive module multiplication. Then there exist a weak* continuous complete isometry $\Phi\colon V^\sharp\to B(\mathcal{H})$ (hence a homeomorphism onto $\Phi(V^\sharp)$ in the weak* topology) and a (not necessarily normal) representation $\phi\colon R\to B(\mathcal{H})$ such that

$$\Phi(ya)=\Phi(y)\,\phi(a) \qquad (y\in V^\sharp, a\in R).$$

For each normal functional θ on $B(\mathcal{H})$ we have

$$\langle\Phi(y)\,\phi(a), \theta\rangle=\langle\Phi(ya), \theta\rangle=\langle ya, \Phi_\sharp(\theta)\rangle \qquad (y\in V^{\sharp^R}, a\in R),$$

where $\Phi_\sharp$ is the pre-adjoint of Φ. Since Φ is weak* continuous and $y\in V^{\sharp^R}$, the right hand side of the above identity is a normal functional in a, therefore a standard argument (see [13, Section 10.1]) shows that we can replace ϕ by its normal part ϕ_{nor}. Thus, $\Phi(y)\,\phi_{\text{nor}}(a)=\Phi(ya)$ for all $y\in V^{\sharp^R}$ and $a\in R$, which shows that $V^{\sharp^R}$ is a normal R-module.

We would like to show that $\Phi(V^{\sharp^R})$ (hence $V^{\sharp^R}$) is in fact a strong R-module. Choose any subsets $\{y_i\}_{i\in\mathbb{I}}\subseteq V^{\sharp^R}$ and $\{a_i\}_{i\in\mathbb{I}}\subseteq R$ so that the sums

$$\sum_{i\in\mathbb{I}}\Phi(y_i)\,\Phi(y_i)^* \qquad \text{and} \qquad \sum_{i\in\mathbb{I}} a_i^* a_i$$

converge in the strong operator topology. Since Φ is a linear homeomorphism in the weak* topology onto the weak* closed subspace $\Phi(V^{\sharp})$ of $B(\mathcal{H})$, the sum

$$y := \sum_{i \in \mathbb{I}} y_i a_i$$

converges to an element $y \in V^{\sharp}$ in the weak* topology and it suffices now to prove that $y \in V^{\sharp^R}$, for then by the weak* continuity of Φ we will have $\sum_{i \in \mathbb{I}} \Phi(y_i)\, \phi_{\mathrm{nor}}(a_i) = \Phi(y) \in \Phi(V^{\sharp^R})$. Thus, we must show that for each $\rho \in V$ the functional $a \mapsto \langle ya, \rho \rangle$ on R is normal. Since Φ is a weak* homeomorphism onto its image, it suffices to show that for each normal functional θ on $B(\mathcal{H})$ the functional

$$\theta_{\Phi}(a) := \langle \Phi(ya), \theta \rangle$$

on R is normal. (Namely, $\langle ya, \rho \rangle = \langle \Phi(ya), \Phi_{\sharp}^{-1}(\rho) \rangle$, where $\Phi_{\sharp}$: $\Phi(V^{\sharp})_{\sharp} \to V$ is the map with the adjoint Φ.) But, $\theta_{\Phi}(a) = \langle \Phi(y)\, \phi_{\mathrm{nor}}(a), \theta \rangle$ and the normality of θ_{Φ} follows easily from the normality of ϕ_{nor} and θ. So we have proved the following proposition.

PROPOSITION 6.2. For each L^1-matricially normed (left) R-module V with a completely contractive module multiplication the corresponding operator R module $V^{\sharp^R}$ is strong (in particular, normal).

Now let X be an arbitrary operator right R-module and apply Proposition 6.2 to the L^1-matricially normed left R-module $V = X^{\sharp^R}$. It follows that the operator R-module

$$\hat{X} := (X^{\sharp^R})^{\sharp^R}$$

is strong. There is a natural completely contractive homomorphism

$$\iota_R : X \to \hat{X}, \qquad \iota_R(x) = \hat{x},$$

where $\hat{x}(\rho) = \rho(x)$ $(\rho \in X^{\sharp^R})$. If ι_R is completely isometric, then X must be normal since $\hat{X}$ is normal. Conversely, if X is normal, then ι_R is completely isometric. (Namely, if $X \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ such that there is a normal representation of R on $\mathcal{H}$ inducing the R-module structure on X, then for each $n \in \mathbb{N}$ the space $\mathcal{M}_n(X)^{\sharp^R}$ contains the restrictions to $\mathcal{M}_n(X)$ of normal functionals on $\mathcal{M}_n(B(\mathcal{H})) = B(\mathcal{H}^n)$, hence $\|x\| = \sup\{|\rho(x)| : \rho \in \mathcal{M}_n(X^{\sharp^R}),\ \|\rho\| \leqslant 1\}$ for each $x \in \mathcal{M}_n(X)$.) This proves the following characterization of normal modules.

COROLLARY 6.3. *An operator R-module X is normal if and only if the natural map* $\iota: X \to \hat{X}$ *is completely isometric. Hence X is strong if and only if* ι *is completely isometric and* $\iota(X)$ *is a closed submodule of* $\hat{X}$ *in the R-topology.*

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