Openness of the Galois image for \(\tau\)-modules of dimension 1

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Abstract

Let \(C\) be a smooth projective absolutely irreducible curve over a finite field \(\mathbb{F}_q\), \(F\) its function field and \(A\) the subring of \(F\) of functions which are regular outside a fixed point \(\infty\) of \(C\). For every place \(\ell\) of \(A\), we denote the completion of \(A\) at \(\ell\) by \(\hat{A}_{\ell}\).

In [Pi2], Pink proved the Mumford–Tate conjecture for Drinfeld modules. Let \(\phi\) be a Drinfeld module of rank \(r\) defined over a finitely generated field \(K\) containing \(F\). For every place \(\ell\) of \(A\), we denote by \(G_{\ell,K}\) the image of the representation \(\rho_{\ell} : \Gamma_K \to \text{Aut}_{\hat{A}_{\ell}}(T_{\ell}(\phi)) \cong \text{GL}_r(\hat{A}_{\ell})\) of the absolute Galois group \(\Gamma_K\) of \(K\) on the Tate module \(T_{\ell}(\phi)\). The Mumford–Tate conjecture states that some subgroup of finite index of \(\Gamma_K\) is open inside a prescribed algebraic subgroup \(H_{\ell,K}\) of \(\text{GL}_r(\hat{A}_{\ell})\). In fact, he proves this result for representations of \(\Gamma_K\) on a finite product of distinct Tate modules.

A \(\tau\)-module over \(A_K\) is a projective \(A \otimes K\)-module of finite type endowed with a \(1 \otimes \varphi\)-semilinear injective homomorphism \(\tau\), where \(\varphi\) denotes the Frobenius morphism on \(K\). Such a \(\tau\)-module is said to have dimension 1, if the \(K\)-vector space \(M/K \cdot \tau(M)\) has dimension 1.

Drinfeld showed how to associate, in a functorial way, to every Drinfeld module over \(K\) a \(\tau\)-module \(M(\phi)\) over \(A_K\) of dimension 1, called the \(t\)-motive of \(\phi\). In this paper, we generalize Pink’s theorem to representations of Tate modules \(T_{\ell}(M)\) of \(\tau\)-modules \(M\) of dimension 1 over \(A_K\). The key result can be formulated as follows: if we suppose that \(\text{End}_K(M) = A\), then

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for every finite place \( \ell \) of \( F \), the image \( \Gamma_\ell \) of the representation

\[
\rho_\ell : \Gamma_K \rightarrow \text{Aut}_{\hat{\mathbb{A}}_\ell} (T_\ell(M))
\]

is open in \( \text{GL}_r(\hat{\mathbb{A}}_\ell) \), where \( r \) denotes the rank of \( M \).

As already demonstrated in the proof of the Tate conjecture for Drinfeld modules by Taguchi and Tamagawa, the relation between \( \tau \)-modules over \( A_K \) and Galois representations with coefficients in \( \mathbb{A} \) is more natural and direct than that between Drinfeld modules (or, more generally, abelian \( \tau \)-modules) and their Tate modules. By this philosophy, the assumption that a \( \tau \)-module \( M \) is pure, or, equivalently, is the \( \tau \)-motive of a Drinfeld module \( \phi \), should be and, indeed, is superfluous in proving a qualitative statement like the above Mumford–Tate conjecture. The main result of this paper is the corresponding statement for \( \tau \)-modules of dimension 1, i.e. whose maximal exterior power is the \( \tau \)-motive of a Drinfeld module.

We stick to the basic outline of Pink’s proof: reducing ourselves to the case where the absolute endomorphism ring of \( M \) equals \( A \), we first show that \( \Gamma_\ell \) is Zariski dense in \( \text{GL}_r(\hat{\mathbb{A}}_\ell) \), and we use his results on compact Zariski dense subgroups of algebraic groups to conclude that \( \Gamma_\ell \) if open in \( \text{GL}_r(\hat{\mathbb{A}}_\ell) \). After a reduction to the case where \( K \) has transcendence degree 1 over \( \mathbb{F}_q \), the essential tools we will use are the Tate and semisimplicity theorem for simple \( \tau \)-modules, Serre’s Frobenius tori and the tori given by inertia at places of good reduction for \( M \).

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1. Introduction

For a finite field \( \mathbb{F}_q \) of \( q = p^m \) elements, we consider a smooth projective absolutely irreducible curve \( \overline{C} \) with field of constants \( \mathbb{F}_q \), and denote its function field by \( F \). Fixing a closed point \( \infty \), we put \( C := \overline{C} \setminus \{ \infty \} \) and let

\[
A := H^0(C, \mathcal{O}_C)
\]

be the ring of regular functions of \( C \) outside \( \infty \).

Let \( K \) be a field containing \( F \) and denote the embedding \( F \hookrightarrow K \) by \( \iota \). On \( K \), we have the Frobenius homomorphism

\[
\varphi \in \text{End}(K) : x \mapsto x^q
\]

(we denote the induced endomorphism in \( \text{End}(\text{Spec} K) \) by \( \varphi \) as well).

Consider the \( \mathbb{F}_q \)-scheme \( C_K := C \otimes K \), and its ring of global regular functions \( A_K := A \otimes \mathbb{F}_q K \), both endowed with the endomorphism \( \sigma := \text{id} \otimes \varphi \). Let \( A_K^\sigma \) be the
ring $A_K$, viewed as an $A_K$-algebra via $\sigma$, and, for any $A_K$-module $M$, put
\[ \sigma^* M := A_K^\sigma \otimes_{A_K} M. \]

A $\tau$-module $(M, \tau)$ (for short: $M$) over $A_K$ of rank $r$ is a projective $A_K$-module $M$ of rank $r$ endowed with an injective $A_K$-linear homomorphism
\[ \tau : \sigma^* M \to M. \]

Equivalently, $M$ can be viewed as a $\tau$-sheaf, i.e. a locally free $C_{C_K}$-module of finite rank $r$, endowed with an injective morphism $\tau : \sigma^* M \to M$.

A morphism of $\tau$-modules is an $A_K$-linear morphism respecting the action of $\tau$. We denote the endomorphism ring of $M$ over $K$ by $\text{End}_K(M)$.

An isogeny between $\tau$-modules is an injective morphism of $\tau$-modules whose cokernel is a torsion $A_K$-module. If all nontrivial sub-$\tau$-modules of a given $\tau$-module $M$ are isogenous to $M$, then $M$ is called simple.

The tensor product $M_1 \otimes M_2$ of two $\tau$-modules over $A_K$ has $M_1 \otimes_R M_2$ as the underlying $R$-module and a $\tau$-action defined by
\[ \tau(m_1 \otimes m_2) = \tau m_1 \otimes \tau m_2 \]
for $m_i \in M_i$.

A $\tau$-module $M$ is said to have (generic) characteristic $i$ if the cokernel of $\tau$ is supported on the point $\Gamma(i)$ of $C_K$ defined as the graph of $i$, and it has dimension $1$ if furthermore the restriction of cokernel $\tau$ on $\Gamma(i)$ has dimension $1$. A $\tau$-module $M$ on $C_K$ is called smooth (or, equivalently, of dimension $0$) if $\tau$ is an isomorphism.

**Example 1.1.** Put $C = \mathbb{A}^1 := \text{Spec } \mathbb{F}_q[t]$. A $\tau$-module $M$ over $A_K \cong K[t]$ can be viewed as a free $K[t]$-module of finite rank endowed with a $\sigma$-semi-linear injective morphism. Fixing a basis for $M$, we express $\tau$ with respect to this basis by means of a matrix $\Delta_M$ in $\text{Mat}_{r \times r}(K[t])$: Putting $m := (m_1, \ldots, m_r)$, this $\Delta_M$ is determined by
\[ \tau(m) = m \cdot \Delta_M. \]

Let the matrix $^\sigma U$ be obtained by applying $\sigma$ to the entries of $U$. If we replace $m$ by another basis $m' = m \cdot U$, with $U \in \text{GL}_r(K[t])$, then $\tau$ is represented by
\[ U^{-1} \cdot \Delta_M \cdot ^\sigma U \]
with respect to the basis $m'$. The determinant of a matrix $\Delta_M$ representing $\tau$ is independent of the choice of a basis $m$ up to a unit in $K$.

The $\tau$-module $M$ is smooth if $\Delta_M$ is invertible. Put $\theta := \iota(t)$. The $\tau$-module $M$ has characteristic $i$ if and only if the annihilator of the $K[t]$-module $\text{coker} \Delta_M$ is contained in the ideal $(t - \theta) \subset K[t]$, i.e. the determinant of $\Delta_M$ equals
\[ h \cdot (t - \theta)^d, \]
where \( h \) is a unit in \( K \). Its dimension is 1 if and only if
\[
\text{coker } A_M \cong K[t]/(t - \theta),
\]
i.e. \( d = 1 \).

Let us recall how Drinfeld modules and \( t \)-motives are related to each other. We consider a Drinfeld \( A \)-module
\[
\phi : A \to \text{End}_{F_q}(\mathbb{G}_{a,K})
\]
whose characteristic homomorphism is the homomorphism \( i \). The \( K \)-vector space
\[
M(\phi) := \text{End}_{F_q}(\mathbb{G}_{a,K})
\]
is endowed with the structure of an \( A_K \)-module via
\[
(a \otimes c) \cdot m := c \cdot m \cdot a,
\]
for all \( m \in M(\phi) \), \( c \in K \) and \( a \in A \). Drinfeld proved that \( M(\phi) \) is a projective \( A \otimes K \)-module, whose rank equals the rank of \( \phi \) as a Drinfeld module.

The endomorphism \( \tau \in \text{End}(\mathbb{G}_{a,K}) \) acting on \( M(\phi) \) via \( \tau \cdot m := \tau \circ m \) commutes with \( A \) and acts as Frobenius on \( K \). We can now view \( M(\phi) \) as a projective \( A_K \)-module, called the \( t \)-motive associated to \( \phi \), and the injective morphism \( \tau : \sigma^* M(\phi) \to M(\phi) \) endows \( M(\phi) \) with the structure of a \( \tau \)-module over \( C_K \) with characteristic \( i \). As \( \text{coker } \tau \cong \text{Lie}(\mathbb{G}_a) \), the \( t \)-motive \( M(\phi) \) has dimension 1. The category of Drinfeld modules and that of the associated \( t \)-motives are antiequivalent.

**Example 1.2** (Drinfeld modules). Put \( C = \mathbb{A}^1_k := \text{Spec } F_q[t] \). Let \( x \in M(\phi) \) denote the identity morphism \( \text{id} : \mathbb{G}_{a,K} \to \mathbb{G}_{a,K} \). The elements \( m_i := \tau^{i-1} x \) for \( i = 1, \ldots, r \) yield a basis for the \( K[t] \)-module \( M(\phi) \) (or, equivalently, for the coherent free sheaf \( M(\phi) \) on \( \mathbb{A}^1_{C_K} \)). If the Drinfeld module \( \phi \) is given by
\[
\phi : t \mapsto \sum_{i=0}^{r} a_i t^i \in \text{End}(\mathbb{G}_{a,K})
\]
(\( a_r \in K^\times \) and \( a_0 = \theta \)), then the action of \( \tau \) with respect to this basis is given by the matrix representation:
\[
\tau \cdot (m_1, \ldots, m_r) = (m_1, \ldots, m_r) \cdot \begin{pmatrix}
0 & \cdots & 0 & t - \theta \\
1 & \cdots & 0 & a_1 \\
& \ddots & \ddots & \ddots \\
0 & & 1 & a_r - a_r^{-1}
\end{pmatrix},
\]
(1)
Remark 1.3 (Purity). Put $C = \mathbb{A}^1 := \text{Spec} \mathbb{F}_q[t]$. Consider the Dieudonné module

$$D_\infty(M) := K((t^{-1})) \otimes_{K[t]} M$$

of $M$. We recall that, following Anderson’s definition (cf. [An1, Section 1.9]), a $t$-motive $M$ over $K[t]$ is pure of weight $w$, if there exists a non-zero positive integer $z$ and a $K[[t^{-1}]]$-lattice $M_\infty$ in $D_\infty(M)$ such that

$$K[[t^{-1}]] \cdot \tau^z(M_\infty) = t^w M_\infty. \quad (2)$$

If a pure $t$-motive has rank $r$, dimension $d$ and weight $w$, then

$$w = \frac{d}{r}.$$

As Anderson shows, the $t$-motive of every Drinfeld module is pure of weight $1/r$. Conversely, if a $t$-module $M$ over $K[t]$ with characteristic $\iota$ is pure (cf. [An1, Section 9]), then, upon replacing $K$ by a finite inseparable extension, it is isomorphic to a $t$-motive and if it has dimension 1, then the associated abelian $t$-module is a Drinfeld module. This in part explains for the nomenclature of ‘dimension 1’. Another reason is that, as we will show in Section 4, if a $t$-module has dimension 1, then the connected part of the formal completion at a finite place $\mathfrak{p}$ of $F$ above which $M$ has good reduction corresponds to a one-dimensional formal $\mathcal{A}_\mathfrak{p}$-module.

Remark 1.4 (Maximal exterior power). A $t$-module $M$ over $K[t]$ with characteristic $\iota$ has dimension 1 if and only if its maximal exterior power $\wedge^r M$ does. Every $t$-module over $K[t]$ of rank 1 and dimension 1 is pure, and therefore isomorphic to the $t$-motive of a Drinfeld module of rank 1. Thus, if $C = \mathbb{A}^1 := \text{Spec} \mathbb{F}_q[t]$, we can reformulate the condition ‘$M$ has dimension 1’ as follows: the maximal exterior power of $M$ is the $t$-motive of a Drinfeld module.

Example 1.5 ($t$-Modules of dimension 1). Putting $C = \mathbb{A}^1 := \text{Spec} \mathbb{F}_q[t]$, we now give pure and non-pure examples of $t$-modules of dimension 1. Taking

$$K = F = \mathbb{F}_q(t)$$

and putting $\theta := \iota(t) \in K$, we define, for every $f \in \mathbb{F}_q[t]$, a rank 2 $t$-module $M_f$ over $K[t]$ as follows: with respect to its $K[t]$-basis $(m_1, m_2)$, we put

$$\tau \cdot (m_1, m_2) := (m_1, m_2) \cdot \begin{pmatrix} 0 & t - \theta \\ -1 & -f \end{pmatrix}.$$ 

As the determinant of this representation matrix equals $(t - \theta)$, the $t$-modules $M_f$ have characteristic $\iota$ and dimension 1 (cf. Example 1.1). One easily sees that the

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\textsuperscript{2}Note that, unlike Anderson, we do not assume $K$ to be perfect, which implies that $\tau^z(M_\infty)$ is not necessarily a $K[[t^{-1}]]$-module. If $K$ is pure, then the above definition coincides with Anderson’s.
maximal exterior power of $M_f$, for any $f$, is the $t$-motive associated to the Carlitz module $\phi_C: t \mapsto \tau + \theta$ (cf. Remark 1.4).

By reducing $M_f$ modulo $\theta$, we obtain a $\tau$-module $\overline{M}_f$ over $\mathbb{F}_q[t]$ (i.e. $M_f$ has good reduction at $\theta$, cf. Section 2). Note that the action of $\sigma$ on $\mathbb{F}_q$ is trivial, so that $\tau$ is a linear endomorphism of $\overline{M}_f$; its characteristic polynomial is

$$P(Z) = Z^2 + f \cdot Z + t.$$ 

Let $x$ be the place of $K$ defined by the ideal $(\theta)$ of $\mathbb{F}_q[\theta]$. As we will explain in Section 3, this characteristic polynomial coincides with the characteristic polynomial of the Frobenius substitution $\text{Frob}_x$ at the place $x$ operating on the Galois modules $V_\ell(M_f)$ associated with $M_f$ (for all finite places $\ell \neq x$).

If $M_f$ is pure, it is the $t$-motive of a Drinfeld module $\phi_f$ over $K$: it is then well known (cf. [Go2, Theorem 3.2.3]) that the eigenvalues $\lambda_i \in \overline{F}$ of the endomorphism $\text{Frob}_x$ of the Galois modules $V_\ell(M_f) \cong V_\ell(\phi_f)$ are pure, i.e.

$$\deg \lambda_i = \frac{1}{2}.$$

From the Newton polygon associated to $P_f(Z)$, we see that this then implies that $\deg f = 0$. In conclusion, $M_f$ is pure (and therefore is the $t$-motive of a Drinfeld module) if and only if $f \in \mathbb{F}_q$.

For any closed point $\ell$ of $C$, i.e. for any finite place of $F$, let $\kappa_\ell$ denote the residue field of $C$ at $\ell$, $\hat{A}_\ell$ the completion of $A$ at $\ell$, and $\hat{F}_\ell$ its fraction field. For every finite set $A$ of finite places of $F$, we also consider the ring

$$\hat{F}_A = \prod_{\ell \in A} \hat{F}_\ell.$$

We denote an algebraic, resp. separable closure of $K$ by $\overline{K}$, resp. $K^{\text{sep}} \subset \overline{K}$, and the absolute Galois group $\text{Gal}(K^{\text{sep}}/K)$ by $\Gamma_K$.

We define the ring

$$\hat{A}_{\ell, K} = \hat{A}_\ell \otimes_{\mathbb{F}_q} K$$

as the completion of $\hat{A}_\ell \otimes_{\mathbb{F}_q} K$ with respect to its maximal ideal. We continuously extend $\sigma \in \text{End}(A_K)$ to an endomorphism of $\hat{A}_{\ell, K}$. For a $\tau$-module $M$ over $A_K$, we put

$$\hat{M}_\ell := \hat{A}_{\ell, K} \otimes_{A_K} M.$$ 

Extending the action of $\tau$ to $\hat{M}_\ell$, the latter should be seen as a formal $\tau$-module over $\hat{A}_{\ell, K}$. We associate to it the module

$$T_\ell(M) = \{ f \in \text{Hom}_{\hat{A}_{\ell, K}}(\hat{M}_\ell, \hat{A}_{\ell, K^{\text{sep}}}); f \circ \tau = \sigma \circ f \}.$$ 

(3)
This $T_\ell(M)$, called the Tate module of $M$ at $\ell$, is a free $\widehat{\mathcal{A}}_\ell$-module of rank $r$, the rank of $M$, endowed with a continuous action of $\Gamma_K$.

**Remark 1.6.** By Drinfeld (cf. [TW, Proposition 6.3]), the map

$$\tilde{M}_\ell \mapsto T_\ell(M)$$

induces an antiequivalence between the categories of smooth $\tau$-modules $\tilde{M}_\ell$ over $\widehat{\mathcal{A}}_{\ell,K}$ and $\widehat{\mathcal{A}}_{\ell}[\Gamma_K]$-modules $T_\ell$ with free underlying $\widehat{\mathcal{A}}_{\ell}$-module of finite type.

**Remark 1.7.** Let $\phi$ be a Drinfeld $\mathcal{A}$-module defined over $K$. For every finite place $\ell$ of $F$, the Tate module $T_\ell(\phi)$ (defined as the projective limit of $\ell$-primary torsion points) is isomorphic to $T_\ell(M(\phi))$ as an $\widehat{\mathcal{A}}_{[\ell]}$-module.

**Example 1.8.** We take up Example 1.1. Let $\ell$ the finite place of $F$ corresponding to the ideal $(t)$. The ring $\widehat{\mathcal{A}}_{\ell}$ is isomorphic to the power series ring $\mathbb{F}_q[[t]]$ and the Tate module $T_\ell(M)$ can be computed as follows: It is the $\mathbb{F}_q[[t]]$-module consisting of vectors \((X_1, \ldots, X_r)\in\mathbb{F}_q[[t]]^\oplus r\) satisfying

\[
(X_1, \ldots, X_r) = (X_1, \ldots, X_r) \cdot \Delta.
\] (4)

We consider the $\mathbb{F}_q[[t]]$-module $V_\ell(M) \coloneqq \mathbb{F}_q \otimes T_\ell(M)$ and its associated Galois representation

$$\rho_\ell : \Gamma_K \to \text{Aut}_{\mathbb{F}_q}(V_\ell(M)) \cong \text{GL}_r(\mathbb{F}_q),$$

whose image we denote by $\Gamma_{\ell}$.

For every finite set $\Lambda$ of finite places $\ell$ of $F$, we put $V_{\Lambda}(M) \coloneqq \prod_{\ell \in \Lambda} V_\ell(M)$, and let $\Gamma_{\Lambda}$ denote the image of the representation

$$\rho_{\Lambda} : \Gamma_K \to \text{Aut}_{\mathbb{F}_q}(V_{\Lambda}(M)) \cong \text{GL}_r(\mathbb{F}_\Lambda).$$

Let $K$ be a finitely generated field containing $F$ and $M$ a simple $\tau$-module $M$ over $\mathcal{A}_K$ of rank $r$, with characteristic $i$ and dimension 1. Our main theorem is the following qualitative statement of the image of $\rho_{\Lambda}$:

**Theorem 1.9.** Suppose that $\text{End}_{\mathbb{F}_q}(M) = \mathcal{A}$. For every finite set $\Lambda$ of finite places of $F$, the group $\Gamma_{\Lambda}$ is open in $\text{GL}_r(\mathbb{F}_\Lambda)$.

The proof of this theorem will be given in Section 6.

There exists a natural embedding of $\text{End}_{\mathbb{F}_q}(M)$ into $\text{End}_{\mathbb{F}_\Lambda}(V_{\Lambda}(M))$. We consider the centralizer

$$H_{\Lambda} \coloneqq \text{Cent}_{\text{GL}_r(\mathbb{F}_\Lambda)}(\text{End}_{\mathbb{F}_q}(M)) \subset \text{GL}_r(\mathbb{F}_\Lambda).$$
As we show in Proposition 2.3, Theorem 1.9 can then be generalized as follows:

**Theorem 1.10** (Openness of the Galois image). *For every finite set Λ of finite places of F, the group Γ_A ∩ H_A is open in both Γ_A and H_A.*

In the case that \( M \) is the \( τ \)-motive of a Drinfeld module \( φ \) defined over \( K \), then, by Remark 1.7, this theorem can be restated in terms of the Tate modules of \( φ \). The theorem in this Drinfeld module formulation was proved by Pink [Pi2, Theorem 0.2]. Although many of the techniques used in his proof will apply to our settings, its arguments make very strongly use of the fact that a \( τ \)-motive associated to a Drinfeld module is pure, more precisely: that the eigenvalues of the Frobenius elements are pure (cf. 3.1).

We stick to the basic outline of Pink’s proof: first, in Section 5, we prove that \( Γ_τ \) is Zariski dense in \( GL_{r,F} \). We then use Pink’s results from [Pi3] on compact Zariski dense subgroups of algebraic groups, to conclude the proof of Theorem 1.9. Reducing ourselves to the case where \( K \) has transcendence degree 1 over \( F_q \), the essential tools we will use are the Tate and semisimplicity theorem (cf. Section 2), Serre’s Frobenius tori (cf. Section 3) and the tori given by the image of inertia groups at a place \( x \) where \( M \) has a good model (cf. Section 4).

At the same time, it is remarkable how a fairly low tech structure as a \( τ \)-module of dimension 1, basically given by a square \( K[t] \)-matrix with a prescribed determinant (cf. Example 1.1), allows such a nice qualitative statement as Theorem 1.10.

**Question 1.11.** What is a good formulation for a conjecture on the image of the Galois representations associated to \( τ \)-modules with characteristic \( i \) and dimension \( d \geq 2 \)? The tensor product of simple \( τ \)-modules of dimension 1 of rank bigger than 1 yields a \( τ \)-module \( M \) with \( \text{End}_{K}(M) = A \) for which \( Γ_τ \cap SL_r(\hat{F}_τ) \) is not open in \( SL_r(\hat{F}_τ) \)?

**Question 1.12.** Does a statement like Theorem 1.10 hold if the characteristic \( i \) of \( M \) is special, i.e. if \( i : A \to K \) is not injective?

**Remark 1.13** (Hodge structures). Suppose that \( M \) is the \( τ \)-motive of a Drinfeld module \( φ \). In [Pi1], Pink associates a Hodge structure to \( φ \). In [Pi1, Theorem 0.6], he establishes that the Hodge group \( H \) of this structure, an algebraic group over \( F \), equals

\[
\text{Cent}_{GL_{r,F}}(\text{End}_{K}(φ)).
\]

The group \( H_A \) can therefore be interpreted as \( H(\hat{F}_A) \). This yields a precise analog of Serre’s conjecture in the classical theory of motives over number fields, cf. [Se6, Section 9–13, Conjecture 11.4]. Up to now, there exists no theory of Hodge structures for general \( τ \)-modules.
Remark 1.14. A generalization of Theorem 1.10 for infinite sets $\Lambda$ of finite places $\ell$ has been conjectured. Let $F_{\text{ad}}$ denote the finite adeles of $F$. The subset

$$V_{\text{ad}}(M) \subseteq \prod_{\ell \neq \infty} V_{\ell}(M)$$

consisting of $(v_{\ell})_{\ell}$ such that $v_{\ell} \in T_{\ell}(M)$ is a free $F_{\text{ad}}$-module endowed with a continuous $\Gamma_K$-action.

We denote by $\Gamma_{\text{ad}}$ the image of the representation

$$\rho_{\text{ad}} : \Gamma_K \rightarrow \text{End}_{F_{\text{ad}}}(V_{\text{ad}}(M))$$

and put

$$H_{\text{ad}} := \text{Cent}_{\text{GL}_r(F_{\text{ad}})}(\text{End}_{F_{\text{ad}}}(M)) \subset \text{GL}_r(F_{\text{ad}}).$$

Conjecture 1.15 (Adelic openness of the Galois image). The group $\Gamma_{\text{ad}} \cap H_{\text{ad}}$ is open in both $\Gamma_{\text{ad}}$ and $H_{\text{ad}}$.

Remark 1.16. For the $t$-motive of a Drinfeld $A$-module $\phi_M$ of rank 1, or equivalently, for a $\tau$-module over $A_K$ of rank 1 and dimension 1, this conjecture, and a fortiori Theorem 1.10, is a well known consequence of the abelian class field theory of $F$ (cf. Theorem [Go1, of Section 7.7]).

In [Ga1, Section 3.III], we explain how Theorem 1.10 implies Conjecture 1.15 in the case where the rank $r$ of $M$ equals 2. The proof of this implication is based on an analog of Serre’s theory of abelian $l$-adic representations (cf. [Se1]) and his famous theorem from [Se3] on the image of Galois on the torsion of elliptic curves over a number field.

2. The endomorphism ring of a $\tau$-module

In this section, we study the endomorphism ring $\text{End}_K(M)$ and recall the Tate and semistability conjectures. We explain how one can reduce the proof of Theorem 1.9 to the case where the field has transcendence degree 1, and we show how Theorem 1.9 implies Theorem 1.10.

Let $K$ be a finitely generated field containing $F$. We fix a model $X$ for $K$, i.e. an $\mathbb{F}_q$-scheme $X$ of finite type over $\text{Spec} \mathbb{F}_q$ with function field $K$. For a point $x$ of $K$, we denote the residue field by $k_x$ and the local ring of functions at $x$ by $R_x$.

We consider the Frobenius homomorphism

$$\phi \in \text{End}_{\mathbb{F}_q}(R_x) : x \mapsto x^q$$
and the endomorphism $\sigma := \text{id} \otimes \phi$ on the ring $A_x \coloneqq A \otimes_{R_q} R_x$. A $\tau$-module $\mathcal{M}$ over $A_x$ is a finitely generated torsion free $A_x$-module endowed with an injective $A_x$-linear homomorphism $\tau : \sigma^* \mathcal{M} \to \mathcal{M}$.

Let $M$ be a $\tau$-module over $A_K$. A model $\mathcal{M}_x$ over $A_x$ of a $\tau$-module $M$ on $A_K$ is a $\tau$-module over $A_x$ such that

$$M \coloneqq A_k \otimes_{A_x} \mathcal{M}_x;$$

in other words, viewed as a $\tau$-sheaf on $C_K$, $M$ is the generic fibre of $\mathcal{M}_x$.

Consider the reduction

$$\overline{\mathcal{M}}_x \coloneqq A_{k_x} \otimes_{A_{R_x}} \mathcal{M}_x$$

of $\mathcal{M}_x$ to $A_{k_x}$, a projective $A_{k_x}$-module endowed with the induced homomorphism

$$\tau : \sigma^* \overline{\mathcal{M}}_x \to \overline{\mathcal{M}}_x.$$

The $\tau$-module $\mathcal{M}$ is called good at the point $x$ if $\overline{\mathcal{M}}_x$ is a $\tau$-module (i.e. $\tau$ is injective). If $\mathcal{M}_x$ is good then it is a maximal model, i.e. it contains every other model of $M$ over $A_x$. There exists an open subscheme $X^\text{good}$ of $X$ such that there exists a good model $\mathcal{M}_x$ for $M$ at $x$.

Let $M$ be a simple $\tau$-module over $A_K$ of rank $r$, with characteristic $i$.

Taguchi (in the case of Drinfeld modules; cf. [Ta1,Ta2]) and Tamagawa (for general $\tau$-modules) [Tam]) proved the so-called Tate and semisimplicity conjectures:

**Theorem 2.1** (Taguchi, Tamagawa). (i) (Tate conjecture.) The homomorphism

$$\hat{\mathcal{F}}_l \otimes A \text{End}_K(M) \to \text{End}_{\hat{\mathcal{F}}_l[F_K]}(V_\ell(M))$$

is an isomorphism.

(ii) (Semisimplicity conjecture.) The $\hat{\mathcal{F}}_l[\Gamma_K]$-module $V_\ell(M)$ is semi-simple.

We now have the following result on the endomorphism ring $\text{End}_K(M)$, as well as on the absolute endomorphism ring $\text{End}_K(M)$:

**Proposition 2.2** (Endomorphism ring). The ring $\text{End}_K(M)$ (resp. $\text{End}_{\overline{\mathcal{M}}}(M)$) is a finitely generated $A$-algebra of rank at most $r^2$. If $M$ has dimension 1, then $\text{End}_K(M)$ (resp. $\text{End}_{\overline{\mathcal{M}}}(M)$) is commutative and has $A$-rank at most $r$.

**Proof.** Let us first prove the result for the ring $E := \text{End}_K(M)$. This $E$ has the structure of an $A$-algebra via the inclusion $A \hookrightarrow E$. We put

$$E^\circ := \text{End}_K^0(M) := F \otimes_A E.$$
We claim that, for every \( a \in E \), there exists an \( \hat{a} \) such that \( \hat{a} \cdot a \in A \). If \( M \) has rank 1, then clearly \( \text{End}_K(M) = A \). In general, if the endomorphism \( a \) is (locally) represented by a matrix \( B \) w.r.t. some basis of \( M \), then its determinant is an element of \( A \). The endomorphism \( \hat{a} \) represented by the adjoint matrix \( B^\text{ad} \) then satisfies the condition. This proves that \( E \) is a torsion free \( A \)-module and that \( E^0 \) is a division \( F \)-algebra.

We choose a model \( \mathcal{M} \) for \( M \) over \( C_X \). For every point \( x \in X^{\text{good}} \), \( \mathcal{M}_x \) is maximal and therefore \( \text{End}_K(M) \)-invariant. By reduction, we obtain a ring homomorphism

\[
\hat{j} : E \rightarrow \text{End}_{K_x}(\mathcal{M}_x).
\]

Without loss of generality, we may reduce ourselves to the case that \( A \cong \mathbb{F}_q[t] \) (cf. 1.1), and, upon replacing \( \tau \) by some power, that \( k_x = \mathbb{F}_q \), as we only risk to increase the endomorphism ring. Then \( \text{End}_{K_x}(\mathcal{M}) \) is just the full matrix ring of rank \( r \times r \) over \( A \), so for sure it is finitely generated as an \( A \)-module.

We now claim that

\[
E^0 \rightarrow F \otimes_A \text{End}_{K_x}(\mathcal{M})
\]

is injective. The ring \( E^0 \) being a division \( F \)-algebra, it suffices to show that it is not a zero morphism. That is obvious because it is non-zero on the subring \( A \subset E \) of multiplication-by-\( a \) endomorphisms. Thus \( j \) is injective, and hence \( \text{End}_K(M) \) is finitely generated as an \( A \)-module.

It follows from the Tate conjecture (Theorem 2.1) that, for every finite place \( \ell \) of \( F \), the \( F \)-algebra \( \widehat{\mathbb{F}}_\ell \otimes_A E \) embeds into \( \text{End}_{\mathbb{F}_\ell}(V_\ell(M)) \), which is an \( \widehat{\mathbb{F}}_\ell \)-algebra of rank \( r \). This shows that \( E \) has \( F \)-dimension at most \( r^2 \); if \( E \) is commutative then its dimension is at most \( r \).

To prove the result for \( \text{End}_{\overline{K}}(M) \), we choose an infinite tower of finitely generated subfield fields \( K_i \subset K \) such that \( \cup K_i = \overline{K} \). For every \( K_i \), the ring \( \text{End}_{K_i}(M) \) is finitely generated over \( A \), of rank \( \leq r^2 \). It follows from the Tate conjecture that the \( \widehat{\mathbb{A}}_r \)-algebra \( \widehat{\mathbb{A}}_r \otimes_A E \) is saturated in \( \text{End}_{\widehat{\mathbb{A}}_r}(T_\ell(M)) \), i.e. the quotient of \( \text{End}_{\widehat{\mathbb{A}}_r}(T_\ell(M)) \) by \( \widehat{\mathbb{A}}_r \otimes_A E \) is torsion free. Therefore, if, for \( K_i \subset K_j \), we have

\[
\text{End}_{K_i}(M) \neq \text{End}_{K_j}(M),
\]

then this implies that

\[
\text{rank}_A \text{End}_{K_i}(M) < \text{rank}_A \text{End}_{K_j}(M).
\]

As these ranks are bounded by \( r^2 \), we obtain a \( K_j \) such that

\[
\text{End}_{K_j}(M) = \text{End}_{\overline{K}}(M),
\]

which, by the above, proves the result.
Finally, suppose that $M$ has dimension 1. Endomorphisms, by definition, commute with $\tau$ and so we have a natural homomorphism:

$$E \rightarrow \text{End}_{\text{coker } \tau} (\text{coker } \tau).$$

As $\text{coker } \tau$ is supported on the closed point $\Gamma(i)$ of $C_K$ and has rank 1 on the point $\Gamma(i)$, we can identify the latter with $K = \text{End}_K(K)$. We claim that

$$j : E \rightarrow K$$

is a injection. For any non-zero $x \in A \subseteq E$, we know, because $M$ has characteristic $i$, that the element $j(x)$ acts on $\text{coker } \tau$ as $i(x) \in K$. Therefore we can extend $j$ to a non-zero ring homomorphism $E/\mathbb{Q} \rightarrow K$, which, the ring $E/\mathbb{Q}$ being a division $F$-algebra, is injective. Hence $E/\mathbb{Q}$ is a (commutative) field extension of $F$, of rank at most $r$, and $E$ is a commutative projective $A$-algebra of rank at most $r$; idem for $\text{End}_K(M)$.  

Suppose now that $M$ has dimension 1. Using the Tate conjecture, we can reduce the proof of Theorem 1.10 to the case where $K$ has transcendence degree 1 and $\text{End}_K(M) = A$.

**Proposition 2.3.** Theorem 1.9 implies Theorem 1.10.

**Proof.** The ideas of this proof originate from [Pi2, Theorem 0.2] (cf. p. 408).

Without loss of generality, we may replace $K$ by a finite extension such that $E := \text{End}_K(M) = \text{End}_K(M)$. If we put $A' := E$ and $F' := F \otimes_A E$, then by Proposition 2.2, $F'$ is a finite extension of $F$. We put $C' := \text{Spec } A'$ and consider the finite morphism $f : C' \rightarrow C$.

The $\tau$-sheaf $M$ on $C_K$, endowed with an action of $A'$, induces a $\tau$-sheaf $M'$ on $C'_K$ such that $f_* M' = M$, denoting the induced morphism $C'_K \rightarrow C_K$ again by $f$. However, $C'$ is not necessarily smooth, so we consider the normalization $\tilde{C}'$ of $C'$ and the morphism $\tilde{f} : \tilde{C}' \rightarrow C'$.

Consider the $\tau$-sheaf

$$M^\tau := \tilde{f}^* M'$$

on $\tilde{C}_K$. We get adjunction morphisms $\tilde{f}_! \tilde{f}^* \rightarrow \text{id}$ and $\text{id} \rightarrow \tilde{f}^* \tilde{f}_*$, which are isomorphisms outside the finite set $S$ of singularities of $C'$. Thus we see that the $\tau$-sheaf $\tilde{f}_* M^\tau$ on $C'_K$ is isogenous to $M'$. As now Tate modules are determined by $\tau$-sheaves up to isogeny (by the Tate conjecture, Theorem 2.1), we can reduce ourselves to the case that $\tilde{C}' = C'$.

By the Tate conjecture, $\Gamma_A \subseteq H_A$, for every finite set $A$ of finite places of $F$. On the other hand, by Theorem 1.9 and the above, this implies, for any finite set $A$ of finite
places of $F$, that the image of the representation

$$\Gamma_K \to V_{A'}(M') := \prod_{\ell \in \mathcal{N}} \text{Aut}(V_{\ell'}(M'))$$

is open, where $A'$ is the set of places of $F'$ above $A$. Finally, the isomorphism $V_{\ell}(M) \cong \bigoplus_{\ell' | \ell} V_{\ell'}(M')$ of Tate modules induces an isomorphism

$$H_A \cong \text{Aut}_{F'}(V_{A'}(M)).$$

This concludes the proof. □

**Proposition 2.4.** If Theorem 1.9 holds for all finite extensions of $F$, then it also holds for every finitely generated field $K$ containing $F$.

**Proof.** We use ideas from [Pi2, Theorem 1.4]. Let $K$ be finitely generated field containing $F$ and $M$ a $\tau$-module on $A_K$ with characteristic $i$ and dimension $1$. By the Tate conjecture (Theorem 2.1) the sub-$\hat{F}_{\ell}$-algebra

$$\hat{F}_{\ell} \Gamma'$$

of $\text{End}_{\hat{F}_{\ell}}(V_{\ell}(M))$ generated by $\Gamma'$ is equal to $\text{End}_{\hat{F}_{\ell}}(V_{\ell}(M))$, and this for any fixed finite place $\ell$ of $F$. By Pink [Pi2, Lemma 1.5], there is an open normal subgroup $\Gamma_1 \subset \Gamma'$ such that for any subgroup $\Omega' \subset \Gamma'$ for which $\Omega' \Gamma_1 = \Gamma'$, we have

$$\hat{F}_{\ell} \Omega' = \hat{F}_{\ell} \Gamma'$$

(denoting by $\hat{F}_{\ell} \Omega'$ the sub-$\hat{F}_{\ell}$-algebra of $\text{End}_{\hat{F}_{\ell}}(V_{\ell}(M))$ generated by $\Omega'$). Taking the extension $\bar{K}$ of $K$ fixed by $\Gamma_1$, one denotes by $\bar{X}$ be the normalization of $X$ in $\bar{K}$ and by $\pi$ the morphism $\bar{X} \to X$.

By Pink [Pi2, Lemma 1.6], there exists a point $x$ in the open subscheme $X^{\text{good}}$ of $X$ such that $k_x$, the residue field of $x$, is a finite extension of $F$, and such that $\pi^{-1}(x)$ is irreducible. Letting $\Omega'_x$ be the image of $\Gamma_{k_x}$ on $V_{\ell}(M)$, seen as a subgroup of $\Gamma'$, we then have

$$\Omega'_x \Gamma_1 = \Gamma'.$$

Hence

$$\hat{F}_{\ell} \Omega'_x = \text{End}(V_{\ell}(M)),$$

and therefore the reduction $\overline{M}_x$ of the good model $\mathcal{M}_x$ of $M$ at $x$ satisfies

$$\text{End}_{\overline{K}}(\overline{M}_x) = A,$$
by the Tate conjecture (Theorem 2.1). As, by assumption, Theorem 1.9 holds for the finite extension of \( k_x \) of \( F \), we get that the image of \( \Gamma_{k_x} \) is open in \( \text{GL}_r(\hat{F}_A) \), for every finite set \( \Lambda \) of finite places of \( F \). A fortiori, \( \Gamma_A \) is open in \( \text{GL}_r(\hat{F}_A) \). \( \square \)

3. Frobenius tori

Let \( K \) be a finite field extension of \( F \) (so \( K \) is a field of transcendence degree 1 over \( \mathbb{F}_q \)). We fix a projective smooth \( \mathbb{F}_q \)-curve \( X \) whose function field is isomorphic to \( K \) to serve as a model for \( K \).

For a place \( x \) of \( K \), i.e. a closed point of \( X \), let \( \ell_x \) be the place of \( F \) below \( x \). Let \( \mathcal{K}_x \) be the completion of \( K \) at \( x \) and \( \mathcal{R}_x \) its valuation ring. Further, we denote the absolute Galois group of \( \mathcal{K}_x \) by \( \mathcal{G}_x \), its inertia group by \( I_x \), and the Frobenius generator of \( \mathcal{G}_x/I_x \) by \( \varphi_x \).

Let \( M \) be a \( t \)-module over \( \mathcal{A}_K \) of rank \( r \), with characteristic \( i \). For a place \( x \in X^{\text{good}} \), we put \( d_x := [k_x : \mathbb{F}_q] \) and we define

\[
P_x(M; Z) := \det_{A \otimes k_x}(\tau^{d_x} - Z | A_{k_x}[Z]). \tag{6}
\]

The identity

\[
P_x(M; U^{d_x}) = \det_{A}(\tau - U | A_{\mathcal{A}_x}[U]) \in A[U]
\]

proves that \( P_x(M; Z) \) actually has coefficients in \( A \).

We consider an element \( t_x \in \text{GL}_r(F) \) with characteristic polynomial \( P_x(M; Z) \). Some power of \( t_x \) is semisimple and lies in a unique conjugacy class. The Zariski closure \( T_x \) of the connected component of the group generated by \( t_x \) is therefore a well-defined diagonalisable subgroup of \( \text{GL}_r(F) \). Following Serre (cf. [Se4]), it is called the Frobenius torus at \( x \). It contains essential information on the action of the Frobenius \( \varphi_x \) on all \( V_\ell(M) \), for \( \ell \neq \ell_x \), as is shown in Proposition 3.1.

Note that the choice of an embedding of \( K^{\text{sep}} \) into a separable closure of \( \mathcal{K}_x \) corresponds to the choice of a conjugate of the group \( \Gamma_x \subset \Gamma_K \). Any lift of \( \varphi_x \) in \( \Gamma_K \) inside \( \Gamma_K \) is called a Frobenius substitution \( \text{Frob}_x \).

**Proposition 3.1** (Strictly compatible system). (i) For every \( x \in X^{\text{good}} \), the \( \Gamma_K \)-module \( V_\ell(M) \) is unramified for all finite places \( \ell \neq \ell_x \) (up to conjugacy, the action of a Frobenius substitution \( \text{Frob}_x \) on \( V_\ell(M) \) is then well defined).

(ii) The characteristic polynomial

\[
P_x(V_\ell(M); Z) := \det_{\overline{\mathbb{F}}_\ell}(\text{Frob}_x - Z | V_\ell(M)) \in \overline{\mathbb{A}}[T] \tag{7}
\]
which is independent of the choice of \( \text{Frob}_x \), satisfies
\[
P_x(V_\ell(M); Z) = P_x(M; Z),
\]
for all \( \ell \neq \ell_x \).

**Remark 3.2.** Following Serre [Se1], the system of Galois modules \( V_\ell(M) \) is thus a strictly compatible system of integral \( \hat{\mathbb{F}}_\ell \)-representations over the set of finite places \( \ell \) of \( F \), with exceptional set \( X \backslash X_{\text{good}} \): i.e. \( P_x(V_\ell(M); Z) \) has coefficients in \( A \) and is independent of \( \ell \), for all \( x \in X_{\text{good}} \) and all finite places \( \ell \neq \ell_x \) of \( F \).

It is also worth mentioning that, by the proposition, the Frobenius torus \( T_\chi \) is a subtorus of \( \text{GL}_{r,F} \), which, upon specialization to \( \hat{\mathbb{F}}_\ell \), is conjugate to the torus spanned by the image of \( \text{Frob}_x \) under the representation \( \rho_\ell \).

We set \( \widehat{A}_{\ell,x} := \widehat{A}_\ell \otimes R_x \) (the completion of \( \widehat{A}_\ell \otimes R_x \)) with respect to its maximal ideal and we continuously extend \( s \in \text{End}(A_x) \) to an endomorphism of \( \widehat{A}_{\ell,x} \).

**Proof of Proposition 3.1.** (i) For every finite place \( x \in X_{\text{good}} \), the cokernel of \( \tau \) on \( \mathcal{M}_x \) is supported on the graph \( \Gamma(\ell) \) of \( \ell : A \rightarrow R_x \) in \( A_x \). If \( x \) is an infinite place in \( X \) good, then \( \mathcal{M}_x \) is smooth, i.e. \( \tau \) operates as an isomorphism on it.

This shows that, for every place \( x \) of \( X_{\text{good}} \) and for every finite place \( \ell \neq \ell_x \) of \( F \), the completion
\[
\widehat{\mathcal{M}}_{\ell,x} := \hat{A}_{\ell,x} \otimes_{A_x} \mathcal{M}_x
\]
is smooth. By the equivalence between smooth formal \( \tau \)-modules and Galois representations mentioned in (cf. 1.6), this implies that it is unramified at \( x \in X_{\text{good}} \) for \( \ell \neq \ell_x \) as it is endowed with a continuous action of \( \Gamma_x \).

(ii) The action of the transpose \( \phi_\ell \) of \( \phi_x \) on the unramified \( \hat{A}_{\ell}[\Gamma_x] \)-module \( T_\ell(M) \) corresponds, by the definition of \( T_\ell(M) \), to the action of \( \tau^{d_\ell} \) on \( \widehat{\mathcal{M}}_{\ell,x} \), which is induced by that on the reduction \( \mathcal{M}_x \). Therefore
\[
P_x(M; Z) = \det(\text{Frob}_x - Z \mid T_\ell(M)) = P_x(V_\ell(M); Z). \quad \square
\]

4. Image of inertia

Let \( K \) be a finite field extension of \( F \) with smooth projective model \( X \). We fix a place \( x \in X_{\text{good}} \) and we adopt the notations from the previous section. Let \( \ell := \ell_x \) be the place of \( F \) below \( x \) and \( \hat{A}_{\ell}^{[s]} \) the unramified extension of degree \( s \) of \( \hat{A}_\ell \). Let \( T_\ell^{[s]} \) denote the torus of \( \text{GL}_{s,\hat{A}_\ell} \) obtained by restriction of scalars from \( \hat{A}_{\ell}^{[s]} \) to \( \hat{A}_\ell \); so
\[
T_\ell^{[s]}(\hat{A}_\ell) = \hat{A}_{\ell}^{[s]} \subseteq \text{GL}_{s}(\hat{A}_\ell).
\]
Let $M$ be a $\tau$-module over $A_K$ of rank $r$, with characteristic $\iota$ and dimension 1. The following theorem gives a lower bound on the image of the representation of the inertia group on $V_{\ell}(M)$:

**Theorem 4.1.** Suppose that the extension $\mathbb{K}_s/\mathbb{F}_\ell$ is unramified. Then there exists an exact sequence

$$0 \to V_\ell^0 \to V_{\ell}(M) \to V_{\ell}^{\text{ct}} \to 0$$

(9)

of $\widehat{\mathbb{F}}_{\ell}[\Gamma_s]$-modules with $s = \dim V_\ell^0 \geq 1$ such that the module $V_{\ell}^{\text{ct}}$ is unramified and such that the image of the inertia group $I_x$ under the representation

$$\rho_\ell: \Gamma_x \to \text{Aut}(V_\ell^0) \cong \text{GL}_s(\widehat{\mathbb{F}}_{\ell})$$

contains a subgroup conjugate to $T^{[s]}(\overline{A}_{\ell})$.

The proof of this theorem will take up the remainder of this section. First, we prove (Lemma 4.2) that (9) corresponds to a ‘connected-étale’ short exact sequence

$$0 \to \mathcal{M}_{\ell}^{\text{cl}} \to \mathcal{M}_{\ell} \to \mathcal{M}_{\ell}^{\circ} \to 0$$

of $\tau$-modules over $\overline{A}_{\ell,x}$, where (via Anderson’s equivalence Proposition 4.3) $\mathcal{M}_{\ell}^{\circ}$ is associated to a formal 1-dimensional $\overline{A}_{\ell}$-module. Finally, we prove in Theorem 4.5 the required result on the image of inertia the Tate module of such a formal module.

A locally free $\overline{A}_{\ell,x}$-module $\mathcal{M}_{\ell,x}$ endowed with an injective morphism

$$\tau: \sigma^\ast \mathcal{M}_{\ell,x} \to \mathcal{M}_{\ell,x}$$

is called a $\tau$-module over $\overline{A}_{\ell,x}$. We consider its reduction $\overline{\mathcal{M}}_{\ell,x}$ modulo the maximal ideal of $\overline{A}_{\ell,x}$.

We call a $\tau$-module $\overline{\mathcal{M}}_{\ell,x}$ over $\overline{A}_{\ell,x}$ connected if its reduction $\overline{\mathcal{M}}_{\ell,x}$ is nilpotent, i.e. $\tau^n: (\sigma^\ast)^n \overline{\mathcal{M}}_{\ell,x} \to \overline{\mathcal{M}}_{\ell,x}$ is the zero morphism for some $n > 0$.

**Lemma 4.2.** For every $\tau$-module $\overline{\mathcal{M}}_{\ell}$ over $\overline{A}_{\ell,x}$, there exists an exact sequence

$$0 \to \overline{\mathcal{M}}_{\ell}^{\text{cl}} \to \overline{\mathcal{M}}_{\ell} \to \overline{\mathcal{M}}_{\ell}^{\circ} \to 0$$

(10)

of $\tau$-modules $\overline{\mathcal{M}}_{\ell}$ over $\overline{A}_{\ell,x}$, where $\overline{\mathcal{M}}_{\ell}^{\text{cl}}$ is smooth and $\overline{\mathcal{M}}_{\ell}^{\circ}$ is connected.

**Proof.** The reduction $\overline{\mathcal{M}}_{\ell,x}$, which is a free $\kappa_{\ell} \otimes k_\chi$-module endowed with a $\sigma$-semilinear endomorphism $\tau$, contains a $\tau$-invariant direct summand $\overline{\mathcal{M}}_1$ on which $\tau$ acts injectively such that $\tau$ acts nilpotently on the quotient $\overline{\mathcal{M}}_{\text{nil}} := \overline{\mathcal{M}}_{\ell,x}/\overline{\mathcal{M}}_1$. We
choose a basis
\[ \overline{m}' := (m_1, \ldots, m_{r_1}) \]
for \( \overline{M}_1 \), which we then extend to a basis \( \overline{m} \) for \( \overline{M}_{\ell,x} \). Any lift \( m = (m_i) \) of \( \overline{m} \) yields a \( \hat{A}_{\ell,x} \)-basis of the latter; let \( m' \) denote the ordered subset of \( m \) which reduces to \( (m_1, \ldots, m_{r_1}) \).

Let
\[ \overline{A}_1 \in \text{Mat}_{r_1 \times r_1}(k_x \otimes k_x) \]
(resp. \( \overline{A} \in \text{Mat}_{r \times r}((\hat{A}_{\ell,k_x})) \)) be the matrix representation of \( \tau \) with respect to the basis \( \overline{m}' \) (resp. \( m \)): i.e. \( \tau(\overline{m}') = \overline{m}' \cdot \overline{A}_1 \). Consider any lift \( \overline{A}_1 \) of \( \overline{A}_1 \) with coefficients in \( \hat{A}_{\ell,x} \).

We construct a basis \( n \) for a sub-\( \tau \)-module \( \overline{M}^{\text{et}}_{\ell} \) of rank \( r_1 \) of \( \overline{M}_{\ell,x} \) such that \( \tau \) operates as \( \overline{A}_1 \) with respect to \( n \). Writing \( n \) as \( m' / \overline{m} \cdot Z \), for some matrix \( Z \), we obtain, upon comparing the action of \( \tau \) with respect to \( n \) and \( m' \), the equation
\[ Z \cdot \overline{A}_1 = A \cdot \sigma Z. \tag{11} \]
For the reduction \( Z \cdot \overline{A}_1 = \overline{A} \cdot \sigma Z \) of this equation, a solution is given by the matrix expressing \( \overline{m}' \) in terms of \( \overline{m} \).

As \( \overline{A}_1 \) is invertible and \( \overline{A}_{\ell,x} \) is complete with respect to its maximal ideal, we can determine a solution \( Z \) for (11) by iteration. The basis \( n \) generates a smooth \( \tau \)-invariant direct summand \( \overline{M}^{\text{et}}_{\ell} \) of \( \overline{M}_{\ell} \) of rank \( r_1 \). The quotient \( \overline{M}^{\text{et}}_{\ell} := \overline{M}_{\ell} / \overline{M}^{\text{et}}_{\ell} \) is connected because its reduction is isomorphic to \( \overline{M}_{\text{nil}} \). \( \square \)

Via the anti-equivalence between formal \( \tau \)-modules and Galois modules (cf. Remark 1.6), sequence (10) yields a short exact sequence
\[ 0 \to T_{\ell}^{\text{et}} \to T_{\ell}(M) \to T_{\ell}^{\text{et}} \to 0 \]
of free \( A_{\ell} \)-modules endowed with a continuous action of \( \Gamma_x \). Tensoring with \( F_{\ell} \) yields a sequence as in (9). As \( \overline{M}^{\text{et}}_{\ell} \) is smooth, the Tate module \( T_{\ell}(M) \) is unramified. What remains to be shown is the statement on the image of inertia on \( V_{\ell}^{\text{et}} \).

Let us fix an isomorphism \( \overline{A}_{\ell} \cong k_{\ell}[[\lambda]] \). Extending \( i \) to an embedding \( \overline{A}_{\ell} \to \hat{R}_x \), we put \( \theta := i(\lambda) \). A \( \tau \)-module \( \overline{M}_{\ell,x} \) over \( \hat{A}_{\ell,x} \) has dimension 1 if
\[ \text{coker } \tau \cong \overline{A}_{\ell,x} / (\lambda - \theta). \]

For our given \( \tau \)-module \( M \) over \( A_K \) with characteristic \( i \) and dimension 1 and a good model \( M_x \) over \( R_x \), the completion \( \overline{M}_{\ell,x} := \hat{A}_{\ell,x} \otimes_{A_{\ell}} M_x \), as well as its connected quotient \( \overline{M}^{\text{et}} \) have dimension 1. We will now explain how \( \overline{M}^{\text{et}} \) is related to a formal 1-dimensional \( \hat{A}_{\ell} \)-module.
Let $\widehat{G}_{a, R_x}$ denote the formal additive group over $\widehat{R}_x$. Its endomorphism ring is isomorphic to the skew power series ring $\widehat{R}_x[[\varphi]]$ generated by the endomorphism $\varphi \in \text{End}(R_x)$ and defined by the relation

$$\varphi \cdot f = \varphi f \cdot \varphi.$$

Following Anderson (cf. [An2, Section 3.4]), a formal 1-dimensional $\widehat{A}_\ell$-module $\mathcal{E}$ over $\widehat{R}_x$ of height $s$ is a continuous homomorphism

$$\mathcal{E} : \widehat{A}_\ell \to \widehat{R}_x[[\varphi]]$$

such that

$$\mathcal{E}(\lambda) \equiv \theta \mod \widehat{R}_x \cdot \varphi(\widehat{R}_x[[\varphi]])$$

and

$$\mathcal{E}(\lambda) \equiv \varphi^{s \deg \ell} \mod \pi \cdot \widehat{R}_x[[\varphi]].$$  \hspace{1cm} (12)

Let us denote the order of the residue field $\kappa_\ell$ by $q_\ell = q^{\deg \ell}$. For every $i$, the kernel

$$\mathcal{E}[\ell^i] := \ker \mathcal{E}(\ell^i)$$

is a finite flat scheme of $\widehat{A}_\ell/\ell^i$-modules of order $q_\ell^i$. One defines the Tate module $T_\ell(\mathcal{E})$ associated to $\mathcal{E}$, a free $\widehat{A}_\ell$-module with continuous action of $\Gamma_K$ as

$$T_\ell(\mathcal{E}) = \lim_{\leftarrow} \mathcal{E}[\ell^i](K^\text{sep}).$$

We now quote:

**Proposition 4.3** (Anderson, [An2, 3.4]). There is an antiequivalence of categories $\mathcal{M}_\ell^\circ$ between the categories of formal one-dimensional $\widehat{A}_\ell$-modules of height $s$ and the category of connected one-dimensional $\tau$-modules over $\widehat{A}_{\ell, x}$ of rank $s$. Moreover,

$$T_\ell(\mathcal{E}) \cong T(\mathcal{M}_\ell^\circ(\mathcal{E})).$$

Let $\mathcal{E}_M$ denote the formal one-dimensional $\widehat{A}_\ell$-module of height $s$ associated to $\mathcal{M}_\ell^\circ$ under this equivalence. As $T_\ell(\mathcal{E}_M) \cong T_\ell^\circ$, the proof of Theorem 4.1 will be concluded by Proposition 4.5, dealing with the image of inertia on the Tate module of a formal one-dimensional $\widehat{A}_\ell$-module.

**Remark 4.4.** The formal one-dimensional $\widehat{A}_\ell$-module $\mathcal{E}_\phi$ associated to the $t$-motive of a Drinfeld $A$-module $\phi$ can also be retrieved by a different method. Fixing a place $x$ of $K$ of good reduction, we set $\ell := \ell_x$. Taguchi proved in [Ta1] that the system
\[ \{ \phi[l^n] \}_{n \in \mathbb{N}} \] forms a \( \lambda \)-divisible group. For such \( \lambda \)-divisible groups there exists a connected-étale exact sequence [Ta1, Remark, p. 296] and an equivalence between connected \( \lambda \)-divisible groups and formal \( A \)-modules [Ta1, Proposition 1.4].

Let \( \mathcal{E} \) be a formal one-dimensional \( \hat{A}_f \)-module of height \( s \). Following ideas of Fontaine (cf. [Abr,Fon]) on the division points of formal groups over unramified extensions of \( \mathbb{Z}_p \), we study the image of the inertia group \( I_x \) under the representation \( \rho_p^e : \Gamma_x \to \text{Aut}(T_f(\mathcal{E})) \cong \text{GL}_s(\hat{A}_f) \).

We set \( G^0 := \text{GL}_s(\hat{A}_f) \) and, for every \( i \geq 1 \), we consider the subgroup \( G^i := 1 + \lambda^i \cdot \text{Mat}_{s \times s}(\hat{A}_f) \) of \( \text{GL}_s(\hat{A}_f) \). We further set \( G^{[i]} := G^i/G^{i+1} \), which is isomorphic to \( \text{Mat}_{s \times s}(\kappa_f) \) via the identification \( \rho_p^e : \text{Mat}_{s \times s}(\hat{A}_f) \to G^{[i]} : y \mapsto 1 + \lambda^i \cdot y \).

We set, for \( i \geq 0 \), \( I^i := \rho_p^e(I_x) \cap G^i \) and
\[ I^{[i]} := I^i/I^{i+1} \subset G^{[i]} . \tag{13} \]

We denote by \( \kappa_f^{[i]} \) the extension of \( \kappa_f \) of degree \( s \) inside \( \hat{A}_f^{[i]} \subset \text{Mat}_{s \times s}(\hat{A}_f) \). We consider the action by conjugation of the finite group \( J := (\kappa_f^{[i]})^\times \) on \( \text{Mat}_{s \times s}(\hat{A}_f) \).

For any integer \( i \in \mathbb{Z}/s\mathbb{Z} \), the isotypical component
\[ U(i) := \{ u \in \text{Mat}_{s \times s}(\hat{A}_f) ; \ \forall j \in J : u^j = j^{1-\phi_f} \cdot u \}, \tag{14} \]
is endowed with the structure of an \( \hat{A}_f^{[i]} \)-module (cf. [Abr, Section 1; Fon]) and we obtain the following decomposition:
\[ U := \text{Mat}_{s \times s}(\hat{A}_f) = \bigoplus_{i \in \mathbb{Z}/s\mathbb{Z}} U(i) . \]

Likewise, we have a decomposition
\[ H := \text{Mat}_{s \times s}(\kappa_f) = \bigoplus_{i \in \mathbb{Z}/s\mathbb{Z}} H(i) , \]
where each \( H(i) \) is a one-dimensional \( \kappa_f^{[i]} \)-vector space on which \( J \) operates through the character \( j \mapsto j^{1-\phi_f} \).

**Proposition 4.5.** Suppose that the extension \( \hat{K}_x/\hat{F}_f \) is unramified. Then there exists a function
\[ v : \mathbb{Z}/s\mathbb{Z} \to \mathbb{N} \cup \{ +\infty \} . \]
satisfying \( v(0) = 1 \) and \( v(i + j) \leq v(i) + v(j) \) such that
\[
\rho^\ell(I_x) = (\kappa^s_\ell)^{\times} \cong \left( 1 + \sum_{i \in \mathbb{Z} / s\mathbb{Z}} \lambda^{v(i)} \cdot U(i) \right).
\] (15)

In particular,
\[
(A^s_\ell)^{\times} \subseteq \rho^\ell(I_x).
\]

Let us, before starting the proof of the proposition, briefly recall the definition of a fundamental character. Fixing a solution \( \pi_s \subseteq \overline{\ell} \) of \( X^{(q^s_\ell)} - \lambda X = 0 \), we define a character of the inertia group
\[
\zeta_{\ell,s} : I_x \rightarrow (\kappa^s_\ell)^{\times} \subseteq \kappa^s_\ell : \sigma \mapsto \sigma(\pi_s) / \pi_s
\]
which factors through the tame inertia group. If the extension \( \overline{\kappa}_s / \overline{\ell} \) is unramified, this character, together with its \( \text{Gal}(\kappa^s_\ell / \kappa_\ell) \)-conjugate
\[
\nu(q_{\ell}^{s-1})
\]
for \( i = 1, \ldots, s \) is called a fundamental character of level \( s \) for \( \kappa_\ell \) (cf. [Se3, Section 1]) (this set of fundamental characters is independent of the choice for \( \pi_s \)).

Consider an \( \mathbb{F}_q \)-linear polynomial \( P(Z) \in \overline{\kappa}_x \) of degree \( q^s_\ell \) satisfying
\[
P(Z) = Z \cdot E(Z),
\]
where \( E(Z) \) is an Eisenstein polynomial. Following Serre (cf. [Se3, Section 1]), the \( \mathbb{F}_q \)-vector space \( W \) of roots of \( P \) is endowed with the structure of a one-dimensional \( \kappa^s_\ell \)-vector space such that the action of \( I_x \) on \( W \) is given by a fundamental character.

**Proof of Proposition 4.5.** The inertia group \( I_x \) is isomorphic to the semidirect product \( I^t_x \rtimes I^w_x \), where \( I^t_x \) is the quotient group of tame inertia and \( I^w_x \) is the subgroup of wild inertia (higher ramification subgroup). We fix a section \( I^t_x \rightarrow I_x \).

**Tame inertia:** By assumption, \( \theta := \nu(\lambda) \) is a uniformizer of \( \overline{\kappa}_x \). By the theory of Newton polygons, the \( \lambda \)-torsion points \( z_1 \in \mathcal{E}[\lambda] \) are roots of a polynomial \( P(Z) = Z \cdot E(Z) \) of degree \( q^s_\ell \), where \( E(Z) \) is an Eisenstein polynomial. Hence the action of \( I_x \) factors through tame inertia, operating via a fundamental character of level \( s \). The image \( I^0_x \) of
\[
I^t_x \rightarrow \text{GL}_s(\kappa_\ell) \subseteq H
\]
can be identified, as a group, with the multiplicative group \( J \) of \( \kappa^s_\ell \).
Moreover, the restriction of the representation $\mathcal{F}_\ell \otimes A' \rho_\ell^\varepsilon$ to $I' \subset I_x \subset \Gamma_x$ on

$$\text{Aut}_{\mathcal{F}_\ell}(\mathcal{F}_\ell \otimes T_\ell(\varepsilon))$$

is then isomorphic to the direct sum:

$$\bigoplus_{i=1}^s \nu_\varepsilon(q_i^{-1}) \cdot I'_x \to \bigoplus_{i=1}^s \mathcal{K}_i^\varepsilon \subset \text{GL}_s(\mathcal{F}_\ell).$$

(16)

**Wild inertia:** Let $L_0 := \mathcal{K}_x^{ur}$ denote the maximal unramified extension of $\mathcal{K}_x$, and, for every $i \geq 1$, set

$$(L_i := L_0(\varepsilon[\lambda^i]) \subset \mathcal{K}_x^{sep})$$

and $L_\infty := \bigcup_{i \geq 1} L_i$. Without any risk of confusion, we will denote the tame inertia subgroup of $\text{Gal}(L_\infty/\mathcal{K}_x^{ur})$ by $J$.

The group $\text{Gal}(L_i/L_{i-1})$ is isomorphic to the sub-$\mathbb{Z}[J]$-module $I[i]$ of $G[i]$. Note that via the identification $G[i] \cong H$, for $i \geq 1$, we have a decomposition

$$G[i] = \bigoplus_{j \in \mathbb{Z}/s\mathbb{Z}} H(j).$$

For $j \neq 0$, the $\mathbb{Z}[J]$-modules $H(j)$ are simple.

For each $i \in \mathbb{N}$, define $\eta(i)$ as the subset of $\mathbb{Z}/s\mathbb{Z}$ such that

$$I[i] \cong \bigoplus_{j \in \eta(i)} H(j).$$

(17)

What we need to show is the following:

(i) for every $i \geq 1$, $0 \in \eta(i)$ and

(ii) for every $i, i' \geq 1$, if $j \in \eta(i)$ and $j' \in \eta(i')$ then $j + j' \in \eta(i + i')$.

Indeed, (i) and (ii) imply that if $j \in \eta(i)$, then

$$1 + \lambda^i U(j) \subset \rho_\ell^\varepsilon(I_x).$$

If we then put

$$v(j) := \inf \{i \in \mathbb{N} ; j \in \eta(i) \} \in \mathbb{N} \cup \{ \infty \},$$

then $v$ clearly is the required function.

(i) For every $i \geq 1$, we recursively fix an element $z_i \in \varepsilon[\lambda^i]$ such that $z_1 \neq 0$ and

$$\varepsilon(\lambda^i)(z_{i+1}) = z_i.$$
and we put
\[ \hat{L}_i := L_0(z_i) \subseteq L_i. \]

The extensions \( \hat{L}_i/\hat{L}_{i-1} \) are Galois: any conjugate \( \sigma(z_i) \) of \( z_i \) is contained in \( z_i + \hat{\mathfrak{e}}[\hat{\lambda}] \), and \( \hat{\mathfrak{e}}[\hat{\lambda}] \subseteq \hat{L}_1 = L_1 \).

Since the group \( \text{Gal}(L_\infty/\hat{L}_i) \) is a subgroup of the higher ramification subgroup of \( \text{Gal}(L_\infty/L_0) \), we have \( J \cap \text{Gal}(L_\infty/\hat{L}_i) = \emptyset \). Let us consider the fixed field \( \hat{L}^0_i \) of the semidirect subgroup \( J \triangleright \text{Gal}(L_\infty/\hat{L}_i) \) of \( \text{Gal}(L_\infty/L_0) \).

Note that the extension \( \hat{L}_i/\hat{L}^0_{i-1} \) is a Galois extension, with Galois group isomorphic to \( J \triangleright \text{Gal}(\hat{L}_i/\hat{L}_{i-1}) \).

The extension \( \hat{L}_i/\hat{L}_0 \) has degree
\[ (q^s - 1) \cdot q^{s(i-1)}, \]
as one sees from the slopes of the Newton polygon. For any \( \sigma \in J \), we have, with respect to the normalized valuation \( v \) of \( \hat{L}_i \), that
\[ v(\sigma(z_i) - z_i) = q^{s(i-1)}. \]

As \( z_{i+1} \) is a uniformizer of \( \hat{L}_i \), the subgroup
\[ \text{Gal}(L_i/L_{i-1}) \subseteq \text{Gal}(L_i/\hat{L}^0_{i-1}) \]
is, by Serre [Se2, IV, Section 1], contained in the \(\mu(i)\)th higher ramification group \(G_{\mu(i)}\) of \(\text{Gal}(L_i/K)\), where we put
\[
\mu(i) := q_i^{e(i-1)} - 1.
\]

Therefore \(J\) acts trivially by conjugation on \(\text{Gal}(\tilde{L}_i/\tilde{L}_{i-1})\), by Serre [Se2, IV, Section 2, Proposition 9, p. 77]).

Let us prove by induction that, for every \(i \geq 1\),
(a) the extensions \(L_i/\tilde{L}_i\) and \(\tilde{L}_{i+1}/\tilde{L}_i\) are linearly disjoint and
(b) \[
\text{Gal}(L_{i+1}/L_i\tilde{L}_{i+1}) \cong \bigoplus_{j \in \eta(i) \setminus \{0\}} H(j). \tag{18}
\]

For \(i = 1\), both statements are trivial. Suppose now that, for a given \(i > 1\), the claim holds for all \(i' < i\). From (18), we see that no element of \(\text{Gal}(L_i/\tilde{L}_i)\) is invariant under the action of \(J\). As \(\text{Gal}(\tilde{L}_{i+1}/\tilde{L}_i)\) is invariant under this action, part (a) follows.

Hence
\[
\text{Gal}(L_i\tilde{L}_{i+1}/L_i) \cong H(0) \subset I^{[0]},
\]

which by (17) proves part (b). This concludes the induction.

In conclusion, the quotient \(\text{Gal}(L_i\tilde{L}_{i+1}/L_i)\) of \(I^{[i]}\), invariant under conjugation by \(J\), has order \(q_i^e\). As a \(\mathbb{Z}[J]\)-module, it must therefore coincide with \(H(0) \subset I^{[0]}\), i.e. \(0 \in \eta(i)\).

(ii) Note that, for the Lie bracket \([\cdot, \cdot]\), we have
\[
[H(j), H(j')] = H(j + j'),
\]

if \((j, j) \neq (0, 0)\) (cf. [Fon, Section 7]). By the commutative diagram
\[
\begin{array}{ccc}
G^{[i]} \times G^{[i']} & \rightarrow & G^{[i+i']} \\
\uparrow & & \uparrow \\
H \times H & \rightarrow & H
\end{array}
\]

this implies that if \(j \in \eta(i)\) and \(j' \in \eta(i')\), then
\[
j + j' \in \eta(i + i')
\]

for \((j, j) \neq (0, 0)\), and so, by (ii), if \(j \in \eta(i)\), then \(j \in \eta(i')\) for every \(i' > i\). \(\square\)
5. Zariski density of $\Gamma_\ell$

Let $K$ be a finite field extension of $F$ and $M$ a $\tau$-module over $A_K$ of rank $r$, with characteristic $i$ and dimension 1. In this section, we will prove the following theorem:

**Theorem 5.1.** Suppose that $\text{End}_K(M) = A$. Then for every finite place $\ell$ of $F$, the group $\Gamma_\ell$ is Zariski dense in $\text{GL}_{r, \overline{F}_\ell}$.

For a finite place $\ell$ of $F$, let $G_\ell$ denote the Zariski closure of $\Gamma_\ell$ inside $\text{GL}_{r, \overline{F}_\ell}$. The following criterion for the connectedness of $G_\ell$ was proved by Serre, independently of the characteristic of the field $K$:

**Theorem 5.2** (Serre [Se4, App.]). The group $G_\ell$ is connected if and only if, for every function $f \in \mathbb{Z}[c_1, \ldots, c_\ell]$, the set of places $x$ of $K$ with $f(\rho_\ell(\text{Frob}_x)) = 0$ has density 0 or 1.

As $f(\rho_\ell(\text{Frob}_x)) \in A$ is independent of $\ell$ (for all but a finite number of places of $K$), this shows that $G_\ell$ is connected for one place $\ell$, if and only if it is connected for all $\ell$. Without loss of generality, we can replace $K$ by a finite separable extension so that $\Gamma_\ell \subset G_\ell^\circ$. Hence, we may assume that $G_\ell$ is connected for all $\ell$.

Without loss of generality, we may replace $K$ by a finite separable extension, so that we can suppose that $\text{End}_K(M) = A$. By the Tate and semisimplicity conjecture, we know that $\Gamma_K$ acts absolutely irreducibly on $V_\ell(M)$, for every finite place $\ell$ of $F$. The tautological representation of $G_\ell$ is absolutely irreducible as well, and it follows from this that $G_\ell$ is a reductive connected group (cf. [Pi2, Fact A.1]).

We want to apply the following result:

**Proposition 5.3** (Pink [Pi2, Proposition A.3]). Let $L$ be an algebraically closed field and $G \subset \text{GL}_{r,L}$ a reductive connected linear algebraic group acting irreducibly on an $r$-dimensional $L$-vector space $V$. Suppose that $G$ possesses a cocharacter which has weight 1 with multiplicity 1 and weight 0 with multiplicity $r - 1$ on $V$. Then $G = \text{GL}_{r,L}$.

Let $\overline{F}_\ell$ denote a fixed algebraic closure of $\overline{F}_\ell$. The rest of this section is devoted to finding a cocharacter for $\overline{F}_\ell \times_{\overline{F}_\ell} G_\ell$ in its representation on $\overline{F}_\ell \otimes V_\ell(M)$ as in Proposition 5.3.

We recall that, in his proof of Theorem 1.9 for $t$-motives of Drinfeld modules, Pink first proves that the set of finite places $x$ of $K$ at which $\phi$ has ordinary reduction has positive density, by an argument involving the purity of the eigenvalues of the Frobenius elements. The Frobenius torus at such a place then yields the desired cocharacter.
In our argument below, we will first prove that the maximal tori for $G_{c}$ look the same for all $c$ and then use the image of tame inertia inside these tori to prove the existence of such a cocharacter.

We define the functions $c_{1}, c_{2}, \ldots, c_{r} : GL_{r} \rightarrow G_{a}$ as the coefficients of the characteristic polynomial of elements in $GL_{r}$, i.e. for every $g \in GL_{r}$, the characteristic polynomial $P_{g}(X)$ of $g$ is equal to

$$X^{r} + c_{1}(g)X^{r-1} + \cdots + c_{r}(g).$$

We also consider the morphism

$$c = (c_{1}, \ldots, c_{r}) : GL_{r} \rightarrow G_{a}^{r-1} \times G_{m}.$$

For every diagonalisable group $T$ in $GL_{r}$, the image $c(T)$ is a closed subvariety of $G_{a}^{r-1} \times G_{m}$ (cf. [Se5, Section 5]).

For every finite place $\ell$ of $F$, we fix a maximal torus $T_{\ell}$ of $G_{\ell}$ and an isomorphism

$$\mathcal{F}_{\ell} \cong \mathcal{F}_{\ell} \otimes V_{\ell}(M)$$

such that

$$\mathcal{F}_{\ell} \otimes F_{\ell} T_{\ell} \subset G_{m,\mathcal{F}_{\ell}} \subset GL_{r,\mathcal{F}_{\ell}}.$$

Also, we choose a torus $\Theta_{\ell} \subset G_{m,\mathcal{F}_{\ell}} \subset GL_{r,\mathcal{F}_{\ell}}$ such that

$$\mathcal{F}_{\ell} \otimes F_{\ell} \Theta_{\ell} \cong \mathcal{F}_{\ell} \otimes F_{\ell} T_{\ell}$$

and such that the representations of these tori on $\mathcal{F}_{\ell} \otimes F_{\ell} = \mathcal{F}_{\ell}^{' \ell}$ coincide.

Note the action of the symmetric group

$$S_{r} \cong \text{Norm}_{GL_{r,\mathcal{F}}}(G_{m,\mathcal{F}})/\text{Cent}_{GL_{r,\mathcal{F}}}(G_{m,\mathcal{F}}),$$

sending the torus $\Theta_{\ell}$ to all of its conjugates inside $GL_{r,\mathcal{F}}$. For every $\sigma \in S_{r}$, we denote the resulting copy of $\Theta_{\ell}$ by $\Theta_{\ell,\sigma}$.

**Proposition 5.4.** The tori $\Theta_{\ell} \subset GL_{r,\mathcal{F}}$ are conjugate for all $\ell$.

**Proof.** This proof was inspired by the arguments of [L-P, Section 8].

Let $\ell$ and $\ell'$ be finite places of $F$. For any place $x \in X^{\text{good}}$ not lying above $\ell$ or $\ell'$, we fix an element $T_{x}$ of the conjugacy class of Frobenius tori for $x$ such that $T_{x} \subset \Theta_{\ell} \subset GL_{r,\mathcal{F}}$. Let $\sigma_{x}$ be the element in $S_{r}$ such that $T_{x} \subset \Theta_{\ell',\sigma_{x}} \subset GL_{r,\mathcal{F}}$.

Let us now assume that

$$\Theta_{\ell} \not\subset \Theta_{\ell',\sigma} \subset G_{m,\mathcal{F}}^{' \ell}.$$
for all $\sigma \in S_r$. Then the image under $c$ of the union of subgroups

$$U := \bigcup_{\sigma \in S_r} (\Theta_\ell \cap \Theta_{\ell',\sigma})$$

is a proper closed subvariety of $c(\Theta_\ell)$.

On the other hand, the set

$$U' = \{c(T_x) : x \in X^{\text{good}} \text{ not above } \ell \text{ or } \ell'\}$$

is a Zariski dense subset of $c(\Theta_\ell)(F)$, the set images $\rho_{\ell}(\text{Frob}_x)$ of Frobenius substitutions being dense in $\Gamma_\ell$ by the Chebotarev density theorem. Therefore, there exists a place $x \in X^{\text{good}}$ such that

$$c(T_x) \notin c(U)(F).$$

This implies that $T_x \not\subset U$, and, in particular, that $T_x \not\subset \Theta_{\ell',\sigma}$, which gives a contradiction.

In conclusion, for any places $\ell$ and $\ell'$ of $F$, we have $\Theta_\ell \subset \Theta_{\ell',\sigma}$, for at least one $\sigma \in S_r$. By symmetry, this shows that the maximal tori $\Theta_\ell$ and $\Theta_{\ell'}$ are conjugate. \qed

Let us fix a subtorus $T \subset \mathbb{G}_{m,F}^{r}$ of $\text{GL}_{r,F}$ which is conjugate to $\Theta_\ell$ for every $\ell$.

**Proposition 5.5.** The torus $T$ possesses, in its representation on $\overline{F}^{r}$, a cocharacter of weight 1 with multiplicity 1 and weight 0 with multiplicity $r - 1$.

**Proof.** Let us fix, for every finite place $\ell$ of $F$, a place $x_\ell$ of $K$ lying above $\ell$. We denote by $\mathcal{A}$ the set of finite places $\ell$ of $F$ which do not ramify in $\overline{K}_{x_\ell}/\overline{F}_{\ell}$ and such that $M$ has good reduction at $x_\ell$. By Theorem 4.1, the image

$$\Delta_\ell := \rho_{\ell}(I_{x_\ell}) \subset \text{GL}_{r}(\overline{F}_{\ell})$$

of the tame inertia group $T_{x_\ell}$ on $V_{\ell}(M)$ is isomorphic to

$$(\kappa_{\ell}^{[s_\ell]} \times \{1\} \subset \text{GL}_{s_{\ell}}(k_{\ell}) \times \{1\}^{r-s_{\ell}} \hookrightarrow G_{\ell}(\overline{F}_{\ell}),$$

where $s_{\ell}$ denotes height of $\mathcal{M}_{\ell,x_{\ell}}$. By (16), the action of inertia on $T_{\ell}$ is given by the direct sum of the $q_{\ell}^{i-1}$th powers of a fundamental character $\zeta_{\ell,s}$ of level $s$, for $i = 1, \ldots, s$.

There exists an $s \leq r$ and an infinite subset $A_1$ of $\mathcal{A}$ such that, for all $\ell \in A_1$, $s_{\ell} = s$.

As $G_{\ell}$ is connected, the cyclic group $\Delta_\ell$ is contained in a maximal torus $T_{\ell}$ of $G_{\ell}$. Under the fixed isomorphism of the character groups $X(T) \cong X(T_{\ell})$, we can now consider the action of the standard basis characters $\chi_{i}$ of $\mathbb{G}_{m,F}^{r}$ on $\Delta_\ell$. For every $\ell \in A_1$, there exists a renumbering $\sigma_{\ell}$ of the standard basis characters $\chi_{i}$ such that

- $\chi_{i}(x) = x_{2^{i-1}}$ for all $x \in (\kappa_{\ell}^{[s_i]} \times \{1\}) \cong \zeta_{\ell,s}(I_{x_{\ell}})$ and $i = 1, \ldots, s$ and
\( \chi_i = 1 \) for all \( i = s + 1, \ldots, r \).

There exists an infinite subset \( A_2 \) of \( A_1 \), such that the above holds with the same renumbering \( \sigma_\ell \) of the characters \( \chi_i \) for all \( \ell \in A_2 \). Let us put

\[
\tilde{T} := T \cap \left( \mathbb{G}_m^s \times \{ e \}^{r-s} \right).
\]

Suppose that \( \tilde{T} \) satisfies a relation

\[
\chi := \sum_{i=1}^{s} a_i \cdot \chi_i,
\]

with \( a_i \in \mathbb{Z} \). Considering the action of \( \chi \) on the elements of \( (\mathbb{k}_{\ell})^{\times} \), we obtain, for all \( \ell \) in \( A_2 \) that

\[
\sum_{i=1}^{s} a_i \cdot q_{\ell}^{-1} \equiv 0 \mod q_{\ell} - 1.
\]

As \( A_2 \) is an infinite set, the \( q_{\ell} \) are unbounded, which yields \( \chi = 0 \). Hence \( \tilde{T} = \mathbb{G}_m^s \times \{ e \}^{r-s} \). In particular, we see that \( T \supseteq \tilde{T} \) possesses a cocharacter of weight 1 with multiplicity 1 and weight 0 with multiplicity \( r - 1 \). \( \square \)

As the representations

\[
\mathcal{F}_\ell \times \mathcal{F}_\ell \Theta_\ell \simeq \mathcal{F}_\ell \times \mathcal{F}_\ell T_\ell
\]

conide on \( \mathcal{F}_\ell \), this theorem now implies that the maximal torus \( \mathcal{F}_\ell \times \mathcal{F}_\ell T_\ell \) of \( \mathcal{F}_\ell \times \mathcal{F}_\ell \) \( G_\ell \) has a cocharacter of weight 1 with multiplicity 1 and weight 0 with multiplicity \( r - 1 \) in its representation on \( \mathcal{F}_\ell \). By Proposition 5.3, it follows that

\[
\mathcal{F}_\ell \times \mathcal{F}_\ell G_\ell = \text{GL}_{r, \mathcal{F}_\ell},
\]

which concludes the proof of Theorem 5.1.

6. Openness of \( \Gamma_A \)

Let \( K \) be a finite field extension of \( F \) and \( M \) a \( \tau \)-module over \( A_K \) of rank \( r \), with characteristic \( i \) and dimension 1. In this section we conclude the proof of the openness of the \( \tau \)-adic representations, Theorem 1.9.

Let \( A \) be a finite set of finite places of \( F \). The maximal exterior power \( \wedge^r M \) is \( \tau \)-module over \( A_K \) of rank 1 and dimension 1, and hence Theorem 1.9 is known for \( \wedge^r M \), by Remark 1.16. As \( V_A(\wedge^r M) \simeq \wedge^r V_A(M) \), it follows from this that the image
of the composite morphism

\[
\Gamma_K \to \text{Aut}_{F_\ell} (V_A(M)) \to \text{Aut}_{F_\ell} (\wedge^r V_A(M))
\]

\[
\cong \to \text{GL}_r(\widehat{F_A}) \quad \stackrel{\text{det}}{\to} \quad \widehat{F_A}^\times
\]

is open. What remains to be shown is that \(\Gamma_A \cap \text{SL}_r(\widehat{F_A})\) is open in \(\text{SL}_r(\widehat{F_A})\).

Let \(\ell\) be a finite place of \(F\). We consider the absolutely simple connected adjoint group \(\text{PGL}_r, F_A\). Without loss of generality, we may replace \(K\) by a finite separable extension, so that we can suppose that \(\text{End}_K(M) = \text{End}_K(M)\). By Theorem 5.1, the image of \(\Gamma_K\) under the composite map

\[
\Gamma_K \overset{\rho_\ell}{\rightarrow} \text{GL}_r(\widehat{F_\ell}) \overset{\Pi}{\rightarrow} \text{PGL}_r(\widehat{F_\ell})
\]

is Zariski dense for every finite place \(\ell\). We use Pink’s results [Pi3] on compact Zariski dense (abstract) subgroups of algebraic groups, which we can formulate as follows:

**Theorem 6.1 (Pink [Pi3, Main Theorem 0.2, Proposition 0.4(a)]).** For every finite set \(\Lambda\) of finite places, there exist

1. a subring \(\widehat{E}_\Lambda \subset \widehat{F}_A\),
2. an algebraic group \(H\) over \(\widehat{E}_A\) such that

\[
\widehat{F}_A \times_{\widehat{E}_A} H \cong \text{PGL}_r(\widehat{F}_A),
\]

3. an inner automorphism \(\varphi\) of \(\text{PGL}_r, \widehat{F}_A\)

such that

- the image of \(\Gamma_A\) in \(\text{PGL}_r(\widehat{F}_A)\) is contained in \(\varphi(H(\widehat{E}_A))\) and
- the commutator subgroup \(\Gamma_A'\) of \(\Gamma_A\) is open in \(\blackhat{H}(\widehat{E}_A)\), where \(\blackhat{H}\) is the maximal covering of \(H\), a model of \(\text{SL}_r, \widehat{F}_A\) over \(\widehat{E}_A\).

An algebraic function \(f: \text{PGL}_r \rightarrow \mathbb{G}_a\) is called central if \(f \circ \varphi = f\) for any inner automorphism \(\varphi\) of \(\text{PGL}_r\). Denoting as before the functions that give the coefficients of the characteristic polynomial of a matrix \(g \in \text{GL}_r\) by \(c_1, c_2, \ldots, c_r\), the functions

\[
f_1 := c_1' \cdot c_r^{-1} \quad \text{and} \quad f_2 := c_1 \cdot c_{r-1} \cdot c_r^{-1}
\]

are examples of such central functions on \(\text{PGL}_r\).
Let \( \mathcal{A} \) be a finite set of finite places of \( F \) and let \( \hat{B}_A \) denote the total ring of quotients of the closure of the subring of \( \hat{F}_A \) generated by 1 and the elements \( f(g) \), where \( f \) is a central function on \( \text{PGL}_r \) and \( g \in \Gamma_A \). Clearly, \( \hat{B}_A \subset \hat{E}_A \).

**Proposition 6.2** (Pink [Pi3, Proposition 0.4(a + c)]). For a finite place \( \ell \) of \( F \), the field \( \hat{B}_\ell \) either equals \( \hat{E}_\ell \) or \( \hat{E}_\ell^p \).

**Lemma 6.3.** For a place \( x \in X^{\text{good}} \), we put \( \ell := \ell_x \). If the field extension \( \hat{K}_x/\hat{F}_\ell \) is unramified, then

(i) we have \( \hat{B}_\ell = \hat{E}_\ell \subset \hat{F}_\ell \) and

(ii) for every finite place \( \ell' \neq \ell \) of \( F \),

\[
\hat{B}_{\ell',\ell} \cap (\hat{F}_\ell \times \{0\}) \neq \{0\}.
\]

**Proof.** Let us fix an isomorphism \( \hat{F}_\ell \cong \kappa_\ell((\mathcal{A})) \). Our proof uses in an essential way the information on the image of inertia from Theorem 4.1.

(i) We will prove that there exists an element \( g \in \Gamma_\ell \) such that, for some central function \( f \) on \( \text{PGL}_r \), we have \( f(g) \notin \hat{F}_\ell^p \). Then \( \hat{B}_\ell \subset \hat{F}_\ell^p \), which implies that \( \hat{B}_\ell = \hat{E}_\ell \), by Proposition 6.2.

First, suppose that \( s > 1 \). Choose a polynomial

\[
P(Z) = Z^s + \sum_{i=0}^{s-1} h_i Z^i \in \kappa_\ell[Z]
\]

which defines the field extension \( \kappa_\ell^{[s]} \) (cf. the notations of Section 4). For the embedding

\[
(\hat{A}_\ell^{[s]})^\times \hookrightarrow \text{GL}_s(\hat{A}_\ell),
\]

we can find an element \( g' \in \hat{A}_\ell^{[s]} \) whose characteristic polynomial equals

\[
P_g(Z) := Z^s + (h_{s-1} + \lambda^p)Z^{s-1} + \sum_{i=1}^{s-2} h_i Z^i + (h_0 + \lambda) \in \hat{F}_\ell[Z],
\]

by Hensel’s lemma.

By Theorem 4.1, there exists an element \( \gamma \in I_\ell \), such that \( g = \rho_\ell(\gamma) \) acts as \( g' \) on the submodule \( T_\ell \) of \( T_{\ell}(M) \) and trivially on the quotient. Thus the characteristic polynomial of \( g \) on \( V_\ell(M) \) equals

\[
P_g(Z) \cdot (Z - 1)^{r-s}.
\]
In particular,
\[ c_1(g) = -\text{Tr}(g) = h^p + h_{-1} - (r - s) \neq 0 \]
and
\[ c_r(g) = (-1)^r \text{det}(g) = (-1)^{r-s}(h_0 + \lambda) \neq 0. \]

But then \( c_1(g) \in (\widehat{\mathbb{F}}_p^\times) \) and \( c_r(g) \notin \widehat{\mathbb{F}}_p^\times \), which shows that \( f_1(g) \notin \widehat{\mathbb{F}}_p^\times \).

If \( s = 1 \), we need a different trick. There exists, by Theorem 4.1, an element \( \gamma \in I_x \), such that \( g = \rho_{\gamma}(\gamma) \) acts as \( h := 1 + \lambda \) on the submodule \( T_r(\mathcal{E}) \) of \( T_r(M) \) and trivially on the quotient. We fix a basis such that \( g \) is in diagonal form:
\[ g = \text{diag}(h, 1, \ldots, 1). \]

As \( \Gamma_K \) is Zariski dense in \( \text{GL}_{\widehat{\mathbb{F}}_p} \), there exists an element \( \alpha = (a_{ij}) \in \Gamma_\epsilon \subset \text{GL}_{\widehat{\mathbb{F}}_p} \), with inverse \( \beta = (\beta_{ij}) \), such that
\[ a_1 := a_{11} \neq 0, \]
\[ a_2 := \sum_{j=2}^r a_{jj} \neq 0, \]
\[ b_1 := \beta_{11} \neq 0, \]
\[ b_2 := \sum_{j=2}^r \beta_{jj} \neq 0. \]

Then
\[ -c_1(\alpha \cdot g^n) = \text{Tr}(\alpha \cdot g^n) = a_1 \cdot h^n + a_2, \]
for all \( n \geq 0 \). Similarly, \( c_r(\alpha \cdot g^n) = -\text{Tr}((\alpha \cdot g^n)^{-1}) = b_1 \cdot h^{-n} + b_2 \). Thus we obtain that
\[ f_2(\alpha \cdot g^n) = w_1 \cdot h^n + w_2 \cdot h^{-n} + w_3 \]
for elements \( w_i \in \widehat{\mathbb{F}}_p \), with \( w_1, w_2 \) non-zero.

We suppose that \( \phi_n := f_2(\alpha \cdot g^n) \in \widehat{\mathbb{F}}_p^\times \), for all \( n \geq 0 \), and aim to find some contradiction. The elements \( w_i \in \widehat{\mathbb{F}}_p \) are in \( \widehat{\mathbb{F}}_p^\times \) as they are the unique solutions of the following system of linear equations with coefficients in \( \widehat{\mathbb{F}}_p^\times \):
\[
\begin{align*}
\left\{ w_1 + w_2 + w_3 &= \phi_0, \\
w_1 \cdot h^p + w_2 \cdot h^{-p} + w_3 &= \phi_p, \\
w_1 \cdot h^{2p} + w_2 \cdot h^{-2p} + w_3 &= \phi_{2p}.
\end{align*}
\]
If $p = 2$, then we choose odd $m$ such that $0 \neq w_1 \cdot h^{2m} + w_2$. As
\[
\varphi_m = (w_1 \cdot h^{2m} + w_2) \cdot h^{-m} + w_3,
\]
where $\varphi_m$, $w_3$ and $w_1 \cdot h^{2m} + w_2$ are elements of $\hat{F}_\ell^2$, we obtain that $h^m \in \hat{F}_\ell^2$, a contradiction.

If $p > 2$, let us put $\omega := w_1 \cdot w_2^{-1} \in \hat{F}_\ell^p$. Expanding $\omega$ as $\sum_{i=0}^{\infty} \alpha_i \lambda^p$, we obtain
\[
w_2^{-1} \cdot \varphi_1 - w_3 = \left( \sum \alpha_i \lambda^p \right) \cdot (1 + \lambda) + (1 - \lambda + \lambda^2 + \ldots).
\]
(22)
In this equation, the left-hand side is an element of $\hat{F}_\ell^p$, whereas the right-hand side clearly is not (cf. the coefficient of $\lambda^2$), which gives a contradiction.

(ii) As $x \in X^{\text{good}}$, the action of $I_x$ on $V_{\ell'}(M)$ is trivial for $\ell' \neq \ell$. For $s > 1$, we immediately see that, with $\gamma \in I_x$ as above,
\[
0 \neq f_1(\rho_A(\gamma)) - f_1(\rho_A(1)) \in \hat{B}_{(\ell', s)} \cap \hat{F}_\ell \times \{0\}.
\]
For $s = 1$, using $\gamma \in I_x$ and $\alpha \in \Gamma_A$ as above, we note that the function
\[
n \mapsto f_2(\alpha \cdot \rho_A(\gamma^m)) - f_2(\alpha \cdot \rho_A(1))
\]
is a non-constant function with values in $\hat{B}_{(\ell', s)} \cap (\hat{F}_\ell \times \{0\})$. □

For every $x \in X^{\text{good}}$ not lying above $\ell$, the eigenvalues of $\rho_{\ell}(\text{Frob}_x)$ lie in $\bar{F}$ and are independent of $\ell \neq \ell_x$, by Proposition 3.1. Consider the subfield $B$ of $F$ generated by 1 and by the set of elements $f(\rho_{\ell}(\text{Frob}_x))$, where $f$ is a central function on $\text{PGL}_2$ and $x \in X^{\text{good}}$. The set of Frobenius substitutions being dense in $\Gamma_K$, the closure of $B$ in $\hat{F}_A$ coincides with $\hat{B}_A$.

**Proposition 6.4.** $B = F$.

**Proof.** By Lemma 6.3(i), $B$ is not a finite field, as $\hat{B}_{(\ell)} \subset \mathbb{F}_q \subset \hat{F}_\ell^p$, for any $\ell$. As $F$ has transcendence degree 1 over $\mathbb{F}_q$, this implies that $F/B$ is a finite extension. Furthermore Lemma 6.3(i) implies that $F/B$ is separable.

Consider the set $\Lambda_B$ of finite places $b$ of $B$ which do not ramify in $K/B$ and such that all places $x$ of $K$ above $b$ are in $X^{\text{good}}$ (this excludes only a finite number of places of $B$). For $b \in \Lambda_B$, let $\hat{B}_b$ denote the completion of $B$ at $b$, and, for a place $x$ above $b$, let $\ell$ denote the place of $F$ below $x$.

If there exists another place $\ell'$ of $F$ above $b$ and $\Lambda = \{\ell, \ell'\}$, then the closure of $B$ in $\hat{F}_{(\ell', \ell)}$ equals $\hat{B}_b$, diagonally embedded; hence
\[
\{0\} = \hat{B}_{(\ell', \ell)} \cap (\hat{F}_\ell \times \{0\}) \subset \hat{F}_{(\ell', \ell)};
\]
which yields a contradiction with Lemma 6.3(ii). In conclusion, only one place of \( F \) can lie above each \( b \in A_B \), which shows that \( B = F \). □

As \( F \) is dense in \( \hat{F}_A \), this implies that \( \hat{B}_A = \hat{F}_A \) and therefore \( \hat{E}_A = \hat{F}_A \). By Theorem 6.1, it follows that \( \Gamma'_A \) is open in \( SL_r(\hat{F}_A) \), which concludes the proof of Theorem 1.9. □

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References

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