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A light-front coupled-cluster method for the nonperturbative solution of quantum field theories

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ABSTRACT

We propose a new method for the nonperturbative solution of quantum field theories and illustrate its use in the context of a light-front analog to the Greenberg–Schweber model. The method is based on light-front quantization and uses the exponential-operator technique of the many-body coupled-cluster method. The formulation produces an effective Hamiltonian eigenvalue problem in the valence Fock sector of the system of interest, combined with nonlinear integral equations to be solved for the functions that define the effective Hamiltonian. The method avoids the Fock-space truncations usually used in nonperturbative light-front Hamiltonian methods and, therefore, does not suffer from the spectator dependence, Fock-sector dependence, and uncanceled divergences caused by such truncations.

1. Introduction

The central problem of a quantum field theory is to compute its mass spectrum and the corresponding eigenstates. All physical quantities can be computed from these. If the theory is quantized in terms of light-front coordinates [1], this spectral problem can be written as a Hamiltonian eigenvalue problem [2], $\mathcal{P}^\mu|\psi(\underline{P})\rangle = P^\mu|\psi(\underline{P})\rangle$, where $\mathcal{P}^- \equiv \mathcal{P}^0 - \mathcal{P}^z$ is the light-front energy operator, $\underline{\mathcal{P}} \equiv (\mathcal{P}^+ = \mathcal{P}^0 + \mathcal{P}^z, \vec{\mathcal{P}}_\perp = (\mathcal{P}^x, \mathcal{P}^y))$ is the light-front momentum operator, and P^μ are the corresponding eigenvalues. For an eigenstate of mass M, the mass-shell condition $P^2 = M^2$ yields $P^- = (M^2 + P_\perp^2)/P^+$. Thus, eigenvalues of $\mathcal{P}^-$ determine the mass spectrum.

The standard light-front Hamiltonian approach is to expand $|\psi(\underline{P})\rangle$ in a set of Fock states, eigenstates of $\underline{\mathcal{P}}$ with definite numbers of constituents. The coefficients in the expansion are the light-front momentum-space wave functions. This takes advantage of two important aspects of light-front coordinates [2]: the relative-momentum coordinates separate from the external momentum, so that the wave functions depend only on the relative momenta, and the positivity of $P^+ = \sqrt{(\vec{P})^2 + M^2} + P^z$ excludes vacuum contributions to the expansion, so that the wave functions represent the properties of the eigenstate only.

Given the Fock-state expansion, the eigenvalue problem becomes an infinite set of coupled integral equations for the wave functions. The expansion and the coupled system are truncated to yield a finite problem, which is then solved, usually by numerical techniques [3].

In more than two dimensions, some form of regularization is required to properly define the integrals of the coupled system. The cancellations that must take place in the regularization scheme are disrupted by the truncation, resulting in uncanceled divergences. A reparameterization of the theory, such as sector-dependent parameterization [4–8], can be arranged to appear to absorb these divergences, but not simultaneously for all physical quantities [8]. The truncation also causes self-energy contributions and vertex functions to be dependent on the momenta of Fock-state constituents that are only spectators to the process in question. This spectator and Fock-state dependence results in great complications for the analysis and solution of the theory.

In particular, the Ward identity of gauge theories is destroyed by truncation. For photon emission in QED, a one-photon truncation keeps only the self-energy correction to the electron leg on the side opposite the photon emission; the self-energy correction on the other leg and the vertex loop correction are eliminated [9,5]. The relevant diagrams are shown in Fig. 1; only the first survives the truncation. Thus, the Ward identity connecting vertex and wave function renormalization is broken for interactions internal to a bound-state problem. This is what drives the renormalization of the charge in a sector-dependent parameterization of

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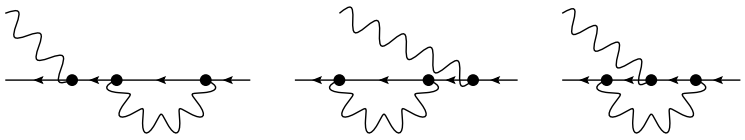

Fig. 1. Graphs contributing to the Ward identity in QED. Only the first contributes in a one-photon truncation of the Fock space.

the theory [5,6], but this is clearly unphysical and has nothing to do with ordinary charge renormalization.[1]

The analog of these difficulties with truncation can be induced in Feynman perturbation theory by separating covariant diagrams into time-ordered diagrams and discarding those time orderings that include intermediate states with more particles than some finite limit. This destroys covariance, disrupts regularization, and induces spectator dependence for subdiagrams. In the nonperturbative case, this happens not just to some finite order in the coupling but to all orders.

2. Light-front coupled-cluster method

To avoid truncation, we introduce the exponential-operator technique of the many-body coupled-cluster (CC) method [12,13]. An eigenstate $|\psi(\underline{P})\rangle$ is written as $\sqrt{Z}e^T|\phi(\underline{P})\rangle$, where $|\phi(\underline{P})\rangle$ is limited to one or a few Fock sectors with the lowest number(s) of constituents, the valence sector(s). The operator T is a sum of operators that increase particle number but conserve momentum $\underline{P}$, the angular momentum component[2] J_z, and all relevant quantum numbers, such as charge and baryon number. The factor $\sqrt{Z}$ is a normalization factor, such that $\langle\phi(\underline{P}')|\phi(\underline{P})\rangle = \langle\psi(\underline{P}')|\psi(\underline{P})\rangle = \delta(\underline{P}' - \underline{P})$. We then construct an effective eigenvalue problem in the valence sector, $P_v\overline{\mathcal{P}^-}|\phi(\underline{P})\rangle = \frac{M^2+P_\perp^2}{P^+}|\phi(\underline{P})\rangle$, where $\overline{\mathcal{P}^-} \equiv e^{-T}\mathcal{P}^-e^T$ and P_v is a projection onto the valence sector. Equations for the functions that determine T are found by the orthogonal projection $(1-P_v)\overline{\mathcal{P}^-}|\phi(\underline{P})\rangle = 0$. Up to this point, no approximation has been made, and the problem remains infinite, because there are infinitely many contributions to T.

To have a finite set of equations, we truncate T to a few operators and truncate the projection $1-P_v$ in a consistent way, such that just enough equations are produced to be able to solve for the functions in the truncated T operator. For example, if T can create one additional particle above the valence state, $1-P_v$ projects onto only this additional Fock sector. After truncation, we have a finite set of nonlinear equations for the functions in T, coupled to the valence-sector wave functions, and a valence-sector eigenvalue problem where the effective Hamiltonian depends on the functions in T. The former are essentially auxiliary equations that help define the latter.

What is not truncated is the exponentiation of T, and thus the full Fock space can be retained, though the wave functions for the higher Fock sectors are clearly only approximate. The effective Hamiltonian is computed from its Baker–Hausdorff expansion $\overline{\mathcal{P}^-} = \mathcal{P}^- + [\mathcal{P}^-, T] + \frac{1}{2}[[\mathcal{P}^-, T], T] + \cdots$. Only a finite number of terms contributes, because each factor of T increases the number of particles created, eventually exceeding the truncation of $(1-P_v)$.

Although this light-front coupled-cluster (LFCC) method uses the mathematics of the traditional CC method [12–14], it is quite different conceptually. In fact, the name coupled cluster does not really apply, but we use it to acknowledge the origin of the LFCC method. The CC method is applied to a single Fock sector, with a large number of constituents. The T operator builds correlated excitations onto a Hartree–Fock type ground state. Within products of T there are no contractions, because every term in T annihilates one or more of the single-particle states in the ground state and creates one or more excited states. In the LFCC method, the valence sector has a small number of constituents, and the method of solution of the eigenvalue problem here is left unspecified. The terms of the T operator do include annihilation, because the positive light-front momentum P^+ cannot be conserved unless one or more particles are annihilated to provide momentum for those that are created. As a consequence, powers of T include contractions, but these are needed in order that T to some power not annihilate the entire valence state, which would effectively truncate the exponentiation of T.

In addition to the fundamental mass eigenvalue problem, the LFCC method must also contend with the evaluation of matrix elements of operators, in order to be able to extract physical quantities from the LFCC eigenstates. This is nontrivial, because a direct calculation of the normalizing factor $\sqrt{Z}$ is impractical, due to the infinite set of terms in the sum over Fock states within $\langle\phi(\underline{P})|e^{T^\dagger}e^T|\phi(\underline{P})\rangle$. This same issue arises in the traditional CC method [13], and there a technique exists for expectation values which can be adapted for the LFCC method and extended to include off-diagonal matrix elements. Some care must be taken, however, in that the LFCC method uses momentum eigenstates with Dirac-delta normalization, unlike the unit normalization of the standard CC states. The normalization factor $\sqrt{Z}$ is introduced to avoid division by $\langle\psi(\underline{P}')|\psi(\underline{P})\rangle = \delta(\underline{P}' - \underline{P})$ in the computation of expectation values.

For an operator $\hat{O}$ we write the expectation value $\langle\hat{O}\rangle$ in the state $\sqrt{Z}e^T|\phi(\underline{P})\rangle$ as $\langle\hat{O}\rangle = Z\langle\phi(\underline{P})|e^{T^\dagger}\hat{O}e^T|\phi(\underline{P})\rangle$ and define $\overline{O} = e^{-T}\hat{O}e^T$ and

$$\langle\widetilde{\psi}(\underline{P})| = Z\langle\phi(\underline{P})|e^{T^\dagger}e^T = \sqrt{Z}\langle\psi(\underline{P})|e^T, \tag{1}$$

so that $\langle\hat{O}\rangle = \langle\widetilde{\psi}(\underline{P})|\overline{O}|\phi(\underline{P})\rangle$. By construction, we have

$$\langle\widetilde{\psi}(\underline{P}')|\phi(\underline{P})\rangle = \langle\psi(\underline{P}')|\psi(\underline{P})\rangle = \delta(\underline{P}' - \underline{P}) \tag{2}$$

and

$$\langle\widetilde{\psi}(\underline{P})|\overline{\mathcal{P}^-} = \sqrt{Z}\langle\psi(\underline{P})|e^Te^{-T}\mathcal{P}^-e^T = \frac{M^2+P_\perp^2}{P^+}\langle\widetilde{\psi}(\underline{P})|. \tag{3}$$

Thus, $\langle\widetilde{\psi}(\underline{P})|$ is a left eigenvector of the (necessarily) non-Hermitian $\overline{\mathcal{P}^-}$, with the same mass eigenvalue, normalized such that the projection onto the valence state is a simple momentum-conserving delta function. Therefore, an expectation value is computed by constructing the effective operator $\overline{O}$ from a Baker–Hausdorff expansion, solving the left-hand eigenvalue problem, and evaluating the inner product $\langle\widetilde{\psi}(\underline{P})|\overline{O}|\phi(\underline{P})\rangle$. As for $\overline{\mathcal{P}^-}$, only a finite number of terms in the Baker–Hausdorff expansion of $\overline{O}$ will contribute. The extension to off-diagonal matrix elements is straightforward.

The left-hand eigenvalue problem must be truncated to an extent consistent with the truncation of T, such that $\langle\widetilde{\psi}(\underline{P})|$ is

[1] For external photon emission, the truncation does not apply and the Ward identity is preserved. Without vacuum polarization, the plus component of the dressed-electron current is not renormalized [10,11].

[2] The other two components of angular momentum are not kinematic [2]. The eigenstates of the Hamiltonian will in general be linear combinations of eigenstates of J^2. Determination of the J^2 eigenstates is a separate dynamical problem.

Fig. 2. Graphical representation of the model Hamiltonian operator $\mathcal{P}^-$ defined in Eq. (4) of the text. Each graph represents an operator that annihilates one or more particles on the right and creates one or more to take their place. The crosses refer to light-front kinetic-energy terms.

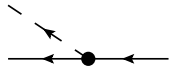

Fig. 3. Graphical representation of the truncated T operator.

limited to the Fock sectors of the valence state plus those created by application of T. To understand the truncation, consider the following. Define an operator $L = (1 - P_v)Ze^{T^\dagger}e^T P_\phi$, with P_ϕ the projection onto the valence eigenstate $|\phi(\underline{P})\rangle$. Because of the projection operators, e^L is simply $1 + L$. The left-hand eigenstate can then be written as $\langle\widetilde{\psi}(\underline{P})| = \langle\phi(\underline{P})|e^{L^\dagger} + Z\langle\phi(\underline{P})|e^{T^\dagger}e^T(P_v - P_\phi)^\dagger$. We see, then, that L plays the role of T, and therefore should be truncated in the same way. The truncated left-hand eigenvalue problem creates a finite set of linear equations for the wave functions of $\langle\widetilde{\psi}(\underline{P})|$.

3. Model application

To illustrate the method, we apply it to a simple model where an analytic solution is known [15]. The model is a light-front analog of the Greenberg–Schweber model [16] for a static fermionic source that emits and absorbs bosons without changing its spin. In updated notation, the Hamiltonian given in [15] can be written as

$$\begin{aligned}\mathcal{P}^- &= \int d\underline{p}\,\frac{M^2 + M_0' p^+}{P^+}\sum_s b_s^\dagger(\underline{p})b_s(\underline{p})\\ &+ \int d\underline{q}\sum_l (-1)^l \frac{\mu_l^2 + q_\perp^2}{q^+}a_l^\dagger(\underline{q})a_l(\underline{q})\\ &+ \frac{g}{P^+}\int \frac{d\underline{p}\,d\underline{q}}{\sqrt{16\pi^3 q^+}}\sum_{ls}\left(\frac{p^+}{p^+ + q^+}\right)^\gamma\\ &\times\left[a_l^\dagger(\underline{q})b_s^\dagger(\underline{q})b_s(\underline{p}+\underline{q}) + b_s^\dagger(\underline{p}+\underline{q})b_s(\underline{p})a_l(\underline{q})\right],\end{aligned} \tag{4}$$

where $a_0^\dagger$ creates a "physical" boson of mass μ_0, $a_1^\dagger$ creates a Pauli–Villars (PV) boson of mass μ_1, and $b_s^\dagger$ creates the fermion with mass M and spin s. The parameter γ can take any positive value; as shown in [15], it controls the longitudinal endpoint behavior of the wave functions. The PV boson provides the necessary ultraviolet regularization, to define the self-energy M_0'. To accomplish the regularization, the PV boson is assigned a negative norm.[3] The (anti)commutation relations are

$$\begin{aligned}&\left\{b_s(\underline{p}), b_{s'}^\dagger(\underline{p}')\right\} = \delta_{ss'}\delta(\underline{p} - \underline{p}'),\\ &\left[a_l(\underline{q}), a_{l'}^\dagger(\underline{q}')\right] = (-1)^l\delta_{ll'}\delta(\underline{q} - \underline{q}').\end{aligned} \tag{5}$$

A graphical representation of the Hamiltonian is given in Fig. 2. The model is not fully covariant, which hides some of the power of the LFCC method, but is sufficient to show how the method can be applied.

We truncate the T operator to include only boson emission from the fermion, as represented in Fig. 3,

$$T = \sum_{ls}\int d\underline{q}\,d\underline{p}\,t_{ls}(\underline{q},\underline{p})a_l^\dagger(\underline{q})b_s^\dagger(\underline{p})b_s(\underline{p}+\underline{q}), \tag{6}$$

with t_{ls} the operator functions to be determined. The effective Hamiltonian $\overline{\mathcal{P}^-}$ is constructed from its Baker–Hausdorff expansion. A graphical representation of the first two commutators is given in Fig. 4. The expression for $\overline{\mathcal{P}^-}$ is

$$\begin{aligned}\overline{\mathcal{P}^-} &= \int d\underline{q}\sum_l (-1)^l\frac{\mu_l^2 + q_\perp^2}{q^+}a_l^\dagger(\underline{q})a_l(\underline{q})\\ &+ \int d\underline{p}\sum_s b_s^\dagger(\underline{p})b_s(\underline{p})\left[\frac{M^2 + M_0' p^+}{P^+}\right.\\ &+ \frac{g}{P^+}\sum_l (-1)^l\int\frac{d\underline{q}}{\sqrt{16\pi^3 q^+}}\left(\frac{p^+ - q^+}{p^+}\right)^\gamma\\ &\left.\times\,\theta(p^+ - q^+)t_{ls}(\underline{q},\underline{p}-\underline{q})\right]\\ &+ \frac{g}{P^+}\int\frac{d\underline{p}\,d\underline{q}}{\sqrt{16\pi^3 q^+}}\sum_{ls}\left(\frac{p^+}{p^+ + q^+}\right)^\gamma b_s^\dagger(\underline{p}+\underline{q})b_s(\underline{p})a_l(\underline{q})\\ &+ \int d\underline{p}\,d\underline{q}\sum_{ls}a_l^\dagger(\underline{q})b_s^\dagger(\underline{p})b_s(\underline{p}+\underline{q})\\ &\times\left\{\frac{g}{P^+}\frac{1}{\sqrt{16\pi^3 q^+}}\left(\frac{p^+}{p^+ + q^+}\right)^\gamma\right.\\ &+ \left(\frac{\mu_l^2 + q_\perp^2}{q^+} - \frac{M_0' q^+}{P^+}\right)t_{ls}(\underline{q},\underline{p})\\ &+ \frac{g}{2P^+}\int\frac{d\underline{q}'}{\sqrt{16\pi^3 q'^+}}\\ &\times\sum_{l'}(-1)^{l'}\left[\theta(p^+ - q'^+)\left(\frac{p^+ - q'^+}{p^+}\right)^\gamma\right.\\ &\times\left\{t_{ls}(\underline{q},\underline{p})t_{l's}(\underline{q}',\underline{p}-\underline{q}') + \theta(p^+ + q^+ - q'^+)\right.\\ &\times t_{ls}(\underline{q},\underline{p}-\underline{q}')t_{l's}(\underline{q}',\underline{p}+\underline{q}-\underline{q}')\Big\}\\ &- 2\theta(p^+ + q^+ - q'^+)\left(\frac{p^+ + q^+ - q'^+}{p^+ + q^+}\right)^\gamma\\ &\left.\left.\times\,t_{ls}(\underline{q},\underline{p})t_{l's}(\underline{q}',\underline{p}+\underline{q}-\underline{q}')\right]\right\}\\ &+ \frac{g}{P^+}\int\frac{d\underline{p}\,d\underline{q}\,d\underline{q}'}{\sqrt{16\pi^3 q^+}}\theta(p^+ + q^+ - q'^+)\\ &\times\sum_{ll's}a_{l'}^\dagger(\underline{q}')b_s^\dagger(\underline{p}+\underline{q}-\underline{q}')b_s(\underline{p})a_l(\underline{q})\\ &\times\left[\theta(p^+ - q'^+)\left(\frac{p^+ - q'^+}{p^+ + q^+ - q'^+}\right)^\gamma t_{l's}(\underline{q}',\underline{p}-\underline{q}')\right.\\ &\left.-\left(\frac{p^+}{p^+ + q^+}\right)^\gamma t_{l's}(\underline{q}',\underline{p}+\underline{q}-\underline{q}')\right],\end{aligned} \tag{7}$$

where we list only terms that connect the lowest Fock sectors. Notice that the self-energy contribution M_0' is the same in all Fock

[3] In [15], the PV cancellations were arranged by use of an imaginary coupling rather than a negative norm.

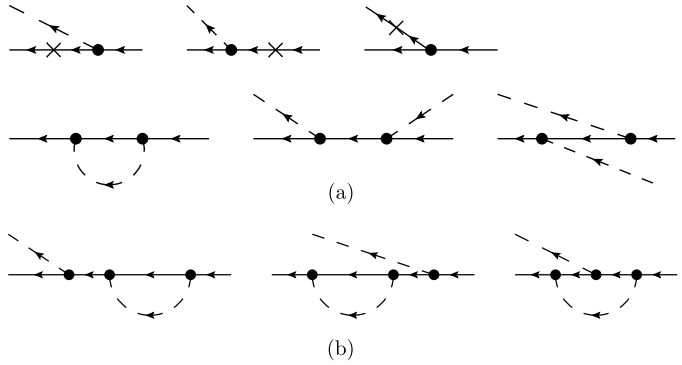

Fig. 4. Graphical representation of the operators (a) $[\mathcal{P}^-,T]$ and (b) $[[\mathcal{P}^-,T],T]$. The crosses indicate light-front kinetic-energy contributions. The self-energy loops in the fourth diagram of (a) and the first and second diagrams of (b) make identical contributions, with no Fock-sector or spectator dependence.

sectors and that Fig. 4(b) contains all three of the diagrams analogous to those for the Ward identity in QED, as discussed in the Introduction, with no truncation in particle number.

The valence state is the bare-fermion state $|\phi^\sigma(\underline{P})\rangle = b^\dagger_\sigma(\underline{P})|0\rangle$. The projection $1-P_v$ is truncated to the one-fermion/one-boson sector. The full eigenstate is $|\psi^\sigma(\underline{P})\rangle = \sqrt{Z}e^T|\phi^\sigma(\underline{P})\rangle$, where we have generalized the basic construction to have one T operator for both spins $\sigma=\pm$ and will solve for both states simultaneously. This allows contractions in powers of T to include sums over both spins. The truncated left-hand eigenvector is

$$\langle\widetilde{\psi}^\sigma(\underline{P})| = \langle\phi^\sigma(\underline{P})| + \sum_{ls}\int d\underline{q}\,\theta\big(P^+-q^+\big)l^{\sigma *}_{ls}(\underline{q},\underline{P})\langle 0|a_l(\underline{q})b_s(\underline{P}-\underline{q}), \tag{8}$$

where l^σ_{ls} is the left-hand one-fermion/one-boson wave function. Due to the lack of covariance in the model, these states are all limited to having a fixed total transverse momentum $\vec{P}_\perp$, which we take to be zero.

The eigenvalue problem in the valence sector $P_v\overline{\mathcal{P}^-}|\phi^\sigma(\underline{P})\rangle = \frac{M^2}{P^+}|\phi^\sigma(\underline{P})\rangle$ becomes

$$\begin{aligned}&\frac{M^2+M'_0P^+}{P^+}\big|\phi^\pm(\underline{P})\big\rangle \\ &\quad+\frac{g}{P^+}\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\theta\big(P^+-q^+\big)\bigg(\frac{P^+-q^+}{P^+}\bigg)^\gamma \\ &\quad\times\sum_l(-1)^l t_{l\pm}(\underline{q},\underline{P}-\underline{q})\big|\phi^\pm(\underline{P})\big\rangle \\ &=\frac{M^2}{P^+}\big|\phi^\pm(\underline{P})\big\rangle,\end{aligned} \tag{9}$$

which reduces to a determination of the self-energy

$$\begin{aligned}M'_0 = &-\frac{g}{P^+}\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\theta\big(P^+-q^+\big)\bigg(\frac{P^+-q^+}{P^+}\bigg)^\gamma \\ &\times\sum_l(-1)^l t_{l\pm}(\underline{q},\underline{P}-\underline{q}).\end{aligned} \tag{10}$$

In the one-fermion/one-boson sector, we have

$$(1-P_v)\overline{\mathcal{P}^-}\big|\phi^\pm(\underline{P})\big\rangle = 0$$

or

$$\begin{aligned}&\int d\underline{q}\,\theta\big(P^+-q^+\big)\sum_l a^\dagger_l(\underline{q})b^\dagger_\pm(\underline{P}-\underline{q})|0\rangle \\ &\times\bigg\{\frac{g}{P^+}\frac{1}{\sqrt{16\pi^3q^+}}\bigg(\frac{P^+-q^+}{P^+}\bigg)^\gamma \\ &+\bigg(\frac{\mu_l^2+q_\perp^2}{q^+}-\frac{M'_0q^+}{P^+}\bigg)t_{ls}(\underline{q},\underline{P}-\underline{q}) \\ &+\frac{g}{2P^+}\int\frac{d\underline{q}'}{\sqrt{16\pi^3q'^+}} \\ &\times\sum_{l'}(-1)^{l'}\bigg[\theta\big(P^+-q^+-q'^+\big)\bigg(\frac{P^+-q^+-q'^+}{P^+-q^+}\bigg)^\gamma \\ &\times\big\{t_{l\pm}(\underline{q},\underline{P}-\underline{q})t_{l'\pm}(\underline{q}',\underline{P}-\underline{q}-\underline{q}') \\ &+\theta\big(P^+-q'^+\big)t_{l\pm}(\underline{q},\underline{P}-\underline{q}-\underline{q}')t_{l'\pm}(\underline{q}',\underline{P}-\underline{q}')\big\} \\ &-2\theta\big(P^+-q'^+\big)\bigg(\frac{P^+-q'^+}{P^+}\bigg)^\gamma \\ &\times t_{l\pm}(\underline{q},\underline{P}-\underline{q})t_{l'\pm}(\underline{q}',\underline{P}-\underline{q}')\bigg]\bigg\}=0.\end{aligned} \tag{11}$$

Thus, the contents of the outer curly brackets must sum to zero. This will occur if the function t_{ls} is

$$t_{ls}(\underline{q},\underline{p}) = \frac{-g}{\sqrt{16\pi^3q^+}}\bigg(\frac{p^+}{p^++q^+}\bigg)^\gamma\frac{q^+/P^+}{\mu_l^2+q_\perp^2}. \tag{12}$$

The fact that the self-energy M'_0 is the same in the valence sector and the one-fermion/one-boson sector plays a critical role; the expression (10) obtained in the valence sector is exactly what is needed to obtain the necessary cancellations in (11). The self-energy can be computed from Eq. (10) as

$$M'_0 = \frac{g^2}{16\pi^3P^+}\frac{\ln(\mu_1/\mu_0)}{\gamma+1/2}, \tag{13}$$

which agrees with the result in [15]. In fact, with t_{ls} as given above, the exponential operator e^T generates the exact solution given in [15].

The solution for t_{ls} provides the input to the left-hand eigenvalue problem, $\langle\widetilde{\psi}^\pm(\underline{P})|\overline{\mathcal{P}^-} = \frac{M^2}{P^+}\langle\widetilde{\psi}^\pm(\underline{P})|$. The effective Hamiltonian $\overline{\mathcal{P}^-}$ simplifies considerably; the square bracket in the $b^\dagger b$ term becomes just M^2/P^+ and the entire $a^\dagger b^\dagger b$ term, which corresponds to the curly brackets in (11), is zero. The remaining terms in $\overline{\mathcal{P}^-}$ yield the following integral equation for the left-hand wave function:

$$\frac{g}{P^+}\frac{1}{\sqrt{16\pi^3q^+}}\bigg(\frac{P^+-q^+}{P^+}\bigg)^\gamma\delta_{s\pm}+\frac{\mu_l^2+q_\perp^2}{q^+}l^\pm_{ls}(\underline{q},\underline{P})$$

$$-\left(\frac{g}{P^+}\right)^2\int\frac{d\underline{q}'}{\sqrt{16\pi^3 q^+}}\theta\big(P^+-q'^+\big)\sqrt{\frac{q'^+}{16\pi^3}}$$
$$\times\sum_{l'}(-1)^{l'}\frac{1}{\mu_{l'}^2+q_\perp'^2}l^{\pm}_{l's}\big(\underline{q}',\underline{P}\big)$$
$$\times\left[\theta\big(P^+-q^+-q'^+\big)\frac{(P^+-q^+-q'^+)^{2\gamma}}{(P^+-q'^+)^{\gamma}(P^+-q^+)^{\gamma}}\right.$$
$$\left.-\left(\frac{P^+-q^+}{P^+}\right)^{\gamma}\left(\frac{P^+-q'^+}{P^+}\right)^{\gamma}\right]=0. \tag{14}$$

Following the pattern of the inhomogeneous term, we can seek a solution of the form

$$l^{\sigma}_{ls}(\underline{q},\underline{P})=\delta_{\sigma s}\frac{-g}{\sqrt{16\pi^3 q^+}}\left(\frac{P^+-q^+}{P^+}\right)^{\gamma}\frac{q^+/P^+}{\mu_l^2+q_\perp^2}\tilde{l}\big(q^+/P^+\big). \tag{15}$$

Substitution yields a one-dimensional integral equation for $\tilde{l}(y)$,

$$\tilde{l}(y)=1+\frac{g^2}{16\pi^2}\frac{\mu_1^2-\mu_0^2}{\mu_0^2\mu_1^2}$$
$$\times\int_0^1 dy'\big(1-y'\big)^{2\gamma}y'\big[(1-y)^2\tilde{l}\big(y'(1-y)\big)-\tilde{l}\big(y'\big)\big]. \tag{16}$$

This equation can be solved iteratively, to generate an expansion in powers of g^2, or numerically. A Gauss–Jacobi quadrature will convert the integral equation into a linear system for the values of $\tilde{l}$ at the chosen quadrature points. The solution then provides the rest of the information needed for the computation of matrix elements.

To consider a particular matrix element as an example, we compute the Dirac form factor for the dressed fermion from a matrix element of the current $J^+=\overline{\psi}\gamma^+\psi$. The current couples to a photon of momentum q. With our normalization, the matrix element is generally [17]

$$\big\langle\psi^{\sigma}(\underline{P}+\underline{q})\big|16\pi^3 J^+(0)\big|\psi^{\pm}(\underline{P})\big\rangle$$
$$=2\delta_{\sigma\pm}F_1\big(q^2\big)\pm\frac{q^1\pm iq^2}{M}\delta_{\sigma\mp}F_2\big(q^2\big), \tag{17}$$

with F_1 and F_2 the Dirac and Pauli form factors. In the present model, the fermion cannot flip its spin; therefore, F_2 is zero, and we investigate only F_1.

In the LFCC method, the form factor is given by the matrix element

$$F_1\big(q^2\big)=8\pi^3\big\langle\widetilde{\psi}^{\pm}(\underline{P}+\underline{q})\big|\overline{J^+(0)}\big|\phi^{\pm}(\underline{P})\big\rangle, \tag{18}$$

with $\overline{J^+(0)}=J^+(0)+[J^+(0),T]+\cdots$. For this model, there are no contributions from fermion-antifermion pairs, so that

$$J^+(0)=2\sum_s\int\frac{d\underline{p}'}{\sqrt{16\pi^3}}\int\frac{d\underline{p}}{\sqrt{16\pi^3}}b_s^{\dagger}(\underline{p}')b_s(\underline{p}), \tag{19}$$

and only the first two terms of the Baker–Hausdorff expansion contribute to the matrix element. The second term is

$$\big[J^+(0),T\big]=2\sum_{ls}\int\frac{d\underline{p}'}{\sqrt{16\pi^3}}\int\frac{d\underline{p}}{\sqrt{16\pi^3}}$$
$$\times\int d\underline{q}'\big[t_{ls}(\underline{q}',\underline{p})-t_{ls}(\underline{q}',\underline{p}')\big]a_l^{\dagger}(\underline{q}')b_s^{\dagger}(\underline{p}')b_s(\underline{p}). \tag{20}$$

The first term contributes $1/8\pi^3$ to the matrix element; the second contributes

$$\big\langle\widetilde{\psi}^{\pm}(\underline{P}+\underline{q})\big|\big[J^+(0),T\big]\big|\phi^{\pm}(\underline{P})\big\rangle$$
$$=\frac{1}{8\pi^3}\sum_l(-1)^l\int d\underline{q}'\theta\big(P^++q^+-q'^+\big)l^{\pm}_{l\pm}\big(\underline{q}',\underline{P}+\underline{q}\big)$$
$$\times\big[\theta\big(P^+-q'^+\big)t_{l\pm}\big(\underline{q}',\underline{P}-\underline{q}'\big)-t_{l\pm}\big(\underline{q}',\underline{P}+\underline{q}-\underline{q}'\big)\big]. \tag{21}$$

Because the model limits calculations to a fixed total transverse momentum, we calculate the matrix element in a frame where $\vec{q}_\perp$ is zero and q^+ is not.[4] With $\alpha\equiv q^+/P^+$ and $P'=P+q$, we have

$$q^2=\big(P'-P\big)^2=2M^2-P'^+P^--P'^-P^+$$
$$=2M^2-M^2(1+\alpha)-\frac{M^2}{1+\alpha}=-\frac{M^2\alpha^2}{1+\alpha}. \tag{22}$$

On substitution of the solutions for the wave functions and evaluation of the transverse integral, the form factor can be written as a function of α

$$F_1\big(q^2\big)=1+\frac{g^2}{16\pi^2}(1+\alpha)\frac{\mu_1^2-\mu_0^2}{\mu_0^2\mu_1^2}$$
$$\times\left[\int_0^{1/(1+\alpha)}dy\,\tilde{l}(y)y(1-y)^{\gamma}\big[1-(1+\alpha)y\big]^{\gamma}\right.$$
$$\left.-\int_0^1 dy\,\tilde{l}(y)y(1-y)^{2\gamma}\right]. \tag{23}$$

The PV dependence is easily removed in the limit of an infinite PV mass ($\mu_1\to\infty$). If $\tilde{l}$ is computed in quadrature, the integrals remaining in F_1 can be computed from the same quadrature rule for any chosen value of α. If $\tilde{l}$ is instead constructed as an expansion in g^2, F_1 can also be constructed as an expansion. In any case, in the limit of $q^2\to 0$, we have from (22) that $\alpha=0$ and, because the two integrals in (23) then cancel, $F_1(0)=1$, consistent with the unit charge in the current $J^+=\overline{\psi}\gamma^+\psi$.

4. Summary

We have proposed a new Hamiltonian method for the nonperturbative solution of quantum field theories that avoids Fock-space truncations. The full eigenstate is constructed from the action of an exponentiated operator T on a valence state $|\phi\rangle$. This yields a valence eigenvalue problem $\overline{\mathcal{P}^-}|\phi\rangle=\frac{M^2+P_\perp^2}{P^+}|\phi\rangle$, with $\overline{\mathcal{P}^-}=e^{-T}\mathcal{P}^-e^T$, and a set of auxiliary equations for the functions in T. Expectation values are computed as $\langle\hat{O}\rangle=\langle\widetilde{\psi}|e^{-T}\hat{O}e^T|\phi\rangle$, with use of the left-hand eigenstate $\langle\widetilde{\psi}|$. The method then generates approximations by truncation of T rather than of Fock space.

The application to the simple model in Section 3 shows that the construction of $\overline{\mathcal{P}^-}$ generates self-energy contributions that are Fock-sector and spectator independent. Thus, the uncanceled divergences that can arise from Fock-space truncations do not occur; the self-energy is the same in every sector. The application also shows that a simple approximation for the T operator can provide a very good approximation to the eigenstate; in this special

[4] In [15] the matrix element was computed in a frame where $q^+=0$. This could be done because the wave functions of the exact solution were taken to be boost invariant. Here, although the right-hand eigenvalue problem has accidentally provided the exact solution, we continue with the LFCC approximation in the calculation of the matrix element, to provide a more complete illustration of the method.

case, the eigenstate is exact. The calculation of a matrix element is demonstrated in the calculation of a Dirac form factor.

The LFCC method is not limited to any particular theory or model, nor to Pauli–Villars regularization. It should be applicable to any regularized field theory. Work on an application to QED is in progress, with some preliminary discussion given in [18]. For theories with symmetry breaking and vacuum structure, modes of zero longitudinal momentum [19] play some role and would require extension of the method to include them; in particular, the contributions to the T operator would not require annihilation, and the exponentiation would produce generalized coherent states. For discrete light-cone quantization (DLCQ) [20,2], where longitudinal momentum fractions are restricted to integer multiples of a fundamental amount $1/K$, truncation to K particles is automatic; however, the method could still be applied as a way of reducing the effective dimension of the underlying matrix eigenvalue problem and allowing higher resolution. Even the supersymmetric form of DLCQ (SDLCQ) [21] should be amenable; instead of constructing $\mathcal{P}^-$ from a discretized supercharge Q^- via $\mathcal{P}^- = \{Q^-, Q^-\}/2\sqrt{2}$, to retain the supersymmetric spectrum, the effective Hamiltonian $\overline{\mathcal{P}^-}$ would be constructed from effective supercharges $\overline{Q^-} \equiv e^{-T} Q^- e^T$. Thus, there is a wide range of applications to consider.

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