Comput. Math. Applic. Vol. 13, No. 8, pp. 677–680, 1987
Printed in Great Britain

0097-4943/87 $3.00 + 0.00
Pergamon Journals Ltd

STABILITY OF SOME CONTINUOUSLY IMBEDDED RUNGE–KUTTA METHODS OF SARAFYAN

S. THOMPSON
Engineering Physics and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831, U.S.A.

(*Received* 22 *October* 1986)

Communicated by E. Y. Rodin

Abstract—Sarafyan and others have recently developed novel explicit Runge–Kutta methods. Associated with each method is an imbedded polynomial which interpolates the Runge–Kutta method and which is itself a Runge–Kutta approximation at non-meshpoints. In this paper, we show that the interpolation polynomials possess the desirable stability properties of the Runge–Kutta methods.

1. INTRODUCTION

Consider the initial-value problem

$$\dot{y}(t) = f(t, y(t)),$$
$$y(t_0) = y_0. \tag{1}$$

Sarafyan and co-workers have developed continuously imbedded explicit Runge–Kutta methods for solving problem (1). Each such method is an $(m + 1)$-stage Runge–Kutta method of the form

$$y_{n+1} = y_n + \sum_{i=0}^{m} \alpha_i k_i, \tag{2}$$

where

$$k_0 = hf(t_n, y_n) \tag{3}$$

and

$$k_j = hf\left(t_n + a_j h, y_n + \sum_{i=0}^{j-1} b_{ji} k_i\right), \quad j = 1, \ldots, m, \tag{4}$$

where h is the integration stepsize.

Denote by r the order of method (2)–(4). Associated with method (2)–(4) is an imbedded polynomial

$$p_{n+1}(c) = y_{n+1}(t_n + ch) = y_n + \sum_{i=1}^{r-1} \gamma_i c^i, \tag{5}$$

for which $p_{n+1}(1)$ yields y_{n+1} in expression (2), and for which $p_{n+1}(c)$ is a Runge–Kutta approximation of order $r - 1$. Expression (5) is constructed in a manner which gives γ_i as a linear function of the weights $k_0, \ldots, k_m$,

$$\langle \gamma_1, \ldots, \gamma_{r-1} \rangle^{\mathrm{T}} = \mathbf{A} \langle k_0, \ldots, k_m \rangle^{\mathrm{T}}, \tag{6}$$

where $\mathbf{A}$ is an $(r - 1) \times (m + 1)$ matrix.

Such interpolants are particularly useful for handling the problem of dense output and for rootfinding purposes [1]. (The methods given below along with other, as yet unpublished, methods of Sarafyan have been implemented in software and evaluated. Results [2] will be reported elsewhere.) However, it is of interest to investigate the stability properties of the imbedded polynomial to ensure that the polynomial behaves in a manner that is consistent with the behavior of the Runge–Kutta method.

In this paper, we will consider the following two continuously imbedded methods. Method 1 is a six-stage, fifth-order method. Method 2 is a nine-stage, sixth-order method.

Method 1 [3]

i	a_i	b_{ij}				
1	1/6	1/6				
2	1/4	1/16	3/16			
3	1/2	1/4	−3/4	4/4		
4	3/4	3/16	0	0	9/16	
5	1	−4/7	3/7	12/7	−12/7	8/7
α_i	7/90	0	32/90	12/90	32/90	7/90

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -89/30 & 0 & 96/30 & 36/30 & -64/30 & 21/30 \\ 142/45 & 0 & -208/45 & -108/45 & 272/45 & -98/45 \\ -10/9 & 0 & 16/9 & 12/9 & -32/9 & 14/9 \end{bmatrix}$$

Method 2 [4]

i	a_i	b_{ij}							
1	$\frac{1}{32}$	$\frac{1}{32}$							
2	$\frac{1}{24}$	$\frac{1}{72}$	$\frac{1}{36}$						
3	$\frac{1}{16}$	$\frac{1}{64}$	0	$\frac{3}{64}$					
4	$\frac{1}{5}$	$\frac{53}{125}$	0	$\frac{-204}{125}$	$\frac{176}{125}$				
5	$\frac{1}{4}$	$\frac{1}{96}$	0	0	$\frac{4}{33}$	$\frac{125}{1056}$			
6	$\frac{1}{2}$	$\frac{-19}{24}$	0	0	$\frac{64}{33}$	$\frac{-875}{264}$	$\frac{8}{3}$		
7	$\frac{3}{4}$	$\frac{-11}{16}$	0	0	$\frac{268}{231}$	$\frac{125}{132}$	$\frac{-17}{12}$	$\frac{251}{336}$	
8	1	$\frac{1211639}{222222}$	0	0	$\frac{-14848}{1617}$	$\frac{125}{154}$	$\frac{16}{3}$	$\frac{-376}{147}$	$\frac{8}{7}$
α_i	$\frac{7}{90}$	0	0	0	0	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -25/6 & 0 & 0 & 0 & 0 & 48/6 & -36/6 & 16/6 & -3/6 \\ 70/9 & 0 & 0 & 0 & 0 & -208/9 & 228/9 & -112/9 & 22/9 \\ -20/3 & 0 & 0 & 0 & 0 & 72/3 & -96/3 & 56/3 & -12/3 \\ 32/15 & 0 & 0 & 0 & 0 & -128/15 & 192/15 & -128/15 & 32/15 \end{bmatrix}.$$

2. ABSOLUTE STABILITY OF THE IMBEDDED POLYNOMIALS

The stability of method (2)–(4) may be studied in the usual manner by applying it to the model equation

$$\dot{y} = \lambda y, \tag{7}$$

where λ is a complex constant. The solution satisfies

$$y_{n+1} = \pi_{m+1}(\alpha)\, y_n, \tag{8}$$

where $\alpha = h\lambda$ and the characteristic polynomial π_{m+1} is given by

$$\pi_{m+1}(\alpha) = \sum_{i=0}^{m+1} d_i \alpha^i \tag{9}$$

and d_i $(i = 0, \ldots, m+1)$ is determined by applying method (2)–(4) to equation (7).

It is straightforward to show that

$$\begin{aligned} d_0 &= 1 \\ d_1 &= \sum_{i=0}^{m} \alpha_i \\ d_k &= \sum_{i=k-1}^{m} \alpha_i c_{i,k-2} \quad (k = 2, \ldots, m+1), \end{aligned} \tag{10}$$

where

$$c_{k,0} = \sum_{i=0}^{k-1} b_{ki} \quad (k = 1, \ldots, m)$$

and

$$c_{kj} = \sum_{i=j}^{k-1} b_{ki} c_{i,j-1} \quad (k = 2, \ldots, m; \;\; j = 1, \ldots, k-1). \tag{11}$$

The region of absolute stability for method (2)–(4) is then the set of points α in the complex plane for which $|\pi_{m+1}(\alpha)| < 1$. Interestingly, Methods 1 and 2 have relatively large regions of absolute stability. In particular, the length of the real stability interval for Method 1 is approx. 3.38. (The length of the corresponding interval for the well-known fifth-order Runge–Kutta–Fehlberg method [5] is 3.67.) The length of the real stability interval for Method 2 is 5.58.

A reasonable question to ask is whether the polynomial approximation (5) has stability properties similar to those of method (2)–(4), that is, whether the polynomial will also decay for $0 < c < 1$, or whether oscillations in the polynomial will propagate in an unstable fashion. Since the polynomial is itself a Runge–Kutta approximation, its stability may be studied in a manner similar to that for method (2)–(4). When applied to the model problem (7), approximation (5) also has a solution of the form (8),

$$y_{n+1}(t_n + ch) = \pi_{m+1}(c, \alpha)\, y_n \tag{12}$$

for each c where the characteristic polynomial $\pi_{m+1}(c, \alpha)$ now depends on c. For a given value of c, the characteristic polynomial satisfies equations (10) and (11) with α_i in expressions (2) and (10) replaced by

$$\sum_{j=1}^{r-1} a_{j,i+1} c^j.$$

Thus, for each value of c, there is an associated region of absolute stability determined by the polynomial (5).

What is of interest is to determine the size of these stability regions for $0 < c < 1$ relative to the size of the region for $c = 1$. This may be done in the usual manner by numerically calculating the regions. This was done for Methods 1 and 2 by calculating the regions (using increments of $\theta = 1°$ to determine the boundaries of the region) for several values of c and noting the maximum reduction in the length of the ray corresponding to θ contained within the stability region. The results are quite interesting. For Method 1, the value of c thus determined is approx. $c = 0.8$ for all values of θ. ($c = 0.75$ gives slightly smaller ray lengths for θ near 90°.) The maximum reduction in any length is approx. 16%. (For most values of c, the corresponding regions are larger than the region for $c = 1.0$. They are roughly twice as large for $c = \frac{1}{2}$.) The stability for values of c between 0 and 1 is thus, not significantly worse than for $c = 1.0$ for this method.

The results for Method 2 are even more interesting. For this method, $c = 1.0$ yields the smallest region in most cases. For θ near 90°, $c = 0.2$ yields slightly smaller ray lengths. The maximum reduction in length is approx. 5%. The polynomial interpolant thus behaves in the same manner as does the solution for Method (2)–(4) for cases of practical interest. We mention that Sarafyan has developed other as yet unpublished continuously imbedded methods (specifically, a seven-stage, fifth-order method and an eight-stage, sixth-order method) and has kindly permitted the author to study the stability of these methods. For the fifth-order method, $c = 0.9$–1.0 gives the smallest regions, with a maximum reduction of approx. 24%. For the sixth-order method, $c = 1.0$ gives the smallest ray length up to about 68°; thereafter, $c = 0.75$–0.80 gives the smallest length, with a maximum reduction of approx. 17° near $\theta = 90°$.

For each of the methods considered, the stability regions for $0 < c < 1$ are at worst only slightly smaller than the region for $c = 1.0$. Thus, only a small reduction in the stepsize is required to ensure the stability of the interpolated solution. Refs [3, 4] give other imbedded methods that may be used with Methods 1 and 2 for error estimation purposes. Since these methods also have associated imbedded polynomials, it is, in fact possible to use values of $c \neq 1.0$ to estimate the error. For example, the code described in Ref. [1] optionally uses the maximum difference between the imbedded polynomials to control the local error. The above stability results illustrate why this more stringent criterion is not unnecessarily restrictive.

3. SUMMARY

In this paper, two continuously imbedded Runge–Kutta methods were considered. By studying the absolute stability of the associated imbedded polynomials, it was seen that the polynomials possess the desirable stability characteristics of the basic Runge–Kutta methods. This enhances their attractiveness for problems requiring interpolation for output and other tasks such as rootfinding.

REFERENCES

1. S. Thompson, Rootfinding and interpolation with Runge–Kutta–Sarafyan methods. *Trans. Soc. Comput. Simuln* **2**(3), 207–218 (1985).
2. S. Thompson, Implementation and evaluation of Sarafyan's continuously imbedded Runge–Kutta methods. Report No. ORNL/TM-10257, Oak Ridge National Lab., Oak Ridge, Tenn. (in press).
3. D. Sarafyan, Continuous approximate solution of ordinary differential equations and their systems. *Comput. Math. Applic.* **10**(2), 139–159 (1984).
4. C. Outlaw, L. Derr and D. Sarafyan, A sixth order imbedded Runge–Kutta algorithm with continuously variable weights. *Comput. Math. Applic.* **12A**(6), 815–824 (1986).
5. L. F. Shampine, Stability regions for extrapolated Runge–Kutta and Adams methods. Report No. SC-RR-72-0223, Sandia Labs, Albuquerque, N.M. (1972).